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FRAMES FOR B($\mathscr{H}$)

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Abstract. The notion of Operator frame for the space $B(\mathscr{H})$ of all bounded linear operators on Hilbert space $\mathscr{H}$ was introduced by Chun-Yan Li and Huai-Xin Cao [1] and the notion of K-frame for an operator $K \in B(\mathscr{H})$ was introduced by L.Guvruta [10]. In this paper, we consider the fusion of the two concepts and introduce K-operator frame as a generalisation of both K-frame and operator frame for $B(\mathscr{H})$ and obtain some results which are more general than the results proved in [1] and [10]. K-dual of a K-operator frame for $B(\mathscr{H})$ is also introduced. Further, we also study perturbation and stability for K-operator frames for $B(\mathscr{H})$.

1. Introduction

Frames for Hilbert spaces were formally introduced by Duffin and Schaeffer [5] who used frames as a tool in the study of non-harmonic Fourier series. Daubechies, Grossmann and Meyer [4] reintroduced frames and observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$. Frames are generalizations of orthonormal bases in Hilbert spaces. Frames are more flexible tools to translate information than bases. Recall that a sequence $\{f_k\} \subset \mathscr{H}$ is called a frame for $\mathscr{H}$ if there exists two positive constants $0 < A \leqslant B < \infty$ such that

$$A\|f\| \leqslant \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leqslant B\|f\|, \ \ f \in \mathscr{H}.$$

For more literature on frame theory, one may refer to [2]. Many generalization of frames for Hilbert spaces have been introduced and studied namely Wavelet Frames [2], Gabor Frames [2], g-frames [12], operator value frames [9], fusion frames [3] and operator frames [1]. The notions like g-frames, operator value frames, fusion frames and operator frames overlap with one another up to some extent. But their approach is independent in nature. Recently, K-frame in a Hilbert space is introduced by L. Gavruta [10] as a generalisation of the notion of frame in Hilbert spaces. K-frames were further studied in [11, 13, 14]. Operator frame for the space $B(\mathscr{H})$ of all bounded linear operators on Hilbert space $\mathscr{H}$ was introduced by Chun-Yan Li and Huai-Xin Cao [1]. In this paper, we consider the fusion of the two concepts and introduce K-operator frame as a generalisation of operator frame for $B(\mathscr{H})$. K-operator frames are more general than

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operator frames in the sense that the lower frame bound holds only for the elements in the range of K, where K is a bounded linear operator in a separable Hilbert space $\mathscr{H}$. We, also study perturbation and stability of K-operator frames for $B(\mathscr{H})$ and obtain a sufficient condition for the stability of K-operator frame under perturbation. Also, we consider finite sum of K-operator frames and obtained a sufficient condition for the finite sum to be a K-operator frame. Finally, we give a result related to the stability of the finite sum of K-operator frames.

2. Preliminaries

Throughout this paper $\mathbb{N}$ denotes the set of natural numbers, and $B(\mathscr{H})$ denotes the set of bounded linear operator on separable Hilbert space $\mathscr{H}$.

Li and Cao [1] defined the notion of operator frame for $B(\mathscr{H})$. They gave the following definition.

DEFINITION 2.1. A family of bounded linear operators $\{T_i\}$ on Hilbert space $\mathscr{H}$ is said to be an operator frame for $B(\mathscr{H})$, if there exists positive constants $A, B > 0$ such that

$$A\|x\|^2 \leqslant \sum_{i\in\mathbb{N}} \|T_i x\|^2 \leqslant B\|x\|^2, \ \ \forall\, x \in \mathscr{H}, \tag{2.1}$$

where A and B are called lower and upper bounds for the operator frame, respectively. An operator frame $\{T_i\}$ is said to be tight if $A = B$. It is called Parseval operator frame if $A = B = 1$. If only upper inequality of (2.1) hold, then $\{T_i\}$ is called an operator Bessel sequence for $B(\mathscr{H})$.

For a separable Hilbert space $\mathscr{H}$, define

$$\ell^2(\mathscr{H}) = \{\{x_i\} : x_i \in \mathscr{H},\ \sum_{i\in\mathbb{N}} \|x_i\|^2 < \infty\}.$$

Define an inner product on $\ell^2(\mathscr{H})$ by

$$\langle \{x_i\}, \{y_i\} \rangle = \sum_{i\in\mathbb{N}} \langle x_i, y_i \rangle.$$

Then $\ell^2(\mathscr{H})$ is a Hilbert space with pointwise operations.

An operator K defined on a Hilbert space $\mathscr{H}$ is said to be hyponormal if $\|K^*x\| \leqslant \|Kx\|$, for all $x \in \mathscr{H}$. Also, for two operator $S, K \in B(\mathscr{H})$, we say that S majorizes K if there exists $C > 0$ such that $\|Kx\| \leqslant C\|Sx\|$, $x \in \mathscr{H}$.

The following terminology is given by Li and Cao [1].

Let e be a unit vector in $\mathscr{H}$. For every $x \in \mathscr{H}$, define $T_x^e y = \langle y, x\rangle e$, for all $y \in \mathscr{H}$. Then T_x^e is a bounded linear operator on $\mathscr{H}$ and T_x^e is called operator response of x with respect to e.

Next, we state a result by Douglas which is popularly known as Douglas' majorization theorem. This result will be used in the subsequent work.

THEOREM 2.2. [6] *Let $\mathscr{H}$ be a Hilbert space and $S,K \in B(\mathscr{H})$. Then the following statements are equivalent:*

1. *$R(K) \subseteq R(S)$.*
2. *$KK^* \leqslant \lambda^2 SS^*$, for some $\lambda > 0$.*
3. *$K = SQ$, for some $Q \in B(\mathscr{H})$.*

The notion of K-frame for Hilbert spaces is introduced and studied by L. Gavruta [10] who gave the following definition.

DEFINITION 2.3. [13] A sequence $\{x_k\} \subset \mathscr{H}$ is called K-frame for $\mathscr{H}$, if there exist constants $A,B > 0$ such that

$$A\|K^*x\| \leqslant \sum_{k\in\mathbb{N}} |\langle x,x_k\rangle|^2 \leqslant B\|x\|^2, \text{ for all } x \in \mathscr{H}. \tag{2.2}$$

We call A,B as lower and upper frame bounds for the K-frame $\{x_k\} \subset \mathscr{H}$, respectively. If only the upper inequality of (2.2) is satisfied, then $\{x_k\}$ is called a Bessel sequence.

Gavruta [10] also proved the following results.

THEOREM 2.4. *Let $\{f_i\} \subset \mathscr{H}$ and $K \in B(\mathscr{H})$. Then following statements are equivalent:*

1. *$\{f_i\}$ is an atomic system for K;*
2. *$\{f_i\}$ is a K-frame for $\mathscr{H}$;*
3. *there exists a Bessel sequence $\{g_i\} \subset \mathscr{H}$ such that*

$$Kx = \sum_{i\in\mathbb{N}} \langle x, g_i\rangle f_i, \ \forall x \in \mathscr{H}.$$

We call the Bessel sequence $\{g_i\} \subset \mathscr{H}$ as the K-dual frame of the K-frame $\{f_i\}$.

3. K-operator frames

We began this section with the following definition.

DEFINITION 3.1. Let $K \in B(\mathscr{H})$. A family of bounded linear operators $\{T_i\}$ on Hilbert space $\mathscr{H}$ is said to be a K-operator frame for $B(\mathscr{H})$, if there exists positive constants $A,B > 0$ such that

$$A\|K^*x\|^2 \leqslant \sum_{i\in\mathbb{N}} \|T_ix\|^2 \leqslant B\|x\|^2, \ \forall\, x \in \mathscr{H}, \tag{3.3}$$

where A and B are called lower and upper bounds for the K-operator frame, respectively. A K-operator frame $\{T_i\}$ is said to be tight if there exists a constant $A > 0$ such that

$$\sum_{i\in\mathbb{N}} \|T_i x\|^2 = A\|K^* x\|^2, \ \forall\, x \in \mathscr{H}. \tag{3.4}$$

It is called Parseval K-operator frame if $A = 1$ in (3.4). If only upper inequality of (3.3) holds, then $\{T_i\}$ is called a K-operator Bessel sequence in $B(\mathscr{H})$. We call $\{T_i\}$ an exact K-operator frame for $B(\mathscr{H})$ if, it ceases to be a K-operator frame whenever any one of its element is removed. If $K = I$, then K-operator frame is an operator frame. Let $K, P \in B(\mathscr{H})$ such that $PK = I$. Then P is called the left inverse of K denoted by K_l^{-1}. If $KP = I$, then P is called the right inverse of K and we write $K_r^{-1} = P$. If $KP = PK = I$, then K and P are inverse of each other. We denote $F_K(\mathscr{H})$ for family of tight K-operator frames for $B(\mathscr{H})$.

Let $\{T_i\}$ be a K-operator frame for $B(\mathscr{H})$. Define an operator $R : \mathscr{H} \to \ell^2(\mathscr{H})$ by

$$Rx = \{T_i x\}, \ x \in \mathscr{H}.$$

Then R is a bounded linear operator called analysis operator of the K-operator frame $\{T_i\}$. The adjoint of the analysis operator R, $R^*(\{x_i\}) : \ell^2(\mathscr{H}) \to \mathscr{H}$ is defined by

$$R^*(\{x_i\}) = \sum_{i\in\mathbb{N}} T_i^* x_i, \ \forall\ \{x_i\} \in \ell^2(\mathscr{H}).$$

The operator R^* is called the synthesis operator of $\{T_i\}$. By composing R and R^*, the frame operator $S : \mathscr{H} \to \mathscr{H}$ for K-operator frame is given by

$$S(x) = R^* R x = \sum_{i\in\mathbb{N}} T_i^* T_i x.$$

Note that frame operator S, in general need not be invertible.

One may ask for the class of operators K which can guarantee the existence of K-operator frame for $B(\mathscr{H})$. The following two results answer this query.

PROPOSITION 3.2. Let $\{T_i\}$ be a K-operator frame for $B(\mathscr{H})$ with frame bounds A and B. Then $\{T_i\}$ is an operator frame for $B(\mathscr{H})$ if K is onto.

Proof. Since K is onto, there exists $\gamma > 0$ such that

$$\|K^* x\| \geqslant \gamma \|x\|, \ x \in \mathscr{H}.$$

Also, since $\{T_i\}$ is a K-operator frame for $B(\mathscr{H})$, we have

$$\gamma^2 A\|x\|^2 \leqslant A\|K^* x\|^2 \leqslant \sum_{i\in\mathbb{N}} \|T_i x\|^2 \leqslant B\|x\|^2, \ x \in \mathscr{H}.$$

Hence $\{T_i\}$ is an operator frame for $B(\mathscr{H})$ with frame bounds $\gamma^2 A$ and B. $\square$

THEOREM 3.3. *Let $\{T_i\}$ be an operator frame for $B(\mathscr{H})$ and let $K \in B(\mathscr{H})$. Then $\{T_i\}$ is a K-operator frame for $B(\mathscr{H})$ if K is hyponormal.*

Proof. Straight forward. $\square$

The advantage of studying K-operator frames is that we can always construct a K-operator frame with the help of a sequence of operator which is not an operator frame for $B(\mathscr{H})$. This is evident from the following examples.

EXAMPLE 3.4. Let $\mathscr{H}$ be a Hilbert space and $\{e_i\}$ be an ONB for $\mathscr{H}$. Define $\{T_i\} \subset B(\mathscr{H})$ by

$$T_i x = \begin{cases} \langle x, e_i \rangle e_i, & \text{if } i \text{ is even} \\ \frac{1}{i}\langle x, e_i \rangle e_i, & \text{if } i \text{ is odd.} \end{cases}$$

Then $\{T_i\}$ is not an operator frame for $B(\mathscr{H})$. Let $K : \mathscr{H} \to \mathscr{H}$ be defined by $Kx = \sum_{i\in\mathbb{N}} \langle x, e_{2i} \rangle e_{2i}$, $x \in \mathscr{H}$. Then

$$\begin{aligned} \|K^* x\|^2 &= \sum_{i\in\mathbb{N}} |\langle x, e_{2i} \rangle|^2 \\ &\leqslant \sum_{i\in\mathbb{N}} \|T_i x\|^2 \\ &= \sum_{i\in\mathbb{N}} |\langle x, e_{2i} \rangle|^2 + \sum_{i\in\mathbb{N}} \frac{1}{(2i-1)^2} |\langle x, e_{2i-1} \rangle|^2 \\ &\leqslant \sum_{i\in\mathbb{N}} |\langle x, e_i \rangle|^2 \\ &= \|x\|^2,\ x \in \mathscr{H}. \end{aligned}$$

Hence $\{T_i\}$ is a K-operator frame for $B(\mathscr{H})$.

EXAMPLE 3.5. Let $\mathscr{H}$ be a Hilbert space and $\{e_i\}$ be an ONB for $\mathscr{H}$. Define $\{T_i\} \subset B(\mathscr{H})$ by

$$T_i x = \frac{1}{i} \langle x, e_i \rangle e_i.$$

Then $\{T_i\}$ is not an operator frame for $B(\mathscr{H})$. Let $K : \mathscr{H} \to \mathscr{H}$ be defined by $Kx = \sum_{i\in\mathbb{N}} \frac{1}{i^2} \langle x, e_{2i} \rangle e_{2i}$, $x \in \mathscr{H}$. Then

$$\begin{aligned} \|K^* x\|^2 &= \sum_{i\in\mathbb{N}} \frac{1}{i^4} |\langle x, e_{2i} \rangle|^2 \\ &\leqslant \sum_{i\in\mathbb{N}} \|T_i x\|^2 \\ &\leqslant \sum_{i\in\mathbb{N}} |\langle x, e_i \rangle|^2 \\ &= \|x\|^2,\ \ x \in \mathscr{H}. \end{aligned}$$

Hence $\{T_i\}$ is a K-operator frame for $B(\mathscr{H})$.

EXAMPLE 3.6. Let $\mathscr{H}$ be a Hilbert space and $\{e_i\}$ be an ONB for $\mathscr{H}$. Define $\{T_i\} \subset B(\mathscr{H})$ by

$$T_i x = \langle x, e_i + e_{i+1}\rangle(e_i + e_{i+1}), \ x \in \mathscr{H}.$$

Then

$$\sum_{i\in\mathbb{N}} \|T_i x\|^2 = 2\sum_{i\in\mathbb{N}} |\langle x, e_i + e_{i+1}\rangle|^2.$$

Hence $\{T_i\}$ is an operator Bessel sequence in $B(\mathscr{H})$ but not an operator frame for $B(\mathscr{H})$. Let $K : \mathscr{H} \to \mathscr{H}$ be defined by $Kx = \sum_{i\in\mathbb{N}}\langle x, e_i\rangle(e_i + e_{i+1})$, $x \in \mathscr{H}$. Then $\{T_i\}$ is a K-operator frame for $B(\mathscr{H})$.

Now, we give an example of an operator Bessel sequence which is not a K-operator frame.

EXAMPLE 3.7. Let $\mathscr{H}$ be a Hilbert space and $\{e_i\}$ be an ONB for $\mathscr{H}$. Define $\{T_i\} \subset B(\mathscr{H})$ by

$$T_i x = \frac{1}{i^2}\langle x, e_{2i}\rangle e_{2i} + \langle x, e_{2i+1}\rangle e_{2i+1}, \ x \in \mathscr{H}.$$

Then

$$\sum_{i\in\mathbb{N}} \|T_i x\|^2 \leqslant \|x\|^2, \ x \in \mathscr{H}, \ x \in \mathscr{H}.$$

Hence $\{T_i\}$ is an operator Bessel sequence in $B(\mathscr{H})$. Let $K : \mathscr{H} \to \mathscr{H}$ be defined by $Kx = \sum_{i\in\mathbb{N}}\langle x, e_{2i}\rangle e_{2i}$, $x \in \mathscr{H}$. Then $\{T_i\}$ is not a K-operator frame for $B(\mathscr{H})$.

In the wake of the above examples, we have the following result.

THEOREM 3.8. *For an operator Bessel sequence $\{T_i\} \subset B(\mathscr{H})$, the following statements are equivalent:*

1. *$\{T_i\}$ is K-operator frame for $B(\mathscr{H})$.*
2. *There exists $A > 0$ such that $S \geqslant AKK^*$,where S is the frame operator for $\{T_i\}$.*
3. *$K = S^{1/2}Q$, for some $Q \in B(\mathscr{H})$.*

Proof. (1) $\Rightarrow$ (2) Note that $\{T_i\}$ is a K-operator frame for $B(\mathscr{H})$ with frame bounds A and B and frame operator S if and only if

$$A\|K^*x\|^2 \leqslant \sum_{i\in\mathbb{N}} \|T_i x\|^2 \leqslant B\|x\|^2, \ \text{for all } x \in \mathscr{H}.$$

Thus, we have

$$\langle AKK^*x, x\rangle \leqslant \langle Sx, x\rangle \leqslant \langle Bx, x\rangle, \ \text{for all } x \in \mathscr{H}.$$

Hence $S \geqslant AKK^*$.

(2) $\Rightarrow$ (3) Suppose there exists $A > 0$ such that $AKK^* \leqslant S^{1/2}S^{1/2^*}$. This gives $\|K^*x\|^2 \leqslant A^{-1}\|S^{1/2}x\|^2$, $x \in \mathscr{H}$. Therefore $S^{1/2}$ majorizes K^*. Then, by Theorem 2.2, $K = S^{1/2}Q$, for some $Q \in B(\mathscr{H})$.

(3) $\Rightarrow$ (1) let $K = S^{1/2}Q$, for some $Q \in B(\mathscr{H})$. Therefore, by Theorem 2.2, $S^{1/2}$ majorizes K^*. Thus, there exists $A > 0$ such that

$$\|K^*x\| \leqslant A\|S^{1/2}x\|, \ \text{for all } \ x \in \mathscr{H}.$$

This gives $KK^* \leqslant A^2S$. Hence $\{T_i\}$ is a K-operator frame for $B(\mathscr{H})$. □

Now, we take up the issue of construction of a K_1-operator frame for $B(\mathscr{H})$ using a K-operator frame.

THEOREM 3.9. *Let $Q \in B(\mathscr{H})$ and $\{T_i\}$ is a K-operator frame for $B(\mathscr{H})$. Then $\{T_iQ\}$ is a Q^*K-operator frame for $B(\mathscr{H})$.*

Proof. Straight forward. □

THEOREM 3.10. *Let $K \in B(\mathscr{H})$ and $\{T_i\} \subset B(\mathscr{H})$ is a tight K-operator frame for $B(\mathscr{H})$ with frame bound A_1. Then $\{T_i\}$ is a tight operator frame for $B(\mathscr{H})$ with frame bound A_2 if and only if $K_r^{-1} = \frac{A_1}{A_2}K^*$.*

Proof. Let $\{T_i\} \subset B(\mathscr{H})$ be a K-tight operator frame for $B(\mathscr{H})$ with frame bound A_1. If $\{T_i\}$ is a tight operator frame for $B(\mathscr{H})$ with frame bound A_2. Then

$$\sum_{i\in\mathbb{N}} \|T_ix\|^2 = A_2\|x\|^2, \text{ for all } x \in \mathscr{H}.$$

So, for each $x \in \mathscr{H}$, we have $A_1\|K^*x\|^2 = A_2\|x\|^2$. This gives

$$\langle KK^*x, x\rangle = \left\langle \frac{A_2}{A_1}x, x \right\rangle \ \text{for all } \ x \in \mathscr{H}.$$

Hence $K_r^{-1} = \frac{A_1}{A_2}K^*$. Conversely, suppose that $K_r^{-1} = \frac{A_1}{A_2}K^*$. Then $KK^* = \frac{A_2}{A_1}I$. Thus

$$\langle KK^*x, x\rangle = \left\langle \frac{A_2}{A_1}x, x \right\rangle, \ \text{for all } \ x \in \mathscr{H}.$$

Since $\{T_i\}$ is a tight K-operator frame for $B(\mathscr{H})$, we have

$$\sum_{i\in\mathbb{N}} \|T_ix\|^2 = A_2\|x\|^2, \ \text{for all } x \in \mathscr{H}.$$

Hence $\{T_i\}$ is a tight operator frame for $B(\mathscr{H})$. □

REMARK 3.11.

1. Let $K \in B(\mathscr{H})$. If $\{T_i\}$ is a K-tight operator frame for $B(\mathscr{H})$ with frame bound A, then $\{T_i(K^N)^*\} \subset B(\mathscr{H})$ is K^{N+1}-tight operator frame for $B(\mathscr{H})$ with frame bound A.

2. If $\{T_i\}$ is a tight operator frame for $B(\mathscr{H})$ with frame bound A, then $\{T_iK^*\}$ is tight K-operator frame for $B(\mathscr{H})$ with frame bound A.

3. Every operator $K \in B(\mathscr{H})$ has K-operator frame. Indeed, if $\{f_k\}$ is a frame for $\mathscr{H}$ with frame bounds A and B, then $T_{f_i}^{e_i}$ is an operator frame. Define $T_i = T_{f_i}^{e_i}K^*$, then $\{T_i\}$ is K-operator frame for $B(\mathscr{H})$ with frame bounds A and B.

Next, we prove that if $\{T_i\}$ is a K_1 as well as K_2-operator frame, then for scalars α and β, it is also a $(\alpha K_1 + \beta K_2)$ and K_1K_2-operator frame.

THEOREM 3.12. *Let $K_1, K_2 \in B(\mathscr{H})$. If $\{T_i\}$ is a K_1 as well as K_2-operator frame for $B(\mathscr{H})$ and α, β are scalars, then $\{T_i\}$ is a $(\alpha K_1 + \beta K_2)$-operator frame and K_1K_2-operator frame for $B(\mathscr{H})$.*

Proof. Let $\{T_i\}$ is a K_1 as well as K_2-operator frame for $B(\mathscr{H})$. Then there exists positive constants $0 \leqslant A_p < \infty$ and $0 \leqslant B_p < \infty$ $(p = 1,2)$ such that

$$A_p\|K_p^*x\|^2 \leqslant \sum_{i\in\mathbb{N}} \|T_ix\|^2 \leqslant B_p\|x\|^2, \ \text{ for all } x \in \mathscr{H}.$$

This gives

$$\frac{A_1A_2}{A_2|\alpha|^2 + A_1|\beta|^2}\|(\alpha K_1 + \beta K_2)^*f\|^2 \leqslant \sum_{i\in\mathbb{N}} \|T_ix\|^2 \leqslant \left(\frac{B_1 + B_2}{2}\right)\|x^2\|, \text{ for all } x \in \mathscr{H}.$$

Therefore, $\{T_i\}$ is a $(\alpha K_1 + \beta K_2)$-operator frame for $B(\mathscr{H})$. Also, for each $x \in \mathscr{H}$, we have

$$\|(K_1K_2)^*x\|^2 = \|K_2^*K_1^*x\|^2 \leqslant \|K_2^*\|^2\|K_1^*x\|^2, \ x \in \mathscr{H}.$$

Since $\{T_i\}$ is a K_1-operator frame for $B(\mathscr{H})$, we have

$$\frac{A_1}{\|K_2^*\|^2}\|(K_1K_2)^*x\|^2 \leqslant \sum_{i\in\mathbb{N}} \|T_ix\|^2 \leqslant B_1\|x\|^2, \ \text{ for all } x \in \mathscr{H}.$$

Hence $\{T_i\}$ is a K_1K_2-operator frame for $B(\mathscr{H})$. $\square$

COROLLARY 3.13. For any $K \in B(\mathscr{H})$, if a sequence of operators $\{T_i\}$ is a K-operator frame for $B(\mathscr{H})$, then $\{T_i\}$ is an $\mathscr{A}$-operator frame for any operator $\mathscr{A}$ in the subalgebra generated by K.

Next, we show that K-operator frame for $\mathscr{H}$ is invariant under a linear homeomorphism, provided K^* commutes with the inverse of a given homeomorphism. A relation between the best bounds of a given K-operator frame and the best bounds of K-operator frame obtained by the action of linear homeomorphism is given in the following theorem, which generalizes Corollary 1 in [7].

THEOREM 3.14. *Let $\{T_i\}$ be a K-operator frame for $\mathscr{H}$ with best frame bounds A and B. If $Q : \mathscr{H} \to \mathscr{H}$ is a linear homeomorphism such that Q^{-1} commutes with K^*, then $\{T_iQ\}$ is a K-operator frame for $\mathscr{H}$ with best frame bounds C and D satisfying the inequalities*

$$A\|Q^{-1}\|^{-2} \leqslant C \leqslant A\|Q\|^2; \quad B\|Q^{-1}\|^{-2} \leqslant D \leqslant B\|Q\|^2. \tag{3.5}$$

Proof. Since B is an upper bound for $\{T_i\}$, for all $x \in \mathscr{H}$, we have

$$\sum_{i\in\mathbb{N}} \|T_iQx\|^2 \leqslant B\|Q\|^2\|x\|^2, \ \ x \in \mathscr{H}.$$

Also, we have

$$\begin{aligned} A\|K^*x\|^2 &= A\|K^*Q^{-1}Qx\|^2 \\ &= A\|Q^{-1}K^*Qx\|^2 \\ &\leqslant \|Q^{-1}\|^2 \sum_{i\in\mathbb{N}} \|T_iQx\|^2, \ \ x \in \mathscr{H}. \end{aligned}$$

Therefore, we obtain

$$A\|Q^{-1}\|^{-2}\|K^*x\|^2 \leqslant \sum_{i\in\mathbb{N}} \|T_iQx\|^2 \leqslant B\|Q\|^2\|x\|^2, \ \ x \in \mathscr{H}.$$

Hence, $\{T_iQ\}$ is a K-operator frame for $\mathscr{H}$ with bounds $A\|Q^{-1}\|^{-2}$ and $B\|Q\|^2$.

Now let C and D be the best bounds of the K-operator frame $\{T_iQ\}$. Then

$$A\|Q^{-1}\|^{-2} \leqslant C \ \text{ and } \ D \leqslant B\|Q\|^2. \tag{3.6}$$

Also, $\{T_iQ\}$ is a K-operator frame for $B(\mathscr{H})$ with frame bounds C and D and

$$\begin{aligned} \|K^*x\|^2 &= \|QQ^{-1}K^*x\|^2 \\ &\leqslant \|Q\|^2\|K^*Q^{-1}x\|^2, \ \text{ for all } \ x \in \mathscr{H}. \end{aligned}$$

Hence

$$\begin{aligned} C\|Q\|^{-2}\|K^*x\|^2 &\leqslant C\|K^*Q^{-1}x\|^2 \\ &\leqslant \sum_{i\in\mathbb{N}} \|T_iQQ^{-1}x\|^2 \ (= \sum_{i\in\mathbb{N}} \|T_ix\|^2) \\ &\leqslant D\|Q^{-1}\|^2\|x\|^2. \end{aligned}$$

Since A and B are the best bounds of K-operator frame $\{T_i\}$,we have

$$C\|Q\|^{-2} \leqslant A, \;\; B \leqslant D\|Q^{-1}\|^2. \tag{3.7}$$

Hence the inequality (3.5) follows from (3.6) and (3.7). $\square$

The following result gives an interplay between a K-frame and K-operator frame. We omit the proof as it can worked out in few steps using the hypothesis.

THEOREM 3.15. *Let $\{f_i\}$ be a sequence in $\mathscr{H}$, $K \in B(\mathscr{H})$ and $\{e_i\}$ be a sequence of standard unit vectors in $\mathscr{H}$. Then*

1. *$\{f_i\}$ is a K-frame for $\mathscr{H}$ if and only if $\{T_{f_i}^{e_i}\}$ is a K-operator frame for $B(\mathscr{H})$.*
2. *$\{f_i\}$ is a tight K-frame for $\mathscr{H}$ if and only if $\{T_{f_i}^{e_i}\}$ is a tight K-operator frame for $B(\mathscr{H})$.*

Motivating from Theorem 3.8 in [14], we define K-dual operator frame for K-operator frames.

DEFINITION 3.16. Let $K \in B(\mathscr{H})$ and $\{T_i\}$ be a K-operator frame for $B(\mathscr{H})$. An operator Bessel sequence $\{R_i\}$ in $B(\mathscr{H})$ is called K-dual operator frame for $\{T_i\}$ if

$$Kx = \sum_{i \in \mathbb{N}} T_i^* R_i x, \; \forall \, x \in \mathscr{H}.$$

REMARK 3.17.

1. Every K-operator frame for $B(\mathscr{H})$ has K-dual operator frame.
2. K-dual operator frame $\{R_i\}$ is K^*-operator frame for $B(\mathscr{H})$.

THEOREM 3.18. *Let $\{f_i\} \subset \mathscr{H}$, $\{\widetilde{f}_i\} \subset \mathscr{H}$ and $\{e_i\}$ be a sequence of standard unit vectors in $\mathscr{H}$. Then the following statements are equivalent:*

1. *$\{\widetilde{f}_i\}$ is a K-dual frame for $\{f_i\}$.*
2. *$\{T_{\widetilde{f}_i}^{e_i}\}$ is a K-dual operator frame for $\{T_{f_i}^{e_i}\}$.*

Proof. $(1) \Rightarrow (2)$. For any $x \in \mathscr{H}$, we have

$$\begin{aligned}
\sum_{i \in \mathbb{N}} T_{f_i}^{e_i *} T_{\widetilde{f}_i}^{e_i} x &= \sum_{i \in \mathbb{N}} T_{f_i}^{e_i *} \langle x, \widetilde{f}_i \rangle e_i \\
&= \sum_{i \in \mathbb{N}} \langle \langle x, \widetilde{f}_i \rangle e_i, e_i \rangle f_i \\
&= Kx.
\end{aligned}$$

Hence $\{T^{e_i}_{\widetilde{f}_i}\}$ is a K-dual operator frame for $\{T^{e_i}_{f_i}\}$.

(2) $\Rightarrow$ (1). For any $x \in \mathscr{H}$, we have

$$\begin{aligned} Kx &= \sum_{i\in\mathbb{N}} T^{e_i\,*}_{f_i} T^{e_i}_{\widetilde{f}_i} x \\ &= \sum_{i\in\mathbb{N}} T^{e_i\,*}_{f_i} \langle x, \widetilde{f}_i\rangle e_i \\ &= \sum_{i\in\mathbb{N}} \langle \langle x, \widetilde{f}_i\rangle e_i, e_i\rangle f_i \\ &= \sum_{i\in\mathbb{N}} \langle x, \widetilde{f}_i\rangle f_i. \end{aligned}$$

Hence $\{\widetilde{f}_i\}$ is a K-dual frame for $\{f_i\}$. $\square$

4. Perturbation of K-operator frames

The theory of perturbation is a very important tool in many area of applied mathematics. In this section, we consider perturbation of K-operator frames by non-zero operators. We begin with the following result that gives a sufficient condition for the perturbed sequence of type $\{T_i + c_iT_0\}$, where $\{T_i\}$ is a K-operator frame for $B(\mathscr{H})$, $\{c_i\}$ is any sequence of scalars and $T_0 \in B(\mathscr{H})$.

THEOREM 4.1. *Let $\{T_i\}$ be a K-operator frame for $B(\mathscr{H})$ with bound A and B. Let $T_0 \neq 0$ be any element in $B(\mathscr{H})$ and $\{c_i\}$ be any sequence of scalars. Then, the perturbed sequence of operators $\{T_i + c_iT_0\}$ is a K-operator frame for $B(\mathscr{H})$ if*

$$\sum_{i\in\mathbb{N}} |c_i|^2 < \frac{A}{\|T_0\|}.$$

Proof. Let $R_i = T_i + c_iT_0$, $i \in \mathbb{N}$. Then, for any $x \in \mathscr{H}$, we have

$$\begin{aligned} \sum_{i\in\mathbb{N}} \|T_ix - R_ix\|^2 &= \sum_{i\in\mathbb{N}} \|c_iT_0x\|^2 \\ &\leqslant \sum_{i\in\mathbb{N}} |c_i|^2\|T_0\|^2\|x\|^2, \\ &= R\|x\|^2, \end{aligned}$$

where $R = \sum_{i\in\mathbb{N}} |c_i|^2\|T_0\|^2$. Therefore, $\{T_i + c_iT_0\}$ is a K-operator frame for $B(\mathscr{H})$ if $R < A$, that is, if

$$\sum_{i\in\mathbb{N}} |c_i|^2 < \frac{A}{\|T_0\|^2}. \quad \square$$

REMARK 4.2. The condition that $\sum_{i\in\mathbb{N}} |c_i|^2 < \dfrac{A}{\|T_0\|^2}$ in the Theorem 4.1 is not necessary. Indeed, let $\mathscr{H}$ be a Hilbert space and $\{e_n\}$ be a sequence of standard unit vectors in $\mathscr{H}$. For each $i \in \mathbb{N}$, define $T_i x = \langle x, e_i \rangle e_i$, $x \in \mathscr{H}$ and $K : \mathscr{H} \to \mathscr{H}$ by $Kx = \sum_{i \in \mathbb{N}} \langle x, e_i \rangle e_i$, $x \in \mathscr{H}$. Then $\{T_i\}$ is a tight K-operator frame for $\mathscr{H}$. Let $T_0 x = \langle x, e_1 \rangle e_1$, $c_1 = 2$ and $c_i = 0$, $n \geqslant 2$, $n \in \mathbb{N}$. Then $\{T_i + c_i T_0\}$ is a K-operator frame for $B(\mathscr{H})$ with $\sum_{i \in \mathbb{N}} |c_i|^2 = 4$.

Next, we consider perturbation of the type $\{\alpha_i T_i - \beta_i R_i\}$, where $\{T_i\} \subset \mathscr{H}$ is a frame for $B(\mathscr{H})$; $\{R_i\} \subset \mathscr{H}$ is any sequence and $\{\alpha_i\}$, $\{\beta_i\}$ are two positively confined sequences and prove the following result in this direction.

THEOREM 4.3. *Let $\{T_i\}$ be a K-operator frame for $B(\mathscr{H})$, $\{R_i\} \subset B(\mathscr{H})$ be any sequence and let $\{\alpha_i\}$, $\{\beta_i\} \subset \mathbb{R}$ be any two positively confined sequences. If there exist constants λ, μ with $0 \leqslant \lambda$, $\mu < \frac{1}{2}$ such that*

$$\sum_{i \in \mathbb{N}} \|(\alpha_i T_i - \beta_i R_i)x\|^2 \leqslant \lambda \sum_{i \in \mathbb{N}} \|\alpha_i T_i x\|^2 + \mu \sum_{i \in \mathbb{N}} \|\beta_i R_i x\|^2, \quad x \in \mathscr{H},$$

then $\{R_i\}$ is a K-operator frame for $B(\mathscr{H})$.

Proof. Suppose that for some constants λ, μ with $0 \leqslant \lambda, \mu < \frac{1}{2}$, we have

$$\sum_{i \in \mathbb{N}} \|(\alpha_i T_i - \beta_i R_i)x\|^2 \leqslant \lambda \sum_{i \in \mathbb{N}} \|\alpha_i T_i x\|^2 + \mu \sum_{i \in \mathbb{N}} \|\beta_i R_i x\|^2, \quad x \in \mathscr{H}.$$

Then, for each $x \in \mathscr{H}$,

$$\begin{aligned} \sum_{i \in \mathbb{N}} \|\beta_i R_i x\|^2 &\leqslant 2\Big(\sum_{i \in \mathbb{N}} \|\alpha_i T_i x\|^2 + \sum_{i \in \mathbb{N}} \|\alpha_i T_i x - \beta_i R_i x\|^2\Big) \\ &\leqslant 2\Big(\sum_{i \in \mathbb{N}} \|\alpha_i T_i x\|^2 + \lambda \sum_{i \in \mathbb{N}} \|\alpha_i T_i x\|^2 + \mu \sum_{i \in \mathbb{N}} \|\beta_i R_i x\|^2\Big) \end{aligned}$$

Therefore

$$(1 - 2\mu) \sum_{i \in \mathbb{N}} \|\beta_i R_i x\|^2 \leqslant 2(1 + \lambda) \sum_{i \in \mathbb{N}} \|\alpha_i T_i x\|^2.$$

This gives

$$(1 - 2\mu)(\inf_{1 \leqslant i < \infty} \beta_i)^2 \sum_{i \in \mathbb{N}} \|R_i x\|^2 \leqslant 2(1 + \lambda)(\sup_{1 \leqslant i < \infty} \alpha_i)^2 \sum_{i \in \mathbb{N}} \|T_i x\|^2.$$

Thus

$$\sum_{i \in \mathbb{N}} \|R_i x\|^2 \leqslant \frac{2(1 + \lambda)(\sup_{1 \leqslant i < \infty} \alpha_i)^2}{(1 - 2\mu)(\inf_{1 \leqslant i < \infty} \beta_i)^2} \sum_{i \in \mathbb{N}} \|T_i x\|^2.$$

Also, for each $x \in \mathscr{H}$, we have

$$\begin{aligned} \sum_{i \in \mathbb{N}} \|\alpha_i T_i x\|^2 &\leqslant 2\Big(\sum_{i \in \mathbb{N}} \|\alpha_i T_i x - \beta_i R_i x\|^2 + \sum_{i \in \mathbb{N}} \|\beta_i R_i x\|^2\Big) \\ &\leqslant 2\Big(\lambda \sum_{i \in \mathbb{N}} \|\alpha_i T_i x\|^2 + \mu \sum_{i \in \mathbb{N}} \|\beta_i R_i x\|^2 + \sum_{i \in \mathbb{N}} \|\beta_i R_i x\|^2\Big), \quad \text{for all,} \quad x \in \mathscr{H}. \end{aligned}$$

Therefore

$$(1-2\lambda)(\inf_{1\leqslant i<\infty}\alpha_i)^2\sum_{i\in\mathbb{N}}\|T_ix\|^2\leqslant 2(1+\mu)(\sup_{1\leqslant i<\infty}\beta_i)^2\sum_{i\in\mathbb{N}}\|R_ix\|^2.$$

This gives

$$\frac{(1-2\lambda)(\inf_{1\leqslant i<\infty}\alpha_i)^2}{2(1+\mu)(\sup_{1\leqslant i<\infty}\beta_i)^2}\sum_{i\in\mathbb{N}}\|T_ix\|^2\leqslant\sum_{i\in\mathbb{N}}\|R_ix\|^2\leqslant\frac{2(1+\lambda)(\sup_{1\leqslant i<\infty}\alpha_i)^2}{(1-2\mu)(\inf_{1\leqslant i<\infty}\beta_i)^2}\sum_{i\in\mathbb{N}}\|T_ix\|^2.$$

Hence, $\{R_i\}$ is a K-operator frame for $B(\mathscr{H})$. $\square$

5. Stability of K-operator frames

We begin this section with the following result.

THEOREM 5.1. *Let $\{T_i\}$ be a K-operator frame for $\mathscr{H}$ with frame bounds A and B. Let $\{R_i\}\subset\mathscr{H}$ and $\alpha,R\geqslant 0$. If $0\leqslant\alpha+\frac{R}{A}<1$ such that*

$$\sum_{i\in\mathbb{N}}\|(T_i-R_i)x\|^2\leqslant\alpha\sum_{i\in\mathbb{N}}\|T_ix\|^2+R\|K^*x\|^2,\ \textit{for all},\ x\in\mathscr{H}.$$

Then $\{R_i\}$ is a K-operator frame with frame bounds $A\Big(1-\sqrt{\alpha+\frac{R}{A}}\Big)^2$ and $B\Big(1+\sqrt{\alpha+\frac{R\|K\|}{B}}\Big)^2$.

Proof. Let $\{T_i\}$ be a K-operator frame for $\mathscr{H}$ with frame bounds A and B. Then for each $x\in\mathscr{H}$, we have

$$\begin{aligned}\|\{T_ix\}\|_{\ell^2(\mathscr{H})}&\leqslant\|\{(T_i-R_i)x\}\|_{\ell^2(\mathscr{H})}+\|\{R_ix\}\|_{\ell^2(\mathscr{H})}\\&\leqslant\sqrt{\alpha\sum_{i\in\mathbb{N}}\|T_ix\|^2+R\|K^*x\|^2}+\sqrt{\sum_{i\in\mathbb{N}}\|R_ix\|^2}\\&\leqslant\sqrt{\alpha\sum_{i\in\mathbb{N}}\|T_ix\|^2+\frac{R}{A}\sum_{i\in\mathbb{N}}\|T_ix\|^2}+\sqrt{\sum_{i\in\mathbb{N}}\|R_ix\|^2}\end{aligned}$$

This gives

$$A\Big(1-\sqrt{\alpha+\frac{R}{A}}\Big)^2\|K^*x\|^2\leqslant\sum_{i\in\mathbb{N}}\|R_ix\|^2.$$

Also, we have

$$\begin{aligned}\|\{R_ix\}\|_{\ell^2(\mathscr{H})}&\leqslant\|\{(T_i-R_i)x\}\|_{\ell^2(\mathscr{H})}+\|\{T_ix\}\|_{\ell^2(\mathscr{H})}\\&\leqslant\sqrt{B}\Big(\alpha+\frac{R\|K\|}{B}\Big)\|x\|.\end{aligned}$$

So we get

$$\sum_{i\in\mathbb{N}} \|R_i x\|^2 \leqslant B\Big(1+\sqrt{\alpha+\frac{R\|K\|}{B}}\Big)^2\|x\|^2.$$

Hence $\{R_i\}$ is a K-operator frame for $\mathscr{H}$. $\square$

COROLLARY 5.2. Let $\{T_i\}$ be a K-operator frame for $\mathscr{H}$ with frame bounds A and B. Let $\{R_i\}\subset\mathscr{H}$. If there is an R with $0<R<A$ such that

$$\sum_{i\in\mathbb{N}} \|(T_i-R_i)x\|^2 \leqslant R\|K^*x\|^2, \text{ for all , } x\in\mathscr{H}.$$

Then $\{R_i\}$ is a K-operator frame with frame bounds $A(1-\sqrt{\frac{R}{A}})^2$ and $B(1+\sqrt{\frac{R}{B}}\|K\|)^2$.

Proof. Follows in view of Theorem 5.1 with $\alpha=0$. $\square$

Next, we give a sufficient condition for the stability of a K-operator frame.

THEOREM 5.3. *Let $\{T_i\}$ be a K-operator frame for $B(\mathscr{H})$ with frame bounds A_1 and B_1. Then a sequence $\{R_i\}\subset B(\mathscr{H})$ is a K-operator frame for $B(\mathscr{H})$ if there exists a constant $M>0$ such that*

$$\sum_{i\in\mathbb{N}} \|(T_i-R_i)x\|^2 \leqslant M\min\Big(\sum_{i\in\mathbb{N}}\|T_ix\|^2,\sum_{i\in\mathbb{N}}\|R_ix\|^2\Big), \;\; x\in\mathscr{H}.$$

Proof. For each $x\in\mathscr{H}$, we have

$$\begin{aligned} A\|K^*x\|^2 &\leqslant \sum_{i\in\mathbb{N}}\|T_ix\|^2\\ &\leqslant 2\Big(\|(T_i-R_i)x\|^2+\sum_{i\in\mathbb{N}}\|R_ix\|^2\Big)\\ &\leqslant \Big(M\sum_{i\in\mathbb{N}}\|R_ix\|^2+\sum_{i\in\mathbb{N}}\|R_ix\|^2\Big)\\ &\leqslant 2(M+1)\sum_{i\in\mathbb{N}}\|R_ix\|^2 \end{aligned}$$

and

$$\begin{aligned} \sum_{i\in\mathbb{N}}\|R_ix\|^2 &\leqslant 2\Big(\|(T_i-R_i)x\|^2+\sum_{i\in\mathbb{N}}\|T_ix\|^2\Big)\\ &\leqslant 2(M+1)B\|x\|^2. \end{aligned}$$

So

$$\frac{A}{2(M+1)}\|K^*x\|^2 \leqslant \sum_{i\in\mathbb{N}}\|R_ix\|^2 \leqslant 2(M+1)B\|x\|^2.$$

Hence $\{R_i\}$ is a K-operator frame for $B(\mathscr{H})$. $\square$

REMARK 5.4. Converse part of Theorem 5.3 is valid for any co-isometry $K \in B(\mathscr{H})$. Indeed, for any $x \in \mathscr{H}$, we have

$$\begin{aligned}\sum_{i\in\mathbb{N}} \|(T_i - R_i)x\|^2 &\leqslant 2\Big(\sum_{i\in\mathbb{N}} \|T_ix\|^2 + \sum_{i\in\mathbb{N}} \|R_ix\|^2\Big)\\ &\leqslant 2\Big(\sum_{i\in\mathbb{N}} \|T_ix\|^2 + B_2\|x\|^2\Big)\\ &\leqslant 2\Big(\sum_{i\in\mathbb{N}} \|T_ix\|^2 + \frac{B_2}{A_1}\sum_{i\in\mathbb{N}} \|T_ix\|^2\Big)\\ &= \Big(1+\frac{B_2}{A_1}\Big)\sum_{i\in\mathbb{N}} \|T_ix\|^2.\end{aligned}$$

Similarly, we have

$$\sum_{i\in\mathbb{N}} \|(T_i - R_i)x\|^2 \leqslant \Big(1+\frac{B_1}{A_2}\Big)\sum_{i\in\mathbb{N}} \|R_ix\|^2.$$

Hence

$$\sum_{i\in\mathbb{N}} \|(T_i - R_i)x\|^2 \leqslant M\min\Big(\sum_{i\in\mathbb{N}} \|T_ix\|^2, \sum_{i\in\mathbb{N}} \|R_ix\|^2\Big), \text{ for all } x \in \mathscr{H}.$$

Next, we consider the sum of K-operator frames for $B(\mathscr{H})$. Let $\{T_{n,i}\}$, $n = 1,2,\ldots,k$ be K-operator frames for $B(\mathscr{H})$. Consider the sequence $\left\{\sum_{n=1}^k T_{n,i}\right\}$ obtained by taking the sum of these K -operator frames. We observe that this sequence $\left\{\sum_{n=1}^k T_{n,i}\right\}$ may not be a K-operator frame for $B(\mathscr{H})$. In this direction, we give the following examples:

EXAMPLE 5.5. Let $K \in B(\mathscr{H})$. Let $\{T_{n,i}\}$, $n = 1,2,\ldots,k$ be K-operator frames for $B(\mathscr{H})$. If for some $1 \leqslant p \leqslant k$,

$$T_{n,i}x = T_{p,i}x, \text{ for all } x \in \mathscr{H},\ n = 1,2,\ldots,k \text{ and } i \in \mathbb{N}.$$

Then $\{\sum_{n=1}^k T_{n,i}x\} = \{kT_{p,i}x\}$, $i \in \mathbb{N}$. Therefore $\left\{\sum_{n=1}^k T_{n,i}\right\}$ is a K-operator frame for $B(\mathscr{H})$.

EXAMPLE 5.6. Let $\{T_{1,i}\}$ and $\{T_{2,i}\}$ be two K-operator frame such that

$$T_{1,i}x = -T_{2,i}x, \text{ for all } x \in \mathscr{H},\ n = 1,2,\ldots,k \text{ and } i \in \mathbb{N}.$$

Let $K : \mathscr{H} \to \mathscr{H}$ be defined by $Kx = \sum_{i\in\mathbb{N}}\langle x, e_i\rangle e_i$, for all $x \in \mathscr{H}$. Since $\|K^*(e_1)\|^2 = 1$ and $\sum_{i\in\mathbb{N}} \|\sum_{n=1}^2 T_{n,i}e_i\|^2 = 0$, $\{\sum_{n=1}^2 T_{n,i}\}$ is not a K-operator frame for $B(\mathscr{H})$.

In the view of the above examples, we give a sufficient condition for the finite sum of K-operator frame to be a K-operator frame.

THEOREM 5.7. *Let $K \in B(\mathscr{H})$. For $n = 1,2,\dots,k$, let $\{T_{n,i}\} \subset B(\mathscr{H})$ be K-operator frames for $B(\mathscr{H})$ and $\{\alpha_n\}_{n=1}^{k}$ be any scalars. Then $\{\sum_{n=1}^{k} \alpha_n T_{n,i}\}$ is a K-operator frame for $B(\mathscr{H})$, if there exists $\beta > 0$ and some $p \in \{1,2,\dots,k\}$ such that*

$$\beta \sum_{i\in\mathbb{N}} \|T_{p,i}x\|^2 \leqslant \sum_{i\in\mathbb{N}} \Big\| \sum_{n=1}^{k} \alpha_n T_{n,i}x \Big\|^2, \ \ x \in \mathscr{H}. \tag{$*$}$$

Proof. For each $1 \leqslant p \leqslant k$, let A_p and B_p be the bounds of the K-operator frame $\{T_{p,i}\}$. Let $\beta > 0$ be a constant satisfying $(*)$. Then

$$\begin{aligned} A_p\beta\|K^*x\|^2 &\leqslant \beta \sum_{i\in\mathbb{N}} \|T_{p,i}x\|^2 \\ &\leqslant \sum_{i\in\mathbb{N}} \Big\| \sum_{n=1}^{k} \alpha_n T_{n,i}x \Big\|^2, \ \ x \in \mathscr{H}. \end{aligned}$$

For any $x \in \mathscr{H}$, we have

$$\begin{aligned} \sum_{i\in\mathbb{N}} \Big\| \sum_{n=1}^{k} \alpha_n T_{n,i}x \Big\|^2 &\leqslant \sum_{i\in\mathbb{N}} k\Big(\sum_{n=1}^{k} \|\alpha_i T_{n,i}\|^2 \Big) \\ &\leqslant k(\max|\alpha_i|^2) \sum_{n=1}^{k} \Big(\sum_{i\in\mathbb{N}} \|T_{n,i}x\|^2 \Big) \\ &\leqslant k(\max|\alpha_i|^2)(\sum_{n=1}^{k} B_i)\|x\|^2. \end{aligned}$$

Hence $\{\sum_{n=1}^{k} \alpha_n T_{n,i}\}$ is a K-operator frame for $B(\mathscr{H})$. □

Finally, we prove the following result related to finite sum of K-operator frames.

THEOREM 5.8. *Let $K \in B(\mathscr{H})$. For each $n \in \{1,2,\dots,k\}$, let $\{T_{n,i}\} \subset B(\mathscr{H})$ be K-operator frame for $B(\mathscr{H})$, $\{R_{n,i}\} \subset B(\mathscr{H})$ be any sequence. Let $Q : \ell^2(\mathscr{H}) \to \ell^2(\mathscr{H})$ be a bounded linear operator such that $Q\Big(\Big\{ \sum_{n=1}^{k} R_{n,i}(x) \Big\}\Big) = \{T_{p,i}(x)\}$, for some $p \in \{1,2,\dots,k\}$. If there exists a non-negative constant λ such that*

$$\sum_{i\in\mathbb{N}} \|(T_{n,i} - R_{n,i})x\|^2 \leqslant \lambda \sum_{i\in\mathbb{N}} \|T_{n,i}x\|^2, \ \ x \in \mathscr{H}, \ n = 1,2,\dots,k.$$

Then $\Big\{ \sum_{n=1}^{k} R_{n,i} \Big\}$ is a K-operator frame for $B(\mathscr{H})$.

Proof. For any $x \in \mathscr{H}$, we have

$$\sum_{i\in\mathbb{N}} \Big\| \sum_{n=1}^{k} R_{n,i}x \Big\|^2 \leqslant 2\sum_{i\in\mathbb{N}} \Big(\sum_{n=1}^{k} \|(T_{n,i}-R_{n,i})x\|^2 + \sum_{n=1}^{k} \|T_{n,i}x\|^2 \Big)$$
$$\leqslant 2k \sum_{n=1}^{k} \Big(\lambda \sum_{i\in\mathbb{N}} \|T_{n,i}x\|^2 + \sum_{i\in\mathbb{N}} \|T_{n,i}x\|^2 \Big)$$
$$\leqslant 2k(1+\lambda)(\sum_{i\in\mathbb{N}} B_i)\|x\|^2.$$

Also, for each $x \in \mathscr{H}$, we have

$$\Big\| Q\Big(\Big\{ \sum_{n=1}^{k} R_{n,i}x \Big\} \Big) \Big\|^2 = \sum_{i\in\mathbb{N}} \|T_{p,i}x\|^2.$$

Therefore, we get

$$A_p\|K^*x\|^2 \leqslant \sum_{i\in\mathbb{N}} \|T_{p,i}x\|^2$$
$$\leqslant \|Q\|^2 \sum_{i\in\mathbb{N}} \Big\| \sum_{n=1}^{k} R_{n,i}x \Big\|^2, \ \ x \in \mathscr{H},$$

where A_p is a lower bound of the K-operator frame $\{T_{p,n}\}$. This gives

$$\frac{A_p}{\|Q\|^2}\|K^*x\|^2 \leqslant \sum_{i\in\mathbb{N}} \Big\| \sum_{n=1}^{k} R_{n,i}x \Big\|^2, \ \ x \in \mathscr{H}.$$

Hence $\left\{ \sum_{n=1}^{k} R_{n,i} \right\}$ is a K-operator frame for $B(\mathscr{H})$. □

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REFERENCES

[1] C. Y. LI, H. X. CAO, *Operator frames for* $B(\mathscr{H})$, in: T. Qian, M. I. Vai, X. Yuesheng (eds.), *Wavelet Analysis and Applications*, Applications of Numerical Harmonic Analysis, 67–82, Springer, Berlin (2006).

[2] O. CHRISTENSEN, *An introduction to Frames and Riesz Bases*, Birkhauser, 2003.

[3] P. CASAZZA, G. KUTYNIOK, S. LI, *Fusion frames and distributed processing*, Appl Comput Harmon Anal **25** (2008), 114–132.

[4] I. DAUBECHIES, A. GROSSMANN AND Y. MEYER, *Painless nonorthogonal expansions*, J. Math. Physics **27** (1986), 1271–1283.

[5] R. J. DUFFIN AND A. C. SCHAEFFER, *A class of nonharmonic Fourier serier*, Trans. Am. Math. Soc. **72** (1952), 341–366.

[6] R. G. DOUGLAS, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415.

[7] S. J. FAVIER AND R. A. ZALIK, *On the stability of frames and Riesz bases*, Appl. Comput. Harmon. Anal. **2** (1995), no. 2, 160–173.

[8] K. GROCHENIG, *Describing functions: atomic decompositions versus frames*, Monatsh. Math. **112** (1991), 1–41.

[9] V. KAFTAL, D. LARSON, S. ZHANG, *Operator valued frames*, Trans Amer Math Soc. **361** (12) (2009), 6349–6385.

[10] L. GAVRUTA, *Frames for operators*, Appl. Comp. Harm. Anal. **32** (2012), 139–144.

[11] L. GAVRUTA, *New results on frame for operators*, Analele Universitatii Oradea Fasc. Matematica, Tom XIX (2) (2012), 55–61.

[12] W. C. SUN, *G-frames and g-Riesz bases*, J. Math. Anal. Appl. **322** (2006), 437–452.

[13] X. C. XIAO, Y. C. ZHU, L. GAVRUTA, *Some properties of K-frames in Hilbert spaces*, Results in mathematics **63** (2013), no. 3–4, 1243–1255.

[14] X. C. XIAO, Y. C. ZHU, Z. B. SHU, M. L. DING, *G-frames with bounded linear operators*, Rocky Mountain J. Math, **45** (2) (2015).

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