

INCLUSION SYSTEMS OF HILBERT MODULES OVER THE C^* -ALGEBRA OF COMPACT OPERATORS

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Abstract. In this note we observe inclusion systems of Hilbert modules over the C^* -algebra of all compact operators acting on a Hilbert space. We prove that if each Hilbert C^* -module in the generated product system is strictly complete, then it is possible to construct a bijection between the set of all units of an inclusion system and a quotient (by a suitable equivalence relation) of a certain set of units in the generated product system. Thereby we obtain a generalization of the result that provides the existence of a bijection between the set of all units in an inclusion system of Hilbert spaces and the set of all units in the generated product system (B. V. R. Bhat and M. Mukherjee [*Inclusion systems and amalgamated products of product systems*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **13** (2010), no. 1, 1–26]).

1. Introduction and preliminary results

Inclusion systems are parametrized families of Hilbert spaces exactly like product systems except that unitaries, as the linking maps, are replaced by isometries. These objects appear in the field of product systems. Associating product systems to CP-semigroups is actually performed using the inclusion systems and the inductive limit procedure, as being done in [7]. In [6] B. V. Rajarama Bhat and Mithun Mukherjee define inclusion systems of Hilbert spaces and use the essence of the above mentioned method from [7] to show that every inclusion system gives rise to a product system in a natural way by taking inductive limits. They also notice that basic properties of product systems such as, for example, existence of units and structure of morphisms can be read off at the level of inclusion systems.

The purpose of this paper is to generalize the notion of inclusion system of Hilbert spaces from [6] and to obtain part of similar results as therein but in a more general context. We observe inclusion systems of two-sided Hilbert modules over the C^* -algebra of all compact operators acting on a Hilbert space and use the concept of extensions of Hilbert C^* -modules (from [3]) to get the result. In detail, we prove that if each Hilbert C^* -module in the generated product system is strictly complete, then it is possible to construct a bijection between the set of all units of an inclusion system and a quotient (by a suitable equivalence relation) of a certain set of units in the generated product system. Also, we notice that it is a generalization of the result that provides the existence

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of a bijection between the set of all units in an inclusion system of Hilbert spaces and the set of all units in the generated product system [6].

A Hilbert C^* -module over a C^* -algebra \mathcal{B} is a right \mathcal{B} -module E equipped with an \mathcal{B} -valued inner product $\langle \cdot, \cdot \rangle$, linear over \mathcal{B} in the second and conjugate linear in the first variable, so that E is complete with respect to the norm $\| \cdot \| = \sqrt{\| \langle \cdot, \cdot \rangle \|}$. Let us denote by $\langle E, E \rangle$ the closed linear span of all elements of the form $\langle x, y \rangle \in \mathcal{B}$, $x, y \in E$. E is said to be a full Hilbert C^* -module if $\langle E, E \rangle = \mathcal{B}$. If E and F are Hilbert \mathcal{B} -modules, we denote by $B^a(E, F)$ the Banach space of all adjointable operators $E \rightarrow F$. The ideal of “compact” operators is denoted by $K(E, F)$. When $E = F$, we write $B^a(E)$ and $K(E)$ instead of $B^a(E, F)$ and $K(E, F)$. A Hilbert $\mathcal{A} - \mathcal{B}$ module is a Hilbert \mathcal{B} -module with a non-degenerate $*$ -representation of a C^* -algebra \mathcal{A} by elements in the C^* -algebra $B^a(E)$ of adjointable (and, therefore, bounded and right linear) homomorphisms on E . For basic facts about Hilbert modules over C^* -algebras we refer the reader to [9], [11], [13].

1.1. Extensions of Hilbert C^* -modules

In [3] the concept of extensions of Hilbert C^* -modules is developed:

DEFINITION 1. Let E be a full Hilbert C^* -module over a C^* -algebra \mathcal{B} . An extension of E is a triple $(\tilde{E}, \mathcal{A}, \Phi)$ so that

1. \mathcal{A} is a C^* -algebra containing \mathcal{B} as an ideal;
2. \tilde{E} is a Hilbert \mathcal{A} -module;
3. $\Phi : E \rightarrow \tilde{E}$ is a map satisfying $\langle \Phi(x), \Phi(y) \rangle = \langle x, y \rangle$ ($x, y \in E$);
4. $\text{Im}\Phi = \tilde{E}\mathcal{B}$, i.e. $\text{Im}\Phi$ is the ideal submodule of \tilde{E} associated to \mathcal{B} . An extension $(\tilde{E}, \mathcal{A}, \Phi)$ is said to be essential if \mathcal{B} is an essential ideal in \mathcal{A} .

By 3, the map Φ is a \mathcal{B} -linear isometry of Hilbert C^* -modules and, hence, E and $\Phi(E)$ are unitarily equivalent Hilbert C^* -modules.

Let us denote by E_d the Hilbert C^* -module over the multiplier algebra $M(\mathcal{B})$ (\mathcal{B} is a non unital C^* -algebra) consisting of all adjointable maps from \mathcal{B} to E with the inner product $\langle r, s \rangle = r^*s$ so that the resulting norm coincides with the operator norm on E_d . Let $\Gamma : E \rightarrow E_d$ be defined by $\Gamma(x) = r_x$, where $r_x(b) = xb$, $b \in \mathcal{B}$. As it can be seen in [3], $(E_d, M(\mathcal{B}), \Gamma)$ is an essential extension of E . There hold

$$\begin{aligned} \Gamma(xb) &= \Gamma(x)b, \quad x \in E, \quad b \in \mathcal{B}; \\ \langle \Gamma(x), \Gamma(y) \rangle &= \langle x, y \rangle, \quad x, y \in E, \end{aligned} \tag{1}$$

i.e. Γ is a \mathcal{B} -linear isometry and E and $\Gamma(E)$ are unitarily equivalent Hilbert C^* -modules. Also, if we use the identification

$$E \ni x \leftrightarrow r_x = \Gamma(x) \in E_d, \tag{2}$$

then $E = E_d\mathcal{B}$ is a submodule of E_d corresponding to the essential ideal \mathcal{B} in $M(\mathcal{B})$.

The strict topology on E_d with respect to $\Gamma(E)$ (or the $\Gamma(E)$ -strict topology) is defined by the family of seminorms $r \mapsto \|\langle r, x \rangle\|$ ($x \in \Gamma(E)$) and $r \mapsto \|rb\|$ ($b \in \mathcal{B}$). A net (r_i) in E_d converges $\Gamma(E)$ -strictly to $r \in E_d$ (which is denoted by $r = (st.) \lim_i r_i$) if and only if the following two conditions are satisfied:

1. $\langle r, \Gamma(x) \rangle = \lim_i \langle r_i, \Gamma(x) \rangle$ ($x \in E$),
2. $rb = \lim_i r_i b$ ($b \in \mathcal{B}$).

Using [3, Definition 1.5], we quote [3, Theorem 1.8] as

THEOREM 1. *Let E be a full Hilbert \mathcal{B} -module. Then $(E_d, M(\mathcal{B}), \Gamma)$ is a E -strictly complete extension of E , i.e. E_d is complete in the strict topology with respect to the ideal submodule $E_d \mathcal{B} = Im\Gamma$.*

DEFINITION 2. A full Hilbert C^* -module E is said to be strictly complete if $E_d = \Gamma(E)$.

If E is a full Hilbert C^* -module over a unital C^* -algebra \mathcal{B} ($1 \in \mathcal{B}$), then E is strictly complete since for each $r \in E_d$ there holds $r = r_{r(1)} = \Gamma(r(1)) \in \Gamma(E)$. Hence, $E_d = \Gamma(E)$. The converse is not true [3, Example 2.6]. Also, each full Hilbert C^* -module so that $K(E)$ is a unital C^* -algebra is strictly complete. The above two classes are the only known classes of strictly complete modules [4]. The question that is raised in [3] is to determine all strictly complete full Hilbert C^* -modules.

1.2. Hilbert C^* -modules over the C^* -algebra of compact operators

In [2] Hilbert C^* -modules over C^* -algebras of compact operators on a Hilbert space are described. The important result is stated in Proposition 1 of the above mentioned paper and in the discussion that precedes it. We quote them here:

Let E be an arbitrary Hilbert C^* -module over the C^* -algebra of all compact operators $K(H)$ with $\dim H = \infty$. Consider the ideal $HS \subset K(H)$ of all Hilbert-Schmidt operators on H and let $E_{HS}^0 = \text{span}(EHS)$ be the linear span of

$$EHS = \{xb \mid x \in E, b \in HS\}.$$

Obviously, E_{HS}^0 is a submodule of E and, also, it is a right module over the H^* -algebra HS . The inner product, defined on E , applied to the elements of E_{HS}^0 takes values in the trace class $\tau \subset HS$. That provides E_{HS}^0 with the inner product $(\cdot, \cdot) = \text{tr}(\langle \cdot, \cdot \rangle)$. (H^* -modules are discussed in [1].)

PROPOSITION 1. [2, Proposition 1] *Let E be a Hilbert C^* -module over $K(H)$. Then there is a Hilbert H^* -module $E_{HS} \subset E$ over the H^* -algebra $HS \subset K(H)$. There is a norm $\|\cdot\|_{HS}$ on E_{HS} defined by $\|x\|_{HS}^2 = \text{tr}(\langle x, x \rangle)$ satisfying*

$$\|x\| \leq \|x\|_{HS}, \quad x \in E_{HS}. \tag{3}$$

The submodule $E_{HS} = \overline{E_{HS}^0}^{\|\cdot\|_{HS}}$ (the completion is with respect to $\|\cdot\|_{HS}$) is dense in E with respect to the original C^* -module norm on E .

REMARK 1. E_{HS} is a Hilbert space with the inner product $(\cdot, \cdot) = \text{tr}(\langle \cdot, \cdot \rangle)$.

Throughout the whole paper, \mathcal{B} denotes the C^* -algebra $K(H)$ of all compact operators on a Hilbert space H . If $\dim H = \infty$, \mathcal{B} is a non unital C^* -algebra. Also, every Hilbert \mathcal{B} -module E is full since $\mathcal{B} = K(H)$ has no nontrivial closed two-sided ideals. It is already known that each bounded \mathcal{B} -linear operator acting on a Hilbert \mathcal{B} -module is adjointable ([10], [8]). So, all the mappings between Hilbert \mathcal{B} -modules that we mention here are adjointable.

2. Inclusion systems

We begin this section with the definition of an inclusion system of Hilbert $\mathcal{B} - \mathcal{B}$ modules and then we show that every inclusion system gives rise to a product system in a natural way, by taking inductive limits. That technique is not significantly different from the one given in [6, Section 2].

REMARK 2. We note that a technical result [13, Proposition A.10.10], that is going to be used in the proof of Theorem 2, is proved for two-sided Hilbert modules where the left C^* -algebra and the right C^* -algebra are different. However, taking into account the definition of the inner tensor product of two-sided Hilbert modules (Definition 3 below), we consider inclusion systems of Hilbert $\mathcal{B} - \mathcal{B}$ modules.

DEFINITION 3. An inclusion system (E, β) is a family of Hilbert $\mathcal{B} - \mathcal{B}$ modules $E = (E_t)_{t>0}$, together with two-sided ($\mathcal{B} - \mathcal{B}$ linear) isometries

$$\beta_{s,t} : E_{s+t} \rightarrow E_s \otimes E_t \quad (s, t > 0),$$

fulfilling the co-associativity condition

$$(\beta_{r,s} \otimes I_{E_t})\beta_{r+s,t} = (I_{E_r} \otimes \beta_{s,t})\beta_{r,s+t} \quad \forall r, s, t > 0.$$

Here \otimes stands for the so-called inner tensor product obtained by identifications $ub \otimes v \sim u \otimes bv$, $u \otimes vb \sim (u \otimes v)b$, $bu \otimes v \sim b(u \otimes v)$, ($u \in E_r$, $v \in E_s$, $b \in \mathcal{B}$) and then completing in the inner product $\langle u \otimes v, u_1 \otimes v_1 \rangle = \langle v, \langle u, u_1 \rangle v_1 \rangle$.

If all $\beta_{s,t}$ are unitaries, then (E, β) is said to be a product system.

REMARK 3. Every product system is an inclusion system. As it has been done in [6, Section 2], we point out a possible problem with terminology since the linking unitary mappings for 'product system' usually map $E_s \otimes E_t$ to E_{s+t} and they are associative. Here we take their adjoint mappings which are co-associative.

Consider an inclusion system (E, β) . For $t > 0$, $J_t = \{(t_n, t_{n-1}, \dots, t_1) \mid t_i > 0, \sum_{i=1}^n t_i = t, n \geq 1\}$. For $\mathfrak{s} = (s_m, s_{m-1}, \dots, s_1) \in J_s$ and $\mathfrak{t} = (t_n, t_{n-1}, \dots, t_1) \in J_t$ the joint tuple $\mathfrak{s} \smile \mathfrak{t} \in J_{s+t}$ is defined by $\mathfrak{s} \smile \mathfrak{t} = (s_m, s_{m-1}, \dots, s_1, t_n, t_{n-1}, \dots, t_1) \in J_{s+t}$. There is a partial order on J_t : $\mathfrak{t} \geq \mathfrak{s} = (s_m, s_{m-1}, \dots, s_1)$ if for each $i \in \{1, 2, \dots, m\}$ there is (unique) $\mathfrak{s}_i \in J_{s_i}$, such that $\mathfrak{t} = \mathfrak{s}_m \smile \mathfrak{s}_{m-1} \smile \dots \smile \mathfrak{s}_1$. For $\mathfrak{t} = (t_n, t_{n-1}, \dots, t_1) \in J_t$, $E_{\mathfrak{t}} = E_{t_n} \otimes E_{t_{n-1}} \otimes \dots \otimes E_{t_1}$. For $\mathfrak{s} = (s_m, s_{m-1}, \dots, s_1) \leq \mathfrak{t} = (\mathfrak{s}_m \smile \mathfrak{s}_{m-1} \smile \dots \smile \mathfrak{s}_1) \in J_t$ the map $\beta_{\mathfrak{t}, \mathfrak{s}} : E_{\mathfrak{s}} \rightarrow E_{\mathfrak{t}}$ is defined by $\beta_{\mathfrak{t}, \mathfrak{s}} = \beta_{\mathfrak{s}_m, s_m} \otimes \beta_{\mathfrak{s}_{m-1}, s_{m-1}} \otimes \dots \otimes \beta_{\mathfrak{s}_1, s_1}$, where $\beta_{\mathfrak{s}, s} : E_{\mathfrak{s}} \rightarrow E_s$ is defined inductively: $\beta_{\mathfrak{s}, s} = I_{E_s}$ and for $\mathfrak{s} = (s_m, s_{m-1}, \dots, s_1) \in J_s$, $\beta_{\mathfrak{s}, s}$ is the composition of maps

$$(\beta_{s_m, s_{m-1}} \otimes I)(\beta_{s_m+s_{m-1}, s_{m-2}} \otimes I) \cdots (\beta_{s_m+\dots+s_3, s_2} \otimes I)\beta_{s_m+\dots+s_2, s_1}. \tag{4}$$

As in the proof of [6, Lemma 4], it can be obtained that the family $(E_{\mathfrak{t}})_{\mathfrak{t} \in J_t}$, together with the family of two-sided isometries $(\beta_{\mathfrak{t}, \mathfrak{s}})_{\mathfrak{s} \leq \mathfrak{t} \in J_t}$, forms an inductive system of Hilbert $\mathcal{B} - \mathcal{B}$ modules in the sense that

$$\begin{aligned} \beta_{\mathfrak{s}, \mathfrak{s}} &= I_{E_{\mathfrak{s}}}, \mathfrak{s} \in J_t; \\ \beta_{\mathfrak{t}, \mathfrak{s}} \beta_{\mathfrak{s}, \mathfrak{r}} &= \beta_{\mathfrak{t}, \mathfrak{r}}, \mathfrak{r} \leq \mathfrak{s} \leq \mathfrak{t} \in J_t. \end{aligned}$$

THEOREM 2. *Let (E, β) be an inclusion system. For $t > 0$, let $\mathcal{E}_t = \text{indlim}_{\mathfrak{s} \in J_t} E_{\mathfrak{s}}$ be an inductive limit of $E_{\mathfrak{s}}$ over J_t . Then $\mathcal{E} = (\mathcal{E}_t)_{t > 0}$ has the structure of a product system of Hilbert $\mathcal{B} - \mathcal{B}$ modules.*

Proof. The proof is obtained similarly as the proof for [6, Theorem 5]. Let $t > 0$. By [13, Proposition A.10.10], \mathcal{E}_t is a Hilbert $\mathcal{B} - \mathcal{B}$ module. By [13, Remark A.10.7] the canonical mappings $i_{\mathfrak{s}} : E_{\mathfrak{s}} \rightarrow \mathcal{E}_t$ ($\mathfrak{s} \in J_t$) are two-sided isometries. They satisfy

$$i_{\mathfrak{s}} \beta_{\mathfrak{s}, \mathfrak{r}} = i_{\mathfrak{r}}, \quad \mathfrak{r} \leq \mathfrak{s} \in J_t. \tag{5}$$

Let $s, t > 0$. Given any element $\mathfrak{r} \in J_{s+t}$, there are $\mathfrak{s} \in J_s$ and $\mathfrak{t} \in J_t$ so that $\mathfrak{s} \smile \mathfrak{t} \geq \mathfrak{r}$. As $J_s \smile J_t \subset J_{s+t}$, by the property of the inductive limit construction, it follows that

$$\mathcal{E}_{s+t} = \text{indlim}_{\mathfrak{r} \in J_{s+t}} E_{\mathfrak{r}} = \text{indlim}_{\mathfrak{s} \smile \mathfrak{t} \in J_s \smile J_t} E_{\mathfrak{s} \smile \mathfrak{t}} = \text{indlim}_{\mathfrak{s} \smile \mathfrak{t} \in J_s \smile J_t} E_{\mathfrak{s}} \otimes E_{\mathfrak{t}}.$$

Consider the map $i_{\mathfrak{s}} \otimes i_{\mathfrak{t}} : E_{\mathfrak{s} \smile \mathfrak{t}} \rightarrow \mathcal{E}_s \otimes \mathcal{E}_t$. There holds that $\mathfrak{s}' \smile \mathfrak{t}' \leq \mathfrak{s} \smile \mathfrak{t} \in J_s \smile J_t$ implies $\mathfrak{s}' \leq \mathfrak{s}$, $\mathfrak{t}' \leq \mathfrak{t}$. Since $\beta_{\mathfrak{s} \smile \mathfrak{t}, \mathfrak{s}' \smile \mathfrak{t}'} = \beta_{\mathfrak{s}, \mathfrak{s}'} \otimes \beta_{\mathfrak{t}, \mathfrak{t}'}$, it follows that $(i_{\mathfrak{s}} \otimes i_{\mathfrak{t}}) \beta_{\mathfrak{s} \smile \mathfrak{t}, \mathfrak{s}' \smile \mathfrak{t}'} = i_{\mathfrak{s}} \beta_{\mathfrak{s}, \mathfrak{s}'} \otimes i_{\mathfrak{t}} \beta_{\mathfrak{t}, \mathfrak{t}'} = i_{\mathfrak{s}'} \otimes i_{\mathfrak{t}'}$. By the universal property of the inductive limit construction, we conclude that there is a unique isometry $B_{s,t} : \mathcal{E}_{s+t} \rightarrow \mathcal{E}_s \otimes \mathcal{E}_t$ so that

$$B_{s,t} i_{\mathfrak{s} \smile \mathfrak{t}} = i_{\mathfrak{s}} \otimes i_{\mathfrak{t}}. \tag{6}$$

For every $t > 0$, $\mathcal{E}_t = \overline{\text{span}}\{i_{\mathfrak{s}}(a) \mid a \in E_{\mathfrak{s}}, \mathfrak{s} \in J_t\}$ (from the inductive limit construction). Hence, it is obvious that $B_{s,t}$ is a unitary map. Also, it is $\mathcal{B} - \mathcal{B}$ linear and by (6) and associativity of $E_{\mathfrak{s}} \otimes E_{\mathfrak{t}} = E_{\mathfrak{s} \smile \mathfrak{t}}$ ($\mathfrak{s} \in J_s$, $\mathfrak{t} \in J_t$) it follows that $(B_{r,s} \otimes I_{\mathcal{E}_t}) B_{r+s,t} = (I_{\mathcal{E}_r} \otimes B_{s,t}) B_{r,s+t} \quad \forall r, s, t > 0. \quad \square$

DEFINITION 4. The product system (\mathcal{E}, B) , constructed in the previous theorem, is called the product system generated by the inclusion system (E, β) .

REMARK 4. If (E, β) is already a product system, then its generated product system is that system itself.

Similarly as in [6], we define morphisms and units of an inclusion system of Hilbert modules:

DEFINITION 5. Let (E, β) and (F, γ) be two inclusion systems. Let $C = (C_t)_{t>0}$ be a family of two-sided mappings $C_t : E_t \rightarrow F_t$ so that there is $p \in \mathbb{R}$ satisfying $\|C_t\| \leq e^{tp}$.

- If $C_{s+t} = \gamma_{s,t}^*(C_s \otimes C_t)\beta_{s,t} \quad \forall s, t > 0$, then C is a weak morphism (or just a morphism);
- If $\gamma_{s,t}C_{s+t} = (C_s \otimes C_t)\beta_{s,t} \quad \forall s, t > 0$, then C is a strong morphism.

Clearly, every strong morphism is a weak morphism but the converse is not true. Also, these two notions coincide for product systems since all the linking maps are unitaries.

DEFINITION 6. Let (E, β) be an inclusion system. Let $u = (u_t)_{t>0}$ be a family of vectors $u_t \in E_t$ so that there is $p \in \mathbb{R}$ satisfying $\|u_t\| \leq e^{tp}$ for all $t > 0$ and there is $t_0 > 0$ so that $u_{t_0} \neq 0$.

- If $u_{s+t} = \beta_{s,t}^*(u_s \otimes u_t) \quad \forall s, t > 0$, then u is a weak unit (or just a unit);
- If $\beta_{s,t}u_{s+t} = u_s \otimes u_t \quad \forall s, t > 0$, then u is a strong unit.

The set of all units in (E, β) is denoted by \mathcal{U}_E .

REMARK 5. Every strong unit is a weak unit, but the converse is not true. Clearly, for product systems these two notions coincide.

3. The result

In this section we present the main results.

Let (E, β) be an inclusion system (Definition 3) and denote its generated product system by (\mathcal{E}, B) . Let $t > 0$. For $t \geq s \in J_t$ and $x \in \mathcal{E}_t$, by (5),

$$\langle (i_t i_t^* - i_s i_s^*)x, x \rangle = \langle (i_t i_t^* - i_t \beta_{t,s} \beta_{t,s}^* i_t^*)x, x \rangle = \langle (I_{E_t} - \beta_{t,s} \beta_{t,s}^*)i_t^* x, i_t^* x \rangle \geq 0,$$

since $\beta_{ts} \beta_{t,s}^* : E_t \rightarrow E_t$ is a projection. Therefore, $\{i_s i_s^* : \mathcal{E}_t \rightarrow \mathcal{E}_t \mid s \in J_t\}$ is an increasing net of projections in $B^a(\mathcal{E}_t)$. By [2, Theorem 5, Remark 5(a)], it follows that $B^a(\mathcal{E}_t)$ and $B^a((\mathcal{E}_t)_{\text{HS}})$ are $*$ -isomorphic C^* -algebras, where the map $A \mapsto A|_{(\mathcal{E}_t)_{\text{HS}}}$

provides an isomorphism. So, according to the Hilbert space structure of the H^* -module $(\mathcal{E}_t)_{\text{HS}}$, the net of projections $i_{\mathfrak{s}} i_{\mathfrak{s}}^* |_{(\mathcal{E}_t)_{\text{HS}}} : (\mathcal{E}_t)_{\text{HS}} \rightarrow (\mathcal{E}_t)_{\text{HS}}$, $\mathfrak{s} \in J_t$, strongly converges to a projection

$$P_t \in B^a((\mathcal{E}_t)_{\text{HS}}), \quad \text{Im}(P_t) = \overline{\text{span} \{i_{\mathfrak{s}} i_{\mathfrak{s}}^* x, x \in (\mathcal{E}_t)_{\text{HS}}, \mathfrak{s} \in J_t\}}^{\|\cdot\|_{\text{HS}}}. \quad (7)$$

There is a unique operator $\tilde{P}_t \in B^a(\mathcal{E}_t)$ that extends P_t . By [2, Lemma 3] there holds

$$\tilde{P}_t = \tilde{P}_t^* \quad (8)$$

and since $(\mathcal{E}_t)_{\text{HS}}$ is dense in \mathcal{E}_t with respect to the original norm $\|\cdot\|$ (Proposition 1), we obtain that

$$\tilde{P}_t^2 = \tilde{P}_t. \quad (9)$$

By (3),

$$\|i_{\mathfrak{s}} i_{\mathfrak{s}}^* x - \tilde{P}_t x\| \leq \|i_{\mathfrak{s}} i_{\mathfrak{s}}^* x - \tilde{P}_t x\|_{\text{HS}} = \|i_{\mathfrak{s}} i_{\mathfrak{s}}^* x - P_t x\|_{\text{HS}}, \quad x \in (\mathcal{E}_t)_{\text{HS}}$$

and, therefore, for every $x \in (\mathcal{E}_t)_{\text{HS}}$, the net $\{i_{\mathfrak{s}} i_{\mathfrak{s}}^* x \mid \mathfrak{s} \in J_t\}$ converges to $\tilde{P}_t x$ with respect to the original norm in \mathcal{E}_t . Since the net $\{\|i_{\mathfrak{s}} i_{\mathfrak{s}}^*\|_{B^a(\mathcal{E}_t)} \mid \mathfrak{s} \in J_t\}$ is bounded and $(\mathcal{E}_t)_{\text{HS}}$ is dense in \mathcal{E}_t , we conclude that

$$\text{for } x \in \mathcal{E}_t, \text{ the net } \{i_{\mathfrak{s}} i_{\mathfrak{s}}^* x \mid \mathfrak{s} \in J_t\} \text{ converges to } \tilde{P}_t x \quad (10)$$

with respect to the original norm $\|\cdot\|$ in \mathcal{E}_t .

Let $u = (u_t)$ be a unit in (E, β) with $\|u_t\| \leq e^{t^p}$ for some $p \in \mathbb{R}$. Fix $t > 0$ and define $u_{\mathfrak{s}} = u_{s_m} \otimes u_{s_{m-1}} \otimes \dots \otimes u_{s_1}$ for $\mathfrak{s} = (s_m, s_{m-1}, \dots, s_1) \in J_t$. For $\mathfrak{s} \leq \mathfrak{t} \in J_t$, it follows easily (from the definition of $\beta_{\mathfrak{t}, \mathfrak{s}}$) that

$$u_{\mathfrak{s}} = \beta_{\mathfrak{t}, \mathfrak{s}}^* u_{\mathfrak{t}}. \quad (11)$$

LEMMA 1. Let $\mathcal{B} = K(H)$ be the C^* -algebra of all compact operators acting on a Hilbert space H . Let (E, β) be an inclusion system with the generated product system (\mathcal{E}, B) . Let $u = (u_t)$ be a unit in (E, β) .

1. For $b \in \mathcal{B}$, $(\tilde{P}_t(i_{\mathfrak{s}} u_{\mathfrak{s}}) b)_{\mathfrak{s} \in J_t}$ is a convergent net in \mathcal{E}_t ($t > 0$).
2. For $x \in \mathcal{E}_t$, $\left(\langle \tilde{P}_t(i_{\mathfrak{s}} u_{\mathfrak{s}}), x \rangle \right)_{\mathfrak{s} \in J_t}$ is a convergent net in \mathcal{B} ($t > 0$).

Proof. There is $p \in \mathbb{R}$ so that $\|u_t\| \leq e^{t^p}$ for every $t > 0$.

1. For $\mathfrak{t} \geq \mathfrak{s} \in J_t$, by (11), there holds

$$\begin{aligned} \langle u_{\mathfrak{t}}, u_{\mathfrak{t}} \rangle - \langle u_{\mathfrak{s}}, u_{\mathfrak{s}} \rangle &= \langle u_{\mathfrak{t}}, u_{\mathfrak{t}} \rangle - \langle \beta_{\mathfrak{t}, \mathfrak{s}}^* u_{\mathfrak{t}}, \beta_{\mathfrak{t}, \mathfrak{s}}^* u_{\mathfrak{t}} \rangle = \langle u_{\mathfrak{t}}, u_{\mathfrak{t}} \rangle - \langle u_{\mathfrak{t}}, \beta_{\mathfrak{t}, \mathfrak{s}} \beta_{\mathfrak{t}, \mathfrak{s}}^* u_{\mathfrak{t}} \rangle \\ &= \langle u_{\mathfrak{t}}, (E_{E_{\mathfrak{t}}} - \beta_{\mathfrak{t}, \mathfrak{s}} \beta_{\mathfrak{t}, \mathfrak{s}}^*) u_{\mathfrak{t}} \rangle \geq 0, \end{aligned}$$

since $\beta_{\mathfrak{t}, \mathfrak{s}} \beta_{\mathfrak{t}, \mathfrak{s}}^* : E_{\mathfrak{t}} \rightarrow E_{\mathfrak{t}}$ is a projection.

Thus, we see that $(\langle u_{\mathfrak{s}}, u_{\mathfrak{s}} \rangle)_{\mathfrak{s} \in J_t}$ is an increasing net of self-adjoint operators in $K(H) = \mathcal{B}$ which is uniformly bounded ($\|\langle u_{\mathfrak{s}}, u_{\mathfrak{s}} \rangle\| \leq e^{2t^p}$) and, therefore, it strongly converges in $B(H)$.

Let $b \in \mathcal{B}$. For $t \geq s \in J_t$, by (5) and (11) we have

$$\begin{aligned} \|i_t u_t b - i_s u_s b\|^2 &= \|\langle i_t u_t b - i_s u_s b, i_t u_t b - i_s u_s b \rangle\| \\ &= \|\langle i_t u_t b, i_t u_t b \rangle - \langle i_t u_t b, i_s u_s b \rangle - \langle i_s u_s b, i_t u_t b \rangle + \langle i_s u_s b, i_s u_s b \rangle\| \\ &= \|b^* \langle i_t u_t, i_t u_t \rangle b - b^* \langle i_t u_t, i_s u_s \rangle b - b^* \langle i_s u_s, i_t u_t \rangle b + b^* \langle i_s u_s, i_s u_s \rangle b\| \\ &= \|b^* \langle u_t, u_t \rangle b - b^* \langle u_s, u_s \rangle b - b^* \langle u_s, u_s \rangle b + b^* \langle u_s, u_s \rangle b\| \\ &= \|b^* \langle u_t, u_t \rangle b - b^* \langle u_s, u_s \rangle b\|. \end{aligned}$$

As $b \in \mathcal{B} = K(H)$ and the net $(\langle u_s, u_s \rangle)_{s \in J_t}$ strongly converges in $B(H)$, it follows that the net $(b^* \langle u_t, u_t \rangle b)_{t \in J_t}$ uniformly converges in \mathcal{B} . Hence, $(i_t u_t b)_{t \in J_t}$ is a convergent net in \mathcal{E}_t . Since

$$\|\tilde{P}_t(i_t u_t) b - \tilde{P}_t(i_s u_s) b\| = \|\tilde{P}_t(i_t u_t) - \tilde{P}_t(i_s u_s)\| \leq \|i_t u_t b - i_s u_s b\|,$$

it follows that the net $(\tilde{P}_t(i_s u_s) b)_{s \in J_t}$ converges in \mathcal{E}_t .

2. For $t \geq s \in J_t$, by (5) and (11), it follows that

$$i_s i_s^* i_t u_t = i_s \beta_{t,s}^* i_t^* i_t u_t = i_s u_s. \tag{12}$$

Let $x \in \mathcal{E}_t$. By (12) there holds

$$\begin{aligned} \|\langle i_t u_t - i_s u_s, \tilde{P}_t x \rangle\| &= \|\langle (I_{\mathcal{E}_t} - i_s i_s^*) i_t u_t, \tilde{P}_t x \rangle\| = \|\langle i_t u_t, (I_{\mathcal{E}_t} - i_s i_s^*) (\tilde{P}_t x) \rangle\| \\ &\leq e^{tP} \|(I_{\mathcal{E}_t} - i_s i_s^*) (\tilde{P}_t x)\| = e^{tP} \|\tilde{P}_t x - i_s i_s^* (\tilde{P}_t x)\|. \end{aligned}$$

By (10), the net $\{i_s i_s^* (\tilde{P}_t x) \mid s \in J_t\}$ converges to $\tilde{P}_t (\tilde{P}_t x) \stackrel{(9)}{=} \tilde{P}_t x$. By (8) it follows that $(\langle \tilde{P}_t(i_s u_s), x \rangle)_{s \in J_t}$ is a Cauchy net in \mathcal{B} and, therefore, it converges. \square

THEOREM 3. *Let $\mathcal{B} = K(H)$ be the C^* -algebra of all compact operators acting on a Hilbert space H . Let (E, β) be an inclusion system with the generated product system (\mathcal{E}, B) .*

1. *The canonical map $i = (i_t)_{t>0}$, $i_t : E_t \rightarrow \mathcal{E}_t$, is an isometric strong morphism of these inclusion systems.*
2. *If each Hilbert C^* -module \mathcal{E}_t is strictly complete, then there is a bijection*

$$f : V / \sim \rightarrow \mathcal{U}_E, \quad f([v]) = (i_t^* (\tilde{P}_t v_t))_{t>0},$$

where $V = \{v \in \mathcal{U}_{\mathcal{E}} \mid (\tilde{P}_t v_t)_{t>0} \in \mathcal{U}_{\mathcal{E}}\}$ and \sim is an equivalence relation on V defined by

$$v \sim w \Leftrightarrow \tilde{P}_t v_t = \tilde{P}_t w_t \quad \forall t > 0.$$

Proof. 1. Let $s, t > 0$. Since $(s+t) \leq (s, t) \in J_{s+t}$, there holds $i_{(s+t)} = i_{(s,t)} \beta_{(s,t), s+t}$ by (5). By (4), $\beta_{(s,t), s+t} = \beta_{s,t}$ and, therefore, using also (6),

$$B_{s,t} i_{s+t} = B_{s,t} i_{(s,t)} \beta_{s,t} = (i_s \otimes i_t) \beta_{s,t} \quad \forall s, t > 0.$$

2. Let $v \in V$. There is $k \in \mathbb{R}$ so that $\|v_t\| \leq e^{tk}$ for all $t > 0$. Therefore, $\|i_t^*(\tilde{P}_t v_t)\| \leq e^{tk}$ for each $t > 0$. For $s, t > 0$ there holds

$$\begin{aligned} \beta_{s,t}^*(i_s^*(\tilde{P}_s v_s) \otimes i_t^*(\tilde{P}_t v_t)) &= \beta_{s,t}^*(i_s^* \otimes i_t^*)(\tilde{P}_s v_s \otimes \tilde{P}_t v_t) = [(i_s \otimes i_t)\beta_{s,t}]^*(\tilde{P}_s v_s \otimes \tilde{P}_t v_t) \\ &= [B_{s,t} i_{s+t}]^*(\tilde{P}_s v_s \otimes \tilde{P}_t v_t) = i_{s+t}^* B_{s,t}^*(\tilde{P}_s v_s \otimes \tilde{P}_t v_t) \\ &= i_{s+t}^*(\tilde{P}_{s+t} v_{s+t}). \end{aligned}$$

The last equality holds since $(\tilde{P}_t v_t)_{t>0} \in \mathcal{U}_{\mathcal{E}}$. Therefore, we obtain that $(i_t^*(\tilde{P}_t v_t))_{t>0}$ is a unit in (E, β) . Also, if $v' \in V$ so that $v' \sim v$, it follows that $\tilde{P}_t v'_t = \tilde{P}_t v_t$ for all $t > 0$, implying $(i_t^*(\tilde{P}_t v'_t))_{t>0} = (i_t^*(\tilde{P}_t v_t))_{t>0}$. We conclude that f is a well defined map.

Let $t > 0$. For $\mathfrak{s} = (s_n, \dots, s_1) \in J_t$, denote $\mathcal{E}_{\mathfrak{s}} = \mathcal{E}_{s_n} \otimes \dots \otimes \mathcal{E}_{s_1}$. Let $i_{\mathfrak{s}} : E_{\mathfrak{s}} \rightarrow \mathcal{E}_t$ be the canonical map and let $B_{\mathfrak{s},t} : \mathcal{E}_t \rightarrow \mathcal{E}_{\mathfrak{s}}$ be the map defined similarly as in (4). By (6) it follows

$$B_{\mathfrak{s},t} i_{\mathfrak{s}} = i_{s_n} \otimes \dots \otimes i_{s_1}. \tag{13}$$

For any unit $v \in \mathcal{U}_{\mathcal{E}}$, $v_{\mathfrak{s}} = v_{s_n} \otimes \dots \otimes v_{s_1}$ and there holds

$$B_{\mathfrak{s},t}^* v_{\mathfrak{s}} = v_t. \tag{14}$$

Injectivity of the mapping f :

Consider $[v], [w] \in V/\sim$ so that $f([v]) = f([w])$, i.e. $i_t^*(\tilde{P}_t v_t) = i_t^*(\tilde{P}_t w_t)$ for all $t > 0$.

Let $t > 0$. For $\mathfrak{s} = (s_n, \dots, s_1) \in J_t$, by (14) and (13),

$$\begin{aligned} i_{\mathfrak{s}}^*(\tilde{P}_t v_t) &= i_{\mathfrak{s}}^* B_{\mathfrak{s},t}^*(\tilde{P}_{\mathfrak{s}} v_{\mathfrak{s}}) = (B_{\mathfrak{s},t} i_{\mathfrak{s}})^*(\tilde{P}_{\mathfrak{s}} v_{\mathfrak{s}}) \\ &= (i_{s_n}^* \otimes \dots \otimes i_{s_1}^*)(\tilde{P}_{s_n} v_{s_n} \otimes \dots \otimes \tilde{P}_{s_1} v_{s_1}) = i_{s_n}^*(\tilde{P}_{s_n} v_{s_n}) \otimes \dots \otimes i_{s_1}^*(\tilde{P}_{s_1} v_{s_1}) \\ &= i_{s_n}^*(\tilde{P}_{s_n} w_{s_n}) \otimes \dots \otimes i_{s_1}^*(\tilde{P}_{s_1} w_{s_1}) = i_{\mathfrak{s}}^*(\tilde{P}_t w_t). \end{aligned}$$

That implies $i_{\mathfrak{s}} i_{\mathfrak{s}}^*(\tilde{P}_t v_t) = i_{\mathfrak{s}} i_{\mathfrak{s}}^*(\tilde{P}_t w_t)$ and, by (10) and (9), $\tilde{P}_t v_t = \tilde{P}_t w_t$. Hence, $v \sim w$, i.e. $[v] = [w]$.

Surjectivity of the mapping f :

Let u be a unit in (E, β) . There is $p \in \mathbb{R}$ so that $\|u_t\| \leq e^{tp}$ for all $t > 0$.

Let $t > 0$. According to Lemma 1, using (1), we conclude that $(\Gamma(\tilde{P}_t(i_{\mathfrak{s}} u_{\mathfrak{s}})))_{\mathfrak{s} \in J_t}$ is a $\Gamma(\mathcal{E}_t)$ -strictly Cauchy net in $(\mathcal{E}_t)_d$. By Theorem 1, there is $g_t \in (\mathcal{E}_t)_d$ so that

$$g_t = (st.) \lim_{\mathfrak{s} \in J_t} \Gamma(\tilde{P}_t(i_{\mathfrak{s}} u_{\mathfrak{s}})),$$

i.e.

1. $\langle g_t, \Gamma(x) \rangle = \lim_{\mathfrak{s} \in J_t} \langle \Gamma(\tilde{P}_t(i_{\mathfrak{s}} u_{\mathfrak{s}})), \Gamma(x) \rangle, x \in \mathcal{E}_t;$
2. $g_t b = \lim_{\mathfrak{s} \in J_t} \Gamma(\tilde{P}_t(i_{\mathfrak{s}} u_{\mathfrak{s}})) b, b \in \mathcal{B}.$

Since \mathcal{E}_t is strictly complete, there is $v_t \in \mathcal{E}_t$ so that $g_t = \Gamma(v_t)$ (Definiton 2). By (1) and (2), the above equalities may be rewritten as

$$1'. \langle v_t, x \rangle = \lim_{\mathfrak{s} \in J_t} \langle \tilde{P}_t(i_{\mathfrak{s}} u_{\mathfrak{s}}), x \rangle, \quad x \in \mathcal{E}_t;$$

$$2'. \quad v_t b = \lim_{\mathfrak{s} \in J_t} \tilde{P}_t(i_{\mathfrak{s}} u_{\mathfrak{s}}) b, \quad b \in \mathcal{B}.$$

Let $\mathfrak{s} \in J_t$ and $x \in (\mathcal{E}_t)_{\text{HS}}$. By (8) and the equation 1', it follows that

$$\langle i_{\mathfrak{s}} i_{\mathfrak{s}}^* v_t, x \rangle = \langle v_t, i_{\mathfrak{s}} i_{\mathfrak{s}}^* x \rangle = \lim_{\mathfrak{r} \in J_t} \langle \tilde{P}_t(i_{\mathfrak{r}} u_{\mathfrak{r}}), i_{\mathfrak{s}} i_{\mathfrak{s}}^* x \rangle = \lim_{\mathfrak{r} \in J_t} \langle i_{\mathfrak{r}} u_{\mathfrak{r}}, \tilde{P}_t(i_{\mathfrak{s}} i_{\mathfrak{s}}^* x) \rangle. \quad (15)$$

As $i_{\mathfrak{s}} i_{\mathfrak{s}}^* : (\mathcal{E}_t)_{\text{HS}} \rightarrow (\mathcal{E}_t)_{\text{HS}}$, it follows that $\tilde{P}_t(i_{\mathfrak{s}} i_{\mathfrak{s}}^* x) = P_t(i_{\mathfrak{s}} i_{\mathfrak{s}}^* x)$. Also, since $i_{\mathfrak{s}} i_{\mathfrak{s}}^* x \in \text{Im}(P_t)$ (by (7)), there holds $P_t(i_{\mathfrak{s}} i_{\mathfrak{s}}^* x) = i_{\mathfrak{s}} i_{\mathfrak{s}}^* x$ (by (9)). Hence, using (5) and (11), it follows that

$$\begin{aligned} \lim_{\mathfrak{r} \in J_t} \langle i_{\mathfrak{r}} u_{\mathfrak{r}}, \tilde{P}_t(i_{\mathfrak{s}} i_{\mathfrak{s}}^* x) \rangle &= \lim_{\mathfrak{r} \in J_t} \langle i_{\mathfrak{r}} u_{\mathfrak{r}}, i_{\mathfrak{s}} i_{\mathfrak{s}}^* x \rangle = \lim_{\mathfrak{r} \in J_t} \langle i_{\mathfrak{s}} i_{\mathfrak{s}}^* i_{\mathfrak{r}} u_{\mathfrak{r}}, x \rangle \\ &= \lim_{\mathfrak{r} \in J_t} \langle i_{\mathfrak{s}} \beta_{\mathfrak{r}, \mathfrak{s}}^* u_{\mathfrak{r}}, x \rangle = \lim_{\mathfrak{r} \in J_t} \langle i_{\mathfrak{s}} u_{\mathfrak{s}}, x \rangle = \langle i_{\mathfrak{s}} u_{\mathfrak{s}}, x \rangle. \end{aligned}$$

Combining (15) and (16), it follows that $\langle i_{\mathfrak{s}} i_{\mathfrak{s}}^* v_t, x \rangle = \langle i_{\mathfrak{s}} u_{\mathfrak{s}}, x \rangle$ for $x \in (\mathcal{E}_t)_{\text{HS}}$. Since $(\mathcal{E}_t)_{\text{HS}}$ is dense in \mathcal{E}_t with respect to the original norm in \mathcal{E}_t , we obtain that $\langle i_{\mathfrak{s}} i_{\mathfrak{s}}^* v_t, x \rangle = \langle i_{\mathfrak{s}} u_{\mathfrak{s}}, x \rangle$ for every $x \in \mathcal{E}_t$ and, hence, $i_{\mathfrak{s}} i_{\mathfrak{s}}^* v_t = i_{\mathfrak{s}} u_{\mathfrak{s}}$. By (10) we see that the net $\{i_{\mathfrak{s}} u_{\mathfrak{s}} \mid \mathfrak{s} \in J_t\}$ converges to $\tilde{P}_t v_t$ in \mathcal{E}_t .

We assert that $(\tilde{P}_t v_t)_{t>0} \in \mathcal{U}_{\mathcal{E}}$:

For $x_1, x_2, \dots, x_k \in \mathcal{E}_s$ and $y_1, y_2, \dots, y_k \in \mathcal{E}_t$ ($k \geq 1$), by (6), there holds

$$\begin{aligned} &\langle B_{s,t}(\tilde{P}_{s+t} v_{s+t}), \sum_i x_i \otimes y_i \rangle = \sum_i \langle \tilde{P}_{s+t} v_{s+t}, B_{s,t}^*(x_i \otimes y_i) \rangle \\ &= \sum_i \lim_{\mathfrak{s} \rightarrow t \in J_s \setminus J_t} \langle i_{\mathfrak{s} \rightarrow t} u_{\mathfrak{s} \rightarrow t}, B_{s,t}^*(x_i \otimes y_i) \rangle = \sum_i \lim_{\mathfrak{s} \rightarrow t \in J_s \setminus J_t} \langle (i_{\mathfrak{s}} \otimes i_t)(u_{\mathfrak{s}} \otimes u_t), x_i \otimes y_i \rangle \\ &= \sum_i \lim_{\mathfrak{s} \rightarrow t \in J_s \setminus J_t} \langle i_{\mathfrak{s}} u_{\mathfrak{s}} \otimes i_t u_t, x_i \otimes y_i \rangle = \sum_i \lim_{\mathfrak{s} \rightarrow t \in J_s \setminus J_t} \langle i_{\mathfrak{s}} u_{\mathfrak{s}}, x_i \rangle \langle y_i, u_t \rangle \\ &= \sum_i \langle \tilde{P}_t v_t, \langle \tilde{P}_s v_s, x_i \rangle y_i \rangle = \sum_i \langle \tilde{P}_s v_s \otimes \tilde{P}_t v_t, x_i \otimes y_i \rangle = \langle \tilde{P}_s v_s \otimes \tilde{P}_t v_t, \sum_i x_i \otimes y_i \rangle. \end{aligned}$$

Therefore, we obtain that $(\tilde{P}_t v_t)_{t>0} \in \mathcal{U}_{\mathcal{E}}$ and, by (9), $(\tilde{P}_t v_t)_{t>0} \in V$.

For $x \in E_t$, by (5) and (11), it follows that

$$\begin{aligned} \langle i_t^*(\tilde{P}_t v_t), x \rangle &= \langle \tilde{P}_t v_t, i_t x \rangle = \lim_{\mathfrak{r} \in J_t} \langle i_{\mathfrak{r}} u_{\mathfrak{r}}, i_t x \rangle = \lim_{\mathfrak{r} \in J_t} \langle i_t^* i_{\mathfrak{r}} u_{\mathfrak{r}}, x \rangle = \lim_{\mathfrak{r} \in J_t} \langle \beta_{\mathfrak{r}, t}^* i_{\mathfrak{r}} u_{\mathfrak{r}}, x \rangle \\ &= \lim_{\mathfrak{r} \in J_t} \langle \beta_{\mathfrak{r}, t}^* u_{\mathfrak{r}}, x \rangle = \lim_{\mathfrak{r} \in J_t} \langle u_{\mathfrak{r}}, x \rangle = \langle u_t, x \rangle, \end{aligned}$$

implying $i_t^*(\tilde{P}_t v_t) = u_t$.

Let us denote $\hat{u} = (\tilde{P}_t v_t)_{t>0} \in V$. Hence, $[\hat{u}] \in V/\sim$ and

$$f([\hat{u}]) = (i_t^*(\tilde{P}_t v_t))_{t>0} = (u_t)_{t>0} = u. \quad \square$$

COROLLARY 1. *Let \mathcal{B} be the C^* -algebra of all bounded operators acting on a Hilbert space H of dimension $n \in \mathbb{N}$ ($\mathcal{B} = M_n(\mathbb{C})$). Let (E, β) be an inclusion system of Hilbert \mathcal{B} - \mathcal{B} modules with the generated product system (\mathcal{E}, B) . The map i^* provides a bijection between the set of all units of (\mathcal{E}, B) and the set of all units of (E, β) by letting it act point-wise on units.*

Proof. We notice that $\mathcal{B} = K(H) = \text{HS}$. Let $t > 0$. By Proposition 1, $\mathcal{E}_t = (\mathcal{E}_t)_{\text{HS}}$ is a Hilbert space with the inner product $(\cdot, \cdot) = \text{tr}(\langle \cdot, \cdot \rangle)$. For $\mathfrak{s} \leq t \in J_t$, by (5) there holds $i_{\mathfrak{s}} i_{\mathfrak{s}}^* i_t i_t^* = i_{\mathfrak{s}} i_{\mathfrak{s}}^*$. Therefore, since $\mathcal{E}_t = \overline{\text{span}}\{i_{\mathfrak{s}}(a) \mid a \in E_{\mathfrak{s}}, \mathfrak{s} \in J_t\}$, the increasing net of projections $(i_{\mathfrak{s}} i_{\mathfrak{s}}^*)_{\mathfrak{s} \in J_t}$ strongly converges to identity operator on \mathcal{E}_t with respect to the original norm in \mathcal{E}_t . By (10), $\tilde{P}_t = I_{\mathcal{E}_t}$. Since \mathcal{B} is a unital C^* -algebra, each \mathcal{E}_t in (\mathcal{E}, B) is a strictly complete Hilbert module. The set V reduces to the set of all units in (\mathcal{E}, B) , i.e. $V = \mathcal{U}_{\mathcal{E}}$, and the equivalence relation \sim reduces to the equality of units. Hence, $V/\sim = \mathcal{U}_{\mathcal{E}}$. Now it follows that $f: \mathcal{U}_{\mathcal{E}} \rightarrow \mathcal{U}_E$, $f(v) = (i_t^* v_t)_{t>0}$, i.e. $f = i^*$ is a bijection. \square

REMARK 6. In particular, we obtain a generalization of the result given by Bhat and Mukherjee in [6, Theorem 10] where they proved the existence of a bijection between the set of all units in an inclusion system of Hilbert spaces ($\mathcal{B} = \mathbb{C}$) and the set of all units in the generated product system.

REFERENCES

- [1] D. BAKIĆ AND B. GULJAŠ, *Operators on Hilbert H^* -modules*, J. Operator Theory **46** (2001), 123–137.
- [2] D. BAKIĆ AND B. GULJAŠ, *Hilbert C^* -modules over C^* -algebras of compact operators*, Acta. Sci. Math. (Szeged) **68** (2002), 249–269.
- [3] D. BAKIĆ AND B. GULJAŠ, *Extensions of Hilbert C^* -modules*, Houston Journal of Mathematics, Vol. **30** (2), (2004), 537–558.
- [4] D. BAKIĆ, *A class of strictly complete Hilbert, C^* -modules*, manuscript (2005).
- [5] S. D. BARRETO, B. V. R. BHAT, V. LIEBSCHER AND M. SKEIDE, *Type I product systems of Hilbert modules*, J. Funct. Anal. **212** (2004), 121–181.
- [6] B. V. R. BHAT AND M. MUKHERJEE, *Inclusion systems and amalgamated products of product system*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. Vol. **13** (1), (2010), 1–26.
- [7] B. V. R. BHAT AND M. SKEIDE, *Tensor product systems of Hilbert modules and dilations of completely positive semigroups*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **3** (2000), 519–575.
- [8] M. FRANK AND V. I. PAULSEN, *Injective and projective Hilbert C^* -modules and C^* -algebras of compact operators*, arXiv:math/0611349v2 [math.OA] (2008).
- [9] E. C. LANCE, *Hilbert C^* -Modules: A toolkit for operator algebraists*, Cambridge University Press (1995).
- [10] B. MAGAJNA, *Hilbert C^* -modules in which all closed submodules are complemented*, Proceedings of the American Mathematical Society Vol. **125** (3), (1997), 849–852.
- [11] V. M. MANUILOV AND E. V. TROITSKY, *Hilbert C^* -Modules*, American Mathematical Society (2005).
- [12] M. SKEIDE, *Dilation theory and continuous tensor product systems of Hilbert modules*, PQQP: Quantum Probability and White Noise Analysis XV (2003), World Scientific.
- [13] M. SKEIDE, *Hilbert modules and application in quantum probability*, Habilitationsschrift, Cottbus (2001).

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