

ASYMPTOTICS OF GENERALIZED VALUE DISTRIBUTION FOR HERGLOTZ FUNCTIONS

Y. CHRISTODOULIDES

(Communicated by F. Gesztesy)

Abstract. Estimates of limiting value distributions for boundary values of Herglotz functions are extended to allow the possibility of value distributions with respect to measures other than Lebesgue measure. We establish a relation between the generalized theory of value distribution and the angle subtended at a point in the upper half-plane, and we carry out an analysis of the corresponding composed Herglotz functions and their measures. The results are applicable to a description of boundary behaviour for the Weyl m -function in Sturm-Liouville theory.

1. Introduction

A value distribution mapping \mathcal{M} for a Lebesgue measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a mapping $\mathcal{M}(A, S) = |A \cap f^{-1}(S)|$, where A, S are any Borel sets and $|\cdot|$ denotes Lebesgue measure. Thus, $\mathcal{M}(A, S)$ is the Lebesgue measure of the points $\lambda \in A$ for which $f(\lambda) \in S$.

A special case of particular interest is when f is the (real) boundary value function of a Herglotz function F , that is, a function which is analytic with positive imaginary part in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im}z > 0\}$. A theory of value distribution in this case was developed in [13], and the theory may be extended to give a meaning to the value distribution mapping where the real function f is replaced by a Herglotz function having real or complex eigenvalues. There are applications of this theory to the spectral analysis of Herglotz measures, and more especially, when F is taken to be the Weyl-Titchmarsh m -function ([7]), to the spectral analysis of Sturm-Liouville differential operators ([3], [2]). In general, ‘averages’ of families of spectral measures and their properties have been studied extensively. See, for example, [8], [9], [10], [12].

The theory of value distribution for boundary values of Herglotz functions was generalized in [5], to allow a description of value distribution in terms of measures other than Lebesgue measure. A close connection between the generalized theory and compositions of Herglotz functions was established. See [6] for results regarding the integral representation of composed Herglotz functions.

In [4], we obtained a description of the asymptotic generalized value distribution of solutions of the Schrödinger equation. That result generalizes the respective result in [3], in the standard case of Lebesgue measure, where the asymptotic value distribution

Mathematics subject classification (2010): 30D35, 34M30.

Keywords and phrases: Asymptotic generalized value distribution, Herglotz functions.

of solutions of the Schrödinger equation was obtained in terms of a limiting integral involving the boundary values of the m -function. We refer the reader to [4] for the precise statement of the result as well as a detailed proof. A key result that was used in [4] was an estimate of generalized value distribution for a family of Herglotz functions translated by an increment $i\delta$ in a direction parallel to the real axis (equation (10) in [4]). In this paper, we give a complete proof of this estimate ((26) in Theorem 1 below). Some of the equations and inequalities that we present are of interest in their own right, and exhibit the interrelation between ideas of the theory of value distribution and geometrical properties of the upper half-plane.

The paper is organized as follows. In Section 2 we state some basic results regarding Herglotz functions, in particular their integral representation. The generalized value distribution associated with a Herglotz function F is introduced in Section 3. In Section 4 we prove the results for the generalized value distribution of translated Herglotz functions. Our main results are presented in Lemma 2, Theorem 1, and Corollary 1.

2. Herglotz function preliminaries

Let F be a Herglotz function, that is, analytic with positive imaginary part in the upper half-plane $\mathbb{C}_+ = \{z : \text{Im } z > 0\}$. Then, F admits the integral representation [11, 1]

$$F(z) = c_1 + c_2 z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\rho(t), \quad (1)$$

where c_1, c_2 are real constants ($c_2 \geq 0$), and the function $\rho(t)$ is non-decreasing, right-continuous, and determined up to an additive constant. For a given Herglotz function F , the constants c_1, c_2 are specified by

$$c_1 = \text{Re } F(i), \quad c_2 = \lim_{s \rightarrow +\infty} \frac{1}{s} \text{Im } F(is).$$

The function $\rho(t)$ gives rise to a measure μ , defined for finite intervals $(a, b]$ by $\mu((a, b]) = \rho(b) - \rho(a)$, and μ extends to Borel sets. The measure μ is referred to as the ‘spectral measure’ corresponding to the Herglotz function F , and satisfies the condition

$$\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu(t) < \infty, \quad (2)$$

which is sufficient for the integral in (1) to converge absolutely.

The decomposition of μ into an absolutely continuous part $\mu_{a.c.}$, and a singular part μ_s , with respect to Lebesgue measure, is determined by the boundary behaviour of F near the real axis ([14]). The boundary value $F_+(\lambda)$ of F at the point $\lambda \in \mathbb{R}$, is defined by $F_+(\lambda) = \lim_{\varepsilon \rightarrow 0^+} F(\lambda + i\varepsilon)$, and exists as a finite number Lebesgue almost everywhere. Then, the support of $\mu_{a.c.}$ is the set $\{\lambda \in \mathbb{R} : 0 < \text{Im } F_+(\lambda) < +\infty\}$, and the density function f of $\mu_{a.c.}$ is given by $f(\lambda) = \frac{1}{\pi} \text{Im } F_+(\lambda)$, whereas the support of μ_s is the set $\{\lambda \in \mathbb{R} : \text{Im } F_+(\lambda) = +\infty\}$.

3. Herglotz functions and generalized value distribution

Given a Herglotz function F , we define a one-parameter family of Herglotz functions F_y ($y \in \mathbb{R}$) by

$$F_y(z) = \frac{1}{y - F(z)}. \quad (3)$$

Let $\{\mu_y\}$ be the measures corresponding to F_y through the integral representation (1). The generalized value distribution associated with the Herglotz function F is defined by

$$v_S(A) = \int_S \mu_y(A) d\sigma(y), \quad (4)$$

for any Borel sets A, S , where the measure σ corresponds to a Herglotz function ϕ , with integral representation

$$\phi(z) = a_\phi + b_\phi z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\sigma(t). \quad (5)$$

(We note that in the case of the standard theory of value distribution of Herglotz functions, the integral in (4) takes place with respect to Lebesgue measure). The measure v_S may also be verified to satisfy condition (2) and hence corresponds to a Herglotz function H , with representation

$$H(z) = a_H + b_H z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} dv_S(t). \quad (6)$$

In the special case when the boundary values of F are real almost everywhere, then the measures μ_y are purely singular, and we have ([5])

$$v_S(A) = v_S(A \cap F_+^{-1}(S)) = v_{\mathbb{R}}(A \cap F_+^{-1}(S)). \quad (7)$$

Thus the measure v_S of the set A is concentrated on the points λ in A at which the boundary value of F is in S , and also it agrees on this set with the measure $v_{\mathbb{R}}$ (for which the integral in (4) takes place over \mathbb{R}).

The measure v_S is closely related with compositions of Herglotz functions. For any Borel set B , we have ([5])

$$v_S(B) = \mu_{(\phi_S \circ F)}(B) - b_\phi \mu(B), \quad (8)$$

where $\mu_{(\phi_S \circ F)}$ is the measure corresponding to the composed Herglotz function $\phi_S \circ F$, and the Herglotz functions $\phi_S, \phi_S \circ F$ have the following representations:

$$\phi_S(z) = a_\phi + b_\phi z + \int_S \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\sigma(t), \quad (9)$$

$$(\phi_S \circ F)(z) = a_{(\phi_S \circ F)} + b_{(\phi_S \circ F)} z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\mu_{(\phi_S \circ F)}(t). \quad (10)$$

Thus, ϕ_S is the Herglotz function having the same representation as ϕ , except that integration takes place over the set S . Hence the case $S = \mathbb{R}$ corresponds to ϕ . Note also that if $b_\phi = 0$, then ν_S is precisely the measure corresponding to the function $\phi_S \circ F$.

In the next Section, we shall obtain a relation between the generalized value distribution and the angle θ subtended at the point $z \in \mathbb{C}_+$ by the set S on the real axis, defined by

$$\theta(z, S) = \int_S \operatorname{Im} \left[\frac{1}{t-z} \right] dt. \tag{11}$$

4. Generalized value distribution and translated Herglotz functions

Given a Herglotz function F , define a Herglotz function F^δ , obtained from F by translation through an increment $i\delta$ parallel to the real axis, thus

$$F^\delta(z) = F(z + i\delta), \quad \delta > 0,$$

and define a family of translated Herglotz functions F_y^δ by

$$F_y^\delta(z) = \frac{1}{y - F^\delta(z)}, \quad \delta > 0, y \in \mathbb{R}. \tag{12}$$

LEMMA 1. *We have $F^\delta(z) \rightarrow F(z)$ uniformly, as $\delta \rightarrow 0^+$, for z on any compact subset of the upper half-plane.*

Proof. Let D be a compact subset of the upper half-plane. There exist constants $K_D, \varepsilon_D > 0$ such that $|\operatorname{Re} z| \leq K_D$ and $0 < \varepsilon_D \leq \operatorname{Im} z \leq K_D$ for all $z \in D$. From the representation of F in (1) we have

$$|F^\delta(z) - F(z)| \leq b_f \delta + \delta \int_{\mathbb{R}} \frac{1}{|t - z - i\delta||t - z|} d\mu(t).$$

For $z \in D$ we have $(|t - z - i\delta||t - z|)^{-1} \leq |t - z|^{-2} \leq \operatorname{const} \cdot (1 + t^2)^{-1}$. Thus the integral above is finite, and taking the limit as $\delta \rightarrow 0^+$ we see that $|F^\delta(z) - F(z)| \rightarrow 0$ uniformly for all $z \in D$. \square

It is straightforward to show that also $F_y^\delta(z) \rightarrow F_y(z)$ uniformly, as $\delta \rightarrow 0^+$, on compact subsets of the upper half-plane. This follows from the relation

$$|F_y^\delta(z) - F_y(z)| = \frac{|F(z + i\delta) - F(z)|}{|y - F(z + i\delta)||y - F(z)|}.$$

The following lemma shows how the generalized value distribution of the translated Herglotz function F^δ may be expressed in terms of an integral of angle subtended with respect to the measure ν_S .

LEMMA 2. Let μ_y^δ be a family of measures corresponding to the Herglotz functions F_y^δ , for $\delta > 0$, $y \in \mathbb{R}$, and σ an arbitrary Herglotz measure. Let A be any bounded Borel set, and S any Borel set. Then, we have

$$\int_S \mu_y^\delta(A) d\sigma(y) = \frac{1}{\pi} |A| (b_{(\phi_S \circ F)} - b_\phi b_F) + \frac{1}{\pi} \int_{\mathbb{R}} \theta(t + i\delta, A) dv_S(t). \tag{13}$$

Moreover, if the measure σ is absolutely continuous, (13) reduces to

$$\int_S \mu_y^\delta(A) d\sigma(y) = \frac{1}{\pi} \int_{\mathbb{R}} \theta(t + i\delta, A) dv_S(t). \tag{14}$$

Proof. Fix $\delta > 0$. Then, the Herglotz function F^δ has boundary values with strictly positive imaginary part, as do the functions F_y^δ . Thus, the measures μ_y^δ are absolutely continuous, with density functions $\frac{1}{\pi} \text{Im } F_y(\lambda + i\delta)$. Hence we have

$$\int_S \mu_y^\delta(A) d\sigma(y) = \int_S \left\{ \frac{1}{\pi} \int_A \text{Im} \left[\frac{1}{y - F(\lambda + i\delta)} \right] d\lambda \right\} d\sigma(y).$$

For $\lambda \in A$ and fixed δ we have $\text{Im} [y - F(\lambda + i\delta)]^{-1} \leq \text{const} \cdot (1 + y^2)^{-1}$. Thus, the double integral above is absolutely convergent and we may change the order of integration. From the integral representation of ϕ_S in (9) we have

$$\text{Im } \phi_S(F(\lambda + i\delta)) - b_\phi \text{Im } F(\lambda + i\delta) = \int_S \text{Im} \left[\frac{1}{t - F(\lambda + i\delta)} \right] d\sigma(t),$$

so that

$$\int_S \mu_y^\delta(A) d\sigma(y) = \frac{1}{\pi} \int_A \left\{ \text{Im } \phi_S(F(\lambda + i\delta)) - b_\phi \text{Im } F(\lambda + i\delta) \right\} d\lambda.$$

Similarly, by using expressions for $\text{Im} (\phi_S \circ F)(\lambda + i\delta)$, $\text{Im } F(\lambda + i\delta)$ from the representations of the functions $\phi_S \circ F$, F in (10), (1) respectively, and equation (8), we obtain

$$\begin{aligned} \int_S \mu_y^\delta(A) d\sigma(y) &= \frac{1}{\pi} \int_A \delta (b_{(\phi_S \circ F)} - b_\phi b_F) d\lambda \\ &\quad + \frac{1}{\pi} \int_A \left\{ \int_{\mathbb{R}} \frac{\delta}{(t - \lambda)^2 + \delta^2} d\mu_{(\phi_S \circ F)}(t) - b_\phi \int_{\mathbb{R}} \frac{\delta}{(t - \lambda)^2 + \delta^2} d\mu(t) \right\} d\lambda \\ &= \frac{1}{\pi} \delta |A| (b_{(\phi_S \circ F)} - b_\phi b_F) + \frac{1}{\pi} \int_A \left\{ \int_{\mathbb{R}} \frac{\delta}{(t - \lambda)^2 + \delta^2} dv_S(t) \right\} d\lambda. \end{aligned}$$

The above double integral is also absolutely convergent, since from the representation of the Herglotz function H in (6) we have

$$\int_{\mathbb{R}} \frac{\delta}{(t - \lambda)^2 + \delta^2} dv_S(t) = \text{Im } H(\lambda + i\delta) - b_H \delta \leq \text{Im } H(\lambda + i\delta),$$

which is uniformly bounded for λ in A . Hence, by changing the order of integration we obtain

$$\int_S \mu_y^\delta(A) d\sigma(y) = \frac{1}{\pi} \delta|A| (b_{(\phi_S \circ F)} - b_\phi b_F) + \frac{1}{\pi} \int_{\mathbb{R}} \theta(t + i\delta, A) d\nu_S(t),$$

from the definition of the angle subtended θ . In the case that σ is absolutely continuous the term $\frac{1}{\pi} \delta|A| (b_{(\phi_S \circ F)} - b_\phi b_F)$ vanishes ([6]), and the lemma is proved. \square

In the remainder of the paper we shall assume that the measure σ is absolutely continuous with density function h_σ . For any Borel set S , we define the sets S_0, S_1 by

$$S_0 = \{y \in S : h_\sigma(y) \leq C\}, \quad S_1 = \{y \in S : h_\sigma(y) > C\}, \tag{15}$$

where $C > 0$ is a constant.

We now define Herglotz functions ϕ_0, ϕ_1 and the corresponding composed Herglotz functions $(\phi_0 \circ F), (\phi_1 \circ F)$ with the following representations:

$$\phi_0(z) = a_\phi + b_\phi z + \int_{S_0} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\sigma(t), \tag{16}$$

$$\phi_1(z) = \int_{S_1} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\sigma(t), \tag{17}$$

$$(\phi_0 \circ F)(z) = a_0 + b_0 z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\nu_0(t), \tag{18}$$

$$(\phi_1 \circ F)(z) = a_1 + b_1 z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\nu_1(t). \tag{19}$$

Since $(\phi_S \circ F)(z) = \phi_0(F(z)) + \phi_1(F(z))$ for all $z \in \mathbb{C}_+$, we have $\mu_{(\phi_S \circ F)}(B) = \nu_0(B) + \nu_1(B)$ for any Borel set B , and also from equation (8) we obtain

$$\nu_S(B) = \nu_0(B) + \nu_1(B) - b_\phi \mu(B). \tag{20}$$

The measure corresponding to ϕ_1 is the restriction of σ to the set S_1 , and the measure α corresponding to ϕ_0 is the restriction of σ to S_0 , and is bounded by Lebesgue measure. For any Borel set B we have $\alpha(B) \leq C|B|$, where C is the constant in (15), and $|\cdot|$ denotes Lebesgue measure. A similar result is given in the next lemma.

LEMMA 3. *For any Borel set B , we have $\nu_0(B) - b_\phi \mu(B) \leq C|B|$. Thus, the measure $(\nu_0 - b_\phi \mu)$ is absolutely continuous with respect to Lebesgue measure, with density function bounded by C .*

Proof. Note first that $(\nu_0 - b_\phi \mu)$ is the measure corresponding to the composed Herglotz function $\phi_0(F)$, in the case when $b_\phi = 0$.

For points a, b ($a < b$) which are not discrete points of $(\nu_0 - b_\phi \mu)$ we have

$$(\nu_0 - b_\phi \mu)((a, b]) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \int_a^b \left\{ \int_{S_0} \operatorname{Im} \left[\frac{1}{t - F(\lambda + i\epsilon)} \right] d\sigma(t) \right\} d\lambda \leq C(b - a),$$

by an application of the Lebesgue dominated convergence theorem, since σ is bounded by Lebesgue measure on the set S_0 , and for any fixed value of $\varepsilon > 0$ we have $\int_{\mathbb{R}} \text{Im} [t - F(\lambda + i\varepsilon)]^{-1} dt = \pi$.

By considering sequences of intervals whose endpoints are not discrete points of $(v_0 - b_\phi\mu)$, this result extends to arbitrary open intervals, and hence to open Borel sets.

Fix $C > 0$. If B is any bounded Borel set, given any $\varepsilon > 0$ there is an open set G containing B such that $|G| < |B| + \frac{\varepsilon}{C}$. Then, we have $(v_0 - b_\phi\mu)(B) \leq (v_0 - b_\phi\mu)(G) \leq C|G| \leq C|B| + \varepsilon$, and since ε was arbitrary we can infer $(v_0 - b_\phi\mu)(B) \leq C|B|$.

Finally, the result generalizes to arbitrary Borel sets through the relation $(v_0 - b_\phi\mu)(B) = (v_0 - b_\phi\mu)\left(\bigcup_N B \cap [-N, N]\right)$, and the lemma is proved. \square

The following lemma provides a useful bound for the angle subtended by an interval.

LEMMA 4. Let $a, b \in \mathbb{R}$ with $a < b$, and fix δ with $0 < \delta < 1$. Then, for $t \in [a - 1, b + 1]^c$ we have

$$\theta(t + i\delta, [a, b]) \leq \delta a_1 \frac{1}{1 + t^2},$$

where $a_1 > 0$ is a constant depending on a and b but independent of δ .

Proof. For $t \in [a - 1, b + 1]^c$, the result follows from the bound

$$0 < \frac{b - a}{\delta^2 + (t - a)(t - b)} \leq a_1 \frac{1}{1 + t^2},$$

where a_1 is a positive constant, on noting the inequality $\tan^{-1} x < x$ for $x > 0$. \square

LEMMA 5. Let $[a, b]$ be a finite closed interval, and suppose that the measure σ is absolutely continuous with respect to Lebesgue measure. Let $\varepsilon > 0$ be given, and take δ with $0 < \delta < 1$. Then, a constant $N = N(\varepsilon)$ can be found such that

$$(i) \frac{1}{\pi} \int_{A^c} \theta(t + i\delta, A) d\nu_1(t) < \varepsilon \quad \text{and} \quad (ii) \frac{1}{\pi} \int_A \theta(t + i\delta, A^c) d\nu_1(t) < \varepsilon$$

hold for all $C > N(\varepsilon)$ and for all Borel sets $A \subseteq [a, b]$. Here C is the constant in (15).

Proof. Since $A \subseteq [a, b]$, we have $\theta(t + i\delta, A) \leq \theta(t + i\delta, [a, b])$, and it follows from lemma 4 that

$$\frac{1}{\pi} \int_{A^c \cap [a-1, b+1]^c} \theta(t + i\delta, A) d\nu_1(t) \leq \frac{1}{\pi} \delta a_1 \int_{\mathbb{R}} \frac{1}{1 + t^2} d\nu_1(t), \tag{21}$$

for some constant a_1 independent of δ . Moreover,

$$\frac{1}{\pi} \int_{A^c \cap [a-1, b+1]} \theta(t + i\delta, A) d\nu_1(t) \leq a_2 \int_{\mathbb{R}} \frac{1}{1 + t^2} d\nu_1(t), \tag{22}$$

where a_2 is a constant satisfying $1+t^2 \leq a_2$ for $t \in [a-1, b+1]$. From the representations of the functions $\phi_1 \circ F$, ϕ_1 in (19), (17) respectively, we have

$$\operatorname{Im}(\phi_1 \circ F)(i) = b_1 + \int_{\mathbb{R}} \frac{1}{1+t^2} dv_1(t), \quad (b_1 \geq 0)$$

$$\operatorname{Im} \phi_1(F(i)) = \int_{S_1} \operatorname{Im} \left[\frac{1}{t-F(i)} \right] d\sigma(t).$$

Therefore, from (21) and (22) we obtain

$$\frac{1}{\pi} \int_{A^c} \theta(t+i\delta, A) dv_1(t) \leq (\delta a_1/\pi + a_2) \int_{S_1} \operatorname{Im} \left[\frac{1}{t-F(i)} \right] d\sigma(t). \quad (23)$$

Since $\operatorname{Im} [t-F(i)]^{-1} \leq \operatorname{const} \cdot (1+t^2)^{-1}$, the constant C in the definition of S_1 can now be chosen ([5]) such that the expression on the right of inequality (23) is less than ε , and the first assertion of the lemma follows.

To verify the second assertion, note that

$$\begin{aligned} \frac{1}{\pi} \int_A \theta(t+i\delta, A^c) dv_1(t) &\leq a_2 \int_{\mathbb{R}} \frac{1}{1+t^2} dv_1(t) \\ &\leq a_2 \int_{S_1} \operatorname{Im} \left[\frac{1}{t-F(i)} \right] d\sigma(t) < \varepsilon \end{aligned}$$

by our choice of C , and the lemma is proved. \square

LEMMA 6. *With the same assumptions as in Lemma 5 we have*

$$(i) \quad \frac{1}{\pi} \int_{A^c} \theta(t+i\delta, A) d(v_0 - b_\phi \mu)(t) \leq CE_A(\delta),$$

and

$$(ii) \quad \frac{1}{\pi} \int_A \theta(t+i\delta, A^c) d(v_0 - b_\phi \mu)(t) \leq CE_A(\delta),$$

where

$$E_A(\delta) = \frac{1}{\pi} \int_{A^c} \theta(t+i\delta, A) dt = \frac{1}{\pi} \int_A \theta(t+i\delta, A^c) dt. \quad (24)$$

Proof. The result follows from lemma 3. Note that

$$\begin{aligned} E_A(\delta) &= \frac{1}{\pi} \int_{A^c} \theta(t+i\delta, A) dt = \frac{1}{\pi} \int_{\mathbb{R}} \theta(t+i\delta, A) dt - \frac{1}{\pi} \int_A \{\pi - \theta(t+i\delta, A^c)\} dt \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \left\{ \int_A \frac{\delta}{(t-\lambda)^2 + \delta^2} d\lambda \right\} dt - |A| + \frac{1}{\pi} \int_A \theta(t+i\delta, A^c) dt \\ &= \frac{1}{\pi} \int_A \theta(t+i\delta, A^c) dt. \quad \square \end{aligned}$$

REMARK 1. Note that in the special case in which σ is Lebesgue measure, we can take $C = 1$ and $\phi_0 = \phi = i\pi$, with $a_\phi = b_\phi = 0$, so that ν_0 and ν are then also Lebesgue measure. In this special case, the two integrals under (i) and (ii) of the lemma are identical.

Note also that in the case when A is a finite interval $[a, b]$, then ([3])

$$E_A(\delta) = \frac{2}{\pi}(b-a) \tan^{-1} \frac{\delta}{(b-a)} + \frac{\delta}{\pi} \ln [(b-a)^2 + \delta^2] - \frac{2}{\pi} \delta \ln \delta. \quad (25)$$

THEOREM 1. Define the measures μ_y^δ as in lemma 2, and $E_A(\delta)$ by (24). With the same assumptions as in lemma 5, and for any given $\varepsilon > 0$, a constant $N = N(\varepsilon)$ can be found such that

$$\left| \int_S \mu_y^\delta(A) d\sigma(y) - \int_S \mu_y(A) d\sigma(y) \right| \leq CE_A(\delta) + \varepsilon, \quad (26)$$

for all $C > N(\varepsilon)$ and for all Borel sets $A \subseteq [a, b]$.

Proof. From equation (14) and the definition of ν_S we have

$$\begin{aligned} & \left| \int_S \mu_y^\delta(A) d\sigma(y) - \int_S \mu_y(A) d\sigma(y) \right| \\ &= \left| \frac{1}{\pi} \int_{\mathbb{R}} \theta(t + i\delta, A) d\nu_S(t) - \nu_S(A) \right| \\ &= \left| \int_A \left\{ \frac{1}{\pi} \theta(t + i\delta, A) - 1 \right\} d\nu_S(t) + \frac{1}{\pi} \int_{A^c} \theta(t + i\delta, A) d\nu_S(t) \right|. \end{aligned} \quad (27)$$

In (27) the integral on the left is negative and the integral on the right is positive. Hence, an upper bound for this expression is

$$\sup \left\{ \frac{1}{\pi} \int_A \theta(t + i\delta, A^c) d\nu_S(t), \frac{1}{\pi} \int_{A^c} \theta(t + i\delta, A) d\nu_S(t) \right\}.$$

From (20) we have $\nu_S(B) = (\nu_0 - b_\phi \mu)(B) + \nu_1(B)$ for any Borel set B , and Theorem 1 now follows from lemmas 5 and 6. \square

COROLLARY 1. With the same assumptions as those in lemma 5 we have

$$\lim_{\delta \rightarrow 0^+} \left| \int_S \mu_y^\delta(A) d\sigma(y) - \int_S \mu_y(A) d\sigma(y) \right| = 0, \quad (28)$$

with the convergence being uniform over all Borel sets S , over all Borel sets $A \subseteq [a, b]$, and over all Herglotz functions F such that $F(i)$ belongs to some compact subset of the upper half-plane.

Proof. The function $E_A(\delta)$ defined in (24) is a non-decreasing function of δ ([3]), and $\lim_{\delta \rightarrow 0^+} E_A(\delta) = 0$ by an application of the Lebesgue dominated convergence theorem. (The specific expression for $E_A(\delta)$ in (25) in the case when $A = [a, b]$ exhibits the convergence of $E_A(\delta)$ to zero in the limit $\delta \rightarrow 0^+$). Corollary 1 now follows from Theorem 1, since $\varepsilon > 0$ was arbitrary. The requirement for $F(i)$ to belong to a compact subset of \mathbb{C}_+ emerges from lemma 5, and in particular inequality (23); if this condition is satisfied then $\text{Im}[t - F(i)]^{-1} \leq \text{const} \cdot (1 + t^2)^{-1}$, and we can choose the constant C as stated. \square

REMARK 2. Since $F_y^\delta(z) \rightarrow F_y(z)$ uniformly, as $\delta \rightarrow 0^+$, on compact subsets of the upper half-plane, we have $\mu_y^\delta((a, b]) \rightarrow \mu_y((a, b])$ for finite intervals $(a, b]$ whose endpoints a, b are not discrete points of the measures μ_y^δ, μ_y ([5]).

REFERENCES

- [1] N. I. AKHIEZER AND I. M. GLAZMAN, *Theory of Linear Operators in Hilbert space I*, Pitman, London, 1981.
- [2] S. V. BREIMESSER, J. D. E. GRANT, AND D. B. PEARSON, *Value distribution and spectral theory of Schrödinger operators with L^2 -sparse potentials*, J. Comput. Appl. Math. **148** (2002), 307–322.
- [3] S. V. BREIMESSER AND D. B. PEARSON, *Asymptotic value distribution for solutions of the Schrödinger equation*, Math. Phys. Anal. Geom. **3** (4) (2000), 385–403.
- [4] Y. T. CHRISTODOULIDES, *Asymptotic generalized value distribution of solutions of the Schrödinger equation*, Oper. Matrices **8** (1) (2014), 279–285.
- [5] Y. T. CHRISTODOULIDES AND D. B. PEARSON, *Generalized value distribution for Herglotz functions and spectral theory*, Math. Phys. Anal. Geom. **7** (4) (2004), 309–331.
- [6] Y. T. CHRISTODOULIDES AND D. B. PEARSON, *Spectral theory of Herglotz functions and their compositions*, Math. Phys. Anal. Geom. **7** (4) (2004), 333–345.
- [7] E. A. CODDINGTON AND N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
- [8] R. DEL RIO, S. JITOMIRSKAYA, Y. LAST, AND B. SIMON, *Operators with singular continuous spectrum. IV. Hausdorff dimensions, rank one perturbations, and localization*, J. Anal. Math. **69** (1996), 153–200.
- [9] R. DEL RIO AND O. TCHEBOTAREVA, *Sturm-Liouville operators in the half-axis with local perturbations*, J. Math. Anal. Appl. **329** (2007), 557–566.
- [10] F. GESZTESY AND A. MAKAROV, *$SL(2, \mathbb{C})$, exponential Herglotz representations, and spectral averaging*, S. Petersburg Math. J. **15** (2004), 393–418.
- [11] G. HERGLOTZ, *Über potenzreihen mit positivem, reellem Teil in Einheitskreis*, Sächs. acad. Wiss. Leipzig **63** (1911), 501–511.
- [12] C. A. MARX, *Continuity of spectral averaging*, P. Am. Math. Soc. **139** (1) (2010), 283–291.
- [13] D. B. PEARSON, *Value distribution and spectral theory*, P. Lond. Math. Soc. **68** (3) (1994), 127–144.
- [14] D. B. PEARSON, *Quantum Scattering and Spectral Theory*, Academic Press, London, 1988.

(Received May 10, 2016)

Y. Christodoulides
 General Department
 Frederick University
 18 Mariou Agathagelou Str., Agios Georgios Havouzas
 Limassol 3080, Cyprus
 e-mail: y.christodoulides@frederick.ac.cy