

ONE-SIDED STAR PARTIAL ORDERS FOR BOUNDED LINEAR OPERATORS

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Abstract. We compare some recent approaches to transferring the notions of left- and right-star partial order, introduced for complex matrices in early 90-ies, to bounded linear Hilbert space operators, and discuss a new version of these orders. The main results state that every initial segment of $\mathcal{B}(H)$ under the (new) left-star order is a complete orthomodular sublattice isomorphic to an initial segment of the lattice of closed subspaces of the underlying Hilbert space H . We also associate a certain orthogonality relation with the order.

The so called logical order on the set of all self-adjoint operators, introduced by S. Gudder in 2006, turns out to be the restriction of any of both one-sided star orders. Various known results concerning the logical order, in particular, characterizations of the join and meet operations, are extended to the left-star order on $\mathcal{B}(H)$.

1. Introduction

In [11], the so called *logical order* \preceq was introduced on the set $\mathcal{S}(H)$ of all self-adjoint operators on a complex Hilbert space H . It was further studied in [14, 4]; see also [7] and references in [4, 7]. According to Lemma 4.3 of [11],

$$A \preceq B \text{ iff } AB = A^2 \text{ iff } A = BP, \tag{1}$$

where P is the projection operator onto the closed range of A ; also some other characteristics of the logical order were given in [11]. It is noticed by various authors that the logical order is the restriction of the *star order*, which is defined on $\mathcal{B}(H)$, the set of all bounded linear operators over H , by

$$A \leq B \text{ iff } A^*A = A^*B \text{ and } AA^* = BA^*;$$

see [1, 8, 10]. In fact, the logical order is a restriction also of the so called left- and right-star orders. We shall return to this point later, and discuss now various definitions of these orders.

One-sided star orders for $m \times n$ complex matrices were introduced in [2] (see also [13]) and have been intensively studied. The corresponding definitions for the left-, resp., right-star order are

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$$A * \leq B \text{ iff } A^*A = A^*B \text{ and } \mathcal{M}(A) \subseteq \mathcal{M}(B) \tag{2}$$

$$A \leq * B \text{ iff } AA^* = BA^* \text{ and } \mathcal{M}(A^*) \subseteq \mathcal{M}(B^*), \tag{3}$$

where $\mathcal{M}(X)$ stands for the column span (known also as the column space, or range) of a matrix X . Both orders have also been transferred to bounded linear Hilbert space operators. For example, the definitions assumed in [8] are direct analogues of those for the matrix case (to unify notations, we borrow here, and in the sequel, those introduced in [11, Sect. 4] and used also in [4, 14]: let $\text{ran}A$, $\overline{\text{ran}}A$ and $\text{null}A$ stand for the range, the closed range and the nullspace of an operator A , respectively):

$$A * \leq B \text{ iff } A^*A = A^*B \text{ and } \text{ran}A \subseteq \text{ran}B, \tag{4}$$

$$A \leq * B \text{ iff } AA^* = BA^* \text{ and } \text{ran}A^* \subseteq \text{ran}B^*. \tag{5}$$

These orders are not independent: evidently, $A \leq * B$ if and only if $A^* * \leq B^*$. As observed in Theorem 2.2 of [8], $A * \leq B$ if and only if there are invertible operators E, F such that $EAF \leq * EBF$. It is known well that range inclusion of operators in a Hilbert space can be characterized algebraically:

$$\text{ran}A \subseteq \text{ran}B \text{ iff there is an operator } C \text{ such that } A = BC$$

(see, e.g. [8, Lemma 2.1]). Therefore, the definitions (4) and (5) can be given a form

$$A * \leq B \text{ iff } A^*A = A^*B \text{ and } A = BC \text{ for some } C,$$

$$A \leq * B \text{ iff } AA^* = BA^* \text{ and } A = CB \text{ for some } C$$

suitable for immediate transferring to rings with involution—see [6].

In [9], the left-star order for operators is defined as follows:

$$A * \leq B \text{ iff } \text{ran}P = \overline{\text{ran}}A, \text{null}A = \text{null}Q, PA = PB, AQ = BQ \tag{6}$$

for some appropriate projection operator P and idempotent operator Q ;

it is then proved that the defined relation is a partial order indeed and that this definition is equivalent to (4). The right-star order $\leq *$ is introduced there similarly, by the same condition (6) with P idempotent and Q a projection. See also [12].

We note that still another extension of (2) and (3) to $\mathcal{B}(H)$ is possible, and introduce two other relations, $\preceq *$ and $* \preceq$ (see [6]):

$$A * \preceq B \text{ iff } A^*A = A^*B \text{ and } \overline{\text{ran}}A \subseteq \overline{\text{ran}}B, \tag{7}$$

$$A \preceq * B \text{ iff } AA^* = BA^* \text{ and } \overline{\text{ran}}A^* \subseteq \overline{\text{ran}}B^*. \tag{8}$$

It will be demonstrated later that they both are order relations. Generally, they are stronger than $* \leq$ and $\leq *$; however, the difference disappears if the underlying Hilbert space is finite dimensional. In the infinite-dimensional case, an operator in $\mathcal{B}(H)$ has a closed range if and only if it is regular (has the Moore-Penrose inverse); therefore,

both versions of one-sided star orders coincide on regular operators. From this point of view, both (4) and (7) are equally appropriate generalizations of the matrix ordering (2), and the same concerns also (5), (8) and (3).

The logical order \preceq is actually a restriction of all orders $*\leq, \leq*, *\preceq, \preceq*$. For example, (4) implies that, for $A, B \in \mathcal{S}(H)$, $A*\leq B$ iff $A^2 = AB$ and $\text{ran}A \subseteq \text{ran}B$ iff $A^2 = AB$ (in virtue of (1), $A^2 = AB$ implies that $\text{ran}A = \text{ran}(BP) \subseteq \text{ran}B$). On the other hand, it is known that the star partial order \leq is an intersection of $*\leq$ and $\leq*$ in the sense that $A \leq B$ iff $A*\leq B$ and $A \leq* B$. Of course, then it is also an intersection of $*\preceq$ and $\preceq*$.

In this paper, we study the structure of $\mathcal{B}(H)$ under the partial order $*\preceq$. We fix in the next section notation and basic algebraic facts concerning projection operators on a Hilbert space. In Section 3, several equivalent definitions of the relations $*\preceq$ and $\preceq*$ are derived. We also prove here that they are partial orders, and state some other elementary properties of these relations. Furthermore, we show that every nonempty subset of $\mathcal{B}(H)$ has the greatest upper bound (equivalently, every subset bounded above has the least upper bound). A stronger result, which says that every initial segment of $\mathcal{B}(H)$ is an orthomodular lattice isomorphic to an initial segment of the lattice of projection operators is stated in Section 4. Explicit descriptions of joins and meets in $\mathcal{B}(H)$ under the ordering $*\preceq$ also are obtained here. Most results in these two sections are more or less similar to results obtained in [11, 14, 4] for the logical order, and the main tool is the observation that $\mathcal{B}(H)$ is order isomorphic to a certain set of partial functions (in fact, restrictions of operators from $\mathcal{B}(H)$) naturally ordered by set inclusion. In Section 5, we introduce orthogonality relations $*\perp$ and $\perp*$ on $\mathcal{B}(H)$ associated in a sense with the left-, resp., right-star order, and discuss the relation $*\perp$ more closely.

2. Preliminaries: projection operators

We fix in this section notation and remind a number of basic algebraic properties of Hilbert space projection operators (i.e., idempotent self-adjoint operators), which will be frequently used below, mostly without explicit references.

An *ortholattice* is a bounded lattice with orthocomplementation, i.e., with a unary operation \perp such that (i) $x^{\perp\perp} = x$, (ii) $x \leq y$ only if $y^\perp \leq x^\perp$, and (iii) $x \wedge x^\perp = 0$. An *orthomodular* lattice is an ortholattice satisfying the weak modularity law (iv) if $x \leq y$, then $y = x \vee (y \wedge x^\perp)$. Every initial segment $[0, x]$ of an orthomodular lattice is a sublattice which also is orthomodular, with the (relative) orthocomplementation \perp_x^\perp given by $a_x^\perp := a^\perp \wedge x$.

Let H be a Hilbert space. The set $\mathcal{P}(H)$ of its projection operators, or just *projections*, is known to be a (complete) orthomodular lattice isomorphic to the lattice of closed subspaces of H (by the transfer $P \mapsto \text{ran}P$); in particular, the corresponding partial ordering \leq on $\mathcal{P}(H)$ is given by

$$P \leq Q \text{ iff } PQ = P \text{ iff } QP = P.$$

We let O and I stand for the zero, resp., unit operator, and denote the join and meet of projections P, Q , by $P \wedge Q$ and $P \vee Q$, respectively. The orthocomplement of a

projection P in $\mathcal{B}(H)$ (i.e., the projection corresponding to the nullspace of P) will be denoted by P^\perp , and its orthocomplement $Q \wedge P^\perp$ in the sublattice $[O, Q]$, which also is orthomodular, by P_Q^\perp .

Recall that the product PQ of projections P and Q belong to $\mathcal{P}(H)$ if and only if they commute. If this is the case, then

$$P \wedge Q = PQ \text{ and } P \vee Q = P + Q - PQ.$$

Projections P and Q are said to be *orthogonal* (in symbols, $P \perp Q$) if the corresponding subspaces are orthogonal: $P \perp Q$ iff $\text{ran} P \subseteq \text{null} Q$ iff $\text{ran} Q \subseteq \text{null} P$ or, equivalently,

$$P \perp Q \text{ iff } PQ = O \text{ iff } QP = O.$$

Moreover

$$P^\perp = I - P, \quad P \leq Q \text{ iff } P \perp Q^\perp, \quad P \perp Q \text{ iff } Q \leq P^\perp.$$

At last, if $P \perp Q$, then $P \wedge Q = O$ and $P \vee Q = P + Q$.

Given an operator X , let us denote by P_X the projection onto $\overline{\text{ran}} X$, the closure of $\text{ran} X$, and by Q_X , that onto $\overline{\text{ran}} X^*$; then P_A^\perp is the projection onto $\text{null} A^*$, while Q_A^\perp is the projection onto $\text{null} A$.

Most properties of projections in the subsequent proposition can easily be verified by translation of them into terms of closed ranges.

PROPOSITION 2.1 *In $\mathcal{B}(H)$,*

- (a) $P_A A = A = A Q_A$,
- (b) $P_A^\perp A = O = A Q_A^\perp$,
- (c) $A P_B = O$ iff $AB = O$, $Q_A B = O$ iff $AB = O$,
- (d) $P_A B = O$ iff $P_A \perp P_B$ iff $P_B A = O$, $A Q_B = O$ iff $Q_A \perp Q_B$ iff $B Q_A = O$,
- (e) $P_{AB} \leq P_A$, $Q_{AB} \leq Q_B$,
- (f) $P_{AB} = P_{A P_B}$, $Q_{AB} = Q_{Q_A B}$,
- (g) if $P \leq P_A$, then $P_{P A} = P$, if $Q \leq Q_A$, then $Q_{Q A} = Q$.

Proof. We shall consider only the “left” case.

(c) As $\overline{\text{ran}} B \subseteq \text{null} A$ iff $\text{ran} B \subseteq \text{null} A$.

(d) By (c).

(e) As $\text{ran}(AB) \subseteq \text{ran} A$.

(f) By (a) and (e), $P_{AB} = P_{A P_B} \leq P_{A P_B}$. On the other hand, $P_{A P_B} \leq P_{AB}$, as this inequality holds iff $(P_{AB})^\perp P_{A P_B} = O$ iff $(P_{AB})^\perp A P_B = O$ iff $(P_{AB})^\perp AB = O$ —see (c) and (b).

(g) By (f). \square

3. Order structure of $\mathcal{B}(H)$

Notice that the defining conditions (7), (8) of $*\preceq$ and \preceq^* also can be rewritten purely in terms of operators, as the lattice of closed subspaces of H is isomorphic to that of projection operators:

$$A * \preceq B \text{ iff } A^*A = A^*B \text{ and } P_A \leq P_B, \quad (9)$$

$$A \preceq^* B \text{ iff } AA^* = BA^* \text{ and } Q_A \leq Q_B. \quad (10)$$

By the way, this form of definitions allows us to transfer them naturally to Rickart *-rings, see [6] (also [5, Remark 2]). The relations \preceq^* and $*\preceq$ can be given also other characterizations.

LEMMA 3.1 *Let $A, B \in \mathcal{B}(H)$. Then*

- (a) $A^*A = A^*B$ iff $A = P_A B$ iff $A = PB$ for some $P \in \mathcal{P}(H)$,
 $AA^* = BA^*$ iff $A = BQ_A$ iff $A = BQ$ for some $Q \in \mathcal{P}(H)$,
- (b) $P_A \leq P_B$ iff $A = P_B A$, $Q_A \leq Q_B$ iff $A = A Q_B$,

Proof. (a) This is Proposition 2.3 in [1].

(b) In virtue of Proposition 2.1(c), $P_B A = A$ iff $(P_B)^\perp A = O$ iff $(P_B)^\perp P_A = O$ iff $P_B P_A = P_A$ iff $P_A \leq P_B$. \square

THEOREM 3.2 *For all $A, B \in \mathcal{B}(H)$,*

- (a) $A * \preceq B$ iff $P_A B = A$ and $P_A \leq P_B$ iff $P_A B = A = P_B A$ iff $A = (P_A \wedge P_B)B$,
- (b) $A \preceq^* B$ iff $BQ_A = A$ and $Q_A \leq Q_B$ iff $BQ_A = A = A Q_B$ iff $A = B(Q_A \wedge Q_B)$.

Proof. Equivalence of the first three conditions in (a) (and in (b)) follows from the previous lemma. Further, the second and the fourth condition also are equivalent. If $A = (P_A \wedge P_B)B$, then, by Proposition 2.1(f), $P_A = P_{(P_A \wedge P_B)P_B} = P_{P_A \wedge P_B} = P_A \wedge P_B$, whence $A = P_A B$ and $P_A \leq P_B$. The converse implication is evident. \square

THEOREM 3.3 *The relations $*\preceq$ and \preceq^* are partial orders on $\mathcal{B}(H)$.*

Proof. We shall use Theorem 3.2. Evidently, both relations are reflexive. They are transitive: for example, if $A * \preceq B$ and $B * \preceq C$, then $P_A \leq P_B \leq P_C$ and $P_A C = P_A P_B C = P_A B = A$; thus, $A * \preceq C$. They are also antisymmetric: if $A * \preceq B$ and $B * \preceq A$, then $P_A = P_B$ and $B = P_B A = P_A A = A$. \square

It follows from (9) and (10) that $A \preceq^* B$ iff $A^* * \preceq B^*$. In view of the preceding theorem, this observation immediately yields the following duality result.

PROPOSITION 3.4 *The map $A \mapsto A^*$ is an order isomorphism of the poset $(\mathcal{B}(H), * \preceq)$ onto $(\mathcal{B}(H), \preceq^*)$, and also conversely.*

We list some elementary properties of $*\preceq$ and $\preceq*$ similar to those of the logical order on $\mathcal{S}(H)$ [11].

LEMMA 3.5 In $\mathcal{B}(H)$,

- (a) O is the least operator w.r.t. both $*\preceq$ and $\preceq*$,
- (b) both orders agree on $\mathcal{P}(H)$ with the usual order of projections,
- (c) $A \in \mathcal{P}(H)$ if and only if $A * \preceq I$ if and only if $A \preceq * I$,
- (d) every right-invertible (resp., left-invertible) operator is maximal w.r.t. $*\preceq$, resp., $\preceq*$.

Proof. Items (a) and (b) are evident.

(c) $A \preceq I$ iff $A = P_A$ iff $A \in \mathcal{P}(H)$.

(d) If $AX = I$, then $I = P_{AX} \leq P_A$ and $P_A = I$ (Proposition 2.1). Thus, if $A * \preceq Y$, then $P_Y = I$ and $Y = A$ by Theorem 3.2(a). \square

It was demonstrated in Section 3 of [4] that every self-adjoint operator $A \in \mathcal{S}(H)$ is completely determined by its restriction to $\overline{\text{ran}}A$ and that the set S_H of such restrictions can be characterized as follows:

$$S_H = \{B|\overline{\text{ran}}C : B|\overline{\text{ran}}C = A|\overline{\text{ran}}A \text{ for some } A \in \mathcal{S}(H)\} \\ = \{B|\overline{\text{ran}}C : P_C = P_A \text{ for some } A \in \mathcal{S}(H) \text{ with } A \preceq B\} = \{B|\overline{\text{ran}}A : A \preceq B\}.$$

Thus, the transformation $A \mapsto A|\overline{\text{ran}}A$ is a bijective mapping from $\mathcal{S}(H)$ onto S_H . This observation was not further developed in [4]. We now take up this idea and show that there is even an order isomorphism between $(\mathcal{B}(H), * \preceq)$, resp., $(\mathcal{B}(H), \preceq *)$, and a similar set \mathcal{B}_H of restrictions of bounded operators ordered by set inclusion—the restricted operators on H being considered as partial functions $H \rightarrow H$, i.e., as sets of ordered pairs $(x, y) \in H^2$, where y is the value of the operator at x . It turns out to be convenient first to investigate the order structure of \mathcal{B}_H , and then transfer the results to $\mathcal{B}(H)$.

The definition of \mathcal{B}_H will be based on the subsequent theorem.

THEOREM 3.6 Suppose that $A, B \in \mathcal{B}(H)$. Then

- (a) $A * \preceq B$ iff $A^*|\overline{\text{ran}}A \subseteq B^*|\overline{\text{ran}}B$,
- (b) $A \preceq * B$ iff $A|\overline{\text{ran}}A^* \subseteq B|\overline{\text{ran}}B^*$.

Proof. Notice that the inclusion $A|\overline{\text{ran}}A^* \subseteq B|\overline{\text{ran}}B^*$ means the following: for every $x \in \overline{\text{ran}}A^*$, there is $y \in \overline{\text{ran}}B^*$ such that $(x, Ax) = (y, By)$.

Recall that $Q_A \leq Q_B$ iff $\overline{\text{ran}}A^* \subseteq \overline{\text{ran}}B^*$. Further,

$$A = BQ_A \text{ iff } A|\overline{\text{ran}}A^* = B|\overline{\text{ran}}A^*. \tag{11}$$

Indeed, if the right-side equality holds, then, for every x , $A(Q_A(x)) = B(Q_A(x))$, whence $A = AQ_A = BQ_A$ in virtue of Proposition 2.1(a). Conversely, if $A = BQ_A$, then, for arbitrary $y \in \overline{\text{ran}}A^*$, $y = Q_A(y)$ and $A(y) = B(Q_A(y)) = B(y)$.

Now Theorem 3.2(b) yields the equivalence

$$A \preceq_* B \text{ iff } \overline{\text{ran}}A^* \subseteq \overline{\text{ran}}B^* \text{ and } A|\overline{\text{ran}}A^* = B|\overline{\text{ran}}A^*,$$

which is evidently a variant of (b). The item (a) is dual to (b). \square

We conclude that

$$A = B \text{ iff } A|\overline{\text{ran}}A^* = B|\overline{\text{ran}}B^*, \tag{12}$$

i.e., every operator A in $\mathcal{B}(H)$ is completely determined by its restriction to $\overline{\text{ran}}A^*$.

Let A^\times and A^\bowtie stand for the partial operators $A^*|\overline{\text{ran}}A$ and $A|\overline{\text{ran}}A^*$, respectively. We denote by \mathcal{B}_H the set of all partial operators of the form A^\times (or, what amounts to the same, of the form A^\bowtie), and consider it as partially ordered by \subseteq . The transformations $A \mapsto A^\times$ and $A \mapsto A^\bowtie$ are injective (see (12)) and, by the selection of \mathcal{B}_H , even bijective; moreover, in view of Theorem 3.6, both transformations are order isomorphisms of the posets $(\mathcal{B}(H), * \preceq)$, resp., $(\mathcal{B}(H), \preceq *)$ onto \mathcal{B}_H . We have obtained the following result.

PROPOSITION 3.7 *The posets $(\mathcal{B}(H), * \preceq)$ and $(\mathcal{B}(H), \preceq *)$ are order isomorphic to \mathcal{B}_H .*

The set \mathcal{B}_H can be given also another description.

LEMMA 3.8 $\mathcal{B}_H = \{B|\overline{\text{ran}}A^* : A = BQ_A\}$.

Proof. By the definition of \mathcal{B}_H , $B|\overline{\text{ran}}C^* \in \mathcal{B}_H$ iff $B|\overline{\text{ran}}C^* = A|\overline{\text{ran}}A^*$ for some $A \in \mathcal{B}(H)$. But

$$\begin{aligned} B|\overline{\text{ran}}C^* = A|\overline{\text{ran}}A^* & \text{ iff } \overline{\text{ran}}C^* = \overline{\text{ran}}A^* \text{ and } A|\overline{\text{ran}}A^* = B|\overline{\text{ran}}A^* \\ & \text{ iff } Q_C = Q_A \text{ and } A = BQ_A \quad (\text{see (11)}). \end{aligned}$$

Thus, $\mathcal{B}_H = \{B|\overline{\text{ran}}C^* : Q_C = Q_A \text{ and } A = BQ_A \text{ for some } A \in \mathcal{B}(H)\}$, whence the desired identity follows. \square

Notice that $Q_C = Q_A$ and $A = BQ_A$ iff $Q_C = Q_A$ and $A = BQ_C$. Therefore, $B|\overline{\text{ran}}C^* \in \mathcal{B}_H$ iff $Q_C = Q_{BQ_C}$ iff $Q_C = Q_{Q_B Q_C}$ (by Proposition 2.1(f)). In particular, evidently

$$\text{if } \overline{\text{ran}}C^* \subseteq \overline{\text{ran}}B^*, \text{ then } B|\overline{\text{ran}}C^* \in \mathcal{B}_H. \tag{13}$$

We shall denote by f, g, h arbitrary elements of \mathcal{B}_H , and by o , the least element $O|\{0\}$ in \mathcal{B}_H . Let the notation $\text{dom } f$ stand for the domain of f . Evidently, if $f \subseteq h$, then $\text{dom } f \subseteq \text{dom } h$. Observe also that

$$f \subseteq h \text{ iff } f = h|\text{dom } f \text{ iff } f = h|G, \tag{14}$$

where G is a (unique) closed subspace of $\text{dom } h$.

For $\mathcal{F} \subseteq \mathcal{B}_H$, we denote by $\bigcap \mathcal{F}$ the intersection of all functions $f \in \mathcal{F}$. As usual, the intersection of an empty subset should be the maximum element of \mathcal{B}_H . However, Lemma 3.5 implies (as I is invertible) that A is the greatest element of $\mathcal{B}(H)$ if and only if $A = I$, i.e., $\mathcal{B}(H) = \mathcal{P}(H)$. Therefore, $\bigcap \emptyset$ generally does not exist in \mathcal{B}_H . However, any other intersection does exist.

LEMMA 3.9 *The set \mathcal{B}_H is closed under arbitrary nonempty intersections.*

Proof. Let \mathcal{F} be a nonempty subset of \mathcal{B}_H . Then

$$(x, y) \in \bigcap \mathcal{F} \text{ iff } x \in \bigcap (\text{dom } f : f \in \mathcal{F}) \text{ and } f(x) = y \text{ for all } f \in \mathcal{F}$$

$$\text{iff } x \in \bigcap (\text{dom } f : f \in \mathcal{F}) \cap \{f(x) - f'(x) = 0 \text{ for all } f, f' \in \mathcal{F}\} \text{ and } f_0(x) = y,$$

where f_0 is any element of \mathcal{F} . There is a unique nonempty subset \mathcal{C} of $\mathcal{B}(H)$ such that $\mathcal{F} = \mathcal{C}^\times = \{C | \overline{\text{ran}} C^* : C \in \mathcal{C}\}$; so $\bigcap \mathcal{F} = \bigcap \mathcal{C}^\times$ and $(x, y) \in \bigcap \mathcal{C}^\times$ iff

$$x \in \bigcap (\overline{\text{ran}} C^* : C \in \mathcal{C}) \cap \bigcap (\text{null}(C - C') : C, C' \in \mathcal{C}) \text{ and } C_0(x) = y,$$

where $(C_0)^\times = f_0$. Let

$$G := \bigcap (\overline{\text{ran}} C^* : C \in \mathcal{C}) \cap \bigcap (\text{null}(C - C') : C, C' \in \mathcal{C});$$

thus, $\bigcap \mathcal{C}^\times = C_0 | G$. As G is evidently a closed subspace of $\overline{\text{ran}} C_0^*$, (13) implies that $\bigcap \mathcal{C}^\times \in \mathcal{B}_H$. \square

It follows that intersection of \mathcal{F} is the greatest lower bound of this subset in \mathcal{B}_H . A poset is said to be *bounded complete* if every its subset bounded above has the join (i.e., the least upper bound) or, equivalently, every its nonempty subset has the meet (the greatest lower bound). Therefore, the poset \mathcal{B}_H is bounded complete.

The next result, which immediately follows due to the Proposition 3.7, is an analogue of that obtained in [14, Corollary 3.6] for the logical order on $\mathcal{S}(H)$.

THEOREM 3.10 *The posets $(\mathcal{B}(H), * \preceq)$ and $(\mathcal{B}(H), \preceq *)$ are bounded complete.*

The union of elements of \mathcal{B}_H may be not a partial function on H at all. Generally, \mathcal{B}_H is not closed under existing unions. For example, the domain of a partial function $f \cup g$ is $\text{dom } f \cup \text{dom } g$, and the latter union is not necessary a closed subspace of H . Even if it is, $f \cup g$ may not be a member of \mathcal{B}_H .

Let $\bigsqcup \mathcal{F}$ stand for the join of \mathcal{F} in \mathcal{B}_H when it exists. We already know that this is the case if and only if \mathcal{F} is bounded from above. Also, if \mathcal{G} is a system of subspaces of H , let $\bigsqcup \mathcal{G}$ stand for the least closed subspace including all members of \mathcal{G} .

LEMMA 3.11 *Let \mathcal{F} be a subset of \mathcal{B}_H , and let h be its upper bound in \mathcal{B}_H . Then*

- (a) $\bigsqcup \mathcal{F} = h | \bigsqcup (\text{dom } f : f \in \mathcal{F})$,
- (b) $\bigcap \mathcal{F} = h | \bigcap (\text{dom } f : f \in \mathcal{F})$.

Proof. Suppose that \mathcal{F} and h satisfy the assumption of the lemma. Then $f = h | \text{dom } f$ for every $f \in \mathcal{F}$. Clearly, $\bigsqcup \mathcal{F} \subseteq h$ and $\text{dom } \bigsqcup \mathcal{F} = \bigsqcup (\text{dom } f : f \in \mathcal{F}) \subseteq \text{dom } h$. Likewise, $\bigcap \mathcal{F} \subseteq h$ and $\text{dom } \bigcap \mathcal{F} = \bigcap (\text{dom } f : f \in \mathcal{F})$. \square

4. Lattice operations in $\mathcal{B}(H)$

In this section we shall deal explicitly only with the left-star order, and regard $\mathcal{B}(H)$ as ordered by $*\preceq$.

By a sublattice of $\mathcal{B}(H)$ we mean a subset L of $\mathcal{B}(H)$ such that every pair of elements of L have a meet and a join (in $\mathcal{B}(H)$) which belong to L . A complete sublattice of $\mathcal{B}(H)$ is defined similarly. Theorem 3.10 now implies the following observation.

PROPOSITION 4.1 *Every initial segment of $\mathcal{B}(H)$ is a complete sublattice of $\mathcal{B}(H)$.*

Lemma 3.5(b,c) further implies that, in particular, $\mathcal{P}(H)$ is a complete sublattice of $\mathcal{B}(H)$. The subsequent refinement of the above proposition is a counterpart of Theorem 4.12 of [11] for $\mathcal{S}(H)$.

THEOREM 4.2 *Every segment $[O, X]$ of $\mathcal{B}(H)$ is isomorphic to $[O, P_X]$; in particular, it is an orthomodular lattice.*

Proof. By (14), the mapping $f \mapsto \text{dom } f$ transforms one-to-one any initial segment $[o, h]$ of \mathcal{B}_H onto the set CS_h of closed subspaces of $\text{dom } h$ (also naturally ordered by set inclusion) and is even an order isomorphism, with $G \mapsto h|G$ the inverse isomorphism.

Now let X be any operator in $\mathcal{B}(H)$ and $h := X^\times = X^*|\overline{\text{ran}} X$. Then the segment $[O, P_X]$ of $\mathcal{P}(H)$ is isomorphic to the poset CS_h . On the other hand, the order isomorphism $^\times : \mathcal{B}(H) \rightarrow \mathcal{B}_H$ maps $[O, X]$ onto $[o, h]$. Indeed, if $A * \preceq X$, then $A^\times \subseteq h$, and if $f \subseteq h$, i.e., $f = h|\text{ran } P$ with $\text{ran } P \subseteq \overline{\text{ran}} X = \text{ran } P_X$, then

$$f = X^*|\text{ran } P = X^*P|\text{ran } P = (PX)^*|\text{ran } P = (PX)^\times$$

(in view of Proposition 2.1(g), $\overline{\text{ran}}(PX) = \text{ran } P$) and $(PX)^\times \subset X^\times$, i.e., $PX * \preceq X$, as needed. So, the restriction of the inverse isomorphism $A^\times \mapsto A$ to $[o, h]$ is an order isomorphism of $[o, h]$ onto $[O, X]$. It follows that the chain of isomorphisms $[O, P_X] \rightarrow \text{CS}_h \rightarrow [o, h] \rightarrow [O, X]$:

$$P \mapsto \text{ran } P \mapsto X^*|\text{ran } P = (PX)^\times \mapsto PX$$

realizes an order isomorphism $[O, P_X] \rightarrow [O, X]$. Hence, it preserves all joins and meets and, being bijective, naturally induces an orthocomplementation on $[O, X]$, making the lattice $[O, X]$ orthomodular. \square

Let us denote by ψ_X the isomorphism $P \mapsto PX$ of $[O, P_X]$ onto $[O, X]$ described in the proof. The mapping $\phi_X : [O, X] \rightarrow [O, P_X]$ which takes every operator $A \in [O, X]$ into the projection P_A is the inverse of ψ_X . Indeed, if $A * \preceq X$, then $\phi_X(A) \leq P_X$ and $\psi_X(\phi_X(A)) = A$ (Theorem 3.2(a)), and if $P \leq P_X$, then $\phi_X(\psi_X(P)) = P$ (Proposition 2.1(g)). Thus, ϕ_X also is an order isomorphism, and therefore preserves arbitrary meets, joins, and also orthocomplements. Moreover,

$$[O, X] = \{PX : P \leq P_X\}, \quad [O, P_X] = \{P_A : A * \preceq X\}.$$

REMARK 1. To prove that $[O, X]$ and $[O, P_X]$ are isomorphic, it actually is not necessary to know in advance that each $[O, X]$ is a lattice. We saw at the end of the proof that the constructed mapping (which was afterwards denoted by ψ_X), transfers orthocomplementation from $[O, P_X]$ to $[O, X]$, and it could likewise transfer also lattice operations. Nevertheless, Theorem 4.2 does not imply Proposition 4.1, in distinction to what could now seem. The point is that the joins induced by ψ_X in the segment $[O, X]$ are really local (i.e., provide suprema in $[O, X]$), and the question if they agree with those existing in the whole poset $\mathcal{B}(H)$ anyway requires a separate consideration.

We are now in position to obtain explicit descriptions of meets and joins also in $\mathcal{B}(H)$. They turn out to be formally the same as for the logical order on $\mathcal{S}(H)$; cf. Section 4 in [4]. We denote the meet (join) of operators A and B by $A * \wedge B$ (resp., $A * \vee B$) when it exists. The notation $*\perp_X$ stands for the orthocomplementation in $[O, X]$.

COROLLARY 4.3 *If $A, B * \preceq X$, then*

- (a) $A * \vee B = (P_A \vee P_B)X$,
- (b) $A * \wedge B = (P_A \wedge P_B)X$,
- (c) $A * \perp_X = (P_X - P_A)X = X - A$.

More generally, if $\mathcal{C} \subseteq \mathcal{B}(H)$ and X is an upper bound of \mathcal{C} , then likewise

- (d) $(\vee(P_C : C \in \mathcal{C}))X$ *is the least upper bound of \mathcal{C} ,*
- (e) $(\wedge(P_C : C \in \mathcal{C}))X$ *is the greatest lower bound of \mathcal{C} .*

Proof. As to (a) and (b), notice that $A * \vee B = \psi_X(\phi_X(A) \vee \phi_X(B))$ and $A * \wedge B = \psi_X(\phi_X(A) \wedge \phi_X(B))$. Further, recall that the orthocomplement $P_X - P_A$ of P_A in $[O, P_X]$ is preserved by the order isomorphism ψ_X . So, $A * \perp_X = \psi_X((\phi_X(A))_{\phi_X(X)}^\perp) = \psi_X(\phi_X(X) - \phi_X(A))$. This proves the first equality in (c); the other one then follows by Proposition 2.1(a) and Theorem 3.2(a). Items (d) and (e) are proved similarly (or either using Lemma 3.11). \square

Theorem 4.2 has also several other useful consequences.

COROLLARY 4.4 *Suppose that A and B have an upper bound. Then*

- (a) $P_{A * \wedge B} = P_A \wedge P_B, \quad P_{A * \vee B} = P_A \vee P_B,$
- (b) $(P_A \wedge P_B)A = A * \wedge B = (P_A \wedge P_B)B.$

If, moreover, $P_A \perp P_B$, then

- (c) $A * \wedge B = O, \quad A * \vee B = A + B,$
- (d) $P_{A+B} = P_A + P_B.$

Proof. Assume that $A, B * \preceq X$.

(a) As ϕ_X is a lattice isomorphism.

(b) For example, (a) and Theorem 3.2(a) imply that $(P_A \wedge P_B)A = P_{A * \wedge B}A = A * \wedge B$.

B .

(c) If $P_A \perp P_B$, then P_A and P_B commute and, by (a), $A * \wedge B = (P_A \wedge P_B)X = P_A P_B X = O$ and $A * \vee B = (P_A \vee P_B)X = (P_A + P_B)X = P_A X + P_B X = A + B$ (see Theorem 3.2(a)).

(d) By (c) and (a), $P_{A+B} = P_{A * \vee B} = P_A \vee P_B = P_A + P_B$. \square

Notice also that, as ϕ preserves orthocomplements,

$$\text{if } A * \preceq X, \text{ then } P_{A * \perp X} = P_{X-A} = P_X - P_A \tag{15}$$

by Corollaries 4.3(c) and 4.4(d).

Finally, we derive also two characterizations of meets of arbitrary pairs of elements of $\mathcal{B}(H)$. Item (b) below is a close analogue of Corollary 7 in [4] for self-adjoint operators; it can also be obtained by an application of Corollary 4.3(d) to the set of all lower bounds of the pair $\{A, B\}$.

THEOREM 4.5 For arbitrary $A, B \in \mathcal{B}(H)$,

$$(a) \ A * \wedge B = (P_A \wedge P_B \wedge (P_{A-B})^\perp)A = (P_A \wedge P_B \wedge (P_{A-B})^\perp)B,$$

$$(b) \ A * \wedge B = \max\{P_C : C * \preceq A \text{ and } C * \preceq B\}A = \max\{P_C : C * \preceq A \text{ and } C * \preceq B\}B.$$

Proof. (a) Let us specify the equality $\bigcap \mathcal{C}^\times = C_0 | G$ derived in the proof of Lemma 3.9 by setting $\mathcal{C} := \{A, B\}$ and $C_0 := A$:

$$\begin{aligned} A^\times \cap B^\times &= A | ((\overline{\text{ran}} A^* \cap \overline{\text{ran}} B^*) \cap \text{null}(A - B)) \\ &= A | (\overline{\text{ran}} A^* \cap \overline{\text{ran}} B^* \cap (\overline{\text{ran}}(A - B)^*)^\perp) \\ &= A | \text{ran}(Q_A \wedge Q_B \wedge (I - Q_{A-B})) \\ &= A(Q_A \wedge Q_B \wedge (Q_{A-B})^\perp) | \text{ran}(Q_A \wedge Q_B \wedge (Q_{A-B})^\perp). \end{aligned}$$

But (see Proposition 2.1(f) and the definition of Q_X)

$$\begin{aligned} \text{ran}(Q_A \wedge Q_B \wedge (Q_{A-B})^\perp) &= \text{ran} Q_{(Q_A \wedge Q_B \wedge (Q_{A-B})^\perp)} = \text{ran} Q_{Q_A(Q_A \wedge Q_B \wedge (Q_{A-B})^\perp)} \\ &= \text{ran} Q_{A(Q_A \wedge Q_B \wedge (Q_{A-B})^\perp)} = \overline{\text{ran}}(A(Q_A \wedge Q_B \wedge (Q_{A-B})^\perp))^*. \end{aligned}$$

Thus, for all $A, B \in \mathcal{B}(H)$, $A^\times \cap B^\times = (A(Q_A \wedge Q_B \wedge (Q_{A-B})^\perp))^\times$. Substituting A^* for A and B^* for B , we obtain the dual identity, $A^\times \cap B^\times = ((P_A \wedge P_B \wedge (P_{A-B})^\perp)A)^\times$, which leads us to the first description of $A * \wedge B$; the other one is obtained similarly.

(b) Notice that

$$\{P_C : C * \preceq A \text{ and } C * \preceq B\} = \{P : P \leq P_A \wedge P_B \wedge (I - P_{A-B})\}.$$

Indeed, if $C * \preceq A, B$, then $P_C A = C = P_C B$, $P_C(A - B) = O$ and $P_C \perp P_{A-B}$ (Proposition 2.1(c)). Moreover, $P_C \leq P_A, P_B$, and P_C belongs to the set at the right. On the other

hand, if P belongs to the set, then $P \perp P_{A-B}$, whence $P(A - B) = O$ (Proposition 2.1(c)) and $PA = PB =: C$. As also $P \leq P_A$, we get from Proposition 2.1(g) that $P = P_C$. Consequently, $P_C A = C$ and $P_C \leq P_A$, i.e., $C * \preceq A$ (Theorem 3.2(a)). Likewise $C * \preceq B$, and P belongs to the set at the left. \square

5. Orthogonality on $\mathcal{B}(H)$ for one-sided star orders

Let us say that operators A and B are *left-star*, resp., *right-star orthogonal* (in symbols, $A * \perp B$, resp., $A \perp * B$), if

$$A * \perp B := A * \preceq A + B, \text{ resp., } A \perp * B := A \preceq * A + B. \tag{16}$$

Adapting the term introduced in [3, Definition 3] in the context of orthomodular groups, we could therefore say that the relations $* \preceq$ and $* \perp$, as well as $\preceq *$ and $\perp *$, are associated. It is easily seen that the relationships (16) are equivalent to

$$A * \preceq B \text{ iff } A * \perp B - A, \text{ resp., } A \preceq * B \text{ iff } A \perp * B - A.$$

For example, $A * \perp B - A$ iff $[A^*(B - A) = O \text{ and } P_A \leq P_{B-A+A}]$ iff $A * \preceq B$. The two equivalences obtained from (16) may further be rewritten as

$$A * \preceq B \text{ iff } B = A + C \text{ for some } C \text{ with } C * \perp A, \tag{17}$$

$$A \preceq * B \text{ iff } B = A + C \text{ for some } C \text{ with } C \perp * A. \tag{18}$$

Observe that if $A, B \in \mathcal{S}(H)$, then $A * \perp B$ iff $A^2 = A(A + B)$ iff $AB = O$. Therefore, $* \perp$ agrees on $\mathcal{S}(H)$ with the orthogonality assumed in [11], and then the equivalence (17) reduces to the initial definition of the logical order in [11]. Of course, the same concerns $\perp *$ and (18).

We shall discuss in detail only the left-star orthogonality. Let us first expand the defining condition of $* \perp$.

PROPOSITION 5.1 *For all $A, B \in \mathcal{B}(H)$,*

$$A * \perp B \text{ iff } P_A \perp P_B \text{ and } P_A \leq P_{A+B}. \tag{19}$$

Proof. By (16) and (9), $A * \perp B$ iff $A^* A = A^*(A + B)$ and $P_A \leq P_{A+B}$. But

$$A^* A = A^*(A + B) \text{ iff } A^* B = O, \quad P_A(A + B) = A \text{ iff } P_A B = O.$$

By virtue of Lemma 3.1(a), we conclude that $A^* B = O$ iff $P_A B = O$, and then Proposition 2.1(d) yields that

$$P_A \perp P_B \text{ iff } A^* B = O \text{ iff } B^* A = O. \tag{20}$$

Now (19) follows. \square

LEMMA 5.2 In $\mathcal{B}(H)$,

- (a) $O * \perp A$,
- (b) if $A * \perp A$, then $A = O$,
- (c) if $A * \perp B$, then $B * \perp A$,
- (d) if $A, B * \preceq C$ and $P_A \perp P_B$, then $A * \perp B$,
- (e) if $A * \preceq B$ and $B * \perp C$, then $A * \perp C$,
- (f) if $A * \perp B$, then $A * \vee B = A + B$,
- (g) if $A * \perp B$, then $A * \wedge B = O$,
- (h) if $A * \perp B, C$ and $B * \preceq A * \vee C$, then $B * \preceq C$.

Proof. Assume that $A, B \in \mathcal{B}(H)$.

(a) Evident.

(b) As $A^*A = O$ implies that $A = O$.

(c) Assume that $P_A \perp P_B$ and $P_A \leq P_{A+B}$. By Proposition 2.1(d), then $P_A B = O$, which implies (in virtue of Proposition 2.1(b)) that $B = (P_A)^\perp A + (P_A)^\perp B = (P_A)^\perp (A + B)$. Consequently, $P_B = P_{(P_A)^\perp(A+B)} = P_{(P_A)^\perp P_{A+B}}$ (see Proposition 2.1(f)). Now observe that P_{A+B} commutes with P_A (by the second supposition) and, therefore, also with $(P_A)^\perp$. Hence, $(P_A)^\perp P_{A+B}$ is a projection, and $P_B = (P_A)^\perp P_{A+B} = (P_A)^\perp_{P_{A+B}} \leq P_{A+B}$. By (19), now $B * \perp A$.

(d) Assume that $A, B * \preceq C$ and $P_A \perp P_B$. Then $P_{A+B} = P_{A * \vee B} = P_A \vee P_B$ by Corollary 4.4(c,a), whence $P_A \leq P_{A+B}$. So, $A * \perp B$ by (19).

(e) Assume that $A * \preceq B$ and $B * \perp C$, i.e., $B * \preceq B + C$. Then $P_A \leq P_B$, $P_B \perp P_C$ and, consequently, $P_A P_C = P_A P_B P_C = O$. On the other hand, $C * \preceq B + C$ by (c); so A and C have an upper bound $B + C$. Now, $A * \perp C$ by (d).

(f) Assume that $A * \perp B$. In virtue of (c), then $A + B$ is an upper bound of A and B . As $P_A \perp P_B$, Corollary 4.4(c) implies that $A * \vee B = A + B$.

(g) Follows from Corollary 4.4(c), since $A + B$ is an upper bound of A and B .

(h) If $A * \perp B$, then $P_A \perp P_B$, i.e., $P_B A = O$ and $P_A B = O$ (Proposition 2.1(d)). If also $A * \perp C$ and $B * \preceq A * \vee C$, then $A, C * \preceq A + C$, $B * \preceq A + C$ (see (f)) and, further, $P_B C = P_B (A + C) = B = P_{A+C} B = (P_A + P_C) B = P_C B$ by Corollary 4.4(d). See also Theorem 3.2(a). \square

REMARK 2. By (17), $A * \preceq B$ implies that $B = A + C$ for some C such that $C * \perp A$. Together with items (c), (e), (a), (f), (h) of the lemma, this allows us to conclude that the system $(\mathcal{B}(H), * \preceq, * \perp)$ is a quasi-orthomodular nearsemilattice (even nearlattice) in the sense of [4, Definition 1]. Then Theorem 9 of [4] provides another proof for the result that every initial segment of $\mathcal{B}(H)$ is an orthomodular lattice which is a sublattice of $\mathcal{B}(H)$ (see Proposition 4.1 and Theorem 4.2 above); this proof does not rest on properties of $\mathcal{S}(H)$ and any analogue of Theorem 3.10. Moreover, Theorem 11 of [4] allows us to conclude that, like $\mathcal{S}(H)$ ([11, Theorem 4.2]), the system $(\mathcal{B}(H), \oplus, O)$, where $A \oplus B = C$ iff $A + B = C$ and $A * \perp B$, is a generalized orthoalgebra (see [11, 4] for a definition). Notice that $\overline{\text{ran}}(A \oplus B) = \overline{\text{ran}}A \oplus \overline{\text{ran}}B$.

We failed to generalize to $\mathcal{B}(H)$ Theorem 4.3 of [14], which states that $\mathcal{S}(H)$ is a weak generalized orthomodular poset. The missing link here is the condition

$$\text{if } A * \perp B, C \text{ and } B * \perp C, \text{ then } A * \perp (B * \vee C);$$

which seemingly does not hold true in any $\mathcal{B}(H)$.

Our last theorem shows, in particular, that the orthogonality relation $*\perp$ agrees on every segment $[O, X]$ with the orthogonality induced in this segment by the corresponding orthocomplementation $*\perp_X$ (see Corollary 4.3(c)).

THEOREM 5.3 *The following conditions on operators $A, B \in \mathcal{B}(H)$ are equivalent:*

- (a) $A * \perp B$,
- (b) *there is an operator $X \in \mathcal{B}(H)$ such that $A * \preceq X$ and $B * \preceq X - A$,*
- (c) *the pair A, B is bounded from above and $P_A \perp P_B$.*

Proof. Suppose that $A, B \in \mathcal{B}(H)$.

(a) \rightarrow (b) If $A * \perp B$, then the inequalities in (b) are fulfilled for $X := A + B$.

(b) \rightarrow (c) If $A * \preceq X$ and $B * \preceq X - A$, then $P_A \leq P_X$, $P_B \leq P_{X-A} = P_X - P_A$ (see (15)), and, consequently, $P_A P_B = P_A (P_X - P_A) P_B = O$. On the other hand, $X - A * \preceq X$ in virtue of Corollary 4.3(c).

(c) \rightarrow (a) See Lemma 5.2(d). \square

It follows from (b) that, for $P, Q \in \mathcal{P}(H)$,

$$P * \perp Q \text{ iff } P \perp Q.$$

Item (c) implies that an orthogonality on \mathcal{B}_H corresponding to $*\perp$ under the isomorphism \times can be introduced by the condition

$$f \perp g \text{ iff } \text{dom } f \perp \text{dom } g \text{ and } f, g \subseteq h \text{ for some } h.$$

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