

POLYNOMIAL AS A NEW VARIABLE — A BANACH ALGEBRA WITH A FUNCTIONAL CALCULUS

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Abstract. Given any square matrix or a bounded operator A in a Hilbert space such that $p(A)$ is normal (or similar to normal), we construct a Banach algebra, depending on the polynomial p , for which a simple functional calculus holds. When the polynomial is of degree d , then the algebra deals with continuous \mathbb{C}^d -valued functions, defined on the spectrum of $p(A)$. In particular, the calculus provides a natural approach to deal with nontrivial Jordan blocks and one does not need differentiability at such eigenvalues.

1. Introduction

There are many situations in which it would be desirable to be able to treat polynomials as new *global* variables. For example, by Hilbert's lemniscate theorem (see e.g [12]) polynomials can be used to map complicated sets of the complex plane onto discs. As polynomials are not one-to-one we represent scalar functions in the original variable by a vector valued function in the polynomial. This leads to *multicentric holomorphic calculus* [9]. In [10] we applied it to generalize the von Neumann theorem on contractions in Hilbert spaces. In such applications one would, given a bounded operator A , search for a polynomial p such that $p(A)$ has a small norm – thus mapping a potentially complicated spectrum into a small disc.

In this paper we study multicentric calculus without assuming the functions to be analytic. As an application we consider situations in which $p(A)$ is diagonalizable or similar to normal. Thus, the aim is to remove the Jordan blocks by moving from A to $p(A)$. To illustrate the goal consider finite dimensional matrices. If $D = \text{diag}\{\alpha_j\}$ is a diagonal matrix and φ is a continuous function, then any reasonable functional calculus satisfies $\varphi(D) = \text{diag}\{\varphi(\alpha_j)\}$. Further, if A is diagonalizable so that with a similarity T we have $A = TDT^{-1}$, then we of course set

$$\varphi(A) = T\varphi(D)T^{-1}. \quad (1)$$

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However, if A has an eigenvalue with a nontrivial Jordan block, then the customary approach is to assume that φ is smooth enough at the eigenvalues so that the off-diagonal elements can be represented by derivatives of φ . For example, if

$$J = \begin{pmatrix} \alpha & 1 \\ & \alpha & 1 \\ & & & \alpha \end{pmatrix} \quad (2)$$

then

$$\varphi(J) = \begin{pmatrix} \varphi(\alpha) & \varphi'(\alpha) & \frac{1}{2}\varphi''(\alpha) \\ & \varphi(\alpha) & \varphi'(\alpha) \\ & & \varphi(\alpha) \end{pmatrix}. \quad (3)$$

A collection of different ways to define $\varphi(A)$ for matrices can be found from [11], where Higham, following Gantmacher [6], says that a function φ is *defined at the spectrum* $\sigma(A) = \{\alpha_j\}$ if the values $\varphi^{(k)}(\alpha_j)$ are known for $0 \leq k \leq n_j$, where $n_j + 1$ are the powers in the minimal polynomial.

This has two obvious drawbacks. First, since it is based on the Jordan form the functional calculus is discontinuous: for diagonalizable matrices it is given for all continuous functions while it requires existence of derivatives at eigenvalues with nontrivial Jordan blocks. Second, the approach cannot conveniently be extended to infinite dimensional spaces. Recall that there is a natural functional calculus for normal operators which easily extends to operators which are similar to normal. If, however, an eigenvalue with a nontrivial Jordan block would exist in the middle of a cluster of other eigenvalues, then one would need to have a way to treat function classes which are continuous and additionally have derivatives at that particular eigenvalue.

We shall show in this paper that there is a simple way to parametrize continuous functions which slow down at those places where some extra smoothness is needed. And it turns out that this allows a functional calculus which agrees with the holomorphic functional calculus if applied to holomorphic functions but is defined for functions which do not need to be differentiable at any point.

The starting point for the calculus is taking $w = p(z)$ as a new variable. Since such a change of variable is only locally injective we compensate this by replacing the scalar function

$$\varphi : z \mapsto \varphi(z) \in \mathbb{C}$$

by a vector valued function

$$f : w \mapsto f(w) \in \mathbb{C}^d$$

where d is the degree of the polynomial p . The *multicentric representation* of φ is then of the form

$$\varphi(z) = \sum_{j=1}^d \delta_j(z) f_j(p(z)), \quad (4)$$

where δ_j 's are the Lagrange interpolation polynomials such that $\delta_j(\lambda_j) = 1$ while $\delta_j(\lambda_k) = 0$ when $k \neq j$, [9].

If now $p(A)$ is diagonalizable, one can apply the known functional calculus to represent $f_j(p(A))$. But since δ_j 's are polynomials, $\delta_j(A)$ is well defined and differentiability of φ is not needed.

The paper is organized as follows. We first consider the Banach space of continuous functions $f: M \rightarrow \mathbb{C}^d$ and associate with it a product, "polyproduct" \odot , such that it becomes a Banach algebra, which we denote by $C_\Lambda(M)$. Here Λ denotes the set of zeros of the polynomial p . Then the functions φ in (4) can be viewed as Gelfand transformations \hat{f} of functions $f \in C_\Lambda(M)$. Towards the end of the paper we discuss the functional calculus for operators in Hilbert spaces H such that $p(A)$ is similar to a normal operator. In particular we study the mapping χ_A which associates to f a bounded operator $\chi_A(f) \in \mathcal{B}(H)$

$$\chi_A(f) = \sum_{j=1}^d \delta_j(A) f_j(p(A))$$

and show that we get a homomorphism $\chi_A(f \odot g) = \chi_A(f)\chi_A(g)$ which, in an appropriate quotient algebra, satisfies a spectral mapping theorem.

2. Construction of the Banach algebra

2.1. Multicentric representation of functions

We assume given a polynomial $p(z) = (z - \lambda_1) \cdots (z - \lambda_d)$ with distinct zeros $\Lambda = \{\lambda_j\}_{j=1}^d$ mapping the z -plane onto w -plane: $w = p(z)$. In addition we denote by $\Lambda_1 = \{z : p'(z) = 0\}$ the set of critical points of p . We call the points of Λ as the local centers of the multicentric calculus. Recall that by the Gauss-Lucas theorem Λ_1 is in the convex hull of Λ .

Suppose $\delta_j(z)$ are the Lagrange interpolation polynomials with interpolation points in Λ so that

$$\delta_j(z) = \frac{p(z)}{p'(\lambda_j)(z - \lambda_j)} = \prod_{k \neq j} \frac{z - \lambda_k}{\lambda_j - \lambda_k}.$$

Assume then that we are given a function f mapping a compact $M \subset \mathbb{C}$ into \mathbb{C}^d . It determines a unique function φ on $K = p^{-1}(M)$ if we set

$$\varphi(z) = \sum_{j=1}^d \delta_j(z) f_j(p(z)) \text{ for } z \in K.$$

We say that φ is given on K by a *multicentric representation* and denote it in short

$$\varphi = \mathcal{L}f.$$

In the reverse direction, suppose we are given a scalar function φ on a set K_0 . Then a necessary condition for f to be determined uniquely is that K_0 is *balanced w.r.t.* Λ in the following sense: $K_0 = p^{-1}(p(K_0))$. We shall assume throughout that $K_0 \subset K = p^{-1}(M)$ is such that $p(K_0) = M$.

Assuming that K is balanced and contains no critical points, then the function f is pointwisely uniquely determined by the values of φ . In order to write down a formula we agree about some additional notation. Denote the roots of $p(z) - w = 0$ by $z_j = z_j(w)$. Away from critical values these are analytic and we assume a fixed numbering so that $z_j(w) \rightarrow \lambda_j$ if $z_1(w) \rightarrow \lambda_1$ (when $w \rightarrow 0$). In the inversion we essentially exchange interpolation and evaluation points. To that end let $\delta_j(\zeta; w)$ denote the interpolation polynomial, with w fixed, which takes the value 1 at $\zeta = z_j(w)$ while vanishing at other $z_k(w)$'s:

$$\delta_j(\zeta; w) = \frac{p(\zeta) - w}{p'(z_j(w))(\zeta - z_j(w))}, \tag{5}$$

so that in particular $\delta_j(\zeta; 0) = \delta_j(\zeta)$.

PROPOSITION 1. *Suppose K is a balanced compact set with respect to local centers Λ . Assume that φ is given pointwisely in K . Then f is uniquely defined for all noncritical values $w \in M \setminus p(\Lambda_1)$ by*

$$f_k(w) = \sum_{j=1}^d \delta_j(\lambda_k; w)\varphi(z_j(w)). \tag{6}$$

The functions f_k inherit the smoothness of φ , and additionally, if $\lambda_c \in \Lambda_1$ is an interior point of K and φ is at that point analytic, then the singularities of each f_k at the critical value $p(\lambda_c)$ are removable.

Proof. See the discussions in [9] and [10]. \square

So, we could use the expression $f = \mathcal{L}^{-1}\varphi$ at least when the components of f are determined by (6) for noncritical values w provided φ is given in a balanced set. In particular this is natural when φ is analytic in a balanced domain. However, the topic of this paper is in functions which are perhaps given only on discrete sets, such as the set of eigenvalues of a matrix and then some extra care is needed in considering the possible lack of injectivity of \mathcal{L} . We shall therefore build a Banach algebra and view \mathcal{L} as performing the Gelfand transformation $\hat{f} = \mathcal{L}f$. We then get many general properties of Gelfand transform to be transported into our situation with relatively small amount of work.

2.2. Multiplication of the vector functions: polyproduct

Consider now continuous functions f mapping M into \mathbb{C}^d . We are aiming to define a Banach algebra structure into $C(M)^d$. Denoting by \odot the multiplication in $C(M)^d$ we then want that \mathcal{L} takes the vector functions into scalar functions in such a way that \mathcal{L} becomes an algebra homomorphism

$$\mathcal{L}(f \odot g) = (\mathcal{L}f)(\mathcal{L}g)$$

where the multiplication of scalar functions $\mathcal{L}f$ is pointwise.

Since $\sum_{j=1}^d \delta_j(z) = 1$ the constant vector $\mathbf{1} = (1, \dots, 1)^t \in \mathbb{C}^d$ serves as the unit in the algebra. In order to define $f \odot g$ we hence need to code the differences between components of f .

DEFINITION 1. For $a \in \mathbb{C}^d$ we set

$$\square : a \mapsto \square a = \begin{pmatrix} 0 & a_1 - a_2 & \dots & a_1 - a_d \\ a_2 - a_1 & 0 & \dots & a_2 - a_d \\ \dots & \dots & \dots & \dots \\ a_d - a_1 & \dots & a_d - a_{d-1} & 0 \end{pmatrix}$$

and call it *boxing* the vector a .

In order to define the product we still need to introduce two “scaling” entities, matrix L and vector ℓ . The matrix L has zero diagonal and $L_{ij} = 1/(\lambda_i - \lambda_j)$ for $i \neq j$, while the vector $\ell \in \mathbb{C}^d$ has components $\ell_j = 1/p'(\lambda_j)$. Now, denoting by \circ the Hadamard (or Schur, elementwise) product we can define the product as follows.

DEFINITION 2. Let f and g be pointwisely defined functions from $M \subset \mathbb{C}$ into \mathbb{C}^d . Then their “polyproduct” $f \odot g$ is a function defined on M , taking values in \mathbb{C}^d such that

$$(f \odot g)(w) = (f \circ g)(w) - w(L \circ \square f(w) \circ \square g(w))\ell.$$

REMARK 1. We shall write this in short, with slight abuse of notation, as

$$f \odot g = f \circ g - w(L \circ \square f \circ \square g)\ell.$$

Further, for the powers we write $f^n = f \odot f^{n-1}$ and the inverse in particular as f^{-1} whenever it exists: $f \odot f^{-1} = \mathbf{1}$.

PROPOSITION 2. *The vector space of functions*

$$f : M \subset \mathbb{C} \rightarrow \mathbb{C}^d$$

equipped with the product \odot becomes a complex commutative algebra with $\mathbf{1}$ as the unit.

Proof. In addition to the obvious properties of scalar multiplication and summation of vectors we observe that the vector product is commutative

$$f \odot g = g \odot f$$

and since $\square \mathbf{1} = 0$ we have $\mathbf{1} \odot f = f$. Further, since $\square(\alpha f + \beta g) = \alpha \square f + \beta \square g$, we get

$$(\alpha f + \beta g) \odot h = \alpha(f \odot h) + \beta(g \odot h).$$

These are enough for the structure to be an algebra. \square

We shall now consider products $\mathcal{L}f\mathcal{L}g$. Denoting

$$\sigma_{ij} = \frac{1}{p'(\lambda_j)} \frac{1}{\lambda_i - \lambda_j} \tag{7}$$

we can write the polynomials $\delta_i\delta_j$ which appear in such products in a simple way.

LEMMA 1. *We have with $w = p(z)$*

$$\delta_i^2(z) = \delta_i(z) - w \sum_{j \neq i} [\sigma_{ij}\delta_i(z) + \sigma_{ji}\delta_j(z)] \tag{8}$$

while for $i \neq j$

$$\delta_i(z)\delta_j(z) = w [\sigma_{ij}\delta_i(z) + \sigma_{ji}\delta_j(z)]. \tag{9}$$

Proof. Let first $i \neq j$. Since

$$\delta_i = \frac{p}{p'(\lambda_i)(z - \lambda_i)}$$

and $p(z) = w$ we can write

$$\delta_i\delta_j = \frac{w}{p'(\lambda_i)} \frac{\delta_j}{z - \lambda_i}.$$

But $\delta_j/(z - \lambda_i)$ is a polynomial of degree $d - 2$ and can thus be written as a linear combination in these basis polynomials. This gives

$$\frac{\delta_j}{z - \lambda_i} = \frac{1}{\lambda_j - \lambda_i} \delta_j + \frac{p'(\lambda_i)}{p'(\lambda_j)(\lambda_i - \lambda_j)} \delta_i$$

which then yields (9). Consider then (8). Since $\sum_j \delta_j = 1$ we can write $\delta_i = 1 - \sum_{j \neq i} \delta_j$ to get

$$\delta_i^2 = \delta_i - \sum_{j \neq i} \delta_i\delta_j,$$

which, with the help of (9), yields the claim and completes the proof of the lemma. \square

THEOREM 1. *Let f and g be defined in M and $K = p^{-1}(M)$. Then if φ and ψ are functions defined on K by $\varphi = \mathcal{L}f$ and $\psi = \mathcal{L}g$, then $\varphi\psi$ is given by*

$$\varphi\psi = \mathcal{L}(f \odot g).$$

Proof. We can now multiply the expressions for φ and ψ :

$$\varphi\psi = \sum_{i,j} \delta_i f_i \delta_j g_j \tag{10}$$

$$= \sum_i \delta_i^2 f_i g_i + \sum_i \sum_{j \neq i} \delta_i \delta_j f_i g_j \tag{11}$$

$$= \sum_i \delta_i f_i g_i - w \sum_i \sum_{j \neq i} [\sigma_{ij}\delta_i(z) + \sigma_{ji}\delta_j(z)] f_i g_j \tag{12}$$

$$+ w \sum_i \sum_{j \neq i} [\sigma_{ij}\delta_i(z) + \sigma_{ji}\delta_j(z)] f_i g_j. \tag{13}$$

Here the term multiplying δ_k appears in the form

$$f_k g_k - w \sum_{j \neq k} \sigma_{kj} (f_k - f_j)(g_k - g_j)$$

and hence the whole expression reads

$$\varphi \psi = \sum_i \delta_i [f_i g_i - w \sum_{j \neq i} \sigma_{ij} (f_i - f_j)(g_i - g_j)].$$

This is easily seen to be of the form $\varphi \psi = \mathcal{L}(f \odot g)$ which completes the proof of the theorem. \square

2.3. The norm in the algebra

We shall be considering continuous functions f from a compact $M \subset \mathbb{C}$ into \mathbb{C}^d and begin with the uniform norm $|f|_M = \max_{w \in M} |f(w)|_\infty$ where $|a|_\infty = \max_{1 \leq j \leq d} |a_j|$. The definition of polyproduct makes this into an algebra, but $|\cdot|_M$ is not an algebra norm in general, so we need to move into the “operator norm”.

DEFINITION 3. For $f \in C(M)^d$ we set

$$\|f\| = \sup_{|g|_M \leq 1} |f \odot g|_M.$$

This is clearly a norm in $C(M)^d$ and it is in fact equivalent with $|\cdot|_M$.

PROPOSITION 3. *There is a constant C , only depending on M and on Λ such that*

$$\|f \odot g\| \leq \|f\| \|g\| \tag{14}$$

$$|f|_M \leq \|f\| \leq C|f|_M. \tag{15}$$

Proof. In fact

$$|f|_M = |f \odot \mathbf{1}|_M \leq \|f\|.$$

On the other hand, from the definition of the polyproduct it is clear that there exists a constant C such that

$$|f \odot g|_M \leq C|f|_M |g|_M.$$

But then

$$\|f\| = \sup_{|g|_M \leq 1} |f \odot g|_M \leq C|f|_M.$$

Finally,

$$|f \odot g \odot h|_M \leq \|f\| |g \odot h|_M \leq \|f\| \|g\| \|h\|_M$$

implies (14). \square

Since the polyproduct \odot is uniquely determined by Λ , we shall denote the algebra in short as $C_\Lambda(M)$.

DEFINITION 4. The vector space $C(M)^d$ of continuous functions f from a compact $M \subset \mathbb{C}$ into \mathbb{C}^d , with the operator norm $\|f\|$ and product \odot is denoted by $C_\Lambda(M)$.

The discussion can be summarized as follows.

THEOREM 2. *The Banach space $C(M)^d$ equipped with polyproduct \odot , and denoted by $C_\Lambda(M)$, is a commutative unital Banach algebra. The algebra-norm $\|\cdot\|$ is equivalent with $|\cdot|_M$ and functions with components given by polynomials $p(w, \bar{w})$ are dense in $C_\Lambda(M)$.*

Proof. Recall that polynomials $p(w, \bar{w})$ are dense in the sup-norm on a compact $M \subset \mathbb{C}$ among continuous functions by Stone-Weierstrass theorem. Applying this on each component of functions $f \in C_\Lambda(M)$ gives the result. \square

2.4. Characters of $C_\Lambda(M)$

In order to be able to apply the Gelfand theory we need to know all characters in the algebra $C_\Lambda(M)$.

DEFINITION 5. A nontrivial linear functional $\chi : C_\Lambda(M) \rightarrow \mathbb{C}$ is called a character if it is additionally multiplicative:

$$\chi(f \odot g) = \chi(f)\chi(g).$$

The set of all characters is the character space, which we denote here by \mathcal{X} .

REMARK 2. In commutative unital Banach algebras all characters – complex homomorphisms – are automatically bounded and of norm 1. Since maximal ideals are kernels of characters, the focus is sometimes on the maximal ideals rather than on the characters, [1], [2], [3], [13].

Because the polyproduct \odot is constructed to yield $\mathcal{L}(f \odot g) = \mathcal{L}f\mathcal{L}g$, we conclude immediately that for each fixed $z_0 \in p^{-1}(M)$ the functional

$$\chi_{z_0} : f \mapsto \sum_{j=1}^d \delta_j(z_0) f_j(p(z_0)) \tag{16}$$

is a character. We show next that there are no others.

THEOREM 3. *The character space \mathcal{X} is*

$$\mathcal{X} = \{\chi_z : z \in p^{-1}(M)\}$$

where χ_z is given in (16).

Proof. We need to show that all characters are of the form (16). Let $\chi \in \mathcal{X}$ be given and apply it within the subalgebra consisting of elements of the form

$$f = \alpha \mathbf{1},$$

where α is a scalar function $\alpha \in C(M)$. Now, it is well known that all multiplicative functionals in $C(M)$ are given by evaluations at some $w_0 \in M$: $\alpha \mapsto \alpha(w_0)$; hence $\chi(\alpha \mathbf{1}) = \alpha(w_0)$ for some $w_0 \in M$.

Next, take an arbitrary $g \in C_\Lambda(M)$. Then we conclude from

$$\chi(\alpha \mathbf{1} \odot g) = \alpha(w_0)\chi(g)$$

that $\chi(g)$ depends on $g(w_0)$, only. In fact $\chi(\alpha \mathbf{1} \odot g) = \chi(\alpha g) = \alpha(w_0)\chi(g)$ and if $\alpha(w_0) = 1$ we have

$$\chi(g) = \chi(\alpha g) + \chi((1 - \alpha)g)$$

so that $\chi((1 - \alpha)g) = 0$.

We assume next that w_0 is chosen and χ is a character $f \mapsto \chi(f)$ such that the value only depends on $f(w_0)$. We may therefore view χ as an arbitrary linear functional in \mathbb{C}^d which is multiplicative with respect to the polyproduct \odot at w_0 . In fact, setting for $a, b \in \mathbb{C}^d$

$$ab = (a \odot b)(w_0)$$

makes \mathbb{C}^d into a Banach algebra, for each fixed w_0 .

Let $a, b \in \mathbb{C}^d$, then χ is of the form

$$\chi(a) = \sum_{j=1}^d \eta_j a_j$$

and we require

$$\chi((a \odot b)(w_0)) = \chi(a)\chi(b).$$

First observe that $\chi(\mathbf{1}) = 1$ gives $\sum_{j=1}^d \eta_j = 1$. Then, comparing with Lemma 1 and using the notation in the proof of it, we see that we must have

$$\eta_i^2 = \eta_i - w_0 \sum_{j \neq i} (\sigma_{ij} \eta_i + \sigma_{ji} \eta_j) \tag{17}$$

while for $j \neq i$

$$\eta_i \eta_j = w_0 (\sigma_{ij} \eta_i + \sigma_{ji} \eta_j). \tag{18}$$

Let first $w_0 = 0$. Then clearly $\eta_i \in \{0, 1\}$ and, since $\sum_i \eta_i = 1$, only one of the η_i 's can differ from zero resulting in exactly d different solutions.

For $w_0 \neq 0$ we have from (18) that $\eta_i \neq 0$ for all i . We take η_1 as an unknown so that for $j > 1$

$$\eta_j = \frac{w_0 \sigma_{1j} \eta_1}{\eta_1 - w_0 \sigma_{j1}}.$$

Substituting these into (17) and dividing with $\eta_1 \neq 0$ yields

$$\eta_1 = 1 - w_0 \sum_{j \neq 1} \sigma_{1j} - w_0^2 \sum_{j \neq 1} \frac{\sigma_{j1} \sigma_{1j}}{\eta_1 - w_0 \sigma_{j1}}.$$

This has, counting multiplicities, exactly d solutions for η_1 . However, we already know d solutions, namely, $\delta_1(z_k(w_0))$, for $k = 1, \dots, d$ where $p(z_k(w_0)) = w_0$, which completes the proof. \square

2.5. Gelfand transform and the spectrum

When φ is holomorphic it is natural to think φ as the “primary” function which is represented or parametrized by the vector function f . However, when dealing with functions with less smoothness it is easier to think their roles to be reversed. This is because f can be taken as any continuous vector function while the behavior of φ is in general complicated near critical points.

We take $C_\Lambda(M)$ as the *defining algebra* while the functions φ appear as *Gelfand transforms*.

Before applying this machinery we recall some basic properties of Gelfand theory. Let \mathcal{A} be a commutative unital Banach algebra with unit e and denote by h a character:

$$h(ab) = h(a)h(b) \text{ for all } a, b \in \mathcal{A}.$$

Let us denote by $\Sigma_{\mathcal{A}}$ the character space of \mathcal{A} . Then every $h \in \Sigma_{\mathcal{A}}$ has norm 1 and $\Sigma_{\mathcal{A}}$ is compact in the *Gelfand topology*: one gives $\Sigma_{\mathcal{A}}$ the relative weak* -topology it has as a subset of the dual of \mathcal{A} .

Then the *Gelfand transform* of $a \in \mathcal{A}$ is

$$\hat{a} : \Sigma_{\mathcal{A}} \rightarrow \mathbb{C} \text{ where } \hat{a}(h) = h(a).$$

The function \hat{a} is then always continuous in the Gelfand topology and this allows one to study the algebra \mathcal{A} by studying continuous functions on $\Sigma_{\mathcal{A}}$.

Since every maximal ideal of \mathcal{A} is of the form $\mathcal{N}_h = \{a \in \mathcal{A} : h(a) = 0\}$, the character space is sometimes called the maximal ideal space of \mathcal{A} . We collect here basic facts on the Gelfand theory, and here we treat $\hat{a} \in C(\Sigma_{\mathcal{A}})$ as a continuous function with the sup-norm. Recall that the *spectrum* $\sigma(a)$ of an element $a \in \mathcal{A}$ consists of those $\lambda \in \mathbb{C}$ for which $\lambda e - a$ does not have an inverse in \mathcal{A} . We denote by $\rho(a)$ the spectral radius of a : $\rho(a) = \max\{|\lambda| : \lambda \in \sigma(a)\}$.

THEOREM 4. (Gelfand representation theorem) *Let \mathcal{A} be a commutative unital Banach algebra. Then for all $a \in \mathcal{A}$*

- (i) $\sigma(a) = \hat{a}(\Sigma_{\mathcal{A}}) = \{\hat{a}(h) : h \in \Sigma_{\mathcal{A}}\}$;
- (ii) $\rho(a) = \|\hat{a}\|_\infty = \lim_{n \rightarrow \infty} \|a^n\|^{1/n} \leq \|a\|$;
- (iii) $a \in \mathcal{A}$ has an inverse if and only if $\hat{a}(h) \neq 0$ for all $h \in \Sigma_{\mathcal{A}}$;

(iv) $\text{rad } \mathcal{A} = \{a \in \mathcal{A} : \hat{a}(h) = 0 \text{ for all } h \in \Sigma_{\mathcal{A}}\}$.

(See any text book treating Banach algebras, e.g. [1], [2], [3], [13]).

We shall now consider $C_{\Lambda}(M)$. In what follows we write $f \odot f^{n-1} = f^n$ and in particular f^{-1} for the inverse of f . Recall that we denote by \mathcal{X} the character space of $C_{\Lambda}(M)$

$$\mathcal{X} = \{\chi_z : z \in p^{-1}(M)\}$$

where

$$\chi_z(f) = \sum_{j=1}^d \delta_j(z) f_j(p(z)).$$

This allows us to identify χ_z with z and consequently \mathcal{X} with $p^{-1}(M)$. Hence we shall view the Gelfand transform \hat{f} as a function of $z \in p^{-1}(M)$.

DEFINITION 6. Given $f \in C_{\Lambda}(M)$ we set

$$\begin{aligned} \hat{f} : p^{-1}(M) &\rightarrow \mathbb{C} \\ \hat{f} : z &\mapsto \hat{f}(z) = \sum_{j=1}^d \delta_j(z) f_j(p(z)). \end{aligned}$$

Thus, we can view the multicentric representation operator \mathcal{L} as performing the Gelfand transformation

$$\mathcal{L} : f \mapsto \hat{f}.$$

We denote this Gelfand transformation by \mathcal{L} to remind that for constant vectors $a \in \mathbb{C}^d$ the transformation \hat{a} is just the Lagrange interpolation polynomial (restricted to $p^{-1}(M)$). We denote $|\hat{f}|_K = \sup_{z \in K} |\hat{f}(z)|$.

We specify now the general Gelfand representation theorem for the algebra $C_{\Lambda}(M)$.

THEOREM 5. (Multicentric representation as Gelfand transform) For $f \in C_{\Lambda}(M)$ the following hold with $K = p^{-1}(M)$:

- (i) $\sigma(f) = \{\hat{f}(z) : z \in K\}$;
- (ii) $\rho(f) = |\hat{f}|_K = \lim_{n \rightarrow \infty} \|f^n\|^{1/n} \leq \|f\|$;
- (iii) f has an inverse if and only if $\hat{f}(z) \neq 0$ for all $z \in K$;
- (iv) $\text{rad } C_{\Lambda}(M) = \{f \in C_{\Lambda}(M) : \hat{f}(z) = 0 \text{ for all } z \in K\}$.

Recall, that an algebra \mathcal{A} is called *semi-simple* if $\text{rad } \mathcal{A} = \{0\}$.

THEOREM 6. $C_{\Lambda}(M)$ is *semi-simple* if and only if M contains no isolated critical values of p .

Proof. Take $f \in \text{rad}(C_\Lambda(M))$ so that $\hat{f}(z) = 0$ for all $z \in K$. Since \hat{f} determines f uniquely outside critical values, we have $f(w) = 0$ away from the critical values. If every critical value is an accumulation point of M then by continuity f vanishes everywhere. On the other hand, if $w_0 \in M$ is an isolated critical value, take critical points $z_i \in p^{-1}(\{w_0\})$. By assumption, they are isolated and all we need to do is to find $0 \neq a \in \mathbb{C}^d$ such that $\sum_j \delta_j(z_i) a_j = 0$ for all i . However, since w_0 is a critical value at least two of the roots z_i coincide and hence the matrix $(\delta_j(z_i))_{ij}$ is not of full rank. So, we conclude that nontrivial solutions f exist and $C_\Lambda(M)$ is not semi-simple. \square

REMARK 3. If s_A is a simplifying polynomial of minimal degree for an $n \times n$ matrix A (see Definition 9), then all critical values of s_A are isolated and inside $\sigma(s_A(A))$.

2.6. Invertible elements of $C_\Lambda(M)$

From Theorem 5 we can conclude that if φ is given by multicentric representation $\varphi = \mathcal{L}f$ where f is continuous and bounded, then $1/\varphi = \mathcal{L}g$ with a bounded and continuous g if and only if $\varphi(z) \neq 0$ for $z \in p^{-1}(M)$. We shall now derive a quantitative version of this.

THEOREM 7. *There exists a constant C depending on M and Λ such that the following holds. If $f \in C_\Lambda(M)$ is such that for all $z \in p^{-1}(M)$*

$$|\mathcal{L}f(z)| \geq \eta > 0,$$

then there exists $g \in C_\Lambda(M)$ such that $f \odot g = \mathbf{1}$ and

$$\|g\| \leq C \frac{\|f\|^{d-1}}{\eta^d}. \tag{19}$$

Before turning to prove this we look at an instructive example and formulate a lemma of some independent interest.

EXAMPLE 1. We shall first consider the degree two case with $w = z^2 - 1$. If we put $\varphi(z) = \mathcal{L}f(z)$, then the inverse $g = f^{-1}$ is given simply as follows:

$$\begin{pmatrix} g_1(w) \\ g_2(w) \end{pmatrix} = \frac{1}{\varphi(z)\varphi(-z)} \begin{pmatrix} f_2(w) \\ f_1(w) \end{pmatrix}. \tag{20}$$

In fact, since

$$g \odot f = g \circ f + \frac{w}{4}(g_1 - g_2)(f_1 - f_2)\mathbf{1}$$

we have

$$g \odot f = \frac{1}{\varphi(z)\varphi(-z)}(f_1 f_2 - \frac{w}{4}(f_1 - f_2)^2)\mathbf{1}$$

and then expanding $\varphi(z)\varphi(-z)$ we obtain

$$\left[\frac{1+z}{2}f_1 + \frac{1-z}{2}f_2 \right] \left[\frac{1-z}{2}f_1 + \frac{1+z}{2}f_2 \right] = f_1 f_2 - \frac{w}{4}(f_1 - f_2)^2.$$

In particular, the constant C in (19) equals 1 in this case.

The example suggests to look at the inverse in the following way. Denote by $z_j = z_j(w)$ the roots of $p(\zeta) - w = 0$ and when needed, we put $z_1(p(z)) = z$. With $\varphi = \mathcal{L}f$ and $\psi = 1/\varphi = \mathcal{L}g$ we then have

$$\psi(z_1) = \frac{1}{\Phi(w)} \prod_{j=2}^d \varphi(z_j)$$

where $\Phi(w) = \prod_{j=1}^d \varphi(z_j)$. In this notation we have the following result.

LEMMA 2. *Suppose that f has analytic components in M . Then*

$$\Phi : w \mapsto \prod_{j=1}^d \varphi(z_j(w)) \tag{21}$$

is analytic in M .

Proof. All roots $z_j(w)$ are analytic except possibly at critical values. Since $\varphi(z_j(w))$ is given by

$$\varphi(z_j(w)) = \sum_{k=1}^d \delta_k(z_j(w)) f_k(w)$$

with f_k 's analytic, we may as well assume that f_k 's are constants as the only source for lack of analyticity at the critical values would come from products of δ_k 's. But if f_k 's are constants, we may put $q(\zeta) = \sum_{k=1}^d f_k \delta_k(\zeta)$. However, then

$$p(\zeta_1, \dots, \zeta_d) = \prod_{j=1}^d q(\zeta_j)$$

is a symmetric polynomial and it can be expressed uniquely by elementary polynomials s_i by Newton's theorem. If we now substitute $\zeta_j = z_j(w)$, where $z_j(w)$'s are the roots of $p(\zeta) - w = 0$, we observe that all elementary polynomials s_i except s_d are constants. For example, $s_1 = -\sum_{j=1}^d z_j(w) = -\sum_{j=1}^d \lambda_j$, while $s_d(w) = (-1)^d (p(0) - w)$. Thus we arrive at a polynomial in w , which completes the proof. \square

Proof of Theorem 7. We turn now to prove the bound for the inverse. From the assumptions it follows immediately that $|\Phi(w)| \geq \eta^d$. We therefore concentrate in estimating

$$\Phi(w)\psi(z) = \prod_{j=2}^d \varphi(z_j) = \prod_{j=2}^d \sum_{k=1}^d \delta_k(z_j) f_k(w)$$

and organize this as a sum of the form

$$\delta_1(z_2) \cdots \delta_1(z_d) f_1(w)^{d-1} + \cdots = \sum_{|\alpha|=d-1} q_\alpha(z) F_\alpha(w).$$

Here $\alpha = (\alpha_1, \dots, \alpha_d)$ and $F_\alpha(w) = \prod_{k=1}^d f_k(w)^{\alpha_k}$ while $q_\alpha(z)$ is a rather complicated sum of products of different δ_k 's evaluated at z_j 's with $j > 1$. Treating $z_j = z_j(p(z))$

as functions of z , $q_\alpha(z)$ are clearly analytic away from the critical points $z \in \Lambda_1$. Individual products of $\delta_k(z_j)$'s within $q_\alpha(z)$ may have branch points at these critical points while the sum $q_\alpha(z)$ itself is however a polynomial. To see this, let $f_k(w) = x_k$ be constants and denote $x = (x_1, \dots, x_d)^t \in \mathbb{C}^d$. Then $F_\alpha(w) = x^\alpha$ and if we put

$$P(z, x) = \sum_{|\alpha|=d-1} q_\alpha(z)x^\alpha,$$

then we can view $P(z, x)$ as a polynomial in $\mathbb{C} \times \mathbb{C}^d$. In fact, $\Phi(p(z))$ is a polynomial and $\varphi(z)$ divides it so $P(z, x)$ must be a polynomial in z . But then, for example by differentiating $P(z, x)$ with $\partial^\alpha = \prod (\frac{\partial}{\partial x_k})^{\alpha_k}$ gives $\partial^\alpha P(z, x) = \alpha! q_\alpha(z)$ showing that each q_α is a polynomial in z .

Finally we write q_α in the multicentric form $q_\alpha(z) = \sum_{j=1}^d \delta_j(z) Q_{\alpha,j}(w)$ so that

$$\Phi(w)\psi(z) = \sum_{j=1}^d \delta_j(z) \sum_{|\alpha|=d-1} F_\alpha(w) Q_{\alpha,j}(w).$$

Here $|F_\alpha(w)| \leq \|f\|^{d-1}$ and $|\Phi(w)| \geq \eta^d$. But $\psi = \mathcal{L}g$ where

$$g = \sum_{|\alpha|=d-1} F_\alpha Q_\alpha / \Phi$$

and hence with $C = \sum_{|\alpha|=d-1} \|Q_\alpha\|$

$$\|g\| \leq C \frac{\|f\|^{d-1}}{\eta^d}. \quad \square$$

2.7. Characteristic function, resolvent estimates and nilpotent elements

From the previous discussion we see that f is invertible in the algebra if and only if Φ does not vanish. This suggests to introduce a characteristic function for f . This gives still another view to the algebra.

Denoting again by $z_j(w)$ the roots of $p(z) - w = 0$ we have $\lambda \in \sigma(f) = \{\hat{f}(z) : z \in K = p^{-1}(M)\}$ if and only if $\prod_{j=1}^d (\lambda - \hat{f}(z_j(w))) = 0$ at some $w \in M$. Expanding the product as a polynomial in λ takes the form

$$\pi_f(\lambda, w) = \prod_{j=1}^d (\lambda - \hat{f}(z_j(w))) = \lambda^d - \Phi_1(w)\lambda^{d-1} + \dots + (-1)^d \Phi_d(w), \quad (22)$$

since the coefficient functions Φ_j are again functions of w by the same argument as in Lemma 2; notice that Φ_d equals the Φ in (21).

DEFINITION 7. Given $f \in C_\Lambda(M)$ we call $\pi_f(\lambda, w)$ the characteristic function of f . Further, we call $(\lambda \mathbf{1} - f)^{-1}$ the resolvent element whenever it exists.

This allows us to formulate a different version of the estimate for the inversion. To that end we denote by $|\cdot|_\infty$ the max-norm in \mathbb{C}^d .

THEOREM 8. *There exists a constant C , depending on M and on Λ , such that for $w \in M$*

$$|(\lambda \mathbf{1} - f)^{-1}(w)|_\infty \leq C \frac{(|\lambda| + \|f\|)^{d-1}}{|\pi_f(\lambda, w)|}.$$

Proof. This follows in an obvious way from Theorem 7 and from the definitions. \square

REMARK 4. From $\rho(f) = |\hat{f}|_K \leq \|f\|$ we have the lower bound

$$\frac{1}{\text{dist}(\lambda, \sigma(f))} = \left| \frac{1}{\lambda - \hat{f}} \right|_K \leq \|(\lambda \mathbf{1} - f)^{-1}\|. \tag{23}$$

This in particular implies that if $f \neq \lambda \mathbf{1}$ and $\lambda \in \partial\sigma(f)$, there exists $g_n \in C_\Lambda(M)$ of unit length such that $(\lambda \mathbf{1} - f) \odot g_n \rightarrow 0$. In other words, $\lambda \mathbf{1} - f$ is a *topological divisor of zero*.

We noted earlier that $f \in \text{rad } C_\Lambda(M)$ if and only if \hat{f} vanishes identically, or, which is the same thing, $\sigma(f) = \{0\}$.

PROPOSITION 4. *If $\sigma(f) = \{0\}$, then f is nilpotent and there exists $n \leq d$ such that $f^n = 0$.*

Proof. In other words, we need to show that all quasinilpotent elements are actually nilpotent. It is clear from Theorem 6 that nontrivial quasinilpotent elements exist when M contains an isolated critical value, say w_0 . We can proceed now as follows. We view the multiplication

$$f : g \mapsto f \odot g$$

as an operator in $C(M, \mathbb{C}^d)$ and hence for each $w \in M$ there is a matrix $B_f(w)$ such that

$$(f \odot g)(w) = B_f(w)g(w).$$

If f is quasinilpotent, it means that each $B_f(w)$ must be for fixed w quasinilpotent. However, a $d \times d$ -matrix is quasinilpotent only when it is nilpotent, from which the claim follows. \square

EXAMPLE 2. Consider $w = z^2 - 1$. Put $h = \frac{w}{4}(f_1 - f_2)$. Then

$$B_f = \begin{pmatrix} f_1 + h & -h \\ h & f_2 - h \end{pmatrix}$$

which for fixed w has the eigenvalues

$$\frac{1}{2}(f_1(w) + f_2(w)) \pm \frac{\sqrt{1+w}}{2}(f_1(w) - f_2(w)).$$

That is, the eigenvalues are simply $\varphi(z)$ and $\varphi(-z)$. Denote

$$E(z) = \begin{pmatrix} \delta_1(z) & \delta_2(z) \\ \delta_1(-z) & \delta_2(-z) \end{pmatrix}$$

so that for $z \neq 0$

$$E(z)^{-1} = \frac{1}{2z} \begin{pmatrix} z+1 & z-1 \\ z-1 & z+1 \end{pmatrix}.$$

Finally,

$$\begin{pmatrix} \varphi(z) & 0 \\ 0 & \varphi(-z) \end{pmatrix} = E(z)B_f(w)E(z)^{-1}$$

and we see that the eigenvectors are independent of the function f . At $z = 0$ the eigenvalues agree, and $E(0)$ is no longer invertible. Put $f(-1) = (1, -1)^t$ so that $\varphi(0) = 0$. Then

$$B_f(-1) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \text{ is similar to } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

2.8. Quotient algebra $C_\Lambda(M)/\mathcal{I}_{K_0}$

When we apply the functional calculus, discussed in the next section, the natural requirement for φ is that it is well defined at the spectrum $\sigma(A)$ of the operator A , which means that f representing φ must be well defined on a set which includes $p(\sigma(A))$. However, $p^{-1}(p(\sigma(A)))$ is likely to be properly larger than $\sigma(A)$ which in practice shows up in lack of uniqueness in representing φ .

Let $K_0 \subset \mathbb{C}$ be compact, put $p(K_0) = M$ and denote as before $K = p^{-1}(M)$. We assume here that the inclusion $K_0 \subset K$ is proper.

Let \mathcal{I}_{K_0} be the closed ideal in $C_\Lambda(M)$

$$\mathcal{I}_{K_0} = \{f \in C_\Lambda(M) : \hat{f}(z) = 0 \text{ for } z \in K_0\}.$$

Then the set of elements we are dealing with can be identified with the cosets $[f]$:

$$C_\Lambda(M)/\mathcal{I}_{K_0} = \{[f] : [f] = f + \mathcal{I}_{K_0}\}.$$

This is a unital Banach algebra with norm defined as

$$\|[f]\| = \inf_{g \in \mathcal{I}_{K_0}} \|f + g\|.$$

We need to identify the character space of this quotient algebra.

DEFINITION 8. Given a closed ideal $J \subset \mathcal{A}$ the hull of the ideal is the set of all characters which vanish at every element in the ideal.

LEMMA 3. (Theorem 6.2 in [5]) *Given a closed ideal J in a commutative Banach algebra \mathcal{A} , the character space of the quotient algebra \mathcal{A}/J is the hull of J .*

COROLLARY 1. *The quotient algebra $C_\Lambda(M)/\mathcal{I}_{K_0}$ is a Banach algebra with unit and the character space can be identified with K_0 , so that the Gelfand transformation becomes $[f] \mapsto \hat{f}|_{K_0}$.*

2.9. Additional remarks on $\mathcal{L}C_\Lambda(M)$

Here we make some observations on the range of the Gelfand transformation. Denoting $\mathcal{L}C_\Lambda(M) = \{\varphi \in C(K) : \exists f \in C_\Lambda(M) \text{ such that } \varphi = \hat{f}\}$ we clearly have a normed subalgebra of $C(K)$ with the sup-norm on $K = p^{-1}(M)$ but the algebra need not be closed.

EXAMPLE 3. Let $\Lambda = \{-1, 1\}$ so that $p(x) = x^2 - 1$, and $M = [-1, 0] = \{x : -1 \leq x \leq 0\}$ so that $K = p^{-1}(M) = [-1, 1]$. Then $\mathcal{L}C_\Lambda(M)$ contains all polynomials as any polynomial $Q(x)$ can uniquely be written as

$$Q(x) = \sum_{j=1}^d \delta_j(x) Q_j(p(x))$$

where Q_j 's are polynomials. Now, polynomials are dense in $C(K)$ and we conclude that the closure of $\mathcal{L}C_\Lambda(M)$ equals $C(K)$ in this case. However, if we take $\varphi \in C(K)$ such that

$$\varphi(x) = \max\{x^\alpha, 0\}$$

then for $0 < \alpha < 1$ we have $\varphi \in C(K) \setminus \mathcal{L}C_\Lambda(M)$. In fact, for $x \neq 0$ we have $\varphi(x) = \mathcal{L}f(x)$ with $f(x^2 - 1)$ becoming unbounded as x tends to 0. Note, that in this example the Gelfand transformation is injective.

EXAMPLE 4. Let $\Lambda = \{-1, 1\}$ but $K = \Lambda_1 = \{0\}$. Then $C_\Lambda(\{-1\})$ is a two-dimensional complex algebra, with nontrivial radical consisting of vectors f such that $f_1(-1) + f_2(-1) = 0$. On the other hand $\mathcal{L}C_\Lambda(\{-1\})$ is one-dimensional, closed and isomorphic with the complex field.

EXAMPLE 5. Let $\Lambda = \{-1, 1\}$ and $K = \{z : \varepsilon \leq |z| \leq 2\}$ with some small positive ε . Then the critical point, the origin, is not in K and the following hold with some constant C

$$\|\mathcal{L}f\|_\infty \leq \|f\| \leq C \|\mathcal{L}f\|_\infty.$$

However, if $\mathcal{L}f(x + iy) = x^\alpha$ with $0 < \alpha < 1$ for $x > 0$ and vanishing on the left half plane, then

$$\|f\| \sim \text{Const}/\varepsilon^{1-\alpha}.$$

It is natural to ask whether $\varphi \in \mathcal{L}C_\Lambda(M)$ shall be differentiable at the interior critical points. After all, we shall be able to apply the functional calculus in such a case for matrices which do have a nontrivial Jordan block and we usually assume that the value on the off-diagonal would be the derivative of φ at the eigenvalue in question.

EXAMPLE 6. Let again $p(z) = z^2 - 1$ but K such that it contains the critical point in the interior: $K = \{z : |p(z)| \leq 2\}$. Then M likewise contains a neighborhood of -1. We have

$$\varphi(z) = \frac{1}{2}[f_1(w) + f_2(w)] + \frac{z}{2}[f_1(w) - f_2(w)].$$

If $\varphi \in \mathcal{L}C_\Lambda(M)$, then $f_i \in C(M)$ and we have

$$\frac{1}{2z}[\varphi(z) - \varphi(-z)] = \frac{1}{2}[f_1(z^2 - 1) - f_2(z^2 - 1)]$$

and hence the limit

$$\lim_{z \rightarrow 0} \frac{1}{2z}[\varphi(z) - \varphi(-z)] = \frac{1}{2}[f_1(-1) - f_2(-1)]$$

always exists. However, it does not imply that φ would be differentiable at the origin. In fact, we have

$$\frac{1}{z}[\varphi(z) - \varphi(0)] \tag{24}$$

$$= \frac{1}{2z}\{[f_1(z^2 - 1) + f_2(z^2 - 1)] - [f_1(-1) + f_2(-1)]\} \tag{25}$$

$$+ \frac{1}{2}[f_1(z^2 - 1) - f_2(z^2 - 1)]. \tag{26}$$

Here the last term is continuous as z tends to origin. Thus the derivative exists depending on the behavior of $f_1 + f_2$ near $w = -1$. In particular, if $f_1 + f_2$ is Hölder continuous with exponent $\alpha > 1/2$, then φ is differentiable.

3. Functional calculi

3.1. Functional calculus for matrices

We discuss first the functional calculus related to $C_\Lambda(M)$ for matrices. Denote by \mathbb{M}_n complex $n \times n$ -matrices with the norm

$$\|A\| = \sup_{|x|_2=1} |Ax|_2.$$

Further, we denote by $\sigma(A) = \{\alpha_k\}$ the eigenvalues of A and by m_A the minimal polynomial of A , that is, the monic polynomial q of smallest degree such that $q(A) = 0$:

$$m_A(z) = \prod_{k=1}^m (z - \alpha_k)^{n_k+1}.$$

Clearly a functional calculus for A can exist only when the functions are defined at the spectrum. As mentioned in the introduction, the usual way is to say that φ is defined at the spectrum $\sigma(A) = \{\alpha_k\}$ if the values $\varphi^{(i)}(\alpha_k)$ are known for $0 \leq i \leq n_k$, [6], [11]. Based on this information one can construct the Hermite interpolation polynomial p for φ and set $\varphi(A) = p(A)$. As we saw in Example 6 the functions in our algebra do not need to be differentiable - but of course when they are the resulting functional calculus yields the same matrices $\varphi(A)$.

DEFINITION 9. Given $A \in \mathbb{M}_n$ we call all monic polynomials p such that $p(A)$ is similar to a diagonal matrix as simplifying polynomials for A .

If \mathcal{K} denotes those indices k for which $n_k > 0$ in the minimal polynomial, then setting

$$s_A(z) = \int_0^z \prod_{k \in \mathcal{K}} (\zeta - \alpha_k)^{n_k} d\zeta + c$$

we have a polynomial of minimal degree such that $s_A^{(j)}(\alpha_k) = 0$ for $j = 1, \dots, n_k$ and $k \in \mathcal{K}$. Clearly then $s_A(A)$ is similar to the diagonal matrix $\text{diag}(s_A(\alpha_k))$. Since we can add an arbitrary constant to s_A we may assume as well that s_A has distinct roots.

Let now p be a simplifying polynomial for A with distinct roots and assume φ is given on $\sigma(A)$ as

$$\varphi(z) = \sum_{j=1}^d \delta_j(z) f_j(p(z)).$$

Denoting $B = p(A)$ we could then define for $f_j \in C(\sigma(B))$ the matrix function $f_j(B)$ either by Lagrange interpolation at $p(\alpha_k)$ or by assuming the similarity transformation to the diagonal form $B = TDT^{-1}$ be given and setting $f_j(B) = T f_j(D) T^{-1}$, both yielding the same matrix $f_j(B)$ which commute with A . Then the following matrix is well defined:

$$\varphi(A) = \sum_{j=1}^d \delta_j(A) f_j(B).$$

It follows immediately that if we have two functions $f, g \in C_\Lambda(\sigma(B))$, and we denote $\varphi = \mathcal{L}f$, $\psi = \mathcal{L}g$ and $\varphi\psi = \mathcal{L}(f \odot g)$, then this definition yields

$$(\varphi\psi)(A) = \varphi(A)\psi(A).$$

However, we formulate the exact statement using a different notation to underline the fact that knowing the values of φ at the spectrum of A need not determine f uniquely, and hence not $\varphi(A)$, either.

DEFINITION 10. Assume p is a simplifying polynomial for $A \in \mathbb{M}_n$ with distinct roots Λ . Then we denote by χ_A the mapping $C_\Lambda(p(\sigma(A))) \rightarrow \mathbb{M}_n$ given by

$$f \mapsto \chi_A(f) = \sum_{j=1}^d \delta_j(A) f_j(B). \tag{27}$$

THEOREM 9. *The mapping χ_A is a continuous homomorphism $C_\Lambda(p(\sigma(A))) \rightarrow \mathbb{M}_n$.*

Proof. That χ_A is a homomorphism is build in the construction and in particular we have

$$\chi_A(f \odot g) = \chi_A(f) \chi_A(g).$$

The continuity of χ_A is seen from

$$\|\chi_A(f)\| \leq \sum_{j=1}^d \|\delta_j(A)\| \|f_j(B)\|$$

combined with $\|f_j(B)\| \leq \varkappa(T)|f|_{\sigma(p(A))}$ and with $|f|_{\sigma(p(A))} \leq \|f\|$, see Proposition 3. Here $\varkappa(T) = \|T\| \|T^{-1}\|$ denotes the condition number of the diagonalizing similarity transformation. \square

We can now also conclude that we can formulate a spectral mapping theorem. Let $M = p(\sigma(A))$ and, as it would likely to be the case, $\sigma(A)$ is a proper subset of $p^{-1}(M)$. Then it follows from Corollary 1 that the spectrum of $[f]$ in $C_\Lambda(M)/\mathcal{I}_{\sigma(A)}$ is $\sigma([f]) = \{\hat{f}(z) : z \in \sigma(A)\}$.

THEOREM 10. *We have for $[f] \in C_\Lambda(p(\sigma(A)))/\mathcal{I}_{\sigma(A)}$ and $\chi_A(f) \in \mathbb{M}_n$*

$$\sigma(\chi_A(f)) = \sigma([f]).$$

Proof. Even so the statement may look rather complicated the proof here can be reduced to the standard spectral mapping theorem for polynomials. However, the statement holds as such in more general setting and then in particular the present simple proof is not available.

Consider $f_j(B)$ where $B = p(A)$ and denote by β_i the eigenvalues of B . There are in general $s \leq m$ different eigenvalues of B . Let q_j be the polynomial of degree $s - 1$ such that

$$q_j(\beta_i) = f_j(\beta_i) \text{ for } i = 1, \dots, s. \tag{28}$$

Then we set $f_j(B) = q_j(B)$. Thus we have

$$\chi_A(f) = P(A) \tag{29}$$

if we set $P(z) = \sum_{j=1}^d \delta_j(z)q_j(p(z))$. The conclusion follows as P is a polynomial. \square

REMARK 5. There are two different steps to be taken when constructing $\chi_A(f)$.

(i) Given $A \in \mathbb{M}_n$ one could for example compute the Schur decomposition of A . From there one must decide what diagonal elements are to be considered as the same and based on that one chooses a simplifying polynomial p such that it has simple roots. Notice in particular that then the eigenvalues α_k for which $n_k > 0$, are distinct from the roots λ_j of p .

(ii) Given f one then computes the Lagrange interpolating polynomials $q_j(w)$ satisfying (28) for each j .

Then $\chi_A(f)$ is given by (29).

REMARK 6. It is natural to ask how this approach is different from the definition based on Hermite interpolation on the spectrum of A . Consider the minimal polynomial m_A as the simplifying polynomial. In the Hermite interpolation one interpolates at the eigenvalues while we add a constant c so that the polynomial $p(z) = m_A(z) + c$ has simple roots. The effect on the differentiability requirement on f and/or φ is then removed and replaced by a balanced limiting behavior of the roots of p near its critical

points - and this happens automatically, independent of the function f as long as it is continuous. To illustrate this, suppose f is holomorphic and $\varphi = \mathcal{L}f$ so that

$$\varphi'(z) = \sum_{j=1}^d [\delta'_j(z)f_j(p(z)) + \delta_j(z)f'_j(p(z))p'(z)].$$

However, at critical points z_c we have $\varphi'(z_c) = \sum_{j=1}^d \delta'_j(z_c)f_j(p(z_c))$ so this value does not depend on whether f is differentiable at critical values or not. See also Example 6.

3.2. Polynomially normal operators in Hilbert spaces

We shall now consider bounded operators A in complex Hilbert spaces H . The operator norm of $A \in \mathcal{B}(H)$ is denoted by $\|A\|$.

DEFINITION 11. We call $A \in \mathcal{B}(H)$ polynomially normal, if there exists a non-constant monic polynomial p such that $p(A)$ is normal. The polynomial p is then called a simplifying polynomial for A .

Polynomially normal operators have been discussed in [4], [7], as operator valued roots for polynomial equations $p(z) - N = 0$ with N normal. We formulate a structure result (see Theorem 3.1, in [7], also Theorem 2 in [8]).

THEOREM 11. *Let H be separable and $A \in \mathcal{B}(H)$ such that $p(A)$ is normal for some nonconstant polynomial p . Then there exist reducing subspaces $\{H_n\}_{n=0}^\infty$ for A , such that $H = \bigoplus_{n=0}^\infty H_n$ and $A|_{H_0}$ is algebraic while $A|_{H_n}$ are for $n \geq 1$ similar to normal.*

We could take use of this structure result but proceed independently of it. We start by assuming that $p(A)$ is normal and then comment the straightforward extension to the case where $p(A)$ is similar to normal.

Let $N = p(A)$ be normal, and as before, we may assume that p has simple roots. Then the first task is to define $f_j(N)$ in a consistent way. Recall the following two results, see e.g. [2].

LEMMA 4. *Let $M \subset \mathbb{C}$ be compact. Then the closure of polynomials of the form $q(w, \bar{w})$ in the uniform norm over M equals $C(M)$.*

Since N commutes with N^* the operator $q(N, N^*)$ is well defined and the following holds.

LEMMA 5. *If $N \in \mathcal{B}(H)$ is normal, then*

$$\|q(N, N^*)\| = \max_{w \in \sigma(N)} |q(w, \bar{w})|.$$

Given now a normal operator N and a continuous function f_j on $\sigma(N)$ one approximates f_j by a sequence $\{q_{j,n}\}$ such that

$$|f_j - q_{j,n}|_\infty = \max_{w \in \sigma(N)} |f_j(w) - q_{j,n}(w, \bar{w})| \rightarrow 0.$$

But then $\{q_{j,n}(N, N^*)\}$ is a Cauchy sequence and we can define

$$f_j(N) = \lim_{n \rightarrow \infty} q_{j,n}(N, N^*). \tag{30}$$

Then $f_j(N) \in \mathcal{B}(H)$ is normal, with $\|f_j(N)\| = |f_j|_\infty \leq \|f\|$.

DEFINITION 12. Assume p is a simplifying polynomial for $A \in \mathcal{B}(H)$ with distinct roots Λ , so that $N = p(A)$ is normal. Then we denote by χ_A the mapping $C_\Lambda(p(\sigma(A))) \rightarrow \mathcal{B}(H)$ given by

$$f \mapsto \chi_A(f) = \sum_{j=1}^d \delta_j(A) f_j(N). \tag{31}$$

Note that here $\delta_j(A)$ and $f_j(N)$ commute. In fact, A commutes with $N = p(A)$ and since N commutes with N^* the operator A commutes with N^* as well, by Fuglede’s theorem, [2]. We combine the construction into the following theorem.

THEOREM 12. Let $A \in \mathcal{B}(H)$ and a simplifying polynomial p be given as in Definition 12. Then the mapping χ_A is a continuous homomorphism from $C_\Lambda(p(\sigma(A)))$ to $\mathcal{B}(H)$. In particular,

$$\chi_A(f \odot g) = \chi_A(f) \chi_A(g)$$

and

$$\|\chi_A(f)\| \leq C \|f\| \text{ with } C = \sum_{j=1}^d \|\delta_j(A)\|.$$

REMARK 7. (The case of $p(A)$ similar to normal) We can extend the construction above to operators which are similar to polynomially normal ones. In short, assume that $A \in \mathcal{B}(H)$ is such that there exists a polynomial p and a bounded T with bounded inverse, such that $N = T^{-1}p(A)T$ is normal. Denote $V = T^{-1}AT$ so that $N = p(V)$ and $B = p(A)$. Then we can define

$$f_j(B) = T f_j(N) T^{-1}$$

and again A commutes with $f_j(B)$ as $A f_j(B) = T [V f_j(p(V))] T^{-1}$. This allows us to define

$$\chi_A(f) = T \chi_V(f) T^{-1} \tag{32}$$

and the extension shares all the natural properties.

REMARK 8. (*Spectral measure*) Recall that if N is normal, then there exists (see e.g. Section 12 in [14]) a spectral measure E from the σ -algebra of all Borel sets of $\sigma(N)$ into $B(H)$ such that if φ is an essentially bounded Borel-measurable function on $\sigma(N)$ then

$$\varphi(N) = \int_{\sigma(N)} \varphi dE.$$

This could in an obvious way be used in defining $f_j(p(A))$, thus extending the functional calculus even further.

3.3. Spectral mapping theorem for operators

If $A \in \mathcal{B}(H)$ is such that $p(A)$ is similar to normal, then we have $\chi_A(f) = T\chi_V(f)T^{-1}$ and therefore $\chi_A(f)$ and $\chi_V(f)$ have the same spectrum. Therefore we may as well assume that A is polynomially normal.

THEOREM 13. *Suppose p has simple zeros and $A \in \mathcal{B}(H)$ is such that $p(A)$ is normal. Then for all $[f] \in C_\Lambda(p(\sigma(A)))/\mathcal{I}_{\sigma(A)}$ we have*

$$\sigma(\chi_A(f)) = \sigma([f]).$$

Proof. Recall that $\sigma([f]) = \{\hat{f}(z) : z \in \sigma(A)\}$. Consider first the inclusion

$$\hat{f}(z) \in \sigma(\chi_A(f)) \text{ for all } z \in \sigma(A). \tag{33}$$

where f is of the form

$$f_j(w) = q_j(w, \bar{w}). \tag{34}$$

We take a $\lambda \in \sigma(A)$ and need to show that $\hat{f}(\lambda) \in \sigma(\chi_A(f))$. The discussion splits into two as to whether

$$\lambda \in \sigma_{ap}(A), \tag{35}$$

or, if that is not the case, then necessarily,

$$\bar{\lambda} \in \sigma_p(A^*). \tag{36}$$

Since $p(A)$ is normal we have in both cases $p(\lambda) \in \sigma_{ap}(p(A))$.

Assuming (35) there exists a sequence of unit vectors x_n such that

$$(A - \lambda)x_n \rightarrow 0 \tag{37}$$

which by writing $p(A) - p(\lambda) = q(A, \lambda)(A - \lambda)$ implies immediately that

$$(p(A) - p(\lambda))x_n \rightarrow 0.$$

But then also

$$(p(A) - p(\lambda))^*x_n \rightarrow 0.$$

In fact, if N is normal and $Ny_n \rightarrow 0$, then

$$(Ny_n, Ny_n) = (N^*y_n, N^*y_n) \rightarrow 0.$$

Denoting $p(A) = N$ and $p(\lambda) = v$ we have

$$\chi_A(f) - \hat{f}(\lambda) \tag{38}$$

$$= \sum_{j=1}^d \delta_j(A) [Q_j(N, N^*) - Q_j(v, \bar{v})] \tag{39}$$

$$+ \sum_{j=1}^d [\delta_j(A) - \delta_j(\lambda)] Q_j(v, \bar{v}). \tag{40}$$

Operating with these at x_n we have

$$[Q_j(N, N^*) - Q_j(v, \bar{v})]x_n \rightarrow 0$$

since both $(N - v)x_n$ and $(N^* - \bar{v})x_n$ tend to 0. In fact, there are polynomials R, S of three variables such that we can write

$$\begin{aligned} Q(N, N^*) - Q(v, \bar{v}) &= [Q(N, N^*) - Q(v, N^*)] + [Q(v, N^*) - Q(v, \bar{v})] \\ &= R(N, v, N^*)(N - v) + S(v, N^*, \bar{v})(N^* - \bar{v}) \end{aligned}$$

Likewise, by (37), $[\delta_j(A) - \delta_j(\lambda)]Q_j(v, \bar{v})x_n \rightarrow 0$, and so $\hat{f}(\lambda) \in \sigma_{ap}(\chi_A(f))$.

Next, assume that $\bar{\lambda} \in \sigma_p(A^*)$ and suppose x is an eigenvector such that

$$A^*x = \bar{\lambda}x.$$

Then clearly

$$\{[\delta_j(A) - \delta_j(\lambda)]Q_j(v, \bar{v})\}^*x = 0.$$

However, we also have

$$Q_j(N, N^*)^*x = \overline{Q_j(v, \bar{v})}x$$

since from $A^*x = \bar{\lambda}x$ we conclude $p(A)^*x = \overline{p(\bar{\lambda})}x$ and so $N = p(A)$ being normal this implies $Nx = vx$ as well. Substituting these into $\chi_A(f)^* - \overline{\hat{f}(\lambda)}$ gives

$$[\chi_A(f)^* - \overline{\hat{f}(\lambda)}]x = 0.$$

Hence $\overline{\hat{f}(\lambda)} \in \sigma_p(\chi_A(f)^*)$ and so $\hat{f}(\lambda) \in \sigma(\chi_A(f))$.

We still need to show (33) when \hat{f} is not of the form (34). To that end assume that \hat{f}_n approximates \hat{f} uniformly in $\sigma(A)$ where \hat{f}_n is of the special form (34).

Take $\mu \in \hat{f}(\sigma(A))$ and we need to show that $\mu \in \sigma(\chi_A(f))$. For some $\lambda \in \sigma(A)$ we thus have $\mu = \hat{f}(\lambda)$. Let $\{\hat{f}_n\}$ be an approximative sequence of the special form (34) such that in particular

$$\sup_{z \in \sigma(A)} |\hat{f}(z) - \hat{f}_n(z)| \rightarrow 0$$

and hence also

$$\chi_A(f) = \lim_n \chi_A(f_n).$$

Fix an arbitrary open set V such that $\sigma(\chi_A(f)) \subset V$. We show that $\mu \in V$ which completes the argument. Fix an open set U such that

$$\sigma(\chi_A(f)) \subset U \subset \text{cl}(U) \subset V.$$

Since the spectrum is upper semicontinuous (e.g. Theorem 3.4.2 in [1]) there exists an $\varepsilon > 0$ such that

$$\sigma(B) \subset U \text{ whenever } \|\chi_A(f) - B\| < \varepsilon.$$

Let n_ε be such that $\|\chi_A(f) - \chi_A(f_n)\| < \varepsilon$ for all $n \geq n_\varepsilon$. Then $\sigma(\chi_A(f_n)) \subset U$. But for \hat{f}_n we then have

$$\hat{f}_n(\lambda) \in \sigma(\chi_A(f_n)) \subset U.$$

Finally, from $\hat{f}_n(\lambda) \rightarrow \hat{f}(\lambda)$ we conclude that

$$\mu = \hat{f}(\lambda) \in \text{cl}(U) \subset V.$$

Consider now the other direction. Here the conclusion follows easily already from Corollary 1 with $K_0 = \sigma(A)$. In fact, suppose $\hat{f}(z) \neq 0$ for $z \in \sigma(A)$. Then there exists $g \in C_\lambda(\sigma(p(A)))$ such that

$$\hat{f}(z)\hat{g}(z) = 1 \text{ for } z \in \sigma(A).$$

By Theorem 5 we then know that $[g]$ is the inverse of $[f]$ and since χ_A is a homomorphism from $C_\lambda(\sigma(p(A)))/\mathcal{I}_{\sigma(A)}$ to $\mathcal{B}(H)$, we have

$$\chi_A(f)\chi_A(g) = I$$

and $0 \notin \sigma(\chi_A(f))$. Thus, if $\mu \in \sigma(\chi_A(f))$, then there must exist $\lambda \in \sigma(A)$ such that $\hat{f}(\lambda) - \mu = 0$. But this simply means that $\sigma(\chi_A(f)) \subset \hat{f}(\sigma(A))$. \square

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