

## SPECTRAL ANALYSIS OF THE DIRAC OPERATOR ON A 3–SPHERE

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*Abstract.* We study the (massless) Dirac operator on a 3-sphere equipped with Riemannian metric. For the standard metric the spectrum is known. In particular, the eigenvalues closest to zero are the two double eigenvalues  $+3/2$  and  $-3/2$ . Our aim is to analyse the behaviour of eigenvalues when the metric is perturbed in an arbitrary smooth fashion from the standard one. We derive explicit perturbation formulae for the two eigenvalues closest to zero, taking account of the second variations. Note that these eigenvalues remain double eigenvalues under perturbations of the metric: they cannot split because of a particular symmetry of the Dirac operator in dimension three (it commutes with the antilinear operator of charge conjugation). Our perturbation formulae show that in the first approximation our two eigenvalues maintain symmetry about zero and are completely determined by the increment of Riemannian volume. Spectral asymmetry is observed only in the second approximation of the perturbation process. As an example we consider a special family of metrics, the so-called generalized Berger spheres, for which the eigenvalues can be evaluated explicitly.

### 1. Introduction

In this paper we study the spectrum of the (massless) Dirac operator on a 3-sphere,  $\mathbb{S}^3$ , equipped with Riemannian metric.

By  $y^\alpha$ ,  $\alpha = 1, 2, 3$ , we denote local coordinates. We specify an orientation, see Appendix A, and use only local coordinates with positive orientation.

We will use the following conventions. Sometimes it will be convenient for us to view the 3-sphere as the hypersurface (A.1) in Euclidean space  $\mathbb{R}^4$ , in which case Cartesian coordinates in  $\mathbb{R}^4$  will be denoted by  $\mathbf{x}^\alpha$ ,  $\alpha = 1, 2, 3, 4$ . Hereinafter we will use bold script for 4-dimensional objects and normal script for 3-dimensional objects. We will use Latin letters for *anholonomic (frame)* indices and Greek letters for *holonomic (tensor)* indices. We will use the convention of summation over repeated indices; this will apply both to Greek and to Latin indices. Also, we will heavily use the analytic concepts of principal and subprincipal symbol of a differential operator; see definitions in [25, subsection 2.1.3] for the case of a scalar operator acting on a single half-density and, more relevantly, [19, Section 1] and [16, Appendix A] for the case of a matrix operator acting on a column of half-densities.

We equip  $\mathbb{S}^3$  with a Riemannian metric tensor  $g_{\alpha\beta}(y)$ ,  $\alpha, \beta = 1, 2, 3$ , and study the corresponding (massless) Dirac operator  $W$ . The Dirac operator on a 3-manifold

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is a particular operator that can be represented as a first order elliptic linear differential operator acting on 2-columns of complex-valued scalar fields (components of a Weyl spinor). It is written down explicitly in Appendix B; note that the definition depends on the choice of orientation, see formula (B.15) or formula (B.7). It is known that the Dirac operator  $W$  is a self-adjoint operator in  $L^2(\mathbb{S}^3; \mathbb{C}^2)$  whose domain is the Sobolev space  $H^1(\mathbb{S}^3; \mathbb{C}^2)$ , and that the spectrum of  $W$  is discrete, accumulating to  $+\infty$  and to  $-\infty$ . Here the inner product in  $L^2(\mathbb{S}^3; \mathbb{C}^2)$  is defined as

$$\langle v, w \rangle := \int_{\mathbb{S}^3} (w^* v \sqrt{\det g_{\alpha\beta}}) dy, \tag{1.1}$$

where the star stands for Hermitian conjugation and  $dy = dy^1 dy^2 dy^3$ . Furthermore, it is known that all eigenvalues have even multiplicity because the linear Dirac operator commutes with the antilinear operator of charge conjugation

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} -\overline{v_2} \\ \overline{v_1} \end{pmatrix} =: C(v),$$

see [12, Appendix A] for details.

A detailed analysis of different definitions of the massless Dirac operator and their equivalence was performed in [17]. Let us emphasise that the underlying reason why we can use three equivalent definitions presented in subsections B.1, B.2 and B.3 is that our manifold is 3-dimensional. One can define the Dirac operator via frames (subsection B.2) or the covariant symbol (subsection B.3) in dimension 3 (Riemannian signature) or in dimension 3 + 1 (Lorentzian signature), see [18] for details, but for higher dimensions these approaches do not seem to work.

The Dirac operator is uniquely defined by the metric modulo the gauge transformation

$$W \mapsto R^* W R, \tag{1.2}$$

where

$$R : \mathbb{S}^3 \rightarrow \text{SU}(2) \tag{1.3}$$

is an arbitrary smooth special unitary matrix-function; see also subsection B.4 for a discussion of spin structure. Obviously, the transformation (1.2), (1.3) does not affect the spectrum.

The Dirac operator  $W$  describes the massless neutrino. We are looking at a single neutrino living in a closed 3-dimensional Riemannian universe. The eigenvalues are the energy levels of the particle. The tradition is to associate positive eigenvalues with the energy levels of the neutrino and negative eigenvalues with the energy levels of the antineutrino. In theoretical physics literature the (massless) Dirac operator is often referred to as the Weyl operator which explains our notation.

In what follows we will mostly conduct a perturbation analysis: starting from the standard metric on the 3-sphere and perturbing this metric in an arbitrary fashion, we will write down explicit asymptotic formulae for the two eigenvalues closest to zero. At a more abstract level the behaviour of eigenvalues of the massless Dirac operator under perturbations of the metric was studied in [11]. Our analysis is very specific in that

- the dimension is three,
- our manifold is a topological sphere,
- the unperturbed metric is the standard metric for the sphere,
- we study only the two eigenvalues closest to zero and
- we calculate the second variations for the perturbed eigenvalues (Theorem 2.2), which is necessary for the observation of spectral asymmetry.

We are motivated by the desire to analyse, in detail and explicitly, the problem in a physically meaningful setting. Namely, we are trying to understand the difference between neutrinos and antineutrinos in curved space. For other related problems in mathematics and physics literature see, e.g., [13, 14].

Finally, in Section 6 and Appendix F we study, non-perturbatively, *all* the eigenvalues for a particular three-parameter family of metrics known as generalized Berger spheres, cf. [20]. This extends previous results of [7, 9].

## 2. Main results

The standard metric  $(g_0)_{\alpha\beta}(y)$  on  $\mathbb{S}^3$  is obtained by restricting the Euclidean metric from  $\mathbb{R}^4$  to  $\mathbb{S}^3$ . For the standard metric the spectrum of the (massless) Dirac operator on  $\mathbb{S}^3$  has been computed by different authors using different methods [26, 27, 8, 10] and reads as follows: the eigenvalues are

$$\pm \left( k + \frac{1}{2} \right), \quad k = 1, 2, \dots,$$

with multiplicity  $k(k+1)$ .

We now perturb the metric, i.e. consider a metric  $g_{\alpha\beta} = g_{\alpha\beta}(y; \varepsilon)$  whose components are smooth functions of local coordinates  $y^\alpha$ ,  $\alpha = 1, 2, 3$ , and small real parameter  $\varepsilon$ , and which turns into the standard metric for  $\varepsilon = 0$ :

$$g_{\alpha\beta}(y; 0) = (g_0)_{\alpha\beta}(y).$$

Let  $\lambda_+(\varepsilon)$  and  $\lambda_-(\varepsilon)$  be the lowest, in terms of absolute value, positive and negative eigenvalues of the Dirac operator  $W(\varepsilon)$ . Our aim is to derive the asymptotic expansions

$$\lambda_\pm(\varepsilon) = \pm \frac{3}{2} + \lambda_\pm^{(1)} \varepsilon + \lambda_\pm^{(2)} \varepsilon^2 + O(\varepsilon^3) \quad \text{as} \quad \varepsilon \rightarrow 0. \quad (2.1)$$

Note that  $\lambda_\pm(\varepsilon)$  are double eigenvalues which cannot split because eigenvalues of the Dirac operator have even multiplicity. Note also that the arguments presented in [15] apply to any double eigenvalue of the Dirac operator on any closed orientable Riemannian 3-manifold, so we know a priori that  $\lambda_\pm(\varepsilon)$  admit the asymptotic expansions (2.1). The issue at hand is the evaluation of the asymptotic coefficients  $\lambda_\pm^{(1)}$  and  $\lambda_\pm^{(2)}$ .

Let

$$V(\varepsilon) := \int_{\mathbb{S}^3} \rho(\varepsilon) dy, \tag{2.2}$$

$dy = dy^1 dy^2 dy^3$ , be the Riemannian volume of our manifold. Here

$$\rho(\varepsilon) = \rho(y; \varepsilon) := \sqrt{\det g_{\mu\nu}(y; \varepsilon)}$$

is the Riemannian density for the perturbed metric.

Then

$$V(\varepsilon) = V^{(0)} + V^{(1)}\varepsilon + O(\varepsilon^2) \quad \text{as} \quad \varepsilon \rightarrow 0, \tag{2.3}$$

where

$$V^{(0)} = \int_{\mathbb{S}^3} \rho_0 dy = 2\pi^2 \tag{2.4}$$

is the volume of the unperturbed 3-sphere,

$$\rho_0 = \rho_0(y) := \sqrt{\det(g_0)_{\mu\nu}(y)}$$

is the standard Riemannian density on the 3-sphere,

$$V^{(1)} = \frac{1}{2} \int_{\mathbb{S}^3} h_{\alpha\beta}(g_0)^{\alpha\beta} \rho_0 dy \tag{2.5}$$

and

$$h_{\alpha\beta} := \left. \frac{\partial g_{\alpha\beta}}{\partial \varepsilon} \right|_{\varepsilon=0}. \tag{2.6}$$

**THEOREM 2.1.** *We have*

$$\lambda_{\pm}^{(1)} = \mp \frac{1}{4\pi^2} V^{(1)}. \tag{2.7}$$

We see that the dependence of the two lowest eigenvalues,  $\lambda_{\pm}(\varepsilon)$ , on the small parameter  $\varepsilon$  is, in the first approximation, very simple: it is determined by the change of volume only. As expected, an increase of the volume of the resonator (volume of our Riemannian manifold) leads to a decrease of the two lowest natural frequencies (absolute values of the two lowest eigenvalues). Furthermore, formulae (2.3), (2.4) and (2.7) imply

$$\frac{\lambda_{\pm}^{(1)}}{\lambda_{\pm}^{(0)}} = -\frac{1}{3} \frac{V^{(1)}}{V^{(0)}}, \tag{2.8}$$

where by  $\lambda_{\pm}^{(0)} = \pm \frac{3}{2}$  we denoted the unperturbed values of the two lowest eigenvalues. Now put  $\ell(\varepsilon) := (V(\varepsilon))^{1/3} = \ell^{(0)} \left( 1 + \frac{1}{3} \frac{V^{(1)}}{V^{(0)}} \varepsilon + O(\varepsilon^2) \right)$ , where  $\ell^{(0)} = \ell(0) = (2\pi^2)^{1/3}$ . The quantity  $\ell(\varepsilon)$  can be interpreted as the characteristic length of our Riemannian manifold. It is easy to see that formula (2.8) is equivalent to the statement

$$\lambda_{\pm}(\varepsilon) = \frac{\lambda_{\pm}^{(0)}}{\ell(\varepsilon)} + O(\varepsilon^2) \quad \text{as} \quad \varepsilon \rightarrow 0,$$

which shows that in the first approximation the two lowest eigenvalues are inversely proportional to the characteristic length of our Riemannian manifold.

An important topic in the spectral theory of first order elliptic systems is the issue of spectral asymmetry [1, 2, 3, 4, 15], i.e. asymmetry of the spectrum about zero. From a physics perspective spectral asymmetry describes the difference between a particle (in our case massless neutrino) and an antiparticle (in our case massless antineutrino). Formulae (2.1) and (2.7) imply

$$\lambda_+(\varepsilon) + \lambda_-(\varepsilon) = (\lambda_+^{(2)} + \lambda_-^{(2)})\varepsilon^2 + O(\varepsilon^3) \quad \text{as} \quad \varepsilon \rightarrow 0,$$

which means that there is no spectral asymmetry in the first approximation in  $\varepsilon$  but there may be spectral asymmetry in terms quadratic in  $\varepsilon$ .

We will now evaluate the asymptotic coefficients  $\lambda_{\pm}^{(2)}$ . We will do this under the simplifying assumption that the Riemannian density does not depend on  $\varepsilon$ :

$$\sqrt{\det g_{\mu\nu}(y; \varepsilon)} = \sqrt{\det(g_0)_{\mu\nu}(y)}, \tag{2.9}$$

so that  $V^{(1)} = 0$ . In mechanics such a deformation is called *shear*. Then Theorem 2.1 implies  $\lambda_{\pm}^{(1)} = 0$ , so formula (2.1) now reads

$$\lambda_{\pm}(\varepsilon) = \pm \frac{3}{2} + \lambda_{\pm}^{(2)}\varepsilon^2 + O(\varepsilon^3) \quad \text{as} \quad \varepsilon \rightarrow 0. \tag{2.10}$$

In order to evaluate the asymptotic coefficients  $\lambda_{\pm}^{(2)}$  we need to introduce triples of special vector fields  $(K_{\pm})_j$ ,  $j = 1, 2, 3$ . For their definitions and properties see Appendix C. Here we mention only that these are triples of orthonormal Killing fields with respect to the standard (unperturbed) metric.

Put

$$(h_{\pm})_{jk} := h_{\alpha\beta} (K_{\pm})_j^{\alpha} (K_{\pm})_k^{\beta}, \tag{2.11}$$

where  $h_{\alpha\beta}$  is the real symmetric tensor from (2.6). Note that the elements of the  $3 \times 3$  real symmetric matrix-function  $(h_{\pm})_{jk}(y)$  are scalars, i.e. they do not change under changes of local coordinates  $y$ . Further on we sometimes raise and lower frame indices (see subsection B.2) and we do this using the Euclidean metric. This means, in particular, that raising a frame index in  $(h_{\pm})_{jk}$  does not change anything.

Put also

$$(L_{\pm})_j := (K_{\pm})_j^{\alpha} \frac{\partial}{\partial y^{\alpha}}, \quad j = 1, 2, 3. \tag{2.12}$$

The operators (2.12) are first order linear differential operators acting on scalar fields over  $\mathbb{S}^3$ . The fact that our  $(K_{\pm})_j$  are Killing vector fields implies that the operators (2.12) are formally anti-self-adjoint with respect to the standard inner product on scalar fields over  $\mathbb{S}^3$ . It is also easy to see that our operators  $(L_{\pm})_j$ ,  $j = 1, 2, 3$ , satisfy the commutator identities

$$[(L_{\pm})_j, (L_{\pm})_k] = \mp 2\varepsilon_{jkl} (L_{\pm})_l, \tag{2.13}$$

where  $\varepsilon_{jkl}$  is the totally antisymmetric quantity,  $\varepsilon_{123} := +1$ .

Let  $\Delta$  be the Laplacian on scalar fields over  $\mathbb{S}^3$  with standard (unperturbed) metric. Our  $\Delta$  is a nonpositive operator, so our definition agrees with the one from basic calculus. By  $(-\Delta)^{-1}$  we shall denote the pseudoinverse of the non-negative differential operator  $-\Delta$ , see Appendix D for explicit definition. Obviously,  $(-\Delta)^{-1}$  is a classical pseudodifferential operator of order minus two.

THEOREM 2.2. *Under the assumption (2.9) we have*

$$\lambda_{\pm}^{(2)} = \frac{1}{2\pi^2} \int_{\mathbb{S}^3} P_{\pm} \rho_0 dy, \tag{2.14}$$

where

$$\begin{aligned} P_{\pm} = & \pm \frac{1}{4} (h_{\pm})_{jk} (h_{\pm})_{jk} \\ & - \frac{1}{16} \varepsilon_{qks} (h_{\pm})_{jq} [(L_{\pm})_s (h_{\pm})_{jk}] \\ & \pm \frac{1}{8} (h_{\pm})_{ks} [(-\Delta)^{-1} (L_{\pm})_s (L_{\pm})_j (h_{\pm})_{jk}] \\ & - \frac{1}{16} \varepsilon_{qks} (h_{\pm})_{rq} [(-\Delta)^{-1} (L_{\pm})_r (L_{\pm})_s (L_{\pm})_j (h_{\pm})_{jk}]. \end{aligned} \tag{2.15}$$

Theorem 2.2 warrants the following remarks.

REMARK 2.3.

- (a) We chose the factor  $\frac{1}{2\pi^2}$  in the RHS of (2.14) based on the observation that the volume of the unperturbed 3-sphere is  $2\pi^2$ , see formula (2.4). This will simplify the comparison with the appropriate formulae previously derived for the 3-torus, see item (f) below, and it will also simplify calculations that will be carried out in subsection 6.3.
- (b) The terms in the RHS of (2.15) are written in such an order that each subsequent term has an extra appearance of a first order differential operator  $L_{\pm}$ .
- (c) The operators  $(-\Delta)^{-1} (L_{\pm})_s (L_{\pm})_j$  and  $(-\Delta)^{-1} (L_{\pm})_r (L_{\pm})_s (L_{\pm})_j$  appearing in the last two terms in the RHS of (2.15) are pseudodifferential operators of order 0 and 1 respectively.
- (d) The fact that the operators  $(L_{\pm})_j, j = 1, 2, 3$ , are formally anti-self-adjoint with respect to the standard inner product on  $\mathbb{S}^3$  implies that for any scalar field  $f : \mathbb{S}^3 \rightarrow \mathbb{C}$  we have

$$\int_{\mathbb{S}^3} [(L_{\pm})_j f] \rho_0 dy = 0. \tag{2.16}$$

Formula (2.16) implies that in the last two terms in the RHS of (2.15) the operator  $(-\Delta)^{-1}$  acts on functions from  $(\text{Ker } \Delta)^{\perp}$ .

- (e) The operators  $(L_{\pm})_j$  commute with the scalar Laplacian, hence, they also commute with  $(-\Delta)^{-1}$ . Therefore, the last two terms in the RHS of (2.15) can be written in a number of equivalent ways.
- (f) The second and fourth terms in the RHS of (2.15) have a structure similar to that of formula (2.5) from [15]. In fact, if one adjusts notation to agree with that of [15], then it turns out that the second and fourth terms in the RHS of (2.15) are, in effect, an equivalent way of writing formula (2.5) from [15]. See Appendix E for more details.
- (g) The first and third terms in the RHS of (2.15) do not have an analogue for the case of the 3-torus [15]. Their appearance is due to the curvature of the 3-sphere.
- (h) The first term in the RHS of (2.15) can be rewritten as

$$\pm \frac{1}{4} h_{\mu\nu} h_{\sigma\tau} (g_0)^{\mu\sigma} (g_0)^{\nu\tau}, \tag{2.17}$$

which means that this term does not feel the Killing vector fields  $(K_{\pm})_j$ ,  $j = 1, 2, 3$ , and, hence, does not contribute to spectral asymmetry. Put

$$\tilde{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{3} \delta_{\mu\nu} h_{\sigma\tau} (g_0)^{\sigma\tau},$$

which is the part of the deformation tensor  $h_{\mu\nu}$  describing shear (deformation preserving Riemannian density). Formula (2.9) implies  $h_{\sigma\tau} (g_0)^{\sigma\tau} = 0$ , so in our case  $\tilde{h}_{\mu\nu} = h_{\mu\nu}$  and the expression (2.17) takes the form

$$\pm \frac{1}{4} \tilde{h}_{\mu\nu} \tilde{h}_{\sigma\tau} (g_0)^{\mu\sigma} (g_0)^{\nu\tau}.$$

Such an expression describes the elastic potential energy generated by shear, see formula (4.3) in [23].

- (i) For a generic perturbation of the metric we expect

$$\lambda_+^{(2)} + \lambda_-^{(2)} \neq 0, \tag{2.18}$$

which means that we expect spectral asymmetry in terms quadratic in  $\varepsilon$ . An example illustrating the inequality (2.18) will be provided in subsection 6.3: see formulae (6.20) and (6.21).

- (j) Let us expand the metric tensor in powers of the small parameter  $\varepsilon$  up to quadratic terms:

$$g_{\alpha\beta}(y; \varepsilon) = (g_0)_{\alpha\beta}(y) + h_{\alpha\beta}(y) \varepsilon + k_{\alpha\beta}(y) \varepsilon^2 + O(\varepsilon^3) \quad \text{as } \varepsilon \rightarrow 0.$$

Here the tensor  $h_{\alpha\beta}$  is defined by (2.6) whereas  $k_{\alpha\beta} := \frac{1}{2} \frac{\partial^2 g_{\alpha\beta}}{\partial \varepsilon^2} \Big|_{\varepsilon=0}$ . One would expect the coefficients  $\lambda_{\pm}^{(2)}$  in the asymptotic expansions (2.10) of the lowest

eigenvalues to depend on the tensor  $k_{\alpha\beta}$ , but Theorem 2.2 tells us that it is not the case. Here a rough explanation is that the only way the tensor  $k_{\alpha\beta}$  can appear in the formulae for  $\lambda_{\pm}^{(2)}$  is linearly, however, condition (2.9) ensures that the linear terms in the map

perturbation of metric  $\rightarrow$  perturbation of lowest eigenvalues

vanish.

### 3. Preparatory material

In this section we present auxiliary results which will be used later in the proofs of Theorems 2.1 and 2.2. Both theorems offer a choice of signs, so for the sake of brevity we present all our preparatory material in a form adapted to the case of upper signs.

#### 3.1. The unperturbed Dirac operator

Suppose that  $\varepsilon = 0$ , i.e. suppose that we are working with the standard (unperturbed) metric. It is convenient to write the (massless) Dirac operator using the triple of vector fields  $(K_+)_j$ ,  $j = 1, 2, 3$ , defined in Appendix C as our frame, see subsection B.2 for the definition of a frame. Straightforward calculations show that in this case the Dirac operator reads

$$W^{(0)} = -is^j(L_+)_j + \frac{3}{2}I, \tag{3.1}$$

where  $s^j$  are the standard Pauli matrices (B.9),  $(L_+)_j$  are the scalar first order linear differential operators (2.12) and  $I$  is the  $2 \times 2$  identity matrix. The superscript in  $W^{(0)}$  indicates that the metric is unperturbed.

Let  $v^{(0)}$  be a normalised eigenfunction corresponding to the eigenvalue  $+\frac{3}{2}$  of the unperturbed Dirac operator (3.1). For example, one can take

$$v^{(0)} = \frac{1}{\sqrt{2}\pi} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.2}$$

Here one can replace  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  by any other constant complex 2-column of unit norm. The freedom in the choice of  $v^{(0)}$  is due to the fact that  $+\frac{3}{2}$  is a double eigenvalue of the unperturbed Dirac operator. The choice of a particular  $v^{(0)}$  does not affect subsequent calculations, what matters is that  $v^{(0)}$  is a constant spinor.

Choosing the optimal frame (gauge) is crucial for our subsequent arguments because we will heavily use the fact that the eigenspinor  $v^{(0)}$  of the unperturbed Dirac operator is a constant spinor. See also Remark C.1(c).

Observe that the triple of vector fields  $(K_+)_j^\alpha$  uniquely defines a triple of covector fields  $(K_+)^j_\alpha$ : the relation between the two is specified by the condition

$$(K_+)_j^\alpha (K_+)^k_\alpha = \delta_j^k, \quad j, k = 1, 2, 3. \tag{3.3}$$

Of course, the covector  $(K_+)^j{}_\alpha$  is obtained by lowering the tensor index in the vector  $(K_+)_j{}^\alpha$  by means of the standard metric on  $\mathbb{S}^3$ . Here the position of the frame index  $j$ , as a subscript or superscript, is irrelevant.

In the next subsection we will make use of both the vector fields  $(K_+)_j{}^\alpha$  and the covector fields  $(K_+)^j{}_\alpha$ .

### 3.2. The perturbed Dirac operator

Let  $e_j{}^\alpha(y; \varepsilon)$  be a frame corresponding to the perturbed metric  $g_{\alpha\beta}(y; \varepsilon)$ . This frame can be written as

$$e_j{}^\alpha(y; \varepsilon) = (c_+)_j{}^k(y; \varepsilon) (K_+)_k{}^\alpha(y), \quad (3.4)$$

where  $(c_+)_j{}^k$  are some real scalar fields. Without loss of generality we choose to work with frames satisfying the symmetry condition

$$(c_+)_j{}^k = (c_+)_k{}^j, \quad (3.5)$$

which can always be achieved by an application of a gauge transformation — multiplication by a  $3 \times 3$  orthogonal matrix-function. Then formulae (2.6), (2.11), (3.4) and (3.5) imply

$$(c_+)_j{}^k(y; \varepsilon) = \delta_j{}^k - \frac{\varepsilon}{2}(h_+)_{jk}(y) + O(\varepsilon^2). \quad (3.6)$$

The frame (3.4) uniquely defines the corresponding coframe  $e^j{}_\alpha$  analogously to (3.3):

$$e^j{}_\alpha e^k{}^\alpha = \delta_j{}^k, \quad j, k = 1, 2, 3. \quad (3.7)$$

Of course, the covector  $e^j{}_\alpha$  is obtained by lowering the tensor index in the vector  $e_j{}^\alpha$  by means of the perturbed metric  $g_{\alpha\beta}(y; \varepsilon)$  on  $\mathbb{S}^3$ . Formulae (3.4), (3.3) and (3.7) imply

$$e^j{}_\alpha(y; \varepsilon) = (d_+)^j{}_k(y; \varepsilon) (K_+)^k{}_\alpha(y), \quad (3.8)$$

where

$$(c_+)^j{}^k (d_+)^l{}_k = \delta_j{}^l, \quad j, l = 1, 2, 3. \quad (3.9)$$

By (3.9) and (3.5), the matrix of scalar coefficients  $(d_+)^j{}_k$  is also symmetric,

$$(d_+)^j{}_k = (d_+)^k{}_j. \quad (3.10)$$

Formulae (3.6), (3.9), (3.5) and (3.10) give

$$(d_+)^j{}_k(y; \varepsilon) = \delta_j{}^k + \frac{\varepsilon}{2}(h_+)_{jk}(y) + O(\varepsilon^2). \quad (3.11)$$

Let  $W(\varepsilon)$  be the perturbed Dirac operator and let  $W_{1/2}(\varepsilon)$  be the corresponding perturbed Dirac operator on half-densities. According to (B.16), the two are related as

$$W(\varepsilon) = (\rho(\varepsilon))^{-1/2} W_{1/2}(\varepsilon) (\rho(\varepsilon))^{1/2}. \quad (3.12)$$

LEMMA 3.1. *The perturbed Dirac operator on half-densities  $W_{1/2}(\varepsilon)$  acts on 2-columns of complex-valued half-densities  $v_{1/2}$  as*

$$v_{1/2} \mapsto -\frac{i}{2} s^j (\rho_0)^{1/2} \left[ (c_+)_j{}^k (L_+)_k + (L_+)_k (c_+)_j{}^k \right] (\rho_0)^{-1/2} v_{1/2} + (W_{1/2}(\varepsilon))_{\text{sub}} v_{1/2}, \tag{3.13}$$

where  $(W_{1/2}(\varepsilon))_{\text{sub}}$  is its subprincipal symbol.

Let us emphasise that the Riemannian density appearing in (3.13) is the unperturbed density  $\rho_0$  and not the perturbed density  $\rho(\varepsilon)$  as in (3.12).

*Proof of Lemma 3.1.* Formulae (B.11) and (3.12) tell us that the principal symbol of the operator  $W_{1/2}(\varepsilon)$  is  $\sigma^\alpha(y; \varepsilon) p_\alpha$ . Using formulae (B.8), (3.4) and (2.12) we can rewrite this principal symbol as

$$-i s^j (c_+)_j{}^k [(L_+)_k]_{\text{prin}}. \tag{3.14}$$

But (3.14) is also the principal symbol of the operator (3.13), so the proof reduces to proving that the operator

$$v_{1/2} \mapsto -\frac{i}{2} s^j (\rho_0)^{1/2} \left[ (c_+)_j{}^k (L_+)_k + (L_+)_k (c_+)_j{}^k \right] (\rho_0)^{-1/2} v_{1/2}$$

has zero subprincipal symbol. By [25, Proposition 2.1.13] it is sufficient to prove that the operators  $(\rho_0)^{1/2} (L_+)_k (\rho_0)^{-1/2}$ ,  $k = 1, 2, 3$ , have zero subprincipal symbols. But the latter is a consequence of (2.12) and the fact that our  $(K_+)_k^\alpha$ , being Killing vector fields with respect to the unperturbed metric, are divergence-free.  $\square$

According to [12, formulae (6.1) and (8.1)] the explicit formula for the subprincipal symbol of the Dirac operator on half-densities reads

$$(W_{1/2}(\varepsilon))_{\text{sub}} = I f(\varepsilon), \tag{3.15}$$

where  $I$  is the  $2 \times 2$  identity matrix and  $f(\varepsilon) = f(y; \varepsilon)$  is the scalar function

$$f(\varepsilon) := \frac{\delta_{kl}}{4\rho(\varepsilon)} \left[ e^k{}_1 \frac{\partial e^l{}_3}{\partial y^2} + e^k{}_2 \frac{\partial e^l{}_1}{\partial y^3} + e^k{}_3 \frac{\partial e^l{}_2}{\partial y^1} - e^k{}_1 \frac{\partial e^l{}_2}{\partial y^3} - e^k{}_2 \frac{\partial e^l{}_3}{\partial y^1} - e^k{}_3 \frac{\partial e^l{}_1}{\partial y^2} \right], \tag{3.16}$$

with  $e^k{}_j = e^k_j(y; \varepsilon)$ .

Combining formulae (3.12), (3.13), (3.15) and (3.16) we conclude that the perturbed Dirac operator  $W(\varepsilon)$  acts on 2-columns of complex-valued scalar fields  $v$  as

$$v \mapsto -\frac{i}{2} s^j \sqrt{\frac{\rho_0}{\rho(\varepsilon)}} \left[ (c_+)_j{}^k (L_+)_k + (L_+)_k (c_+)_j{}^k \right] \sqrt{\frac{\rho(\varepsilon)}{\rho_0}} v + f(\varepsilon) v. \tag{3.17}$$

Of course, when  $\varepsilon = 0$  formulae (3.17) and (3.16) turn into formula (3.1) with  $W^{(0)} = W(0)$ .

### 3.3. Half-densities versus scalar fields

Given a pair of 2-columns of complex-valued half-densities,  $v_{1/2}$  and  $w_{1/2}$ , we define their inner product as

$$\langle v_{1/2}, w_{1/2} \rangle := \int_{\mathbb{S}^3} (w_{1/2})^* v_{1/2} dy. \tag{3.18}$$

The advantage of (3.18) over (1.1) is that the inner product (3.18) does not depend on the metric. Consequently, if we work with half-densities, perturbations of the metric will not change our Hilbert space. And, unsurprisingly, the perturbation process described in [15, Section 4] was written in terms of half-densities.

The explicit formula for the action of the operator  $W_{1/2}(\varepsilon)$  reads

$$v_{1/2} \mapsto -\frac{i}{2} s^j (\rho_0)^{1/2} \left[ (c_+)_j{}^k (L_+)_k + (L_+)_k (c_+)_j{}^k \right] (\rho_0)^{-1/2} v_{1/2} + f(\varepsilon) v_{1/2}, \tag{3.19}$$

where  $f(\varepsilon)$  is the scalar function (3.16).

Formulae (3.19) and (3.16) give us a convenient explicit representation of the perturbed Dirac operator on half-densities  $W_{1/2}(\varepsilon)$ . We will use this representation in the next two sections when proving Theorems 2.1 and 2.2.

When  $\varepsilon = 0$  formulae (3.19) and (3.16) turn into

$$v_{1/2} \mapsto -i s^j (\rho_0)^{1/2} (L_+)_k (\rho_0)^{-1/2} v_{1/2} + \frac{3}{2} v_{1/2},$$

which is the action of the unperturbed Dirac operator on half-densities  $W_{1/2}^{(0)} = W_{1/2}(0)$ .

The normalised eigenfunction of the operator  $W_{1/2}^{(0)}$  corresponding to the eigenvalue  $+\frac{3}{2}$  reads

$$v_{1/2}^{(0)} = \rho_0^{1/2} v^{(0)}, \tag{3.20}$$

where  $v^{(0)}$  is given by formula (3.2).

### 3.4. Asymptotic process

Let us expand our Dirac operator on half-densities in powers of  $\varepsilon$ ,

$$W_{1/2}(\varepsilon) = W_{1/2}^{(0)} + \varepsilon W_{1/2}^{(1)} + \varepsilon^2 W_{1/2}^{(2)} + \dots \tag{3.21}$$

Then, according to [15, formulae (4.12) and (4.13)], formula (2.1) holds with

$$\lambda_+^{(1)} = \langle W_{1/2}^{(1)} v_{1/2}^{(0)}, v_{1/2}^{(0)} \rangle, \tag{3.22}$$

$$\lambda_+^{(2)} = \langle W_{1/2}^{(2)} v_{1/2}^{(0)}, v_{1/2}^{(0)} \rangle - \langle (W_{1/2}^{(1)} - \lambda_+^{(1)} I) Q_{1/2} (W_{1/2}^{(1)} - \lambda_+^{(1)} I) v_{1/2}^{(0)}, v_{1/2}^{(0)} \rangle, \tag{3.23}$$

where  $Q_{1/2}$  is the pseudoinverse of the operator  $W_{1/2}^{(0)} - \frac{3}{2} I$ . See [15, Section 3] for definition of pseudoinverse.

LEMMA 3.2. *We have*

$$\langle W_{1/2}(\varepsilon)v_{1/2}^{(0)}, v_{1/2}^{(0)} \rangle = \langle f(\varepsilon)v_{1/2}^{(0)}, v_{1/2}^{(0)} \rangle = \frac{1}{2\pi^2} \int_{\mathbb{S}^3} f(\varepsilon)\rho_0 dy. \tag{3.24}$$

*Proof.* Substituting (3.19), (3.20) and (3.2) into the LHS of (3.24) and using Remark 2.3(d), we see that the terms with  $(L_+)_k$  integrate to zero, which leaves us with the RHS of (3.24).  $\square$

Let us now expand the scalar function  $f(\varepsilon)$  in powers of our  $\varepsilon$ ,

$$f(\varepsilon) = f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \dots \tag{3.25}$$

Here, of course,  $f^{(0)} = f(0) = \frac{3}{2}$ .

Formulae (3.24), (3.21) and (3.25) imply

$$\langle W_{1/2}^{(n)}v_{1/2}^{(0)}, v_{1/2}^{(0)} \rangle = \langle f^{(n)}v_{1/2}^{(0)}, v_{1/2}^{(0)} \rangle = \frac{1}{2\pi^2} \int_{\mathbb{S}^3} f^{(n)}\rho_0 dy, \quad n = 0, 1, \dots$$

Then formulae (3.22) and (3.23) become

$$\lambda_+^{(1)} = \frac{1}{2\pi^2} \int_{\mathbb{S}^3} f^{(1)}\rho_0 dy, \tag{3.26}$$

$$\begin{aligned} \lambda_+^{(2)} &= \frac{1}{2\pi^2} \int_{\mathbb{S}^3} f^{(2)}\rho_0 dy - \langle (W_{1/2}^{(1)} - \lambda_+^{(1)}I)Q_{1/2}(W_{1/2}^{(1)} - \lambda_+^{(1)}I)v_{1/2}^{(0)}, v_{1/2}^{(0)} \rangle \\ &= \frac{1}{2\pi^2} \int_{\mathbb{S}^3} f^{(2)}\rho_0 dy - \langle Q_{1/2}(W_{1/2}^{(1)} - \lambda_+^{(1)}I)v_{1/2}^{(0)}, (W_{1/2}^{(1)} - \lambda_+^{(1)}I)v_{1/2}^{(0)} \rangle. \end{aligned} \tag{3.27}$$

#### 4. Proof of Theorem 2.1

We prove Theorem 2.1 for the case of upper signs.

We have

$$\rho(\varepsilon) = \rho_0 \left( 1 + \frac{\varepsilon}{2} h_{\alpha\beta}(g_0)^{\alpha\beta} + O(\varepsilon^2) \right). \tag{4.1}$$

Using formulae (3.8), (3.10) and (3.11), we get

$$\begin{aligned} \delta_{kl} \left[ e^k_1 \frac{\partial e^l_3}{\partial y^2} + e^k_2 \frac{\partial e^l_1}{\partial y^3} + e^k_3 \frac{\partial e^l_2}{\partial y^1} - e^k_1 \frac{\partial e^l_2}{\partial y^3} - e^k_2 \frac{\partial e^l_3}{\partial y^1} - e^k_3 \frac{\partial e^l_1}{\partial y^2} \right] \\ = 6\rho_0 \left( 1 + \frac{\varepsilon}{3} (h_+)_{jj} + O(\varepsilon^2) \right) = 6\rho_0 \left( 1 + \frac{\varepsilon}{3} h_{\alpha\beta}(g_0)^{\alpha\beta} + O(\varepsilon^2) \right). \end{aligned} \tag{4.2}$$

Substitution of (4.1) and (4.2) into (3.16) gives us

$$f^{(1)} = -\frac{1}{4} h_{\alpha\beta}(g_0)^{\alpha\beta}. \tag{4.3}$$

Finally, substituting (4.3) into (3.26) and using (2.5), we arrive at (2.7).  $\square$

### 5. Proof of Theorem 2.2

We prove Theorem 2.2 for the case of upper signs.

Recall also that we are proving this theorem under the assumption (2.9). This implies, in particular, that

$$\lambda_+^{(1)} = 0. \quad (5.1)$$

With account of (5.1), in order to use formula (3.27) we require the expressions for the scalar function  $f^{(2)}$  and for  $W_{1/2}^{(1)} v_{1/2}^{(0)}$ .

Substituting (3.8) and (3.11) into (3.16) and using (2.9) and (3.10), we get

$$f^{(1)} = 0, \quad (5.2)$$

$$f^{(2)} = \frac{1}{4}(h_+)_{jk}(h_+)_{jk} - \frac{1}{16}\varepsilon_{qks}(h_+)_{jq}[(L_+)_{s}(h_+)_{jk}]. \quad (5.3)$$

Examination of formulae (3.19), (3.6) and (5.2) gives us the explicit formula for the action of the operator  $W_{1/2}^{(1)}$ :

$$v_{1/2} \mapsto \frac{i}{4}s^j(\rho_0)^{1/2}[(h_+)_{jk}(L_+)_{k} + (L_+)_{k}(h_+)_{jk}](\rho_0)^{-1/2}v_{1/2}. \quad (5.4)$$

Acting with the operator (5.4) on the eigenfunction (3.20) of the unperturbed massless Dirac operator on half-densities, we obtain

$$W_{1/2}^{(1)}v_{1/2}^{(0)} = \frac{i}{4}(\rho_0)^{1/2}s^jv^{(0)}[(L_+)_{k}(h_+)_{jk}]. \quad (5.5)$$

Using formula (5.5), we get

$$\begin{aligned} & -\langle Q_{1/2}W_{1/2}^{(1)}v_{1/2}^{(0)}, W_{1/2}^{(1)}v_{1/2}^{(0)} \rangle \\ &= -\frac{1}{16}\int_{\mathbb{S}^3}[(L_+)_{r}(h_+)_{qr}]\left(\left[[v^{(0)}]^*s^q(\rho_0)^{-1/2}Q_{1/2}(\rho_0)^{1/2}s^jv^{(0)}\right][(L_+)_{k}(h_+)_{jk}]\right)\rho_0 dy. \end{aligned} \quad (5.6)$$

But  $(\rho_0)^{-1/2}Q_{1/2}(\rho_0)^{1/2} = Q$ , the pseudoinverse of the operator  $W^{(0)} - \frac{3}{2}I$ . Hence, formula (5.6) simplifies and reads now

$$\begin{aligned} & -\langle Q_{1/2}W_{1/2}^{(1)}v_{1/2}^{(0)}, W_{1/2}^{(1)}v_{1/2}^{(0)} \rangle \\ &= -\frac{1}{16}\int_{\mathbb{S}^3}[(L_+)_{r}(h_+)_{qr}]\left(\left[[v^{(0)}]^*s^qQs^jv^{(0)}\right][(L_+)_{k}(h_+)_{jk}]\right)\rho_0 dy. \end{aligned} \quad (5.7)$$

Observe now that we have the identity

$$\left(W^{(0)} - \frac{1}{2}I\right)^2 = (-\Delta + 1)I, \quad (5.8)$$

where  $I$  is the  $2 \times 2$  identity matrix and  $\Delta$  is the Laplacian on scalar fields over  $\mathbb{S}^3$  with standard metric. Formula (5.8) can be established by direct substitution of (3.1)

and the use of the commutator formula (2.13). Formula (5.8) appears also as Lemma 2 in [8].

Formula (5.8) implies

$$Q = (-\Delta)^{-1} \left( W^{(0)} + \frac{1}{2}I \right) = (-\Delta)^{-1} \left( -is^l(L_+)l + 2I \right). \tag{5.9}$$

Formula (5.9), in turn, gives us the following representation for the scalar pseudodifferential operator  $[v^{(0)}]^* s^q Q s^j v^{(0)}$ :

$$\begin{aligned} & [v^{(0)}]^* s^q Q s^j v^{(0)} \\ &= 2 \left( [v^{(0)}]^* s^q s^j v^{(0)} \right) (-\Delta)^{-1} - i \left( [v^{(0)}]^* s^q s^l s^j v^{(0)} \right) (-\Delta)^{-1} (L_+)l. \end{aligned} \tag{5.10}$$

Substituting (5.10) into (5.7), we get

$$\begin{aligned} & -\langle Q_{1/2} W_{1/2}^{(1)} v_{1/2}^{(0)}, W_{1/2}^{(1)} v_{1/2}^{(0)} \rangle \\ &= -\frac{1}{8} \left( [v^{(0)}]^* s^q s^j v^{(0)} \right) \int_{\mathbb{S}^3} [(L_+)r(h_+)_{qr}] [(-\Delta)^{-1}(L_+)k(h_+)_{jk}] \rho_0 dy \\ &\quad + \frac{1}{16} \left( i [v^{(0)}]^* s^q s^l s^j v^{(0)} \right) \int_{\mathbb{S}^3} [(L_+)r(h_+)_{qr}] [(-\Delta)^{-1}(L_+)l(L_+)k(h_+)_{jk}] \rho_0 dy \\ &= \frac{1}{8} \operatorname{Re} \left( [v^{(0)}]^* s^q s^j v^{(0)} \right) \int_{\mathbb{S}^3} (h_+)_{qr} [(-\Delta)^{-1}(L_+)r(L_+)k(h_+)_{jk}] \rho_0 dy \\ &\quad - \frac{1}{16} \operatorname{Re} \left( i [v^{(0)}]^* s^q s^l s^j v^{(0)} \right) \int_{\mathbb{S}^3} (h_+)_{qr} [(-\Delta)^{-1}(L_+)r(L_+)l(L_+)k(h_+)_{jk}] \rho_0 dy. \end{aligned} \tag{5.11}$$

But

$$\begin{aligned} \operatorname{Re} \left( [v^{(0)}]^* s^q s^j v^{(0)} \right) &= \frac{1}{2} \left( [v^{(0)}]^* (s^q s^j + s^j s^q) v^{(0)} \right) \\ &= \delta^{qj} \left( [v^{(0)}]^* I v^{(0)} \right) = \frac{1}{2\pi^2} \delta^{qj}, \\ \operatorname{Re} \left( i [v^{(0)}]^* s^q s^l s^j v^{(0)} \right) &= \frac{i}{2} \left( [v^{(0)}]^* (s^q s^l s^j - s^j s^l s^q) v^{(0)} \right) \\ &= -\varepsilon^{qlj} \left( [v^{(0)}]^* I v^{(0)} \right) = -\frac{1}{2\pi^2} \varepsilon^{qlj}, \end{aligned}$$

where we made use of (3.2). Hence, formula (5.11) simplifies and reads now

$$\begin{aligned} & -\langle Q_{1/2} W_{1/2}^{(1)} v_{1/2}^{(0)}, W_{1/2}^{(1)} v_{1/2}^{(0)} \rangle \\ &= \frac{1}{16\pi^2} \int_{\mathbb{S}^3} (h_+)_{jr} [(-\Delta)^{-1}(L_+)r(L_+)k(h_+)_{jk}] \rho_0 dy \\ &\quad + \frac{1}{32\pi^2} \varepsilon_{qlj} \int_{\mathbb{S}^3} (h_+)_{qr} [(-\Delta)^{-1}(L_+)r(L_+)l(L_+)k(h_+)_{jk}] \rho_0 dy. \end{aligned} \tag{5.12}$$

Substituting (5.3) and (5.12) into (3.27) we arrive at (2.14), (2.15) with upper signs.  $\square$

## 6. Generalized Berger spheres

A left-handed generalized Berger sphere is a 3-sphere equipped with metric

$$g_{\alpha\beta} = C_{jk}(K_+)^j{}_{\alpha}(K_+)^k{}_{\beta}, \quad (6.1)$$

where  $C = (C_{jk})_{j,k=1}^3$  is a **constant**  $3 \times 3$  positive real symmetric matrix and  $(K_+)^j{}_{\alpha}$ ,  $j = 1, 2, 3$ , are our special covector fields defined in accordance with Section 2 and formula (3.3). One can, of course, define in a similar fashion right-handed generalized Berger spheres: these involve the covector fields  $(K_-)^j{}_{\alpha}$ ,  $j = 1, 2, 3$ . However, in this paper, as in [21], we restrict our analysis to left-handed ones.

One can always perform a rotation in  $\mathbb{R}^4$  so that (6.1) turns to

$$g_{\alpha\beta} = \sum_{j=1}^3 a_j^2 (K_+)^j{}_{\alpha} (K_+)^j{}_{\beta}, \quad (6.2)$$

where  $a_j$ ,  $j = 1, 2, 3$ , are some positive constants. In formula (6.2) the  $(K_+)^j{}_{\alpha}$ ,  $j = 1, 2, 3$ , are new covector fields defined in the new Cartesian coordinate system in accordance with formulae (C.1) and (3.3). Of course,  $a_j^2$  are the eigenvalues of the matrix  $C$ . Further on we assume that our generalized Berger metric has the form (6.2).

To the authors' knowledge, metrics of the type (6.2) were first considered in Section 3 of [21]. The expression "generalized Berger sphere" first appears in [20]. The standard (as opposed to the generalized) Berger sphere corresponds to the case  $a_2 = a_3 = 1$ , and the standard sphere corresponds to the case  $a_1 = a_2 = a_3 = 1$ .

For future reference, let us give the formula for the Riemannian volume (2.2) of the generalized Berger sphere:

$$V = 2\pi^2 a_1 a_2 a_3. \quad (6.3)$$

### 6.1. Dirac operator on generalized Berger spheres

The remarkable feature of generalized Berger spheres is that for these metrics the calculation of eigenvalues of the (massless) Dirac operator reduces to finding roots of polynomials.

The Dirac operator (3.17) corresponding to the generalized Berger metric reads

$$W = -i \sum_{j=1}^3 \frac{1}{a_j} s^j (L_+)_j + \nu I, \quad (6.4)$$

where

$$\nu = \frac{a_1^2 + a_2^2 + a_3^2}{2a_1 a_2 a_3}. \quad (6.5)$$

In writing (6.4) we followed the convention of choosing the symmetric gauge, see formulae (3.5) and (3.10). The constant (6.5) was written down by means of a careful application of formula (3.16).

Note that formula (6.4) appears also in Proposition 3.1 of [21].

Examination of formula (6.4) shows that  $\lambda = \nu$  is an eigenvalue of the Dirac operator, with the corresponding eigenspinors being constant spinors.

In order to calculate other eigenvalues of the Dirac operator it is convenient to extend our spinor field from  $\mathbb{S}^3$  to a neighbourhood of  $\mathbb{S}^3$  in  $\mathbb{R}^4$  and rewrite the operator in Cartesian coordinates. Substituting (C.1) into (6.4), we get

$$\mathbf{W} = -i \sum_{j=1}^3 \frac{1}{a_j} s^j (\mathbf{L}_+)_j + \nu I, \quad (6.6)$$

where

$$\begin{aligned} (\mathbf{L}_+)_1 &= -\mathbf{x}^4 \partial_1 - \mathbf{x}^3 \partial_2 + \mathbf{x}^2 \partial_3 + \mathbf{x}^1 \partial_4, \\ (\mathbf{L}_+)_2 &= \mathbf{x}^3 \partial_1 - \mathbf{x}^4 \partial_2 - \mathbf{x}^1 \partial_3 + \mathbf{x}^2 \partial_4, \\ (\mathbf{L}_+)_3 &= -\mathbf{x}^2 \partial_1 + \mathbf{x}^1 \partial_2 - \mathbf{x}^4 \partial_3 + \mathbf{x}^3 \partial_4. \end{aligned} \quad (6.7)$$

Here the way to work with the Cartesian representation of the Dirac operator is to act with (6.6), (6.7) on a spinor field defined in a neighbourhood of  $\mathbb{S}^3$  and then restrict the result to (A.1). It is easy to see that under this procedure the resulting spinor field on  $\mathbb{S}^3$  does not depend on the way we extended our original spinor field from  $\mathbb{S}^3$  to a neighbourhood of  $\mathbb{S}^3$  in  $\mathbb{R}^4$ .

The operators (6.7) commute with the scalar Laplacian in  $\mathbb{R}^4$ . This implies that these operators map homogeneous harmonic polynomials of degree  $k$  to homogeneous harmonic polynomials of degree less than or equal to  $k$ . Hence, the eigenspinors of the Dirac operator can be written in terms of homogeneous harmonic polynomials. Of course, the restriction of homogeneous harmonic polynomials to the 3-sphere (A.1) gives spherical functions, but we find working with polynomials in Cartesian coordinates more convenient than working with spherical functions in spherical coordinates (A.2).

Let us seek an eigenspinor which is linear in Cartesian coordinates  $\mathbf{x}^\alpha$ ,  $\alpha = 1, 2, 3, 4$ . Such an eigenspinor is determined by eight complex constants and finding the corresponding eigenvalues reduces to finding the eigenvalues of a particular  $8 \times 8$  Hermitian matrix. Explicit calculations (which we omit for the sake of brevity) show that the characteristic polynomial of this  $8 \times 8$  Hermitian matrix is the square of a polynomial of degree four whose roots are

$$\nu - \frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3}, \quad (6.8)$$

$$\nu - \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}, \quad (6.9)$$

$$\nu + \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3}, \quad (6.10)$$

$$\nu + \frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{a_3}. \quad (6.11)$$

One can repeat the above procedure for homogeneous harmonic polynomials of degree  $n = 2, 3, \dots$ , thus reducing the problem of finding eigenvalues of the Dirac operator on a generalized Berger sphere to finding roots of polynomials. See Appendix F for further details.

Note that for the standard Berger sphere ( $a_2 = a_3 = 1$ ) the spectrum of the Dirac operator was previously calculated in [7, 9].

## 6.2. Testing Theorem 2.1 on generalized Berger spheres

From now on we will assume that the positive constants  $a_j$  are close to 1. This assumption will allow us to identify the lowest, in terms of modulus, positive and negative eigenvalues of the Dirac operator.

When  $a_1 = a_2 = a_3 = 1$  the expression (6.5) takes the value  $+\frac{3}{2}$  and the expression (6.8) takes the value  $-\frac{3}{2}$ . Hence,

$$\lambda_+ = \nu \tag{6.12}$$

is the lowest positive eigenvalue of the Dirac operator and

$$\lambda_- = \nu - \frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} \tag{6.13}$$

is the lowest, in terms of modulus, negative eigenvalue of the Dirac operator. Recall that  $\nu$  is given by formula (6.5). As for the expressions (6.9)–(6.11), their values are close to  $+\frac{5}{2}$ .

In this subsection and the next one we assume that the constants  $a_j$  appearing in formula (6.2) are smooth functions of the small parameter  $\varepsilon$  and that  $a_j(0) = 1$ .

Expanding (6.3), (6.12) and (6.13) in powers of  $\varepsilon$ , we get

$$V(\varepsilon) = 2\pi^2 (1 + (a'_1 + a'_2 + a'_3)\varepsilon + O(\varepsilon^2)), \tag{6.14}$$

$$\lambda_{\pm}(\varepsilon) = \pm \frac{3}{2} \mp \frac{1}{2}(a'_1 + a'_2 + a'_3)\varepsilon + O(\varepsilon^2), \tag{6.15}$$

where

$$a'_j := \left. \frac{da_j}{d\varepsilon} \right|_{\varepsilon=0}.$$

Formulae (2.1), (2.3), (2.4), (6.14) and (6.15) imply (2.7). Thus, we are in agreement with Theorem 2.1.

## 6.3. Testing Theorem 2.2 on generalized Berger spheres

In this subsection we make the additional assumption

$$a_1(\varepsilon)a_2(\varepsilon)a_3(\varepsilon) = 1, \tag{6.16}$$

which ensures the preservation of Riemannian volume (6.3) under perturbations. But generalized Berger spheres are homogeneous Riemannian spaces, so preservation of

Riemannian volume is equivalent to preservation of Riemannian density. Hence, (6.16) implies (2.9), which is required for testing Theorem 2.2.

For future reference note that formula (6.16) implies

$$a'_1 + a'_2 + a'_3 = 0, \tag{6.17}$$

$$a''_1 + a''_2 + a''_3 + 2(a'_1 a'_2 + a'_2 a'_3 + a'_3 a'_1) = 0, \tag{6.18}$$

where

$$a''_j := \left. \frac{d^2 a_j}{d\varepsilon^2} \right|_{\varepsilon=0}. \tag{6.19}$$

Expanding (6.12) and (6.13) in powers of  $\varepsilon$  and using formulae (6.17) and (6.18), we get

$$\lambda_+(\varepsilon) = \frac{3}{2} + ((a'_1)^2 + (a'_2)^2 + (a'_3)^2) \varepsilon^2 + O(\varepsilon^3), \tag{6.20}$$

$$\lambda_-(\varepsilon) = -\frac{3}{2} + \frac{1}{2} ((a'_1)^2 + (a'_2)^2 + (a'_3)^2) \varepsilon^2 + O(\varepsilon^3). \tag{6.21}$$

Note that the second derivatives (6.19) do not appear in formulae (6.20) and (6.21), which is in agreement with Remark 2.3(j).

We first test whether formula (6.20) agrees with Theorem 2.2. Calculating the scalars (2.11) with upper sign, we get

$$(h_+)_{jk} = 2 \sum_{l=1}^3 a'_l \delta_{lj} \delta_{lk}. \tag{6.22}$$

Formulae (2.15) and (6.22) imply

$$P_+ = (a'_1)^2 + (a'_2)^2 + (a'_3)^2. \tag{6.23}$$

Substituting (6.23) into (2.14) and using (2.4), we get  $\lambda_+^{(2)} = (a'_1)^2 + (a'_2)^2 + (a'_3)^2$ , which is in agreement with (6.20).

In the remainder of this subsection we test whether formula (6.21) agrees with Theorem 2.2. This is trickier because the scalar fields  $(h_-)_{jk}$  are not constant.

Consider the matrix-function

$$\begin{pmatrix} (\mathbf{x}^1)^2 - (\mathbf{x}^2)^2 - (\mathbf{x}^3)^2 + (\mathbf{x}^4)^2 & 2(\mathbf{x}^1 \mathbf{x}^2 - \mathbf{x}^3 \mathbf{x}^4) & 2(\mathbf{x}^1 \mathbf{x}^3 + \mathbf{x}^2 \mathbf{x}^4) \\ 2(\mathbf{x}^1 \mathbf{x}^2 + \mathbf{x}^3 \mathbf{x}^4) & -(\mathbf{x}^1)^2 + (\mathbf{x}^2)^2 - (\mathbf{x}^3)^2 + (\mathbf{x}^4)^2 & 2(\mathbf{x}^2 \mathbf{x}^3 - \mathbf{x}^1 \mathbf{x}^4) \\ 2(\mathbf{x}^1 \mathbf{x}^3 - \mathbf{x}^2 \mathbf{x}^4) & 2(\mathbf{x}^1 \mathbf{x}^4 + \mathbf{x}^2 \mathbf{x}^3) & -(\mathbf{x}^1)^2 - (\mathbf{x}^2)^2 + (\mathbf{x}^3)^2 + (\mathbf{x}^4)^2 \end{pmatrix}$$

whose elements are homogeneous harmonic quadratic polynomials. Let  $O$  be the restriction of the above matrix-function to the 3-sphere (A.1). Note that the matrix-function  $O$  is orthogonal. Let us denote the elements the matrix-function  $O$  by  $O_{jk}$ , with the first subscript enumerating rows and the second enumerating columns. The two sets of scalar fields,  $(h_+)_{jk}$  and  $(h_-)_{jk}$ , are related as

$$(h_-)_{il} = O_{ij}(h_+)_{jk} O_{lk}. \tag{6.24}$$

Substitution of (6.22) into (6.24) gives us explicit formulae for the scalar fields  $(h_-)_{jk}$ .

We now need to substitute (6.24) into the formula for  $P_-$ , see (2.15).

Observe that the (spherical) functions  $O_{jk}$  satisfy the identity

$$(L_-)_i O_{jk} = 2\epsilon_{ijl} O_{lk}. \tag{6.25}$$

Formulae (6.24) and (6.25) and the fact that the matrix of constants  $(h_+)_{jk}$  is symmetric imply  $(L_-)_s(L_-)_j(h_-)_{jk} = 0$ , so the last two terms in the RHS of (2.15) vanish, giving us

$$P_- = -\frac{1}{4}(h_-)_{jk}(h_-)_{jk} - \frac{1}{16}\epsilon_{qks}(h_-)_{jq} [(L_-)_s(h_-)_{jk}]. \tag{6.26}$$

We examine the two terms in the RHS of (6.26) separately. As the matrix  $O$  is orthogonal, we have, with account of (6.22),

$$-\frac{1}{4}(h_-)_{jk}(h_-)_{jk} = -\frac{1}{4}(h_+)_{jk}(h_+)_{jk} = -[(a'_1)^2 + (a'_2)^2 + (a'_3)^2]. \tag{6.27}$$

The other term is evaluated by substituting (6.24), using the identity (6.25) and the fact that our perturbation of the metric is pointwise trace-free  $(h_{\pm})_{jj} = 0$ , which gives us

$$-\frac{1}{16}\epsilon_{qks}(h_-)_{jq} [(L_-)_s(h_-)_{jk}] = \frac{3}{8}(h_+)_{jk}(h_+)_{jk} = \frac{3}{2}[(a'_1)^2 + (a'_2)^2 + (a'_3)^2]. \tag{6.28}$$

Substituting (6.27) and (6.28) into the RHS of (6.26), we arrive at

$$P_- = \frac{1}{2}[(a'_1)^2 + (a'_2)^2 + (a'_3)^2]. \tag{6.29}$$

Substituting (6.29) into (2.14) and using (2.4), we get  $\lambda_-^{(2)} = \frac{1}{2}[(a'_1)^2 + (a'_2)^2 + (a'_3)^2]$ , which is in agreement with (6.21).

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### Appendix A Orientation

The unit 3-sphere,  $\mathbb{S}^3$ , is the hypersurface in  $\mathbb{R}^4$  defined by the equation

$$\|\mathbf{x}\| = 1, \tag{A.1}$$

where  $\|\cdot\|$  is the standard Euclidean norm. Spherical coordinates

$$\begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \\ \mathbf{x}^4 \end{pmatrix} = \begin{pmatrix} \cos y^1 \\ \sin y^1 \cos y^2 \\ \sin y^1 \sin y^2 \cos y^3 \\ \sin y^1 \sin y^2 \sin y^3 \end{pmatrix}, \quad y^1, y^2 \in (0, \pi), \quad y^3 \in [0, 2\pi), \tag{A.2}$$

are an example of local coordinates on  $\mathbb{S}^3$ . We define the orientation of spherical coordinates (A.2) to be positive.

## Appendix B The Dirac operator

### B.1 Classical geometric definition

Unlike the rest of the paper, in this subsection we work in a more general setting. Namely, we do not assume our base manifold to be 3-dimensional.

The material presented in this subsection can be found in many classical books on spin geometry. We follow the notation from [22]. Let  $X$  be an  $m$ -dimensional connected manifold and  $E$  be an  $n$ -dimensional oriented Riemannian vector bundle with a spin structure. Recall that a complex spin bundle of  $E$  is given by  $S_{\mathbb{C}}(E) = P_{\text{Spin}}(E) \times_{\mu} M_{\mathbb{C}}$ , where  $P_{\text{Spin}}(E)$  is the principal  $\text{Spin}_n$  bundle associated with  $E$ ,  $M_{\mathbb{C}}$  is an  $N$ -dimensional left complex module for  $Cl(\mathbb{R}^n) = Cl(\mathbb{R}^n) \otimes \mathbb{C}$  and  $\mu : \text{Spin}_n \mapsto \text{SO}(M_{\mathbb{C}})$  is the representation induced by left multiplication by elements of  $\text{Spin}_n \subset Cl^0(\mathbb{R}^n) \subset Cl(\mathbb{R}^n)$ .

**THEOREM B.1.** ([22, Chapter II, Section 4]) *Let  $\omega$  be a connection 1-form on the bundle of  $P_{\text{SO}}(E)$  oriented orthonormal bases of  $E$ , which can be expressed as  $\omega = \sum_{i < j} \omega_{ij} e_i \wedge e_j$ . Here  $e_i \wedge e_j$  is the elementary skew-symmetric  $(i, j)$  matrix. Then the covariant derivative  $\nabla^s$  on  $S(E)$  is given locally by the formula*

$$\nabla^s b_{\alpha} = \frac{1}{2} \sum_{i < j} \tilde{\omega}_{ij} \otimes e_i e_j \cdot b_{\alpha}, \tag{B.1}$$

where  $\mathcal{E} = (e_1, \dots, e_n)$  is a local section of  $P_{\text{SO}}(E)$  on  $U$ ,  $\tilde{\omega} = \mathcal{E}^*(\omega)$ , and  $(b_1, \dots, b_N)$  is a local section of  $P_{\text{SO}}(S(E))$ .

**REMARK B.2.** A local section  $(b_1, \dots, b_N)$  corresponds to the choice of basis in the module  $M_{\mathbb{C}}$ , that is, a basis in the representation  $\mu$ . Once the representation  $\mu$  and basis  $(b_1, \dots, b_N)$  are chosen, spinor fields  $\varphi$  can be written locally as

$$\varphi = \varphi^{\alpha} b_{\alpha},$$

where  $\varphi^{\alpha} \in C^{\infty}(U)$ . We also have

$$\mu(e_j) b_{\alpha} = \mu(e_j)^{\beta}_{\alpha} b_{\beta}.$$

Hence, formula (B.1) is equivalent to

$$\nabla^s \varphi^{\alpha} = d\varphi^{\alpha} + \frac{1}{2} \sum_{i < j} \tilde{\omega}_{ij} \otimes \mu(e_i e_j)^{\alpha}_{\beta} \varphi^{\beta},$$

or abbreviated as

$$\nabla^s \varphi = d\varphi + \frac{1}{2} \sum_{i < j} \tilde{\omega}_{ij} \otimes \mu(e_i e_j) \cdot \varphi. \tag{B.2}$$

The (massless) Dirac operator acting on  $S(E)$  is defined as

$$D^s \varphi = e_i \nabla_{e_i}^s \varphi. \tag{B.3}$$

It is well known that any spinor bundle of  $E$  on a connected manifold can be decomposed into a direct sum of irreducible spinor bundles. However, the most interesting case is when  $E = TX$ , which implies  $n = m$ . Therefore, we will only consider the irreducible complex spinor bundle on  $TX$ , which further implies that the module has complex dimension  $N = 2^{\lfloor \frac{n}{2} \rfloor}$ , where  $\lfloor \cdot \rfloor$  stands for the integer part.

Let  $\Delta_n^{\mathbb{C}}$  be the representation of  $\text{Spin}_n$  given by restricting an irreducible complex representation  $\mathbb{C}l(\mathbb{R}^n) \mapsto \text{Hom}_{\mathbb{C}}(S_N, S_N)$  to  $\text{Spin}_n \subset Cl^0(\mathbb{R}^n) \subset \mathbb{C}l(\mathbb{R}^n)$ .

REMARK B.3. When  $n$  is odd, this representation of  $\text{Spin}_n$  is independent of which irreducible representations of  $\mathbb{C}l(\mathbb{R}^n)$  are used.

As we only work in odd dimension(s), we focus on the spin bundle  $\Delta_n(X) := P_{\text{Spin}}(TX) \times_{\Delta_n^{\mathbb{C}}} S_N$ . Furthermore, we consider the Levi-Civita connection on  $P_{\text{SO}}(TM)$ , which induces a connection on  $P_{\text{Spin}}(TM)$  as given in Theorem B.1. We use  $\nabla$  to denote covariant derivatives induced by the Levi-Civita connection. In this case the  $\omega_{ij}$  in Theorem B.1 are given by

$$\omega_{ij} = g(\nabla e_i, e_j). \tag{B.4}$$

Hence, formulae (B.2) and (B.3) become

$$\nabla \varphi = d\phi + \frac{1}{4} g(\nabla e_i, e_j) \otimes \Delta_n^{\mathbb{C}}(e_i e_j) \cdot \varphi \tag{B.5}$$

and

$$W \varphi = \Delta_n^{\mathbb{C}}(e_i) \nabla_{e_i} \varphi, \tag{B.6}$$

where we use  $W$  to denote the Dirac operator.

Now, we further assume that our manifold is parallelizable. In particular, this assumption is satisfied for any 3-dimensional oriented manifold. Then  $P_{\text{SO}}(TX)$  is trivial. Thus, there exists a global section  $\mathcal{E} = (e_1, \dots, e_n)$  of  $P_{\text{SO}}(TX)$ . This implies that formulae (B.5) and (B.6) can be extended globally.

### B.2 Definition via frames

In this subsection we set  $n = 3$ . Hence,  $N = 2$  and  $\Delta_3(X) := P_{\text{Spin}}(TX) \times_{\Delta_3^{\mathbb{C}}} S_2$ . Consider a triple of orthonormal (with respect to the given metric  $g$ ) smooth real vector fields  $e_j$ ,  $j = 1, 2, 3$ . Each vector  $e_j(y)$  has coordinate components  $e_j^\alpha(y)$ ,  $\alpha = 1, 2, 3$ . The triple of vector fields  $e_j$ ,  $j = 1, 2, 3$ , is called an *orthonormal frame*. We assume that

$$\det e_j^\alpha > 0, \tag{B.7}$$

which means that the orientation of our frame agrees with the orientation of our local coordinates.

Define Pauli matrices

$$\sigma^\alpha(y) := s^j e_j^\alpha(y), \tag{B.8}$$

where

$$s^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.9}$$

Note that formula (B.8) is equivalent to choosing a particular representation of  $\Delta_3^{\mathbb{C}}$  which is given by

$$\Delta_3^{\mathbb{C}}(e_j) = -is_j. \tag{B.10}$$

It is not hard to see that this representation is an irreducible representation of  $\mathbb{C}l(\mathbb{R}^3)$ .

Let

$$\left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} := \frac{1}{2} g^{\beta\delta} \left( \frac{\partial g_{\gamma\delta}}{\partial y^\alpha} + \frac{\partial g_{\alpha\delta}}{\partial y^\gamma} - \frac{\partial g_{\alpha\gamma}}{\partial y^\delta} \right)$$

be the Christoffel symbols.

Using formulae (B.4) to (B.6), we conclude that the massless Dirac operator is the  $2 \times 2$  matrix first order linear differential operator given by

$$W := -i\sigma^\alpha \left( \frac{\partial}{\partial y^\alpha} + \frac{1}{4} \sigma_\beta \left( \frac{\partial \sigma^\beta}{\partial y^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^\gamma \right) \right) \tag{B.11}$$

acting on sections of  $\Delta_3$ . Note that the standard basis for the representation (B.10) is used here and hence  $W$  can be thought of as acting on 2-columns of complex-valued scalar fields. See also Remark B.2 and [12, Appendix A].

### B.3 Analytic definition

Since we are working on a connected oriented 3-manifold, by picking a global section of  $P_{SO}(TX)$  we can regard the operator  $W$  as an operator acting on 2-columns of complex-valued scalar fields. Now we shall extend it to an operator on half-densities.

The *massless Dirac operator on half-densities*,  $W_{1/2}$ , corresponding to the given metric  $g$  is a particular  $2 \times 2$  matrix first order linear differential operator acting on 2-columns of complex-valued half-densities. It is defined by the following four conditions:

$$\text{tr}(W_{1/2})_{\text{prin}} = 0, \tag{B.12}$$

$$\det(W_{1/2})_{\text{prin}}(y, p) = -g^{\alpha\beta}(y) p_\alpha p_\beta, \tag{B.13}$$

$$(W_{1/2})_{\text{sub}} = \frac{i}{16} g_{\alpha\beta} \{ (W_{1/2})_{\text{prin}}, (W_{1/2})_{\text{prin}}, (W_{1/2})_{\text{prin}} \}_{p_\alpha p_\beta}, \tag{B.14}$$

$$-i \text{tr} [ ((W_{1/2})_{\text{prin}})_{p_1} ((W_{1/2})_{\text{prin}})_{p_2} ((W_{1/2})_{\text{prin}})_{p_3} ] > 0. \tag{B.15}$$

Here  $y = (y^1, y^2, y^3)$  denotes local coordinates,  $p = (p_1, p_2, p_3)$  denotes the dual variable (momentum),  $(W_{1/2})_{\text{prin}}(y, p)$  is the principal symbol,  $(W_{1/2})_{\text{sub}}(y)$  is the subprincipal symbol, curly brackets denote the generalized Poisson bracket on matrix-functions

$$\{F, G, H\} := F_{y^\alpha} G H_{p_\alpha} - F_{p_\alpha} G H_{y^\alpha},$$

and the subscripts  $y^\alpha$  and  $p_\alpha$  indicate partial derivatives.

The *massless Dirac operator*,  $W$ , is defined as

$$W := (\det g_{\kappa\lambda})^{-1/4} W_{1/2} (\det g_{\mu\nu})^{1/4}. \quad (\text{B.16})$$

It acts on 2-columns of complex-valued scalar fields.

The analytic definition of the massless Dirac operator given in this subsection originates from [5, Section 8] and is equivalent to the traditional geometric definition presented in subsection B.1.

## B.4 Spin structure

The definitions from subsections B.2 and B.3 work for any connected oriented Riemannian 3-manifold and are equivalent. Note, however, that they do not define the massless Dirac operator uniquely. Namely, let  $W$  be a massless Dirac operator and let  $R(y)$  be an arbitrary smooth  $2 \times 2$  special unitary matrix-function (1.3). One can check that then  $R^*WR$  is also a massless Dirac operator.

Let us now look at the issue of non-uniqueness of the massless Dirac operator the other way round. Suppose that  $W$  and  $\tilde{W}$  are two massless Dirac operators. Does there exist a smooth matrix-function (1.3) such that  $\tilde{W} = R^*WR$ ? If the operators  $W$  and  $\tilde{W}$  are in a certain sense ‘close’ then the answer is yes, but in general there are topological obstructions and the answer is no. This motivates the introduction of the concept of spin structure, see [5, Section 7] and [6] for details.

Fortunately, for the purposes of our paper the issue of spin structure is irrelevant because it is known [10, Section 5], that the 3-sphere admits a unique spin structure. In other words, when we work on  $\mathbb{S}^3$  equipped with a Riemannian metric  $g$  the constructions from subsections B.3 and B.2 define the massless Dirac operator uniquely modulo the gauge transformation (1.2), (1.3).

## Appendix C Special vector fields on the 3-sphere

Working in  $\mathbb{R}^4$  and using Cartesian coordinates, consider the triple of vector fields  $(\mathbf{K}_\pm)_j^\alpha$ ,  $j = 1, 2, 3$ ,  $\alpha = 1, 2, 3, 4$ , defined as

$$\begin{aligned} (\mathbf{K}_\pm)_1 &= (-\mathbf{x}^4 \mp \mathbf{x}^3 \pm \mathbf{x}^2 \mathbf{x}^1), \\ (\mathbf{K}_\pm)_2 &= (\pm \mathbf{x}^3 - \mathbf{x}^4 \mp \mathbf{x}^1 \mathbf{x}^2), \\ (\mathbf{K}_\pm)_3 &= (\mp \mathbf{x}^2 \pm \mathbf{x}^1 - \mathbf{x}^4 \mathbf{x}^3). \end{aligned} \quad (\text{C.1})$$

Observe that the vector fields (C.1) are tangent to the 3-sphere (A.1), so let us denote by  $(K_\pm)_j^\alpha$  the restrictions of the vector fields (C.1) to the 3-sphere. Here the tensor index  $\alpha = 1, 2, 3$  corresponds to local coordinates  $y^\alpha$  on  $\mathbb{S}^3$ . Note that we have  $\det\{(K_\pm)_j^\alpha\} > 0$ , which is in agreement with (B.7).

REMARK C.1. The vector fields  $(K_\pm)_j$ ,  $j = 1, 2, 3$ , constructed above are special because with the standard metric on  $\mathbb{S}^3$  they possess the following properties.

- (a) The vector fields  $(K_{\pm})_j$  are orthonormal,
- (b) The vector fields  $(K_{\pm})_j$  are Killing vector fields.
- (c) If we write down the Dirac operator  $W_{\pm}$  using  $(K_{\pm})_j$  as a frame, then the eigenspinors corresponding to the eigenvalue  $\pm\frac{3}{2}$  are constant spinors. Of course, for a given operator  $W_+$  or  $W_-$  one cannot have constant eigenspinors for eigenvalues  $+\frac{3}{2}$  **and**  $-\frac{3}{2}$  because this would contradict the fact that eigenspinors corresponding to different eigenvalues are orthogonal.

Note that the operators  $W_+$  and  $W_-$  defined in Remark C.1(c) are related as  $W_- = R^*W_+R$ , where  $R : \mathbb{S}^3 \rightarrow \text{SU}(2)$  is the restriction of the matrix-function

$$\pm \begin{pmatrix} \mathbf{x}^4 + i\mathbf{x}^3 & \mathbf{x}^2 + i\mathbf{x}^1 \\ -\mathbf{x}^2 + i\mathbf{x}^1 & \mathbf{x}^4 - i\mathbf{x}^3 \end{pmatrix}$$

to the 3-sphere (A.1).

The construction of the unperturbed Dirac operator by means of a triple of orthonormal Killing vector fields and an immersion of  $\mathbb{S}^3$  in  $\mathbb{R}^4$  was previously used in [24].

### Appendix D The scalar Laplacian and its pseudoinverse

In this appendix we work on the 3-sphere equipped with standard metric  $(g_0)_{\alpha\beta}(y)$ .

Let  $f$  be a smooth scalar function on  $\mathbb{S}^3$ . Then there exists a unique sequence of homogeneous harmonic polynomials  $p_n(\mathbf{x})$  of degree  $n = 0, 1, 2, \dots$  such that the series  $\sum_{n=0}^{+\infty} p_n(\mathbf{x})$  converges uniformly, together with all its partial derivatives, on the closed unit ball in  $\mathbb{R}^4$ , and coincides with  $f$  on  $\mathbb{S}^3$ .

It is known that the eigenvalues of the operator  $-\Delta$  acting on  $\mathbb{S}^3$  are  $n(n+2)$ ,  $n = 0, 1, 2, \dots$ , and their multiplicity is  $(n+1)^2$ , which is the dimension of the vector space of homogeneous harmonic polynomials of degree  $n$ . The explicit formula for the action of the operator  $(-\Delta)^{-1}$ , the pseudoinverse of  $-\Delta$ , on our function  $f$  is

$$(-\Delta)^{-1}f = \sum_{n=1}^{+\infty} \frac{p_n(\mathbf{x})}{n(n+2)} \Big|_{\|\mathbf{x}\|=1}.$$

### Appendix E Comparison with the 3-torus

If we leave only the second and fourth terms in the RHS of (2.15), substitute this expression into (2.14), drop the subscripts  $\pm$  and use (2.4), we get

$$\lambda^{(2)} = -\frac{1}{16V(0)} \varepsilon_{qks} \int_{\mathbb{S}^3} (h_{jq} [L_s h_{jk}] + h_{rq} [(-\Delta)^{-1} L_r L_s L_j h_{jk}]) \rho_0 dy. \tag{E.1}$$

Formula (E.1) coincides with the result from [15, Theorem 2.1] if we put  $V^{(0)} = (2\pi)^3$  (volume of the unperturbed torus),  $\rho_0 = 1$  and  $L_j = \delta_j^\alpha \partial_\alpha$ , with  $\partial_\alpha$  denoting partial differentiation in the  $\alpha$ th cyclic coordinate on the 3-torus.

## Appendix F Eigenvalues for generalized Berger spheres

Here we give further explicit expressions for the eigenvalues using the procedure from Section 6.1 where we apply the operator to harmonic polynomials of degree  $n$ . For convenience we seek eigenvalues  $\mu$  of the operator  $\widetilde{\mathbf{W}} = \mathbf{W} - \nu I$  obtained by dropping the constant term from (6.6).

Let  $\kappa = (\kappa_1, \kappa_2, \kappa_3) \in \{\pm 1\}^3$ , and let

$$N_+ := \{\kappa \in \{\pm 1\}^3 : \kappa_1 \kappa_2 \kappa_3 = +1\} = \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}.$$

For each  $n \leq 4$  we give below an explicit formula for the characteristic polynomial  $\chi_n(\mu)$  whose roots give the eigenvalues of  $\widetilde{\mathbf{W}}$ . For  $n \geq 5$  formulae become too cumbersome to list, and we do not have a general formula yet.

*Degree  $n = 0$ .*

$$\chi_0(\mu) = \mu^2.$$

*Degree  $n = 1$ .*

$$\chi_1(\mu) = \prod_{\kappa \in N_+} \left[ \mu + \sum_{j=1}^3 \frac{\kappa_j}{a_j} \right]^2.$$

See also formulae (6.8)–(6.11).

*Degree  $n = 2$ .*

$$\chi_2(\mu) = \left[ \mu^3 - \left( 4 \sum_{j=1}^3 a_j^{-2} \right) \mu + \frac{16}{\prod_{j=1}^3 a_j} \right]^6.$$

*Degree  $n = 3$ .*

$$\chi_3(\mu) = \prod_{\kappa \in N_+} \left[ \mu^2 - \left( \sum_{j=1}^3 \frac{\kappa_j}{a_j} \right) \mu - 3 \left( \sum_{j=1}^3 a_j^{-2} - \sum_{\substack{j,k=1 \\ j \neq k}}^3 \frac{\kappa_j \kappa_k}{a_j a_k} \right) \right]^4.$$

*Degree  $n = 4$ .*

$$\begin{aligned} \chi_4(\mu) = & \left[ \mu^5 - \left( 20 \sum_{j=1}^3 a_j^{-2} \right) \mu^3 + \left( \frac{80}{\prod_{j=1}^3 a_j} \right) \mu^2 \right. \\ & \left. + 64 \left( \sum_{j=1}^3 a_j^{-4} + 2 \sum_{\substack{j,k=1 \\ j \neq k}}^3 a_j^{-2} a_k^{-2} \right) \mu - 768 \frac{\sum_{j=1}^3 a_j^{-2}}{\prod_{j=1}^3 a_j} \right]^{10}. \end{aligned}$$

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