

## SPECTRA OF INFINITE GRAPHS VIA SCHUR COMPLEMENT

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*Abstract.* The goal of the paper is to apply the general operator theoretic construction known as the Schur complement for computation of the spectrum of certain infinite graphs which can be viewed as finite graphs with rays attached to them. The examples of a multiple star and a flower with infinite rays are considered.

### Introduction and preliminaries

#### 0.1. Graph theory

We begin with some basics of the graph theory. For the sake of simplicity we restrict ourselves with simple, connected, undirected, finite or infinite (countable) graphs, although the main result holds for weighted multigraphs and graphs with loops. We will label the vertex set  $\mathcal{V}(\Gamma)$  by positive integers  $\mathbb{N} = \{1, 2, \dots\}$ ,  $\{v\}_{v \in \mathcal{V}} = \{j\}_{j=1}^{\omega}$ ,  $\omega \leq \infty$ . The symbol  $i \sim j$  means that the vertices  $i$  and  $j$  are incident, i.e.,  $\{i, j\}$  belongs to the edge set  $\mathcal{E}(\Gamma)$ .

The degree (valency) of a vertex  $v \in \mathcal{V}(\Gamma)$  is a number  $\gamma(v)$  of edges emanating from  $v$ . A graph  $\Gamma$  is said to be locally finite, if  $\gamma(v) < \infty$  for all  $v \in \mathcal{V}(\Gamma)$ , and uniformly locally finite, if  $\sup_{v \in \mathcal{V}} \gamma(v) < \infty$ .

The spectral graph theory studies the spectra and spectral properties of certain matrices related to graphs (more precisely, operators acting on the Hilbert spaces  $\mathbb{C}^n$  or  $\ell^2 = \ell^2(\mathbb{N})$ , generated by such matrices in the standard basis  $\{e_k\}_{k=1}^{\omega}$ ). One of the most notable of them is the *adjacency matrix*  $A(\Gamma)$

$$A(\Gamma) = \|a_{ij}\|_{i,j=1}^{\omega}, \quad a_{ij} = \begin{cases} 1, & \{i, j\} \in \mathcal{E}(\Gamma); \\ 0, & \text{otherwise.} \end{cases} \quad (0.1)$$

The corresponding adjacency operator will be denoted by the same symbol. For locally finite graphs it acts as

$$A(\Gamma)e_k = \sum_{j \sim k} e_j, \quad k = 1, 2, \dots, \omega. \quad (0.2)$$

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Clearly,  $A(\Gamma)$  is a symmetric, densely-defined linear operator, whose domain is the set of all finite linear combinations of the basis vectors. The operator  $A(\Gamma)$  is bounded and self-adjoint on  $\ell^2$  as long as the graph  $\Gamma$  is uniformly locally finite.

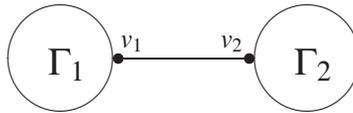
Whereas the spectral theory of finite graphs is very well established (see, e.g., [2, 3, 4]), the corresponding theory for infinite graphs is in its infancy. We refer to [9, 10] for the basics of this theory. In contrast to the general consideration in [10], our goal is to compute the spectra of certain infinite graphs (precisely, the spectra of the corresponding adjacency operators) which loosely speaking can be called “finite graphs with tails attached to them”.

To make the notion precise, we define first an operation of coupling well known for finite graphs (see, e.g., [4, Theorem 2.12]).

DEFINITION 1. Let  $\Gamma_k, k = 1, 2$ , be two graphs with no common vertices, with the vertex sets and edge sets  $\mathcal{V}(\Gamma_k)$  and  $\mathcal{E}(\Gamma_k)$ , respectively, and let  $v_k \in \mathcal{V}(\Gamma_k)$ . A graph  $\Gamma = \Gamma_1 + \Gamma_2$  will be called a *coupling by means of the bridge*  $\{v_1, v_2\}$  if

$$\mathcal{V}(\Gamma) = \mathcal{V}(\Gamma_1) \cup \mathcal{V}(\Gamma_2), \quad \mathcal{E}(\Gamma) = \mathcal{E}(\Gamma_1) \cup \mathcal{E}(\Gamma_2) \cup \{v_1, v_2\}. \quad (0.3)$$

So we join  $\Gamma_2$  to  $\Gamma_1$  by the new edge between  $v_2$  and  $v_1$ .



In general, the adjacency matrix  $A(\Gamma_1 + \Gamma_2)$  takes the form of a block operator matrix

$$A(\Gamma) = \begin{bmatrix} A(\Gamma_1) & E \\ E^* & A(\Gamma_2) \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (0.4)$$

If the graph  $\Gamma_1$  is finite,  $\mathcal{V}(\Gamma_1) = \{1, 2, \dots, n\}$ , and  $\mathcal{V}(\Gamma_2) = \{j\}_{j=1}^\omega$ , we can with no loss of generality put  $v_1 = 1, v_2 = n + 1$ , so the adjacency matrix  $A(\Gamma)$  can be written as the block operator matrix

$$A(\Gamma) = \begin{bmatrix} A(\Gamma_1) & E_n \\ E_n^* & A(\Gamma_2) \end{bmatrix}, \quad E_n = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \dots \end{bmatrix} \quad (0.5)$$

has  $n$  rows. If  $\Gamma_2 = \mathbb{P}_\infty$ , the one-sided infinite path, we can view the coupling  $\Gamma = \Gamma_1 + \mathbb{P}_\infty$  as a finite graph with the tail. Now

$$A(\Gamma_2) = J_0 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{0.6}$$

is a Jacobi matrix called a *discrete Laplacian* or a *free Jacobi matrix*. It is of particular interest in the sequel.

The spectral theory of infinite graphs with one or several rays attached to certain finite graphs was initiated in [7, 8, 11] wherein a number of particular examples of graphs was examined. In [5] this collection was enlarged considerably. A canonical form of the adjacency matrix  $A(\Gamma_1 + \mathbb{P}_\infty)$  suggested there enabled one to compute the spectrum of such graph by using the spectral theory of Jacobi matrices of finite rank.

Our argument here is based on a general construction from the block operator matrices theory known as the Schur complement. As a matter of fact, the procedure applies not only to adjacency matrices, but to both Laplacians on graphs of such type.

The examples in the next section rely heavily on the formula of Schwenk [13] (see [4, Problem 2.7.9]) for characteristic polynomials of finite graphs  $F$

$$P(\lambda, F) := \det(\lambda I - A(F)).$$

Given a graph  $F$  and  $V \subset \mathcal{V}(F)$  we denote by  $F \setminus V$  the subgraph of  $F$  induced by the vertices of  $\mathcal{V}(F) \setminus V$ .

**Theorem** (Schwenk). For a given finite graph  $F$  and  $v \in \mathcal{V}(F)$ , let  $\mathcal{C}(v)$  denote the set of all simple cycles  $Z$  which contain  $v$ . Then

$$P(\lambda, F) = \lambda P(\lambda, F \setminus v) - \sum_{v' \sim v} P(\lambda, F \setminus \{v', v\}) - 2 \sum_{Z \in \mathcal{C}(v)} P(\lambda, F \setminus Z).$$

**0.2. Schur complement**

Let

$$\mathcal{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{0.7}$$

be a block operator matrix which acts on the orthogonal sum  $\mathcal{H}_1 \oplus \mathcal{H}_2$  of two Hilbert spaces. If  $A_{11}$  is invertible, the matrix  $\mathcal{A}$  can be factorized as

$$\mathcal{A} = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}, \tag{0.8}$$

$I$  is the unity operator on the corresponding Hilbert space. Similarly, if  $A_{22}$  is invertible, one can write

$$\mathcal{A} = \begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} C_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}. \tag{0.9}$$

Here

$$C_{22} := A_{22} - A_{21}A_{11}^{-1}A_{12}, \quad C_{11} := A_{11} - A_{12}A_{22}^{-1}A_{21} \tag{0.10}$$

are usually referred to as the *Schur complements* [12]. Both equalities can be checked by direct multiplication.

The result below follows immediately from the formulae (0.8) and (0.9).

PROPOSITION 1. Given a block operator matrix  $\mathcal{A}$  (0.7), let  $A_{11}$  ( $A_{22}$ ) be invertible. Then  $\mathcal{A}$  is invertible if and only if so is  $C_{22}$  ( $C_{11}$ ).

Note that in the premises of Proposition 1 the inverse  $\mathcal{A}^{-1}$  takes the form

$$\mathcal{A}^{-1} = \begin{bmatrix} C_{11}^{-1} & -C_{11}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}C_{11}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}C_{11}^{-1}A_{12}A_{22}^{-1} \end{bmatrix}$$

and, respectively,

$$\mathcal{A}^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}C_{22}^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}C_{22}^{-1} \\ -C_{22}^{-1}A_{21}A_{11}^{-1} & C_{22}^{-1} \end{bmatrix}.$$

Denote by  $\rho(T)$  the resolvent set of a bounded, linear operator  $T$ , i.e., the set of complex numbers  $\lambda$  so that  $\lambda I - T$  is boundedly invertible. We apply the latter result to the block matrix

$$\lambda I - \mathcal{A} = \begin{bmatrix} \lambda I - A_{11} & -A_{12} \\ -A_{21} & \lambda I - A_{22} \end{bmatrix}, \quad \lambda \in \mathbb{C}, \tag{0.11}$$

to obtain

PROPOSITION 2. Given a block operator matrix  $\mathcal{A}$  (0.7), let  $\lambda \in \rho(A_{11})$  ( $\lambda \in \rho(A_{22})$ ). Then  $\lambda \in \rho(\mathcal{A})$  if and only if the operator

$$\begin{aligned} C_{22}(\lambda) &= \lambda I - A_{22} - A_{21}(\lambda I - A_{11})^{-1}A_{12} \\ (C_{11}(\lambda) &= \lambda I - A_{11} - A_{12}(\lambda I - A_{22})^{-1}A_{21}) \end{aligned} \tag{0.12}$$

is invertible.

### 1. Spectra of infinite graphs with tails

The adjacency matrices (0.4) and (0.5) of couplings  $\Gamma = \Gamma_1 + \Gamma_2$  are block operator matrices (0.7), and, moreover,  $\lambda I - \mathcal{A}$  is of the form (0.11),

$$\lambda I - A(\Gamma) = \begin{bmatrix} \lambda I - A(\Gamma_1) & -E \\ -E^* & \lambda I - A(\Gamma_2) \end{bmatrix} = (\lambda I - A(\Gamma_1)) \oplus (\lambda I - A(\Gamma_2)) + \Delta$$

with  $\text{rank}(\Delta) = 2$ . So we can gather some information about the spectrum of  $A(\Gamma)$  for arbitrary graphs  $\Gamma_j$ ,  $j = 1, 2$ , from the general perturbation theory of finite rank (see, e.g., [1, Section 9.2]). For instance, according to the famous result of Kato

$$\sigma_{\text{ess}}(A(\Gamma)) = \sigma_{\text{ess}}(A(\Gamma_1)) \cup \sigma_{\text{ess}}(A(\Gamma_2)),$$

(recall that a number  $\lambda \in \sigma(T)$  belongs to the essential spectrum  $\sigma_{ess}(T)$  of a self-adjoint operator  $T$  if it is not an eigenvalue of finite multiplicity). Moreover,  $\lambda$  is the eigenvalue of  $A(\Gamma)$  as soon as  $\lambda$  is the isolated eigenvalue of either  $A(\Gamma_1)$  or  $A(\Gamma_2)$  of multiplicity at least 3.

The Schur complements (0.12) take now the form

$$C_{ii}(\lambda) = \lambda I - A(\Gamma_i) - G(\lambda, \Gamma_j) \tilde{e}_i, \quad i \neq j, \quad i, j = 1, 2, \tag{1.1}$$

where

$$G(\lambda, \Gamma_j) := \left( (\lambda I - A(\Gamma_j))^{-1} \right)_{11}, \quad \lambda \in \rho(A(\Gamma_j)), \quad j = 1, 2 \tag{1.2}$$

is the so-called Green’s function of  $A(\Gamma_j)$ .  $\tilde{e}_i$  is a square matrix of the same order as  $A(\Gamma_i)$ , with the only nonzero entry  $(\tilde{e}_i)_{11} = 1$ .

We say that  $\lambda$  is a regular point of the graph  $\Gamma$  if  $\lambda \in \rho(A(\Gamma))$ .

**THEOREM 1.** *Let  $\Gamma = \Gamma_1 + \Gamma_2$  be the coupling of a finite graph  $\Gamma_1$  and an arbitrary graph  $\Gamma_2$ , and let  $\lambda$  be a regular point of  $\Gamma_2$ . The point  $\lambda$  belongs to the spectrum of  $\Gamma$  if and only if it solves the equation*

$$P(\lambda, \Gamma_1) - G(\lambda, \Gamma_2)P(\lambda, \Gamma_1 \setminus v_1) = 0. \tag{1.3}$$

*Proof.* By Proposition 2 and (1.1),  $\lambda \in \sigma(A(\Gamma))$  if and only if

$$\det C_{11}(\lambda) = \det((\lambda I - A(\Gamma_1) - G(\lambda, \Gamma_2) \tilde{e}_1)) = 0, \quad \tilde{e}_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

The result follows by expanding the latter determinant along the first row.  $\square$

The basic example for us is  $\Gamma_2 = \mathbb{P}_\infty$ . In this case

$$\sigma(\Gamma) = \sigma_{ess}(\Gamma) \cup \sigma_d(\Gamma) = [-2, 2] \cup \sigma_d(\Gamma),$$

where the discrete spectrum  $\sigma_d(\Gamma)$  is the set of eigenvalues of finite multiplicity off  $[-2, 2]$ . Theorem 1 can be applied, and the Green’s function is known explicitly (see, e.g., [6])

$$\left( \lambda I - A(\mathbb{P}_\infty) \right)^{-1} = \|r_{ij}(z)\|_{i,j=1}^\infty, \quad r_{ij}(z) = \frac{z^{i+j} - z^{i-j}}{z - z^{-1}}, \quad \lambda = z + \frac{1}{z}, \tag{1.4}$$

$|z| < 1$ , and so

$$G(\lambda, \mathbb{P}_\infty) = r_{11}(z) = z. \tag{1.5}$$

The discrete spectrum agrees with the zero set of the algebraic equation (1.3)

$$\lambda \in \sigma_d(\Gamma) \Leftrightarrow P(\lambda, \Gamma_1) - xP(\lambda, \Gamma_1 \setminus v_1) = 0, \quad \lambda = x + \frac{1}{x}, \quad x \in (-1, 1). \tag{1.6}$$

So the problem amounts to computation of two characteristic polynomials and solving the equation (1.6).

EXAMPLE 1. “A multiple star”.

Let  $\kappa := (k_1, k_2, \dots, k_n)$ ,  $k_j \in \mathbb{N}$  be an  $n$ -tuple of positive integers. Denote by  $S(\kappa)$  the graph obtained from the standard star graph  $K_{1,n}$  with  $n \geq 2$  edges by inserting  $k_j - 1$  new vertices into  $j$ 's edge, so this edge contains  $k_j + 1$  vertices altogether. Put  $\Gamma_1 = S(\kappa)$  and consider the coupling  $\Gamma = \Gamma_1 + \mathbb{P}_\infty$  obtained from  $S(\kappa)$  by attaching the infinite path to its root  $v_1$ .

Denote by  $Q(\cdot, m)$  the characteristic polynomial of the finite path  $\mathbb{P}_m$  with  $m$  vertices, so

$$Q(\lambda, m) = U_m\left(\frac{\lambda}{2}\right), \quad U_m(\cos t) := \frac{\sin(m+1)t}{\sin t}$$

is the standard Chebyshev polynomial of the second kind.

To compute the characteristic polynomial of  $\Gamma_1$  we apply the result of Schwenk which now looks

$$P(\lambda, \Gamma_1) = \lambda Q(\lambda) - Q(\lambda) \sum_{j=1}^n \frac{Q(\lambda, k_j - 1)}{Q(\lambda, k_j)}, \quad Q(\lambda) := \prod_{j=1}^n Q(\lambda, k_j).$$

It is clear that

$$P(\lambda, \Gamma_1 \setminus v_1) = Q(\lambda),$$

so the equation in (1.6) takes the form

$$Q(\lambda) \left\{ \frac{1}{x} - \sum_{j=1}^n \frac{Q(\lambda, k_j - 1)}{Q(\lambda, k_j)} \right\} = 0, \quad \lambda = x + \frac{1}{x}.$$

Since  $Q \neq 0$  off  $[-2, 2]$  we come to

$$\sum_{j=1}^n \frac{Q(\lambda, k_j - 1)}{Q(\lambda, k_j)} - \frac{1}{x} = 0, \quad \lambda = x + \frac{1}{x}, \quad -1 < x < 1.$$

The function in the left-hand side is odd (as a function of  $x$ ), so we can restrict ourselves with the values  $0 < x < 1$ . Putting  $x = e^{-t}$ ,  $t > 0$ , we obtain after a bit of calculation

$$\varphi(t) := \sum_{j=1}^n \frac{\sinh k_j t}{\sinh(k_j + 1)t} = e^t, \quad t > 0. \tag{1.7}$$

Since the function

$$\frac{\sinh at}{\sinh bt}, \quad 0 < a < b, \quad t > 0$$

is easily seen to be monotone decreasing (it follows, e.g., from the infinite product expansion of  $\sinh z$ ), then so is  $\varphi$  in the left-hand side of (1.7). Moreover,  $\varphi$  vanishes at infinity. Next,

$$\lim_{t \rightarrow 0} \varphi(t) = \sum_{j=1}^n \frac{k_j}{k_j + 1} > 1$$

(we discard the trivial configuration  $n = 2, k_1 = k_2 = 1$ ). So (1.7) has a unique solution  $t_+ > 0$ . Finally, the discrete spectrum is

$$\sigma_d(\Gamma) = \pm\lambda_+, \quad \lambda_+ := 2 \cosh t_+. \tag{1.8}$$

In particular case  $k_j = p, 1 \leq j \leq n$ , equation (1.7) looks

$$n \frac{\sinh pt}{\sinh(p+1)t} = e^t, \quad n(e^{pt} - e^{-pt}) = e^{(p+2)t} - e^{-pt},$$

which is equivalent to

$$(n-1)x^{2p+2} - nx^2 + 1 = 0.$$

Note that in this case the more detailed description of the spectrum is available, see [5, Example 2.3]. Precisely, there are  $p$  eigenvalues lying on the essential spectrum  $[-2, 2]$ .

EXAMPLE 2. “A flower with  $n$  petals”.

In this example  $\Gamma_1$  is composed of  $n$  cycles  $\{\mathbb{C}_j\}_{j=1}^n$ , glued together at one common vertex (root)  $\mathcal{O}$ . Put  $\Gamma = \Gamma_1 + \mathbb{P}_\infty$  with the infinite path attached to the root  $\mathcal{O}$ . Assume that the cycle  $\mathbb{C}_j$  contains  $k_j + 1$  vertices.

To compute the characteristic polynomial of  $\Gamma_1$  we apply again the Schwenk theorem which gives

$$P(\lambda, \Gamma_1) = Q(\lambda) \left\{ \lambda - 2 \sum_{j=1}^n \frac{Q(\lambda, k_j - 1) + 1}{Q(\lambda, k_j)} \right\}.$$

As in the above example,  $P(\cdot, \Gamma_1 \setminus \mathcal{O}) = Q$ , and we come to the following equation

$$2 \sum_{j=1}^n \frac{Q(\lambda, k_j - 1) + 1}{Q(\lambda, k_j)} - \frac{1}{x} = 0, \quad \lambda = x + \frac{1}{x}, \quad -1 < x < 1.$$

Putting  $x = e^{-t}, t > 0$ , we obtain

$$2 \sum_{j=1}^n \frac{\sinh k_j t}{\sinh(k_j + 1)t} = e^t, \quad t > 0. \tag{1.9}$$

The same argument as above shows that (1.9) has a unique solution  $t_+ > 0$ , so the point  $\lambda_+ = 2 \cosh t_+ \in \sigma_d(\Gamma)$  for all configurations in  $\Gamma_1$  (with no exceptions). Putting  $x = -e^{-t}, t > 0$ , we come to

$$\varphi(t) := 2 \sum_{j=1}^n \frac{\sinh k_j t + (-1)^{k_j+1} \sinh t}{\sinh(k_j + 1)t} = e^t, \quad t > 0. \tag{1.10}$$

Since

$$\frac{\sinh mt - \sinh t}{\sinh(m+1)t} = \frac{\sinh \frac{m-1}{2}t}{\sinh \frac{m+1}{2}t},$$

$\varphi$  is a monotone decreasing function vanishing at infinity, and  $\varphi(0+) > 1$  for all configurations in  $\Gamma_1$ . Hence (1.10) has a unique solution  $t_- > 0$ , and  $\lambda_- = -2 \cosh t_- \in \sigma_d(\Gamma)$ . Finally, the discrete spectrum is

$$\sigma_d(\Gamma) = \lambda_{\pm}, \quad \lambda_{\pm} := \pm 2 \cosh t_{\pm}. \quad (1.11)$$

Note that in particular case  $n = 2$ ,  $k_1 = k_2$  (the propeller with equal blades), a complete description of the spectrum is given in [5, Example 3.4].

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