

ORTHONORMAL SEQUENCES AND TIME FREQUENCY LOCALIZATION RELATED TO THE RIEMANN–LIOUVILLE OPERATOR

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(Communicated by D. Han)

Abstract. For every real number $p > 0$, we define the p -dispersion $\rho_{p, \nu_\alpha}(f)$ of a measurable function f on $[0, +\infty[\times \mathbb{R}$, where ν_α is some positive measure. We prove that for every orthonormal basis $(\varphi_{m,n})_{(m,n) \in \mathbb{N}^2}$ of $L^2(d\nu_\alpha)$, the sequences $(\rho_{p, \nu_\alpha}(\varphi_{m,n}))_{(m,n) \in \mathbb{N}^2}$, $(\rho_{p, \nu_\alpha}(\widetilde{\mathcal{F}}_\alpha(\varphi_{m,n})))_{(m,n) \in \mathbb{N}^2}$ can not be simultaneously bounded, where $\widetilde{\mathcal{F}}_\alpha$ is some Fourier transform. The main tool is a time frequency localization inequality for orthonormal sequences in $L^2(d\nu_\alpha)$.

On the other hand, we construct an orthonormal sequence $(\psi_{m,n})_{(m,n) \in \mathbb{N}^2} \subset L^2(d\nu_\alpha)$ such that the sequence $(\rho_{p, \nu_\alpha}(\psi_{m,n})\rho_{p, \nu_\alpha}(\widetilde{\mathcal{F}}_\alpha(\psi_{m,n})))_{(m,n) \in \mathbb{N}^2}$ is bounded.

1. Introduction

The uncertainty principles play an important role in harmonic analysis. These principles state that a nonzero function f and its Fourier transform \widehat{f} can not be simultaneously and sharply localized at the same time. Many mathematical formulations of this fact have been checked in the last decades [8, 10, 11, 16, 17, 25, 26]. In [30], Shapiro has studied the localization for an orthonormal sequence $(\varphi_k)_{k \in \mathbb{N}}$. He showed that if the means and the dispersions of the orthonormal sequence $(\varphi_k)_{k \in \mathbb{N}}$ and their Fourier transforms $(\widehat{\varphi}_k)_{k \in \mathbb{N}}$ are uniformly bounded, then $(\varphi_k)_{k \in \mathbb{N}}$ is finite. In [22], the authors gave a quantitative version of the precedent theorem, that is if $(\varphi_k)_{k \in \mathbb{N}}$ is an orthonormal sequence in $L^2(\mathbb{R})$, then for every $n \in \mathbb{N}$,

$$\sum_{k=0}^n \left(\|x\varphi_k\|_2^2 + \|y\widehat{\varphi}_k\|_2^2 \right) \geq \frac{(n+1)^2}{2\pi}.$$

Recently, in [24], the author obtains a quantitative multivariables version of Shapiro's theorem for generalized dispersion, in fact the author showed that if $(\varphi_k)_{k \in \mathbb{N}}$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$; then for every positive real number p and for every $n \in \mathbb{N}^*$

$$\sum_{k=1}^n \left(\| |x|^{\frac{p}{2}} \varphi_k \|_2^2 + \| |y|^{\frac{p}{2}} \widehat{\varphi}_k \|_2^2 \right) \geq C n^{1+\frac{p}{2d}}$$

Mathematics subject classification (2010): 42A38, 44A35.

Keywords and phrases: Orthonormal basis, Hilbert Schmidt operator, frequency localization, Riemann-Liouville operator, Fourier transform.

where C is a constant which does not depend on p . The author obtains also a multiplicative form of the above theorem by showing that if $(\varphi_k)_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$, then for every positive real number p ,

$$\sup_{k \in \mathbb{N}} \left(\| |x|^{\frac{p}{2}} \varphi_k \|_2 \| |y|^{\frac{p}{2}} \widehat{\varphi}_k \|_2 \right) = +\infty.$$

On the other hand, in [5], the authors have defined the Riemann-Liouville operator \mathcal{R}_α ; $\alpha \geq 0$, by

$$\mathcal{R}_\alpha(f)(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^1 \int_{-1}^1 f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}} \\ \quad \times (1-s^2)^{\alpha-1} dt ds, & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^1 f(r\sqrt{1-t^2}, x+rt) \frac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha = 0; \end{cases} \quad (1.1)$$

where f is any continuous function on \mathbb{R}^2 , even with respect to the first variable.

The dual operator ${}^t\mathcal{R}_\alpha$ is defined by

$${}^t\mathcal{R}_\alpha(g)(r, x) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{1}{2^\alpha \Gamma(\alpha+1)} \int_r^{+\infty} \int_{-\sqrt{u^2-r^2}}^{\sqrt{u^2-r^2}} g(u, x+v) \\ \quad \times (u^2-v^2-r^2)^{\alpha-1} u du dv, & \text{if } \alpha > 0, \\ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\sqrt{r^2+(x-y)^2}, y) dy, & \text{if } \alpha = 0; \end{cases} \quad (1.2)$$

where g is any continuous function on \mathbb{R}^2 , even with respect to the first variable and with compact support.

In particular, for $\alpha = 0$ and by a change of variables, we get

$$\mathcal{R}_0(f)(r, x) = \frac{1}{2\pi} \int_0^{2\pi} f(r \cos \theta, x + r \sin \theta) d\theta.$$

This means that $\mathcal{R}_0(f)(r, x)$ is the mean value of f on the circle centered at $(0, x)$ and with radius r .

The mean operator \mathcal{R}_0 and its dual ${}^t\mathcal{R}_0$ play an important role and have many applications, for example, in image processing of the so-called synthetic aperture radar (SAR) data [19, 20] or in the linearized inverse scattering problem in acoustics [15].

The operators \mathcal{R}_α and its dual ${}^t\mathcal{R}_\alpha$ have the same properties as the Radon transform [18], for this reason, \mathcal{R}_α is called sometimes the generalized Radon transform.

The Fourier transform \mathcal{F}_α associated with the operator \mathcal{R}_α is defined by

$$\begin{aligned} \forall (\lambda_0, \lambda) \in \Upsilon, \mathcal{F}_\alpha(f)(\lambda_0, \lambda) &= \int_0^\infty \int_{\mathbb{R}} f(r, x) \mathcal{R}_\alpha(\cos(\lambda_0 \cdot) e^{-i\lambda \cdot})(r, x) d\nu_\alpha(r, x) \\ &= \int_0^\infty \int_{\mathbb{R}} f(r, x) j_\alpha(r\sqrt{\lambda_0^2 + \lambda^2}) e^{-i\lambda x} d\nu_\alpha(r, x), \end{aligned}$$

where Υ is the set given by

$$\Upsilon = \mathbb{R}^2 \cup \{(i\lambda_0, \lambda); (\lambda_0, \lambda) \in \mathbb{R}^2; |\lambda_0| \leq |\lambda|\}. \quad (1.3)$$

$d\nu_\alpha(r, x)$ is the measure defined on $[0, +\infty[\times \mathbb{R}$ by

$$d\nu_\alpha(r, x) = \frac{r^{2\alpha+1} dr}{2^\alpha \Gamma(\alpha+1)} \otimes \frac{dx}{\sqrt{2\pi}}. \quad (1.4)$$

j_α is the modified Bessel function that will be defined in the second section.

Many harmonic analysis results have been established for the Fourier transform \mathcal{F}_α [1, 4, 5, 6, 7, 28, 29]. Also, many uncertainty principles related to the Fourier transform \mathcal{F}_α have been proved [2, 3, 21, 26, 27].

Our investigation in this work is to prove a generalized quantitative version of the mean-dispersion Shapiro's theorem related to the Riemann-Liouville operator.

This paper is arranged as follows. In the second section, we collect some harmonic analysis results for the Riemann-Liouville operator \mathcal{R}_α and its connected Fourier transform \mathcal{F}_α . The third section contains the main results of this work, we will prove a quantitative version of the mean-dispersion shapiro's theorem. Next, we establish a multiplicative form of this theorem.

2. The Riemann-Liouville transform

In this section, we recall some harmonic analysis results related to the convolution product and the Fourier transform associated with the Riemann-Liouville operator.

Let D and Ξ be the singular partial differential operators defined by

$$\begin{cases} D = \frac{\partial}{\partial x}; \\ \Xi = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}; \end{cases} (r, x) \in]0, +\infty[\times \mathbb{R}, \alpha \geq 0.$$

For all $(\lambda_0, \lambda) \in \mathbb{C}^2$, the system

$$\begin{cases} Du(r, x) = -i\lambda u(r, x); \\ \Xi u(r, x) = -\lambda_0^2 u(r, x); \\ u(0, 0) = 1, \quad \frac{\partial u}{\partial r}(0, x) = 0; \forall x \in \mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\lambda_0, \lambda}$ given by

$$\forall (r, x) \in [0, +\infty[\times \mathbb{R}, \quad \varphi_{\lambda_0, \lambda}(r, x) = j_\alpha(r\sqrt{\lambda_0^2 + \lambda^2}) e^{-i\lambda x}, \quad (2.1)$$

where j_α is the modified Bessel function defined by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(\alpha+k+1)} \left(\frac{z}{2}\right)^{2k},$$

and J_α is the Bessel function of first kind and index α [13, 14, 23, 35]. The modified Bessel function j_α has the integral representation

$$j_\alpha(z) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} \exp(-izt) dt. \quad (2.2)$$

Consequently, for every $k \in \mathbb{N}$ and $z \in \mathbb{C}$, we have

$$|j_{\alpha}^{(k)}(z)| \leq e^{|\operatorname{Im}(z)|}. \quad (2.3)$$

PROPOSITION 2.1. *The eigenfunction $\varphi_{\lambda_0, \lambda}$ satisfies the following properties*

i. *The function $\varphi_{\lambda_0, \lambda}$ is bounded on \mathbb{R}^2 if, and only if $(\lambda_0, \lambda) \in \Upsilon$, where Υ is the set given by the relation (1.3) and in this case,*

$$\sup_{(r,x) \in \mathbb{R}^2} |\varphi_{\lambda_0, \lambda}(r, x)| = 1. \quad (2.4)$$

ii. *The function $\varphi_{\lambda_0, \lambda}$ has the following Mehler integral representation*

$$\varphi_{\lambda_0, \lambda}(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^1 \int_{-1}^1 \cos(\lambda_0 r s \sqrt{1-t^2}) \exp(-i\lambda(x+rt)) \\ \quad \times (1-t^2)^{\alpha-\frac{1}{2}} (1-s^2)^{\alpha-1} dt ds; & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^1 \cos(r\lambda_0 \sqrt{1-t^2}) \exp(-i\lambda(x+rt)) \\ \quad \times \frac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha = 0. \end{cases} \quad (2.5)$$

REMARK 2.2. The Mehler integral representation (2.5) of the eigenfunction $\varphi_{\lambda_0, \lambda}$ allows us to define the integral transform \mathcal{R}_{α} by

$$\mathcal{R}_{\alpha}(f)(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^1 \int_{-1}^1 f(rs\sqrt{1-t^2}, x+rt) (1-t^2)^{\alpha-\frac{1}{2}} \\ \quad \times (1-s^2)^{\alpha-1} dt ds, & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^1 f(r\sqrt{1-t^2}, x+rt) \frac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha = 0; \end{cases} \quad (2.6)$$

where f is any continuous function on \mathbb{R}^2 ; even with respect to the first variable. Then, the relations (2.5) and (2.6) show that

$$\varphi_{\lambda_0, \lambda}(r, x) = \mathcal{R}_{\alpha}(\cos(\lambda_0 \cdot) e^{-i\lambda \cdot})(r, x), \quad (2.7)$$

which gives the mutual connection between the functions $\varphi_{\lambda_0, \lambda}$ and $\cos(\lambda_0 \cdot) e^{-i\lambda \cdot}$.

For this reason, the operator \mathcal{R}_{α} is called the Riemann-Liouville transform associated with the operators D and Ξ .

The partial differential operators D and Ξ satisfy the intertwining properties with the Riemann-Liouville operator and its dual

$$\begin{aligned} {}^t \mathcal{R}_{\alpha} \Xi(f) &= \frac{\partial^2}{\partial r^2} {}^t \mathcal{R}_{\alpha}(f), & {}^t \mathcal{R}_{\alpha} D(f) &= D {}^t \mathcal{R}_{\alpha}(f), \\ \Xi \mathcal{R}_{\alpha}(f) &= \mathcal{R}_{\alpha} \frac{\partial^2}{\partial r^2}(f), & D \mathcal{R}_{\alpha}(f) &= \mathcal{R}_{\alpha} D(f), \end{aligned}$$

where f is a sufficiently smooth function.

We denote by $L^p(dv_\alpha)$; $p \in [1, +\infty]$, the Lebesgue space formed by the measurable functions f on $[0, +\infty[\times \mathbb{R}$ such that $\|f\|_{p, v_\alpha} < +\infty$, with

$$\|f\|_{p, v_\alpha} = \begin{cases} \left(\int_0^{+\infty} \int_{\mathbb{R}} |f(r, x)|^p dv_\alpha(r, x) \right)^{\frac{1}{p}}, & \text{if } p \in [1, +\infty[, \\ \text{ess sup}_{(r, x) \in [0, +\infty[\times \mathbb{R}} |f(r, x)|, & \text{if } p = +\infty, \end{cases} \quad (2.8)$$

and dv_α is given by the relation (1.4).

$\langle \cdot | \cdot \rangle_{v_\alpha}$ the inner product on the Hilbert space $L^2(dv_\alpha)$ defined by

$$\langle f | g \rangle_{v_\alpha} = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x) \overline{g(r, x)} dv_\alpha(r, x).$$

$\mathcal{C}_{0, e}(\mathbb{R}^2)$ the space of continuous function on \mathbb{R}^2 , even with respect to the first variable such that

$$\lim_{r^2+x^2 \rightarrow +\infty} f(r, x) = 0,$$

the space $\mathcal{C}_{0, e}(\mathbb{R}^2)$ is equipped with the norm

$$\|f\|_{\infty, v_\alpha} = \sup_{(r, x) \in [0, +\infty[\times \mathbb{R}} |f(r, x)|.$$

To define the translation operator associated with the Riemann-Liouville transform, we use the product formula for the eigenfunction $\varphi_{\lambda_0, \lambda}$, that is for $(r, x), (s, y) \in [0, +\infty[\times \mathbb{R}$,

$$\varphi_{\lambda_0, \lambda}(r, x) \varphi_{\lambda_0, \lambda}(s, y) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_0^\pi \varphi_{\lambda_0, \lambda}(\sqrt{r^2 + s^2 + 2rs \cos \theta}, x + y) \sin^{2\alpha}(\theta) d\theta.$$

DEFINITION 2.3. i) For every $(r, x) \in [0, +\infty[\times \mathbb{R}$, the translation operator $\tau_{(r, x)}$ associated with the Riemann-Liouville transform is defined on $L^p(dv_\alpha)$; $p \in [1, +\infty]$, by

$$\tau_{(r, x)} f(s, y) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_0^\pi f(\sqrt{r^2 + s^2 + 2rs \cos \theta}, x + y) \sin^{2\alpha}(\theta) d\theta. \quad (2.9)$$

ii) The convolution product of $f, g \in L^1(dv_\alpha)$ is defined for every $(r, x) \in [0, +\infty[\times \mathbb{R}$, by

$$f * g(r, x) = \int_0^{+\infty} \int_{\mathbb{R}} \tau_{(r, -x)}(\check{f})(s, y) g(s, y) dv_\alpha(s, y), \quad (2.10)$$

where $\check{f}(s, y) = f(s, -y)$.

The set $[0, +\infty[\times \mathbb{R}$ equipped with the convolution product $*$ is an hypergroup in the sense of [9].

PROPOSITION 2.4.

i. For every $f \in L^p(dv_\alpha)$; $p \in [1, +\infty]$, and for every $(r, x) \in [0, +\infty[\times \mathbb{R}$, the function $\tau_{(r,x)}(f)$ belongs to $L^p(dv_\alpha)$ and we have

$$\|\tau_{(r,x)}(f)\|_{p, v_\alpha} \leq \|f\|_{p, v_\alpha}. \quad (2.11)$$

ii. For every $f \in L^1(dv_\alpha)$ and $(r, x) \in [0, +\infty[\times \mathbb{R}$,

$$\int_0^\infty \int_{\mathbb{R}} \tau_{(r,x)}(f)(s, y) dv_\alpha(s, y) = \int_0^\infty \int_{\mathbb{R}} f(s, y) dv_\alpha(s, y). \quad (2.12)$$

iii. For every $f \in L^p(dv_\alpha)$; $p \in [1, +\infty[$, we have

$$\lim_{(r,x) \rightarrow (0,0)} \|\tau_{(r,x)}(f) - f\|_{p, v_\alpha} = 0. \quad (2.13)$$

iv. For every $f \in \mathcal{C}_{0,e}(\mathbb{R}^2)$ and every $(r, x) \in \mathbb{R}^2$, the function $\tau_{(r,x)}(f)$ belongs to $\mathcal{C}_{0,e}(\mathbb{R}^2)$ and

$$\lim_{(r,x) \rightarrow (0,0)} \|\tau_{(r,x)}(f) - f\|_{\infty, v_\alpha} = 0. \quad (2.14)$$

v. Let φ be a nonnegative measurable function on $\mathbb{R} \times \mathbb{R}$, even with respect to the first variable, such that

$$\int_0^{+\infty} \int_{\mathbb{R}} \varphi(r, x) dv_\alpha(r, x) = 1.$$

Then the family $(\varphi_{(a,b)})_{(a,b) \in (\mathbb{R}_+^*)^2}$ defined by

$$\forall (r, x) \in \mathbb{R} \times \mathbb{R}, \varphi_{(a,b)}(r, x) = \frac{1}{a^{2\alpha+2b}} \varphi\left(\frac{r}{a}, \frac{x}{b}\right)$$

is an approximation of the identity in $L^p(dv_\alpha)$; $p \in [1, +\infty[$, that is for every $f \in L^p(dv_\alpha)$, we have

$$\lim_{(a,b) \rightarrow (0^+, 0^+)} \|f * \varphi_{(a,b)} - f\|_{p, v_\alpha} = 0. \quad (2.15)$$

vi. For every $f \in \mathcal{C}_{0,e}(\mathbb{R}^2)$,

$$\lim_{(a,b) \rightarrow (0^+, 0^+)} \|f * \varphi_{(a,b)} - f\|_{\infty, v_\alpha} = 0. \quad (2.16)$$

vii. If $1 \leq p, q, r \leq +\infty$ are such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and if $f \in L^p(dv_\alpha)$, $g \in L^q(dv_\alpha)$, then the function $f * g$ belongs to $L^r(dv_\alpha)$, and we have the Young's inequality

$$\|f * g\|_{r, v_\alpha} \leq \|f\|_{p, v_\alpha} \|g\|_{q, v_\alpha}. \quad (2.17)$$

In the following, we need the notations Υ_+ is the subset of Υ given by

$$\Upsilon_+ = \mathbb{R}_+ \times \mathbb{R} \cup \{(it, x); (t, x) \in \mathbb{R}^2; 0 \leq t \leq |x|\}.$$

\mathcal{B}_{Υ_+} is the σ -algebra defined on Υ_+ by

$$\mathcal{B}_{\Upsilon_+} = \{\theta^{-1}(B), B \in \mathcal{B}_{\text{Or}}([0, +\infty[\times\mathbb{R}])\},$$

where θ is the bijective function defined on the set Υ_+ by

$$\theta(\lambda_0, \lambda) = (\sqrt{\lambda_0^2 + \lambda^2}, \lambda), \quad (2.18)$$

and $\mathcal{B}_{\text{Or}}([0, +\infty[\times\mathbb{R})$ is the usual Borel σ -algebra on $[0, +\infty[\times\mathbb{R}$.
 $d\gamma_\alpha$ is the measure defined on \mathcal{B}_{Υ_+} by

$$\forall A \in \mathcal{B}_{\Upsilon_+}, \gamma_\alpha(A) = \nu_\alpha(\theta(A)).$$

$L^p(d\gamma_\alpha)$; $p \in [1, +\infty]$, the Lebesgue space consisting of measurable function g on Υ_+ such that

$$\|g\|_{p, \gamma_\alpha} < +\infty.$$

$\langle \cdot | \cdot \rangle_{\gamma_\alpha}$ the inner product on the Hilbert space $L^2(d\gamma_\alpha)$ given by

$$\langle f | g \rangle_{\gamma_\alpha} = \int \int_{\Upsilon_+} f(\lambda_0, \lambda) \overline{g(\lambda_0, \lambda)} d\gamma_\alpha(\lambda_0, \lambda).$$

PROPOSITION 2.5.

i. For all nonnegative measurable function g on Υ_+ , we have

$$\begin{aligned} & \int \int_{\Upsilon_+} g(\lambda_0, \lambda) d\gamma_\alpha(\lambda_0, \lambda) \\ &= \frac{1}{2^\alpha \Gamma(\alpha + 1) \sqrt{2\pi}} \left(\int_0^{+\infty} \int_{\mathbb{R}} g(\lambda_0, \lambda) (\lambda_0^2 + \lambda^2)^\alpha \lambda_0 d\lambda_0 d\lambda \right. \\ & \quad \left. + \int_{\mathbb{R}} \int_0^{|\lambda|} g(i\lambda_0, \lambda) (\lambda^2 - \lambda_0^2)^\alpha \lambda_0 d\lambda_0 d\lambda \right). \end{aligned}$$

ii. For all nonnegative measurable function f on $[0, +\infty[\times\mathbb{R}$ (respectively integrable on $[0, +\infty[\times\mathbb{R}$ with respect to the measure $d\nu_\alpha$), $f \circ \theta$ is a nonnegative measurable function on Υ_+ (respectively integrable on Υ_+ with respect to the measure $d\gamma_\alpha$) and we have

$$\int \int_{\Upsilon_+} (f \circ \theta)(\lambda_0, \lambda) d\gamma_\alpha(\lambda_0, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x) d\nu_\alpha(r, x). \quad (2.19)$$

Now, using the eigenfunction $\varphi_{\lambda_0, \lambda}$ given by the relation (2.1), we can define the Fourier transform.

DEFINITION 2.6. The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(d\nu_\alpha)$ by

$$\forall (\lambda_0, \lambda) \in \Upsilon, \quad \mathcal{F}_\alpha(f)(\lambda_0, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x) \varphi_{\lambda_0, \lambda}(r, x) d\nu_\alpha(r, x).$$

PROPOSITION 2.7.

i. For every $f \in L^1(d\nu_\alpha)$, the function $\mathcal{F}_\alpha(f)$ belongs to the space $L^\infty(d\gamma_\alpha)$ and we have

$$\|\mathcal{F}_\alpha(f)\|_{\infty, \gamma_\alpha} \leq \|f\|_{1, \nu_\alpha}. \quad (2.20)$$

ii. Let $f \in L^1(d\nu_\alpha)$. For every $(r, x) \in [0, +\infty[\times \mathbb{R}$, we have

$$\forall (\lambda_0, \lambda) \in \Upsilon, \quad \mathcal{F}_\alpha(\tau_{(r,x)}(f))(\lambda_0, \lambda) = \overline{\varphi_{\lambda_0, \lambda}(r, x)} \mathcal{F}_\alpha(f)(\lambda_0, \lambda).$$

iii. For $f, g \in L^1(d\nu_\alpha)$, we have

$$\forall (\lambda_0, \lambda) \in \Upsilon, \quad \mathcal{F}_\alpha(f * g)(\lambda_0, \lambda) = \mathcal{F}_\alpha(f)(\lambda_0, \lambda) \mathcal{F}_\alpha(g)(\lambda_0, \lambda). \quad (2.21)$$

vi. For $f \in L^1(d\nu_\alpha)$, we have

$$\forall (\lambda_0, \lambda) \in \Upsilon, \quad \mathcal{F}_\alpha(f)(\lambda_0, \lambda) = \widetilde{\mathcal{F}}_\alpha(f) \circ \theta(\lambda_0, \lambda), \quad (2.22)$$

where for every $(\lambda_0, \lambda) \in \mathbb{R}^2$,

$$\widetilde{\mathcal{F}}_\alpha(f)(\lambda_0, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x) j_\alpha(r\lambda_0) \exp(-i\lambda x) d\nu_\alpha(r, x), \quad (2.23)$$

and θ is the function defined by the relation (2.18).

Also, the Fourier transform \mathcal{F}_α satisfies the following properties

THEOREM 2.8.

i. Let $f \in L^1(d\nu_\alpha)$ such that the function $\mathcal{F}_\alpha(f)$ belongs to the space $L^1(d\gamma_\alpha)$, then we have the following inversion formula for \mathcal{F}_α , for almost every $(r, x) \in [0, +\infty[\times \mathbb{R}$,

$$\begin{aligned} f(r, x) &= \int \int_{\Upsilon_+} \mathcal{F}_\alpha(f)(\lambda_0, \lambda) \overline{\varphi_{\lambda_0, \lambda}(r, x)} d\gamma_\alpha(\lambda_0, \lambda) \\ &= \int_0^{+\infty} \int_{\mathbb{R}} \widetilde{\mathcal{F}}_\alpha(f)(\lambda_0, \lambda) j_\alpha(r\lambda_0) e^{i\lambda x} d\nu_\alpha(\lambda_0, \lambda). \end{aligned} \quad (2.24)$$

ii. (Plancherel theorem) The Fourier transform \mathcal{F}_α can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto $L^2(d\gamma_\alpha)$ and for every $f \in L^2(d\nu_\alpha)$,

$$\|\mathcal{F}_\alpha(f)\|_{2, \gamma_\alpha} = \|f\|_{2, \nu_\alpha}. \quad (2.25)$$

In particular, we have the Parseval equality; for all $f, g \in L^2(d\nu_\alpha)$,

$$\langle f | g \rangle_{\nu_\alpha} = \langle \mathcal{F}_\alpha(f) | \mathcal{F}_\alpha(g) \rangle_{\gamma_\alpha}. \quad (2.26)$$

Using the relations (2.20), (2.25) and the Riesz-Thorin theorem's [31, 32], we deduce that for every $f \in L^p(d\nu_\alpha)$; $p \in [1, 2]$, the function $\mathcal{F}_\alpha(f)$ lies in $L^{p'}(d\gamma_\alpha)$; $p' = \frac{p}{p-1}$, and we have

$$\|\mathcal{F}_\alpha(f)\|_{p', \gamma_\alpha} \leq \|f\|_{p, \nu_\alpha}. \tag{2.27}$$

We denote by $\mathcal{S}_e(\mathbb{R}^2)$ the space of infinitely differentiable functions on \mathbb{R}^2 , rapidly decreasing together with all their derivatives, even with respect to the first variable.

The space $\mathcal{S}_e(\mathbb{R}^2)$ is endowed with the topology generated by the family of norms

$$\rho_m(\varphi) = \sup_{\substack{(r,x) \in [0, +\infty[\times \mathbb{R} \\ k+|\beta| \leq m}} (1+r^2+x^2)^k |D^\beta(\varphi)(r,x)|. \tag{2.28}$$

$\mathcal{D}_e(\mathbb{R}^2)$ the subspace of $\mathcal{S}_e(\mathbb{R}^2)$ formed by the functions with compact support.

From [33, 34], the transform $\widetilde{\mathcal{F}}_\alpha$ given by the relation (2.23) is a topological isomorphism from $\mathcal{S}_e(\mathbb{R}^2)$ onto itself and we have

$$\widetilde{\mathcal{F}}_\alpha^{-1}(f)(r,x) = \int_0^\infty \int_{\mathbb{R}} f(\lambda_0, \lambda) j_\alpha(r\lambda_0) e^{i\lambda x} d\nu_\alpha(\lambda_0, \lambda) = \widetilde{\mathcal{F}}_\alpha(\check{f})(r,x).$$

3. Main results

In this section we shall prove the dispersion principle and one multiplicative form related to the Riemann-Liouville operator. For this, we need some intermediate results.

DEFINITION 3.1. Let p be a positive real number.

- i. For every measurable function f on $[0, +\infty[\times \mathbb{R}$, the p -dispersion of f with respect to the measure $d\nu_\alpha$ is defined by

$$\rho_{p, \nu_\alpha}(f) = \left(\int_0^\infty \int_{\mathbb{R}} |(r,x)|^p |f(r,x)|^2 d\nu_\alpha(r,x) \right)^{\frac{1}{p}}.$$

- ii. For every measurable function g on Υ_+ , the p -dispersion of g with respect to the measure $d\gamma_\alpha$ is defined by

$$\rho_{p, \gamma_\alpha}(g) = \left(\int \int_{\Upsilon_+} |\theta(\lambda_0, \lambda)|^p |g(\lambda_0, \lambda)|^2 d\gamma_\alpha(\lambda_0, \lambda) \right)^{\frac{1}{p}}.$$

DEFINITION 3.2. Let ε be a positive real number and let f be a square integrable function on $[0, +\infty[\times \mathbb{R}$ with respect to the measure $d\nu_\alpha$.

- i. We say that f is ε -concentrated in the ball $B_\rho^+ = \{(r,x) \in [0, +\infty[\times \mathbb{R}; r^2 + x^2 \leq \rho^2\}$ if

$$\left(\int \int_{(B_\rho^+)^c} |f(r,x)|^2 d\nu_\alpha(r,x) \right)^{\frac{1}{2}} \leq \varepsilon \|f\|_{2, \nu_\alpha}.$$

- ii. We say that f is ε -bandlimited in the ball $\tilde{B}_\rho^+ = \{(\lambda_0, \lambda) \in \Upsilon_+; |\theta(\lambda_0, \lambda)|^2 = \lambda_0^2 + 2\lambda^2 \leq \rho^2\}$ if

$$\left(\int \int_{(\tilde{B}_\rho^+)^c} |\mathcal{F}_\alpha(f)(\lambda_0, \lambda)|^2 d\gamma_\alpha(\lambda_0, \lambda) \right)^{\frac{1}{2}} \leq \varepsilon \|f\|_{2, v_\alpha}.$$

Let S be a measurable subset of $[0, +\infty[\times \mathbb{R}$ and let Σ be a measurable subset of Υ_+ such that

$$v_\alpha(S) < +\infty \text{ and } \gamma_\alpha(\Sigma) < +\infty.$$

We denote by P_S and P_Σ the bounded self adjoint operators defined on $L^2(dv_\alpha)$ respectively by $P_S(f) = \mathbf{1}_S \cdot f$ and $P_\Sigma(f) = \mathcal{F}_\alpha^{-1}(\mathbf{1}_\Sigma \mathcal{F}_\alpha(f))$.

We have the following interesting result

THEOREM 3.3. *The operators $P_S P_\Sigma$ and $P_\Sigma P_S$ are Hilbert-Schmidt operators such that*

$$\|P_S P_\Sigma\|_{HS} \leq \sqrt{v_\alpha(S) \gamma_\alpha(\Sigma)} \quad \text{and} \quad \|P_\Sigma P_S\|_{HS} \leq \sqrt{v_\alpha(S) \gamma_\alpha(\Sigma)},$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm.

Proof. Since $v_\alpha(S) < +\infty$ and $\gamma_\alpha(\Sigma) < +\infty$, then for every $f \in L^2(dv_\alpha)$, $\mathbf{1}_S \cdot f$ belongs to $L^1(dv_\alpha) \cap L^2(dv_\alpha)$ and for every $g \in L^2(d\gamma_\alpha)$, $\mathbf{1}_\Sigma \cdot g$ belongs to $L^1(d\gamma_\alpha) \cap L^2(d\gamma_\alpha)$. Consequently, for every $f \in L^2(dv_\alpha)$,

$$\begin{aligned} P_\Sigma P_S(f)(r, x) &= \mathcal{F}_\alpha^{-1}(\mathbf{1}_\Sigma \mathcal{F}_\alpha(\mathbf{1}_S \cdot f))(r, x) \\ &= \int \int_{\Upsilon_+} \mathbf{1}_\Sigma(\lambda_0, \lambda) \mathcal{F}_\alpha(\mathbf{1}_S \cdot f)(\lambda_0, \lambda) \overline{\varphi_{\lambda_0, \lambda}(r, x)} d\gamma_\alpha(\lambda_0, \lambda) \\ &= \int \int_{\Upsilon_+} \mathbf{1}_\Sigma(\lambda_0, \lambda) \overline{\varphi_{\lambda_0, \lambda}(r, x)} \\ &\quad \times \left(\int_0^\infty \int_{\mathbb{R}} \mathbf{1}_S(t, y) f(t, y) \varphi_{\lambda_0, \lambda}(t, y) dv_\alpha(t, y) \right) d\gamma_\alpha(\lambda_0, \lambda). \end{aligned}$$

Applying Fubini's theorem, we get

$$\begin{aligned} P_\Sigma P_S(f)(r, x) &= \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_S(t, y) f(t, y) \\ &\quad \times \left(\int \int_{\Upsilon_+} \mathbf{1}_\Sigma(\lambda_0, \lambda) \varphi_{\lambda_0, \lambda}(t, y) \overline{\varphi_{\lambda_0, \lambda}(r, x)} d\gamma_\alpha(\lambda_0, \lambda) \right) dv_\alpha(t, y) \\ &= \int_0^\infty \int_{\mathbb{R}} f(t, y) K((r, x), (t, y)) dv_\alpha(t, y), \end{aligned} \tag{3.1}$$

where K is the kernel given by

$$\begin{aligned} K((r, x), (t, y)) &= \mathbf{1}_S(t, y) \left(\int \int_{\Upsilon_+} \mathbf{1}_\Sigma(\lambda_0, \lambda) \varphi_{\lambda_0, \lambda}(t, y) \overline{\varphi_{\lambda_0, \lambda}(r, x)} d\gamma_\alpha(\lambda_0, \lambda) \right) \\ &= \mathbf{1}_S(t, y) \mathcal{F}_\alpha^{-1}(\mathbf{1}_\Sigma \varphi_{\cdot, \cdot}(t, y))(r, x). \end{aligned}$$

Using the Plancherel theorem for \mathcal{F}_α , Fubini's theorem and the relation (2.4), we get

$$\begin{aligned} \|K\|_{2, \nu_\alpha \otimes \nu_\alpha}^2 &= \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_S(t, y) \left(\int \int_{\Gamma_+} \mathbf{1}_\Sigma(\lambda_0, \lambda) |\varphi_{\lambda_0, \lambda}(t, y)|^2 d\gamma_\alpha(\lambda_0, \lambda) \right) d\nu_\alpha(t, y) \\ &\leq \nu_\alpha(S) \gamma_\alpha(\Sigma). \end{aligned} \quad (3.2)$$

The relations (3.1) and (3.2) show that $P_\Sigma P_S$ is an Hilbert Schmidt operator and that

$$\|P_\Sigma P_S\|_{HS} = \|K\|_{2, \nu_\alpha \otimes \nu_\alpha} \leq \sqrt{\nu_\alpha(S) \gamma_\alpha(\Sigma)}. \quad (3.3)$$

As the same way, for every $f \in L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$ and for every $(r, x) \in [0, +\infty[\times \mathbb{R}$,

$$\begin{aligned} P_S P_\Sigma(f)(r, x) &= \mathbf{1}_S(r, x) \mathcal{F}_\alpha^{-1}(\mathbf{1}_\Sigma \mathcal{F}_\alpha(f))(r, x) \\ &= \mathbf{1}_S(r, x) \int \int_{\Gamma_+} \mathbf{1}_\Sigma(\lambda_0, \lambda) \mathcal{F}_\alpha(f)(\lambda_0, \lambda) \overline{\varphi_{\lambda_0, \lambda}(r, x)} d\gamma_\alpha(\lambda_0, \lambda) \\ &= \mathbf{1}_S(r, x) \int \int_{\Gamma_+} \mathbf{1}_\Sigma(\lambda_0, \lambda) \left(\int_0^\infty \int_{\mathbb{R}} f(s, y) \varphi_{\lambda_0, \lambda}(s, y) d\nu_\alpha(s, y) \right) \\ &\quad \times \overline{\varphi_{\lambda_0, \lambda}(r, x)} d\gamma_\alpha(\lambda_0, \lambda). \end{aligned}$$

Applying Fubini's theorem, we have

$$\begin{aligned} P_S P_\Sigma(f)(r, x) &= \mathbf{1}_S(r, x) \int_0^\infty \int_{\mathbb{R}} f(s, y) \left(\int \int_{\Gamma_+} \mathbf{1}_\Sigma(\lambda_0, \lambda) \varphi_{\lambda_0, \lambda}(r, -x) \right. \\ &\quad \left. \times \overline{\varphi_{\lambda_0, \lambda}(s, -y)} d\gamma_\alpha(\lambda_0, \lambda) \right) d\nu_\alpha(s, y) \\ &= \int_0^\infty \int_{\mathbb{R}} f(s, y) H((r, x), (s, y)) d\nu_\alpha(s, y), \end{aligned}$$

where H is the kernel given by

$$H((r, x), (s, y)) = \mathbf{1}_S(r, x) \mathcal{F}_\alpha^{-1}(\mathbf{1}_\Sigma \varphi_{\cdot, \cdot}(r, -x))(s, -y).$$

Applying again Fubini-Tonelli theorem and the Plancherel theorem for \mathcal{F}_α , we get

$$\begin{aligned} &\int_{([0, +\infty \times \mathbb{R}]^2)} |H((r, x), (s, y))|^2 d\nu_\alpha(r, x) d\nu_\alpha(s, y) \\ &= \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_S(r, x) \left(\int_0^\infty \int_{\mathbb{R}} \left| \mathcal{F}_\alpha^{-1}(\mathbf{1}_\Sigma \varphi_{\cdot, \cdot}(r, -x))(s, y) \right|^2 d\nu_\alpha(s, y) \right) d\nu_\alpha(r, x) \\ &= \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_S(r, x) \left(\int \int_{\Gamma_+} |\mathbf{1}_\Sigma(\lambda_0, \lambda) \varphi_{\lambda_0, \lambda}(r, -x)|^2 d\gamma_\alpha(\lambda_0, \lambda) \right) d\nu_\alpha(r, x) \\ &\leq \nu_\alpha(S) \gamma_\alpha(\Sigma). \end{aligned} \quad (3.4)$$

The last inequality shows that the operator $P_S P_\Sigma$ is an Hilbert-Schmidt operator and that

$$\|P_S P_\Sigma\|_{HS} = \|H\|_{2, \nu_\alpha \otimes \nu_\alpha} \leq \sqrt{\nu_\alpha(S) \gamma_\alpha(\Sigma)}. \quad \square$$

THEOREM 3.4. (Time frequency localization) *Let $S \subset [0, +\infty[\times \mathbb{R}$; $\Sigma \subset \Upsilon_+$ such that $v_\alpha(S) < +\infty$ and $\gamma_\alpha(\Sigma) < +\infty$. Let \mathcal{X} be a finite subset of \mathbb{N}^2 and let $(\varphi_{m,n})_{(m,n) \in \mathcal{X}}$ be an orthonormal sequence in $L^2(dv_\alpha)$. Then*

$$\sum_{(m,n) \in \mathcal{X}} \left(1 - \frac{3}{2}a_{m,n}(S) - \frac{3}{2}b_{m,n}(\Sigma)\right) \leq v_\alpha(S)\gamma_\alpha(\Sigma), \quad (3.5)$$

where

$$a_{m,n}(S) = \|I_{S^c} \varphi_{m,n}\|_{2, v_\alpha} \text{ and } b_{m,n}(\Sigma) = \|I_{\Sigma^c} \mathcal{F}_\alpha(\varphi_{m,n})\|_{2, \gamma_\alpha}. \quad (3.6)$$

Proof. For every $(m, n) \in \mathbb{N}^2$; we put

$$e_{m,n}^\alpha(r, x) = \left(\frac{2^{\alpha+1} m! \Gamma(\alpha+1)}{2^{n-\frac{1}{2}} n! \Gamma(m+\alpha+1)} \right)^{\frac{1}{2}} e^{-\frac{r^2+x^2}{2}} L_m^\alpha(r^2) H_n(x)$$

where $(L_m^\alpha)_{m \in \mathbb{N}}$ are the Laguerre polynomials and $(H_n)_{n \in \mathbb{N}}$ are the Hermite polynomials. Then, $(e_{m,n}^\alpha)_{(m,n) \in \mathbb{N}^2}$ is an Hilbert basis of $L^2(dv_\alpha)$ [27]. Moreover, the family $(\mathcal{E}_{m,n}^\alpha)_{(m,n) \in \mathbb{N}^2}$ defined by

$$\mathcal{E}_{m,n}^\alpha(\lambda_0, \lambda) = e_{m,n}^\alpha \circ \theta(\lambda_0, \lambda)$$

is an Hilbert basis of $L^2(d\gamma_\alpha)$ such that

$$\mathcal{F}_\alpha(e_{m,n}^\alpha) = (-i)^{2m+n} \mathcal{E}_{m,n}^\alpha. \quad (3.7)$$

On the other hand, for every bounded operator T on $L^2(dv_\alpha)$, we denote by T^* the adjoint operator of T defined by

$$\langle T(f) | g \rangle_{v_\alpha} = \langle f | T^*(g) \rangle_{v_\alpha}; \quad f, g \in L^2(dv_\alpha).$$

Then, the operators P_S and P_Σ are self adjoint and satisfy $P_S^2 = P_S$, $P_\Sigma^2 = P_\Sigma$. Let ϕ be the self adjoint operator defined by $\phi = (P_\Sigma P_S)^*(P_\Sigma P_S)$, then ϕ can be written $\phi = P_S P_\Sigma P_S$ and ϕ is an operator with trace such that

$$\begin{aligned} tr(\phi) &= \sum_{(m,n) \in \mathbb{N}^2} \|P_\Sigma P_S(e_{m,n}^\alpha)\|_{2, v_\alpha}^2 \\ &= \sum_{(m,n) \in \mathbb{N}^2} \langle \phi(e_{m,n}^\alpha) | e_{m,n}^\alpha \rangle_{v_\alpha} \\ &= \|P_\Sigma P_S\|_{HS}^2. \end{aligned}$$

Applying Theorem 3.3, we get

$$\sum_{(m,n) \in \mathbb{N}^2} \langle \phi(e_{m,n}^\alpha) | e_{m,n}^\alpha \rangle_{v_\alpha} \leq v_\alpha(S)\gamma_\alpha(\Sigma).$$

Now, let $(\varphi_{m,n})_{(m,n) \in \mathcal{X}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$, then $(\varphi_{m,n})_{(m,n) \in \mathcal{X}}$ can be completed to an Hilbert basis of $L^2(d\nu_\alpha)$ denoted by $(\varphi_{m,n})_{(m,n) \in \mathbb{N}^2}$. So,

$$\begin{aligned} \sum_{(m,n) \in \mathbb{N}^2} \|P_\Sigma P_S(\varphi_{m,n})\|_{2,\nu_\alpha}^2 &= \sum_{(m,n) \in \mathbb{N}^2} \|P_\Sigma P_S(e_{m,n}^\alpha)\|_{2,\nu_\alpha}^2 \\ &= \text{tr}(\phi) \leq \nu_\alpha(S) \gamma_\alpha(\Sigma). \end{aligned}$$

In particular,

$$\begin{aligned} \sum_{(m,n) \in \mathcal{X}} \|P_\Sigma P_S(\varphi_{m,n})\|_{2,\nu_\alpha}^2 &= \sum_{(m,n) \in \mathbb{N}^2} \langle \phi(\varphi_{m,n}) | \varphi_{m,n} \rangle_{\nu_\alpha} \\ &\leq \nu_\alpha(S) \gamma_\alpha(\Sigma). \end{aligned} \quad (3.8)$$

On the other hand, for every $(m,n) \in \mathbb{N}^2$,

$$\begin{aligned} \langle \phi(\varphi_{m,n}) | \varphi_{m,n} \rangle_{\nu_\alpha} &= \langle P_S P_\Sigma P_S(\varphi_{m,n}) | \varphi_{m,n} \rangle_{\nu_\alpha} = \langle P_\Sigma P_S(\varphi_{m,n}) | P_S(\varphi_{m,n}) \rangle_{\nu_\alpha} \\ &= \langle P_\Sigma P_S(\varphi_{m,n}) | \varphi_{m,n} \rangle_{\nu_\alpha} - \langle P_\Sigma P_S(\varphi_{m,n}) | \varphi_{m,n} - P_S(\varphi_{m,n}) \rangle_{\nu_\alpha}, \end{aligned}$$

but,

$$\begin{aligned} \langle P_\Sigma P_S(\varphi_{m,n}) | \varphi_{m,n} \rangle_{\nu_\alpha} &= \langle P_S(\varphi_{m,n}) | P_\Sigma(\varphi_{m,n}) \rangle_{\nu_\alpha} \\ &= \langle P_S(\varphi_{m,n}) | \varphi_{m,n} \rangle_{\nu_\alpha} - \langle P_S(\varphi_{m,n}) | \varphi_{m,n} - P_\Sigma(\varphi_{m,n}) \rangle_{\nu_\alpha} \\ &= 1 - \langle \varphi_{m,n} - P_S(\varphi_{m,n}) | \varphi_{m,n} \rangle_{\nu_\alpha} - \langle P_S(\varphi_{m,n}) | \varphi_{m,n} - P_\Sigma(\varphi_{m,n}) \rangle_{\nu_\alpha}. \end{aligned}$$

The relations (3.9) and (3.9) imply that

$$\begin{aligned} \langle \phi(\varphi_{m,n}) | \varphi_{m,n} \rangle_{\nu_\alpha} &= 1 - \langle \varphi_{m,n} - P_S(\varphi_{m,n}) | \varphi_{m,n} \rangle_{\nu_\alpha} - \langle P_S(\varphi_{m,n}) | \varphi_{m,n} - P_\Sigma(\varphi_{m,n}) \rangle_{\nu_\alpha} \\ &\quad - \langle P_\Sigma P_S(\varphi_{m,n}) | \varphi_{m,n} - P_S(\varphi_{m,n}) \rangle_{\nu_\alpha}. \end{aligned}$$

Thus,

$$\begin{aligned} \langle \phi(\varphi_{m,n}) | \varphi_{m,n} \rangle_{\nu_\alpha} &= \left| \langle \phi(\varphi_{m,n}) | \varphi_{m,n} \rangle_{\nu_\alpha} \right| \\ &\geq 1 - \left| \langle \varphi_{m,n} - P_S(\varphi_{m,n}) | \varphi_{m,n} \rangle_{\nu_\alpha} \right| - \left| \langle P_S(\varphi_{m,n}) | \varphi_{m,n} - P_\Sigma(\varphi_{m,n}) \rangle_{\nu_\alpha} \right| \\ &\quad - \left| \langle P_\Sigma P_S(\varphi_{m,n}) | \varphi_{m,n} - P_S(\varphi_{m,n}) \rangle_{\nu_\alpha} \right|. \end{aligned} \quad (3.9)$$

However, for every $(m,n) \in \mathbb{N}^2$,

$$\begin{aligned} a_{m,n}(S) &= \|\mathbf{1}_{S^c} \varphi_{m,n}\|_{2,\nu_\alpha} = \|\varphi_{m,n} - P_S \varphi_{m,n}\|_{2,\nu_\alpha} \\ &\geq \left| \langle \varphi_{m,n} - P_S \varphi_{m,n} | \varphi_{m,n} \rangle_{\nu_\alpha} \right|, \end{aligned} \quad (3.10)$$

$$\begin{aligned} \left| \langle P_\Sigma P_S \varphi_{m,n} | \varphi_{m,n} - P_S \varphi_{m,n} \rangle_{\nu_\alpha} \right| &\leq \|P_\Sigma P_S\| \|\varphi_{m,n} - P_S \varphi_{m,n}\|_{2,\nu_\alpha} \\ &\leq \|\varphi_{m,n} - P_S \varphi_{m,n}\|_{2,\nu_\alpha} = a_{m,n}(S), \end{aligned} \quad (3.11)$$

$$\begin{aligned} \left| \langle P_S \varphi_{m,n} | \varphi_{m,n} - P_\Sigma \varphi_{m,n} \rangle_{v_\alpha} \right| &\leq \|P_S\| \| \varphi_{m,n} - P_\Sigma \varphi_{m,n} \|_{2, v_\alpha} \\ &\leq \| \mathcal{F}_\alpha(\varphi_{m,n} - P_\Sigma \varphi_{m,n}) \|_{2, \gamma_\alpha} = b_{m,n}(\Sigma). \end{aligned} \quad (3.12)$$

Combining the relations (3.9), (3.10), (3.11) and (3.12), we get

$$\langle \phi(\varphi_{m,n}) | \varphi_{m,n} \rangle_{v_\alpha} \geq 1 - 2a_{m,n}(S) - b_{m,n}(\Sigma). \quad (3.13)$$

Similarly, let R be the self adjoint operator defined by

$$R = (P_S P_\Sigma)^* (P_S P_\Sigma) = P_\Sigma P_S P_\Sigma,$$

by Theorem 3.3, we have

$$\begin{aligned} tr(R) &= \sum_{(m,n) \in \mathbb{N}^2} \|P_S P_\Sigma(\varphi_{m,n})\|_{2, v_\alpha}^2 \\ &= \sum_{(m,n) \in \mathbb{N}^2} \langle R(\varphi_{m,n}) | \varphi_{m,n} \rangle_{v_\alpha} \\ &= \|P_S P_\Sigma\|_{HS}^2 \leq v_\alpha(S) \gamma_\alpha(\Sigma). \end{aligned} \quad (3.14)$$

As the same way, for every $(m,n) \in \mathbb{N}^2$, we have

$$\langle R(\varphi_{m,n}) | \varphi_{m,n} \rangle_{v_\alpha} \geq 1 - a_{m,n}(S) - 2b_{m,n}(\Sigma). \quad (3.15)$$

Using the relations (3.8), (3.13), (3.14) and (3.15), we deduce that

$$\begin{aligned} &\sum_{(m,n) \in \mathcal{X}} (2 - 3a_{m,n}(S) - 3b_{m,n}(\Sigma)) \\ &\leq \sum_{(m,n) \in \mathcal{X}} \langle \phi(\varphi_{m,n}) | \varphi_{m,n} \rangle_{v_\alpha} + \sum_{(m,n) \in \mathcal{X}} \langle R(\varphi_{m,n}) | \varphi_{m,n} \rangle_{v_\alpha} \\ &\leq \sum_{(m,n) \in \mathbb{N}^2} \langle \phi(\varphi_{m,n}) | \varphi_{m,n} \rangle_{v_\alpha} + \sum_{(m,n) \in \mathbb{N}^2} \langle R(\varphi_{m,n}) | \varphi_{m,n} \rangle_{v_\alpha} \\ &\leq 2 v_\alpha(S) \gamma_\alpha(\Sigma). \end{aligned} \quad (3.16)$$

The proof of theorem is complete. \square

COROLLARY 3.5. *Let ε , ρ and η be positive real numbers such that $0 < \varepsilon < \frac{1}{3}$. Let $\mathcal{X} \subset \mathbb{N}^2$ be a nonempty subset and let $(\varphi_{m,n})_{(m,n) \in \mathcal{X}}$ be an orthonormal sequence in $L^2(dv_\alpha)$. If for every $(m,n) \in \mathcal{X}$, $\varphi_{m,n}$ is ε -concentrated in the ball B_ρ^+ and $\varphi_{m,n}$ is ε -bandlimited in the ball \widehat{B}_η^+ , then the subset \mathcal{X} is finite and*

$$card(\mathcal{X}) \leq \frac{\rho^{2\alpha+3} \eta^{2\alpha+3}}{(1-3\varepsilon)^{2^{2\alpha+3}} (\Gamma(\alpha + \frac{5}{2}))^2}. \quad (3.17)$$

Proof. Let \mathcal{X}_1 be a finite subset of \mathcal{X} . From the hypothesis, for every $(m,n) \in \mathcal{X}_1$,

$$a_{m,n}(B_\rho^+) = \left(\int \int_{(B_\rho^+)^c} |\varphi_{m,n}(r,x)|^2 dv_\alpha(r,x) \right)^{\frac{1}{2}} \leq \varepsilon$$

and

$$b_{m,n}(\tilde{B}_\eta^+) = \left(\int \int_{(\tilde{B}_\eta^+)^c} |\mathcal{F}_\alpha(\varphi_{m,n})(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) \right)^{\frac{1}{2}} \leq \varepsilon,$$

and consequently, for every $(m, n) \in \mathcal{K}_1$,

$$1 - \frac{3}{2}a_{m,n}(B_\rho^+) - \frac{3}{2}b_{m,n}(\tilde{B}_\eta^+) \geq 1 - 3\varepsilon.$$

According to Theorem 3.4, it follows that

$$\begin{aligned} (1 - 3\varepsilon)\text{card}(\mathcal{K}_1) &\leq \sum_{(m,n) \in \mathcal{K}_1} \left(1 - \frac{3}{2}a_{m,n}(B_\rho^+) - \frac{3}{2}b_{m,n}(\tilde{B}_\eta^+) \right) \\ &\leq v_\alpha(B_\rho^+) \gamma_\alpha(\tilde{B}_\eta^+). \end{aligned}$$

However, we have

$$v_\alpha(B_\rho^+) = \frac{\rho^{2\alpha+3}}{2^{\alpha+\frac{3}{2}} \Gamma(\alpha + \frac{5}{2})} \text{ and } \gamma_\alpha(\tilde{B}_\eta^+) = v_\alpha(\theta(\tilde{B}_\eta^+)) = v_\alpha(B_\eta^+) = \frac{\eta^{2\alpha+3}}{2^{\alpha+\frac{3}{2}} \Gamma(\alpha + \frac{5}{2})}.$$

This involves that for every finite subset \mathcal{K}_1 of \mathcal{K} , we have

$$\text{card}(\mathcal{K}_1) \leq \frac{\rho^{2\alpha+3} \eta^{2\alpha+3}}{(1 - 3\varepsilon) 2^{2\alpha+3} (\Gamma(\alpha + \frac{5}{2}))^2}.$$

Consequently, \mathcal{K} is a finite subset and

$$\text{card}(\mathcal{K}) \leq \frac{\rho^{2\alpha+3} \eta^{2\alpha+3}}{(1 - 3\varepsilon) 2^{2\alpha+3} (\Gamma(\alpha + \frac{5}{2}))^2}. \quad \square$$

COROLLARY 3.6. *Let a, p be positive real numbers. Let \mathcal{K} be a nonempty subset of \mathbb{N}^2 and let $(\varphi_{m,n})_{(m,n) \in \mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. Assume that for every $(m, n) \in \mathcal{K}$,*

$$\rho_{p, v_\alpha}(\varphi_{m,n}) \leq a \text{ and } \rho_{p, \gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})) \leq a.$$

Then \mathcal{K} is a finite subset and

$$\text{card}(\mathcal{K}) \leq \frac{2^{\frac{8}{p}(2\alpha+3)-2\alpha-1} a^{2(2\alpha+3)}}{(\Gamma(\alpha + \frac{5}{2}))^2}.$$

Proof. Let ρ, η be positive real numbers, from the hypothesis, we deduce that for every $(m, n) \in \mathcal{K}$,

$$\int \int_{(B_\rho^+)^c} |\varphi_{m,n}(r, x)|^2 d\nu_\alpha(r, x) \leq \frac{1}{\rho^p} \int_0^\infty \int_{\mathbb{R}} |\varphi_{m,n}(r, x)|^2 |(r, x)|^p d\nu_\alpha(r, x) \leq \left(\frac{a}{\rho}\right)^p$$

and

$$\begin{aligned} & \int \int_{(\tilde{B}_\eta)^c} |\mathcal{F}_\alpha(\varphi_{m,n})(\lambda_0, \lambda)|^2 d\gamma_\alpha(\lambda_0, \lambda) \\ & \leq \frac{1}{\eta^p} \int \int_{\Upsilon_+} |\theta(\lambda_0, \lambda)|^p |\mathcal{F}_\alpha(\varphi_{m,n})(\lambda_0, \lambda)|^2 d\gamma_\alpha(\lambda_0, \lambda) \\ & \leq \left(\frac{a}{\eta}\right)^p. \end{aligned}$$

In particular, if we pick $\rho = \eta = a 2^{\frac{4}{p}}$, we deduce that for every $(m, n) \in \mathcal{K}$; $\varphi_{m,n}$ is $\frac{1}{4}$ -concentrated in the ball $B_{a 2^{\frac{4}{p}}}^+$ and $\frac{1}{4}$ -bandlimited in the ball $\tilde{B}_{a 2^{\frac{4}{p}}}^+$.

Applying Corollary 3.5, it follows that \mathcal{K} is a finite subset of \mathbb{N}^2 and that

$$\text{card}(\mathcal{K}) \leq \frac{(a 2^{\frac{4}{p}})^{2\alpha+3}}{(1 - \frac{3}{4})^{2^{2\alpha+3}}} \frac{(a 2^{\frac{4}{p}})^{2\alpha+3}}{(\Gamma(\alpha + \frac{5}{2}))^2} = \frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)} a^{2(2\alpha+3)}}{(\Gamma(\alpha + \frac{5}{2}))^2}. \quad \square$$

LEMMA 3.7. *Let $p > 0$ and let $(\varphi_{m,n})_{(m,n) \in \mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. Then, there exists $j_0 \in \mathbb{Z}$ such that*

$$\forall (m, n) \in \mathbb{N}^2, \max \{ \rho_{p, \nu_\alpha}(\varphi_{m,n}), \rho_{p, \gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})) \} \geq 2^{j_0}.$$

Proof. For every $j \in \mathbb{Z}$, let

$$P_j = \left\{ (m, n) \in \mathbb{N}^2; 2^j \leq \max \{ \rho_{p, \nu_\alpha}(\varphi_{m,n}), \rho_{p, \gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})) \} < 2^{j+1} \right\}.$$

Then, $\mathbb{N}^2 = \bigcup_{j \in \mathbb{Z}} P_j$, $P_{j_1} \cap P_{j_2} = \emptyset$ if $j_1 \neq j_2$ and for every $(m, n) \in P_j$,

$$\rho_{p, \nu_\alpha}(\varphi_{m,n}) < 2^{j+1} \text{ and } \rho_{p, \gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})) < 2^{j+1}.$$

Applying Corollary 3.6, we deduce that P_j is finite and

$$\text{card}(P_j) \leq \frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{(\Gamma(\alpha + \frac{5}{2}))^2} (2^{j+1})^{4\alpha+6}. \quad (3.18)$$

Thus, for j negative and $|j|$ sufficiently large, we get $\text{card}(P_j) = 0$ or $P_j = \emptyset$. This means that there exists $j_0 \in \mathbb{Z}$ such that $\forall j < j_0$, $P_j = \emptyset$. So,

$$\mathbb{N}^2 = \bigcup_{j \in \mathbb{Z}} P_j = \bigcup_{j=j_0}^{+\infty} P_j.$$

This implies that

$$\forall (m, n) \in \mathbb{N}^2, \max \{ \rho_{p, \nu_\alpha}(\varphi_{m,n}), \rho_{p, \gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})) \} \geq 2^{j_0}. \quad \square$$

THEOREM 3.8. (Quantitative version of the mean-dispersion Shapiro's theorem) *Let $(\varphi_{m,n})_{(m,n) \in \mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$, then for every positive real number p and for every nonempty finite subset $\mathcal{K} \subset \mathbb{N}^2$, we have*

$$\begin{aligned} & \sum_{(m,n) \in \mathcal{K}} \left((\rho_{p,\nu_\alpha}(\varphi_{m,n}))^p + (\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))^p \right) \\ & \geq (\text{card}(\mathcal{K}))^{1+\frac{p}{4\alpha+6}} \left(\frac{\Gamma^2(\alpha + \frac{5}{2})(2^{4\alpha+6} - 1)}{2^{(2\alpha+3)(\frac{10}{p}+3)+3}} \right)^{\frac{p}{4\alpha+6}}. \end{aligned} \quad (3.19)$$

Proof. Let j_0 be defined in Lemma 3.7. Then for every $(m,n) \in \mathbb{N}^2$,

$$\max \{ \rho_{p,\nu_\alpha}(\varphi_{m,n}), \rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})) \} \geq 2^{j_0}.$$

For every $k \geq j_0$, we put

$$Q_k = \bigcup_{j=j_0}^k P_j$$

From the relation (3.18),

$$\begin{aligned} \text{card}(Q_k) &= \sum_{j=j_0}^k \text{card}(P_j) \\ &\leq \frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2} \sum_{j=j_0}^k (2^{4\alpha+6})^{j+1} \\ &= \frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2} 2^{(4\alpha+6)(j_0+1)} \frac{2^{(4\alpha+6)(k-j_0+1)} - 1}{2^{4\alpha+6} - 1} \\ &\leq \frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)} 2^{(4\alpha+6)(k+2)}. \end{aligned} \quad (3.20)$$

i) If $\text{card}(\mathcal{K}) > 2 \frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)} 2^{(4\alpha+6)(j_0+2)}$; let $k > j_0$ such that

$$2 \frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)} 2^{(4\alpha+6)(k+1)}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)} \leq \text{card}(\mathcal{K}) < 2 \frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)} 2^{(4\alpha+6)(k+2)}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)}. \quad (3.21)$$

From the relations (3.20) and (3.21), we have

$$\text{card}(Q_{k-1}) \leq \frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)} 2^{(4\alpha+6)(k+1)}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)} \leq \frac{\text{card}(\mathcal{K})}{2}. \quad (3.22)$$

On the other hand,

$$\begin{aligned}
& \sum_{(m,n) \in \mathcal{K}} \left((\rho_{p,v_\alpha}(\varphi_{m,n}))^p + (\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))^p \right) \\
&= \sum_{(m,n) \in \mathcal{K} \cap Q_{k-1}} \left((\rho_{p,v_\alpha}(\varphi_{m,n}))^p + (\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))^p \right) \\
&\quad + \sum_{(m,n) \in \mathcal{K} \setminus Q_{k-1}} \left((\rho_{p,v_\alpha}(\varphi_{m,n}))^p + (\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))^p \right) \\
&\geq \sum_{(m,n) \in \mathcal{K} \setminus Q_{k-1}} \left((\rho_{p,v_\alpha}(\varphi_{m,n}))^p + (\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))^p \right).
\end{aligned}$$

But, for every $(m,n) \in \mathcal{K} \setminus Q_{k-1}$,

$$\begin{aligned}
(\rho_{p,v_\alpha}(\varphi_{m,n}))^p + (\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))^p &\geq \left(\max \{ \rho_{p,v_\alpha}(\varphi_{m,n}), \rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})) \} \right)^p \\
&\geq 2^{kp}
\end{aligned}$$

So,

$$\sum_{(m,n) \in \mathcal{K}} \left((\rho_{p,v_\alpha}(\varphi_{m,n}))^p + (\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))^p \right) \geq 2^{kp} \text{card}(\mathcal{K} \setminus Q_{k-1}).$$

Then, from the relation (3.22), we deduce that

$$\sum_{(m,n) \in \mathcal{K}} \left((\rho_{p,v_\alpha}(\varphi_{m,n}))^p + (\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))^p \right) \geq 2^{kp} \frac{\text{card}(\mathcal{K})}{2}. \quad (3.23)$$

Now, from the relation (3.21), we have

$$\begin{aligned}
\frac{\text{card}(\mathcal{K}) 2^{kp-1}}{(\text{card}(\mathcal{K}))^{1+\frac{p}{4\alpha+6}}} &= 2^{kp-1} (\text{card}(\mathcal{K}))^{-\frac{p}{4\alpha+6}} \\
&> 2^{kp-1} \left(2 \frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)} 2^{(4\alpha+6)(k+2)} \right)^{-\frac{p}{4\alpha+6}} \\
&= \left(\frac{2^{\frac{(1-kp)(4\alpha+6)}{p} + 1 + \frac{8}{p}(2\alpha+3)-(2\alpha+1)+(k+2)(4\alpha+6)}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)} \right)^{-\frac{p}{4\alpha+6}} \\
&= \left(\frac{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)}{2^{(2\alpha+3)(3+\frac{10}{p})+3}} \right)^{\frac{p}{4\alpha+6}}
\end{aligned}$$

which means that

$$2^{kp-1} \text{card}(\mathcal{K}) > (\text{card}(\mathcal{K}))^{1+\frac{p}{4\alpha+6}} \left(\frac{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)}{2^{(2\alpha+3)(3+\frac{10}{p})+3}} \right)^{\frac{p}{4\alpha+6}}. \quad (3.24)$$

Combining the relations (3.23) and (3.24), we get

$$\begin{aligned} & \sum_{(m,n) \in \mathcal{K}} \left((\rho_{p,\nu_\alpha}(\varphi_{m,n}))^p + (\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))^p \right) \\ & \geq (\text{card}(\mathcal{K}))^{1+\frac{p}{4\alpha+6}} \left(\frac{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)}{2^{(2\alpha+3)(3+\frac{10}{p})+3}} \right)^{\frac{p}{4\alpha+6}}. \end{aligned}$$

ii) If $\text{card}(\mathcal{K}) \leq 2 \frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)} 2^{(4\alpha+6)(j_0+2)}$. By Lemma 3.7, we

have

$$\begin{aligned} & \sum_{(m,n) \in \mathcal{K}} \left((\rho_{p,\nu_\alpha}(\varphi_{m,n}))^p + (\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))^p \right) \\ & \geq \sum_{(m,n) \in \mathcal{K}} \left(\max \{ \rho_{p,\nu_\alpha}(\varphi_{m,n}), \rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})) \} \right)^p \\ & \geq \text{card}(\mathcal{K}) 2^{j_0 p}. \end{aligned}$$

As the same way,

$$\begin{aligned} \frac{\text{card}(\mathcal{K}) 2^{j_0 p}}{(\text{card}(\mathcal{K}))^{1+\frac{p}{4\alpha+6}}} &= 2^{j_0 p} (\text{card}(\mathcal{K}))^{-\frac{p}{4\alpha+6}} \\ & \geq 2^{j_0 p} \left(2 \frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)} 2^{(4\alpha+6)(j_0+2)} \right)^{-\frac{p}{4\alpha+6}} \\ &= \left(\frac{2^{-j_0(4\alpha+6)+1+\frac{8}{p}(2\alpha+3)-(2\alpha+3)+2+(j_0+2)(4\alpha+6)}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)} \right)^{-\frac{p}{4\alpha+6}} \\ &= \left(\frac{2^{(2\alpha+3)(3+\frac{8}{p})+3}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)} \right)^{-\frac{p}{4\alpha+6}} \\ & \geq \left(\frac{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 (2^{4\alpha+6} - 1)}{2^{(2\alpha+3)(3+\frac{10}{p})+3}} \right)^{\frac{p}{4\alpha+6}}. \end{aligned}$$

Then,

$$\begin{aligned} & \sum_{(m,n) \in \mathcal{K}} \left((\rho_{p,v_\alpha}(\varphi_{m,n}))^p + (\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))^p \right) \\ & \geq (\text{card}(\mathcal{K}))^{1+\frac{p}{4\alpha+6}} \left(\frac{\Gamma^2(\alpha + \frac{5}{2})(2^{4\alpha+6} - 1)}{2^{(2\alpha+3)(\frac{10}{p}+3)+3}} \right)^{\frac{p}{4\alpha+6}}. \end{aligned}$$

REMARK 3.9.

- i. The relation (3.19) shows in particular that for every orthonormal basis $(\varphi_{m,n})_{(m,n) \in \mathbb{N}^2}$ of $L^2(dv_\alpha)$ and for every $p > 0$,

$$\sum_{(m,n) \in \mathbb{N}^2} \left((\rho_{p,v_\alpha}(\varphi_{m,n}))^p + (\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))^p \right) = +\infty.$$

- ii. In [27, Remark 1], the authors have established the well known Heisenberg Pauli-Weyl uncertainty principle for the Riemann-Liouville transform, that is for every $f \in L^2(dv_\alpha)$, we have

$$\| |(r,x)| f \|_{2,v_\alpha}^2 + \| |\theta(\lambda_0, \lambda)| \mathcal{F}_\alpha(f) \|_{2,\gamma_\alpha}^2 \geq (2\alpha + 3) \|f\|_{2,v_\alpha}^2. \quad (3.25)$$

Let $f \in L^2(dv_\alpha) \setminus \{0\}$ and let $\varphi_{0,0} = \frac{f}{\|f\|_{2,v_\alpha}}$. The set $\{\varphi_{0,0}\}$ can be completed to an Hilbert basis $(\varphi_{m,n})_{m,n \in \mathbb{N}^2}$ of $L^2(dv_\alpha)$. Taking $\mathcal{K} = \{(0,0)\}$ in the relation (3.19), we deduce that for every $p > 0$,

$$\begin{aligned} & \| |(r,x)|^{\frac{p}{2}} f \|_{2,v_\alpha}^2 + \| |\theta(\lambda_0, \lambda)|^{\frac{p}{2}} \mathcal{F}_\alpha(f) \|_{2,\gamma_\alpha}^2 \\ & \geq \left(\frac{\Gamma^2(\alpha + \frac{5}{2})(2^{4\alpha+6} - 1)}{2^{(2\alpha+3)(\frac{10}{p}+3)+3}} \right)^{\frac{p}{4\alpha+6}} \|f\|_{2,v_\alpha}^2. \end{aligned} \quad (3.26)$$

The relation (3.26) generalizes the relation (3.25). However, in the relation (3.25), the constant $2\alpha + 3$ is optimal (the best). \square

LEMMA 3.10. *For every positive real number a , there exists a non zero function $f \in L^2(dv_\alpha)$ which vanishes almost everywhere on $B_a^+ = \{(r,x) \in [0, +\infty[\times \mathbb{R}; r^2 + x^2 \leq a^2\}$ and such that $\mathcal{F}_\alpha(f)$ vanishes almost every where on $\tilde{B}_a^+ = \{(\lambda_0, \lambda) \in \Upsilon_+; |\theta(\lambda_0, \lambda)|^2 = \lambda_0^2 + 2\lambda^2 \leq a^2\}$.*

Proof. Let \mathcal{H}_α be the Hankel transform defined on $L^1(d\mu_\alpha) \cap L^2(d\mu_\alpha)$ by

$$\mathcal{H}_\alpha(f)(\lambda_0) = \int_0^\infty f(r) j_\alpha(r\lambda_0) d\mu_\alpha(r)$$

where $d\mu_\alpha$ is the measure defined on $[0, +\infty[$ by $d\mu_\alpha(r) = \frac{r^{2\alpha+1} dr}{2^\alpha \Gamma(\alpha+1)}$. Let $a > 0$, according to [12], there exists a nonzero function $g \in L^2(d\mu_\alpha)$ such that g and $\mathcal{H}_\alpha(g)$

vanish almost everywhere on $[0, a]$. As the same way and using [24], there exists a nonzero function $h \in L^2(\mathbb{R}, dx)$ such that h and \widehat{h} vanish almost everywhere on $[-a, a]$ where

$$\widehat{h}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(x) e^{-i\lambda x} dx.$$

We consider the function $f(r, x) = g(r)h(x)$. By construction, f belongs to $L^2(d\nu_\alpha)$ and f vanishes on B_a^+ . Moreover, for every $(\lambda_0, \lambda) \in \Upsilon$,

$$\mathcal{F}_\alpha(f)(\lambda_0, \lambda) = \mathcal{H}_\alpha(g)(\sqrt{\lambda_0^2 + \lambda^2})\widehat{h}(\lambda),$$

consequently, $\mathcal{F}_\alpha(f)$ vanishes almost everywhere on

$$\widetilde{B}_a^+ = \{(\lambda_0, \lambda) \in \Upsilon_+; |\theta(\lambda_0, \lambda)|^2 = \lambda_0^2 + 2\lambda^2 \leq a^2\}. \quad \square$$

THEOREM 3.11. (Multiplicative version of the mean-dispersion Shapiro's theorem) *Let $(\varphi_{m,n})_{(m,n) \in \mathbb{N}^2}$ be an orthonormal basis of $L^2(d\nu_\alpha)$. For every $p > 0$, the sequence $(\rho_{p, \nu_\alpha}(\varphi_{m,n})\rho_{p, \gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))_{(m,n) \in \mathbb{N}^2}$ is not bounded, that is*

$$\sup_{(m,n) \in \mathbb{N}^2} \left(\rho_{p, \nu_\alpha}(\varphi_{m,n})\rho_{p, \gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})) \right) = +\infty.$$

Proof. Suppose that

$$\sup_{(m,n) \in \mathbb{N}^2} \left(\rho_{p, \nu_\alpha}(\varphi_{m,n})\rho_{p, \gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})) \right) < +\infty.$$

Then, there exists a positive constant C such that

$$\forall (m, n) \in \mathbb{N}^2, \rho_{p, \nu_\alpha}(\varphi_{m,n})\rho_{p, \gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})) \leq C^2.$$

For every $k \in \mathbb{Z}$, we put

$$A_k = \left\{ (m, n) \in \mathbb{N}^2; 2^{-k} C \leq \rho_{p, \nu_\alpha}(\varphi_{m,n}) < 2^{-k+1} C \right\},$$

then $A_{k_1} \cap A_{k_2} = \emptyset$ if $k_1 \neq k_2$ and $\bigcup_{k \in \mathbb{Z}} A_k = \mathbb{N}^2$.

Moreover, for every $(m, n) \in A_k$,

$$\rho_{p, \nu_\alpha}(\varphi_{m,n}) \leq 2^{-k+1} C \text{ and } \rho_{p, \gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})) \leq 2^k C. \quad (3.27)$$

On the other hand, for all $\rho, \eta > 0$ and $(m, n) \in A_k$, we have

$$\begin{aligned} \int \int_{(B_\rho^+)^c} |\varphi_{m,n}(r, x)|^2 d\nu_\alpha(r, x) &\leq \frac{1}{\rho^p} \int_0^\infty \int_{\mathbb{R}} |(r, x)|^p |\varphi_{m,n}(r, x)|^2 d\nu_\alpha(r, x) \\ &= \left(\frac{\rho_{p, \nu_\alpha}(\varphi_{m,n})}{\rho} \right)^p \leq \left(\frac{2^{-k+1} C}{\rho} \right)^p \end{aligned} \quad (3.28)$$

and

$$\begin{aligned} \int \int_{(\widetilde{B}_\eta^+)^c} |\mathcal{F}_\alpha(\varphi_{m,n})(\mu, \lambda)|^2 d\gamma_\alpha(\mu, \lambda) &\leq \frac{1}{\eta^p} (\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\varphi_{m,n})))^p \\ &\leq \left(\frac{2^k C}{\eta}\right)^p. \end{aligned} \quad (3.29)$$

In particular, for $\rho = C 2^{\frac{4}{p}+1} 2^{-k}$ and $\eta = C 2^{\frac{4}{p}} 2^k$, we deduce that for every $(m, n) \in A_k$, the function $\varphi_{m,n}$ is $\frac{1}{4}$ -concentrated in the ball B^+ $C 2^{\frac{4}{p}+1} 2^{-k}$ and $\frac{1}{4}$ -bandlimited in the ball \widetilde{B}^+ $C 2^{\frac{4}{p}} 2^k$. Using Corollary 3.5, we conclude that the set A_k is finite and that

$$\begin{aligned} \text{card}(A_k) &\leq \frac{(C 2^{\frac{4}{p}+1} 2^{-k})^{2\alpha+3} (C 2^{\frac{4}{p}} 2^k)^{2\alpha+3}}{2^{2\alpha+3} \left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2 \left(1 - \frac{3}{4}\right)} \\ &= \frac{C^{2(2\alpha+3)} 2^{\frac{8}{p}(2\alpha+3)+2}}{\left(\Gamma\left(\alpha + \frac{5}{2}\right)\right)^2}. \end{aligned} \quad (3.30)$$

Now, let $R > 0$. By Lemma 3.10, there exists $f \in L^2(d\nu_\alpha)$; $\|f\|_{2,\nu_\alpha} = 1$ such that f vanishes on B_R^+ and $\mathcal{F}_\alpha(f)$ vanishes on \widetilde{B}_R^+ . For every $(m, n) \in A_k$, we have

$$\begin{aligned} |\langle f | \varphi_{m,n} \rangle_{\nu_\alpha}|^2 &\leq \left(\int \int_{(B_R^+)^c} \frac{|f(r,x)|}{|(r,x)|^{\frac{p}{2}}} |(r,x)|^{\frac{p}{2}} |\varphi_{m,n}(r,x)| d\nu_\alpha(r,x) \right)^2 \\ &\leq \left(\int \int_{(B_R^+)^c} \frac{|f(r,x)|^2}{|(r,x)|^p} d\nu_\alpha(r,x) \right) \left(\int_0^\infty \int_{\mathbb{R}} |(r,x)|^p |\varphi_{m,n}(r,x)|^2 d\nu_\alpha(r,x) \right) \\ &\leq \frac{1}{R^p} \rho_{p,\nu_\alpha}^p(\varphi_{m,n}). \end{aligned}$$

Using the relation (3.27), we get

$$|\langle f | \varphi_{m,n} \rangle_{\nu_\alpha}|^2 \leq \frac{(2C)^p 2^{-kp}}{R^p}. \quad (3.31)$$

Similarly, for every $(m, n) \in A_k$,

$$\begin{aligned} |\langle \mathcal{F}_\alpha(f) | \mathcal{F}_\alpha(\varphi_{m,n}) \rangle_{\gamma_\alpha}|^2 &\leq \left(\int \int_{(\widetilde{B}_R^+)^c} \frac{|\mathcal{F}_\alpha(f)(\lambda_0, \lambda)|^2}{|\theta(\lambda_0, \lambda)|^p} d\gamma_\alpha(\lambda_0, \lambda) \right) \\ &\quad \times \left(\int \int_{\Gamma_+} |\theta(\lambda_0, \lambda)|^p |\mathcal{F}_\alpha(\varphi_{m,n})(\lambda_0, \lambda)|^2 d\gamma_\alpha(\lambda_0, \lambda) \right) \\ &\leq \frac{1}{R^p} \rho_{p,\gamma_\alpha}^p(\mathcal{F}_\alpha(\varphi_{m,n})). \end{aligned}$$

Again, by the relation (3.27), we have

$$|\langle \mathcal{F}_\alpha(f) | \mathcal{F}_\alpha(\varphi_{m,n}) \rangle_{\gamma_\alpha}|^2 \leq \frac{(2C)^p}{R^p} 2^{kp}. \quad (3.32)$$

Using the relations (3.31), (3.32) and the Parseval equality for \mathcal{F}_α , it follows that for every $(m, n) \in A_k$,

$$\begin{aligned} |\langle f | \varphi_{m,n} \rangle_{v_\alpha}|^2 &= |\langle \mathcal{F}_\alpha(f) | \mathcal{F}_\alpha(\varphi_{m,n}) \rangle_{\gamma_\alpha}|^2 \\ &\leq \left(\frac{2C}{R}\right)^p \min\{2^{kp}, 2^{-kp}\} = \left(\frac{2C}{R}\right)^p 2^{-|k|p}. \end{aligned} \quad (3.33)$$

On the other hand, we know that

$$\begin{aligned} \|f\|_{2, v_\alpha}^2 = 1 &= \sum_{(m,n) \in \mathbb{N}^2} |\langle f | \varphi_{m,n} \rangle_{v_\alpha}|^2 \\ &= \sum_{k \in \mathbb{Z}} \left(\sum_{(m,n) \in A_k} |\langle f | \varphi_{m,n} \rangle_{v_\alpha}|^2 \right). \end{aligned} \quad (3.34)$$

By the relations (3.30) and (3.33), we have

$$\begin{aligned} 1 &\leq \left(\frac{2C}{R}\right)^p \sum_{k \in \mathbb{Z}} 2^{-|k|p} \text{card}(A_k) \\ &\leq \frac{C^{2(2\alpha+3)+p} 2^{\frac{8}{p}(2\alpha+3)+p+2}}{R^p \left(\Gamma(\alpha + \frac{5}{2})\right)^2} \left(2 \sum_{k=0}^{\infty} 2^{-kp} - 1\right) \\ &= \frac{C^{2(2\alpha+3)+p} 2^{\frac{8}{p}(2\alpha+3)+p+2}}{R^p \left(\Gamma(\alpha + \frac{5}{2})\right)^2} \left(\frac{2^p + 1}{2^p - 1}\right). \end{aligned} \quad (3.35)$$

This gives a contradiction because we can choose R sufficiently large. \square

PROPOSITION 3.12. *There exists an orthonormal sequence $(\psi_{m,n})_{(m,n) \in \mathbb{N}^2}$ such that for every $p > 0$,*

$$\sup_{(m,n) \in \mathbb{N}^2} \left(\rho_{p, v_\alpha}(\psi_{m,n}) \rho_{p, \gamma_\alpha}(\mathcal{F}_\alpha(\psi_{m,n})) \right) < +\infty.$$

Proof. Let $\psi \in \mathcal{D}_e(\mathbb{R}^2)$; $\text{supp}(\psi) \subset \{(r, x) \in \mathbb{R}^2; 1 \leq |(r, x)| \leq 2\}$ such that $\|\psi\|_{2, v_\alpha} = 1$. Since the transform \mathcal{F}_α is a topological isomorphism from $\mathcal{S}_e(\mathbb{R}^2)$ onto itself, we deduce that $\widetilde{\mathcal{F}_\alpha}(\psi)$ belongs to $\mathcal{S}_e(\mathbb{R}^2)$. Consequently, from Definition 3.1 and the relation (2.19), for every $p > 0$,

$$\rho_{p, v_\alpha}(\psi) < +\infty \text{ and } \rho_{p, \gamma_\alpha}(\mathcal{F}_\alpha(\psi)) = \rho_{p, v_\alpha}(\widetilde{\mathcal{F}_\alpha}(\psi)) < +\infty.$$

Let $\theta : \mathbb{N}^2 \rightarrow \mathbb{N}$ be a bijective application, we define the sequence $(\psi_{m,n})_{(m,n) \in \mathbb{N}^2}$ by

$$\forall (r, x) \in [0, +\infty[\times \mathbb{R}, \psi_{m,n}(r, x) = 2^{(\alpha + \frac{3}{2})\theta(m,n)} \psi(2^{\theta(m,n)} r, 2^{\theta(m,n)} x).$$

For every $(m, n) \in \mathbb{N}^2$;

$$\|\psi_{m,n}\|_{2, v_\alpha} = 1,$$

and

$$\text{supp}(\Psi_{m,n}) \subset \left\{ (r,x) \in \mathbb{R}^2; 2^{-\theta(m,n)} \leq |(r,x)| \leq 2^{1-\theta(m,n)} \right\}. \quad (3.36)$$

Let $(m,n), (m',n') \in \mathbb{N}^2$; $(m,n) \neq (m',n')$, then $\theta(m,n) \neq \theta(m',n')$, for example

$$\theta(m,n) < \theta(m',n'), \text{ or } 1 + \theta(m,n) \leq \theta(m',n').$$

Then, from the relation (3.36),

$$\forall (r,x) \in [0, +\infty[\times \mathbb{R}; \Psi_{m,n}(r,x) \Psi_{m',n'}(r,x) = 0,$$

In particular $\langle \Psi_{m,n} | \Psi_{m',n'} \rangle_{v_\alpha} = 0$. Consequently, $(\Psi_{m,n})_{(m,n) \in \mathbb{N}^2}$ is an orthonormal sequence in $L^2(dv_\alpha)$. On the other hand, by a standard computation, we have

$$\begin{aligned} \rho_{p,v_\alpha}(\Psi_{m,n}) &= 2^{-\theta(m,n)} \rho_{p,v_\alpha}(\Psi) \\ \mathcal{F}_\alpha(\Psi_{m,n})(\lambda_0, \lambda) &= 2^{-(\alpha + \frac{3}{2})\theta(m,n)} \mathcal{F}_\alpha(\Psi)(2^{-\theta(m,n)}\lambda_0, 2^{-\theta(m,n)}\lambda) \end{aligned} \quad (3.37)$$

$$\rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\Psi_{m,n})) = 2^{\theta(m,n)} \rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\Psi)). \quad (3.38)$$

The relations (3.37) and (3.38) show that

$$\forall (m,n) \in \mathbb{N}^2, \rho_{p,v_\alpha}(\Psi_{m,n}) \rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\Psi_{m,n})) = \rho_{p,v_\alpha}(\Psi) \rho_{p,\gamma_\alpha}(\mathcal{F}_\alpha(\Psi)). \quad \square$$

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(Received September 14, 2017)

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