

## SPECTRAL ESTIMATES FOR DIRICHLET LAPLACIAN ON TUBES WITH EXPLODING TWISTING VELOCITY

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*Abstract.* We study the spectrum of the Dirichlet Laplacian on an unbounded twisted tube with twisting velocity exploding to infinity. If the tube cross section does not intersect the axis of rotation, then its spectrum is purely discrete under some additional conditions on the twisting velocity (D. Krejčířik, 2015). In the current work we prove a Berezin type upper bound for the eigenvalue moments.

### 1. Introduction

Advances in mesoscopic physics have given rise to study spectral properties of unbounded regions of tubular shape. The Dirichlet Laplacian in such domains is a reasonable model for the Hamiltonian in quantum-waveguide nanostructures. One of the peculiarities of such domains is that they may possess geometrically-induced bound states, which was noticed first in the two-dimensional situation by P. Exner and P. Šeba [12], and studied intensively since then, see, e.g., the papers [5, 6, 13, 21] and the recent monograph [11].

In the above mentioned papers bound states are generated by a local *bending* of a straight waveguide. In the present work we deal with another class of unbounded tubular domains – the so-called *twisted tubes*.

*Twisted tube* is a set which is obtained by translating and rotating a bounded open connected set  $\omega \subset \mathbb{R}^2$  about a straight line in  $\mathbb{R}^3$ . More precisely, for a given  $x_1 \in \mathbb{R}$  and  $x := (x_2, x_3) \in \omega$  we define the mapping  $\mathcal{L} : \mathbb{R} \times \omega \rightarrow \mathbb{R}^3$  by

$$\mathcal{L}(x_1, x) = (x_1, x_2 \cos \theta(x_1) + x_3 \sin \theta(x_1), x_3 \cos \theta(x_1) - x_2 \sin \theta(x_1)). \quad (1.1)$$

Here  $\theta : \mathbb{R} \rightarrow \mathbb{R}$  is the *rotation angle* which is assumed to be a sufficiently regular function. Then the region  $\Omega := \mathcal{L}(\mathbb{R} \times \omega) \subset \mathbb{R}^3$  is a twisted tube unless the function  $\theta$  is constant or  $\omega$  is rotationally symmetric with respect to the origin in  $\mathbb{R}^2$  (i.e., either a disk or an annulus with the center at the origin).

In what follows for  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 1$ , we denote by  $-\Delta_D^\Omega$  the Dirichlet Laplacian in  $L^2(\Omega)$ . If  $\Omega$  is bounded, the spectrum of  $-\Delta_D^\Omega$  is purely discrete. However, for unbounded domains the discreteness of the spectrum is no longer guaranteed. A necessary

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condition is the so called *quasi-boundedness* of  $\Omega$  (see, e.g., [7]) which is satisfied, by definition, if  $\lim_{x \in \Omega, |x| \rightarrow \infty} \text{dist}(x, \partial\Omega) = 0$ .

It is easy to see that the twisted tube  $\Omega$  is not a quasi-bounded domain if the cross-section  $\omega$  contains the origin in  $\mathbb{R}^2$ . Consequently, in this case the essential spectrum of  $-\Delta_D^\Omega$  is non-empty. For example, if  $\theta$  vanishes at infinity then [17]  $\sigma(-\Delta_D^\Omega) = [\lambda_1, \infty)$ , where  $\lambda_1$  is the first eigenvalue of  $-\Delta_D^\omega$  in  $L^2(\omega)$ . Another interesting example was treated in [10]:  $\theta(x_1)$  is a constant (we denote it  $\beta$ ). In this case  $\sigma(-\Delta_D^\Omega) = [\lambda_1(\beta), \infty)$ , where  $\lambda_1(\beta)$  is the spectral threshold of the two-dimensional operator  $-\Delta_D^\omega - \beta^2 \partial_\tau^2$  in  $L^2(\omega)$ ,  $\partial_\tau$  is the angular momentum operator  $\partial_\tau := x_3 \partial_2 - x_2 \partial_3$ . Of course, if  $\omega$  contains the origin it does not mean that the spectrum is purely essential: for instance, if we perturb locally a twisted tube with constant  $\theta(x_1) = \beta$ , then eigenvalues may appear below  $\lambda_1(\beta)$ , see [10] for more details.

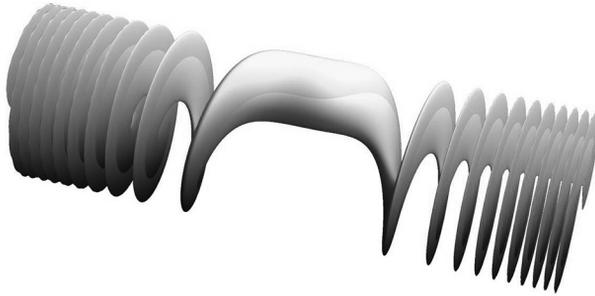


Figure 1: *The twisted tube  $\Omega$*

The picture changes drastically if

$$\omega \subset \{(x_2, x_3) \in \mathbb{R}^2 \mid x_2 > 0\}. \tag{1.2}$$

The corresponding twisted tube is depicted on Figure 1. In this case it turns out that  $\sigma(-\Delta_D^\Omega)$  is purely discrete provided

$$\lim_{x_1 \rightarrow \pm\infty} |\dot{\theta}(x_1)| = \infty, \tag{1.3}$$

and therefore  $\Omega$  becomes quasi-bounded [16].

In the present note we study some properties of the discrete eigenvalues in the model considered in [16]. Our main result is the Berezin type bound for eigenvalue moments of order  $\sigma \geq 0$ .

Recall, that the classical Berezin bound is the estimate from above for the moments of eigenvalues of the Dirichlet Laplacian  $-\Delta_D^\Omega$  on a *bounded* domain  $\Omega$  lying below a fixed  $\Lambda > 0$  [1]:

$$\text{tr}(-\Delta_D^\Omega - \Lambda)_-^\sigma := \sum_k (\lambda_k - \Lambda)_-^\sigma \leq L_{\sigma,d}^{\text{cl}} |\Omega| \Lambda^{\sigma+d/2}, \quad \sigma \geq 1. \tag{1.4}$$

Here  $\{\lambda_k\}_{k \in \mathbb{N}}$  is a sequence of eigenvalues of  $-\Delta_D^\Omega$ , numbered in the ascending order with account of their multiplicities,  $|\Omega|$  stands for the measure of  $\Omega$ ,  $L_{\sigma,d}^{\text{cl}}$  is the so-called semiclassical constant given by

$$L_{\sigma,d}^{\text{cl}} = \frac{\Gamma(\sigma + 1)}{(4\pi)^{d/2} \Gamma(\sigma + 1 + d/2)}, \quad (1.5)$$

and, finally,  $(\cdot)_-$  is the negative part of the enclosed quantity (cf. (2.8)). A similar inequality holds also for  $0 \leq \sigma < 1$  with some, probably non-sharp, constant instead of  $L_{\sigma,d}^{\text{cl}}$  [19].

Unfortunately, for tubular domains we consider in the current work these estimates are meaningless since their right-hand sides become infinite due to  $|\Omega| = \infty$ . Nevertheless, we are able to derive a Berezin type bound for twisted tubes whose rotation velocity explodes at infinity (see (1.3)) and additional technical assumptions (2.6)-(2.7) hold (see also Subsection 4.1). The role of  $|\Omega|\Lambda^{\sigma+3/2}$  will be played by  $|\omega|$  times certain expression involving  $\Lambda$ ,  $\theta(x_1)$  and  $\dot{\theta}(x_1)$ .

Eigenvalue bounds for twisted tubes were also treated by P. Exner and the first author in [9], where a locally perturbed twisted tube with constant rotation velocity was considered. The authors derived Lieb-Thirring-type inequalities for eigenvalue moments of order  $\sigma > 1/2$ . Other spectral aspects of twisted tubes were treated in [14, 15] (existence/non-existence of bound states), [2, 8] (Hardy type inequalities), [4] (asymptotic behavior of the spectrum as the thickness of the tube cross section goes to zero), [3] (eigenvalue asymptotics in the case when the rotation velocity decays slowly at infinity).

The paper is organized as follows. In Section 2 we present our main result (Theorem 2.1). Its proof is given in Section 3. Finally, in Section 4 we discuss the obtained result.

## 2. Main result

Recall, that we are given with the domain  $\Omega := \mathfrak{L}(\mathbb{R} \times \omega) \subset \mathbb{R}^3$ , where  $\mathfrak{L}$  is defined by (1.1) and the domain  $\omega \subset \mathbb{R}^2$  satisfies (1.2). The rotational angle  $\theta(x_1)$  is assumed to be a continuously differentiable function satisfying condition (1.3). Additionally, we assume that

$$\dot{\theta}(x_1) \text{ is a monotonically increasing function,} \quad (2.6)$$

$$\dot{\theta}(x_1) \geq 0 \text{ on } \mathbb{R}_+, \quad \dot{\theta}(x_1) \leq 0 \text{ on } \mathbb{R}_- \quad (2.7)$$

(for example, one can choose  $\theta(x_1) = \sum_{k=0}^m A_k x_1^{2k}$  with  $m \in \mathbb{N}$ ,  $A_k \geq 0$ ,  $A_m \neq 0$ ). Another functions also can be treated, see Subsection 4.1. Note, that (1.3), (2.7) imply

$$\lim_{|x_1| \rightarrow \infty} \theta(x_1) = \infty.$$

We set for  $\alpha \in \theta[0, \infty)$ ,  $\beta \in \theta(-\infty, 0]$ :

$$\theta_+^{-1}(\alpha) := \{z \geq 0 : \theta(z) = \alpha\}, \quad \theta_-^{-1}(\beta) := \{z \leq 0 : \theta(z) = \beta\}.$$

In what follows for  $z \in \mathbb{R}$  we denote

$$(z)_\pm := |z \pm |z||/2 \tag{2.8}$$

(i.e., the negative and positive parts of  $z$ ).

Our main result is the following theorem.

**THEOREM 2.1.** *Let  $\sigma \geq 0$ . Under the above assumptions on  $\theta$  and  $\omega$ , for any  $0 < \varepsilon < 1$  and  $\Lambda \geq 0$  the following inequality holds true,*

$$\text{tr} \left( -\Delta_D^\Omega - \Lambda \right)_-^\sigma \leq \frac{L_\sigma}{(1 - \varepsilon)^{3/2}} |\omega| \int_{\mathbb{R}} (\varepsilon f(x_1) - \Lambda)_-^{\sigma+3/2} dx_1, \tag{2.9}$$

where

$$\begin{aligned} f(x_1) = & \left( \dot{\theta} \left( \theta_+^{-1}(\theta(x_1) - \pi) \right) \right)^2 \chi_{\{x_1 \geq \theta_+^{-1}(\theta(0)+2\pi)\}}(x_1) \\ & + \left( \dot{\theta} \left( \theta_-^{-1}(\theta(x_1) - \pi) \right) \right)^2 \chi_{\{x_1 \leq \theta_-^{-1}(\theta(0)+2\pi)\}}(x_1), \end{aligned} \tag{2.10}$$

and  $L_\sigma$  is a constant depending on  $\sigma$ . For  $\sigma \geq 3/2$  (2.9) is valid with  $L_\sigma = L_{\sigma,3}^{\text{cl}}$ , where  $L_{\sigma,3}^{\text{cl}}$  is given by (1.5).

**REMARK 2.1.** It follows easily from the assumptions on the function  $\theta(x_1)$  that

$$f(x_1) \rightarrow \infty \text{ as } |x_1| \rightarrow \infty,$$

which implies the finiteness of the integral at the right-hand-side of (2.9).

### 3. Proof of Theorem 2.1

We fix a point  $x = (x_2, x_3) \subset \mathbb{R}^2$  and denote

$$\omega_x = \{x_1 \in \mathbb{R} : (x_1, x_2, x_3) \in \Omega\}.$$

It is easy to see that  $\omega_x$  is either the empty set or a sequence of segments  $(a_k(x), b_k(x))_{k=-\infty}^\infty$  satisfying

$$\begin{aligned} a_k(x) < b_k(x) < a_{k+1}(x), \quad \forall k \in \mathbb{Z}, \\ a_k(x) \rightarrow \pm\infty \text{ as } k \rightarrow \pm\infty. \end{aligned}$$

We assume that these intervals are renumbered in such a way that

$$b_{-1}(x) < 0, \quad a_1(x) > 0.$$

In what follows we use the same notation for  $u \in \mathcal{H}_0^1(\Omega)$  and its extension by zero to the whole  $\mathbb{R}^3$  (the resulting function will belong to  $\mathcal{H}^1(\mathbb{R}^3)$ ).

LEMMA 3.1. For each  $u \in \mathcal{H}_0^1(\Omega)$

$$\int_{\Omega} \left| \frac{\partial u}{\partial x_1}(x_1, x) \right|^2 dx_1 dx \geq \int_{\Omega} f(x_1) |u|^2 dx_1 dx, \quad (3.11)$$

where  $f(x_1)$  is defined by (2.10).

*Proof.* Let us fix  $x = (x_2, x_3) \subset \mathbb{R}^2$ . Since  $u(a_k(x), x) = u(b_k(x), x) = 0$  one has the following Friedrich inequality for each  $k \in \mathbb{Z}$ :

$$\int_{a_k(x)}^{b_k(x)} \left| \frac{\partial u}{\partial x_1} \right|^2 dx_1 \geq \frac{\pi^2}{(b_k(x) - a_k(x))^2} \int_{a_k(x)}^{b_k(x)} |u|^2 dx_1,$$

whence

$$\int_{\omega_x^{\pm}} \left| \frac{\partial u}{\partial x_1} \right|^2 dx_1 \geq \sum_{k \in \mathbb{Z}: \pm k \geq 1} \frac{\pi^2}{(b_k(x) - a_k(x))^2} \int_{a_k(x)}^{b_k(x)} |u|^2 dx_1, \quad (3.12)$$

where  $\omega_x^{\pm} := \omega_x \cap \mathbb{R}_{\pm}$ . Our aim is to establish a *uniform* (with respect to  $x$ ) estimate from below for the right hand side of (3.12). We will do this for  $\omega_x^+$ , for  $\omega_x^-$  the arguments are similar.

At first we notice that on the way from  $a_k(x)$  to  $b_k(x)$  the cross-section  $\omega$  turns by the angle which is not greater than  $\pi$ , i.e.

$$\theta(b_k(x)) - \theta(a_k(x)) \leq \pi. \quad (3.13)$$

This follows easily from (1.2) and the definition of  $a_k$  and  $b_k$ . Then, using the mean value theorem and (1.3), we obtain from (3.13):

$$b_k(x) - a_k(x) \leq \frac{\pi}{\min_{s \in [a_k(x), b_k(x)]} (\dot{\theta}(s))} \leq \frac{\pi}{\dot{\theta}(a_k(x))}, \quad k \geq 1. \quad (3.14)$$

Also from (3.13) we get, using the monotonicity of  $\theta$  (see (2.7)),

$$a_k(x) \geq \theta_+^{-1}(\theta(b_k(x)) - \pi) \geq \theta_+^{-1}(\theta(x_1) - \pi), \quad x_1 \in [a_k(x), b_k(x)], \quad k \geq 1 \quad (3.15)$$

provided

$$\theta(a_k(x)) \geq \theta(0) + \pi. \quad (3.16)$$

Condition (3.16) is required to guarantee

$$\theta(x_1) - \pi \in \text{dom}(\theta_+^{-1}) = [\theta(0), \infty) \quad \text{as } x_1 \in [a_k(x), b_k(x)].$$

Then, again using (1.3), we conclude from (3.15):

$$\dot{\theta}(a_k(x)) \geq \dot{\theta}(\theta_+^{-1}(\theta(x_1) - \pi)), \quad x_1 \in [a_k(x), b_k(x)], \quad k \geq 1. \quad (3.17)$$

Using inequalities (3.14) and (3.17) we can estimate from below the summands in the right-hand side of (3.12) (recall, that now we consider its “+” part) which correspond to  $k$  satisfying (3.17); the remaining summands we estimate by zero. As a result we obtain

$$\int_{\omega_x^+} \left| \frac{\partial u}{\partial x_1} \right|^2 dx_1 \geq \sum_{k: \theta(a_k(x)) \geq \theta(0) + \pi} \int_{a_k(x)}^{b_k(x)} (\dot{\theta}(\theta_+^{-1}(\theta(x_1) - \pi)))^2 |u|^2 dx_1. \tag{3.18}$$

We need more information on the location of the smallest  $a_k(x)$  satisfying (3.16). Let  $k_0$  be such that  $\theta(a_{k_0}(x)) \geq \theta(0) + \pi$ , while  $\theta(a_{k_0-1}(x)) < \theta(0) + \pi$ . There are two possibilities: either

- all intervals  $(a_k(x), b_k(x))$ ,  $k < k_0$  belong to  $(0, \theta_+^{-1}(\theta(0) + \pi)]$ , or
- $(a_k(x), b_k(x)) \subset (0, \theta_+^{-1}(\theta(0) + \pi))$ ,  $k \leq k_0 - 2$ ,  $a_{k_0-1}(x) < \theta_+^{-1}(\theta(0) + \pi)$ ,  $b_{k_0-1}(x) > \theta_+^{-1}(\theta(0) + \pi)$ .

In the first case  $u$  vanishes on  $[\theta_+^{-1}(\theta(0) + \pi), a_{k_0}(x)]$  and, therefore one can replace  $\sum_{k: \theta(a_k(x)) \geq \theta(0) + \pi} \int_{a_k(x)}^{b_k(x)}$  in (3.18) by  $\int_{\theta_+^{-1}(\theta(0) + \pi)}^\infty$ . In the second case we use the following observation: on the way from  $b_k(x)$  to  $a_{k+1}(x)$  the cross-section  $\omega$  turns on the angle which is larger than  $\pi$ , i.e.  $\theta(a_{k+1}(x)) - \theta(b_k(x)) \geq \pi$ . Therefore

$$\theta(a_{k_0}(x)) \geq \theta(b_{k_0-1}(x)) + \pi \geq \theta(0) + 2\pi,$$

while in view of (3.13)

$$\theta(b_{k_0-1}(x)) \leq \theta(a_{k_0-1}(x)) + \pi \leq \theta(0) + 2\pi.$$

Consequently, in the second case the right-hand side of (3.18) is not smaller than

$$\int_{\theta_+^{-1}(\theta(0) + 2\pi)}^\infty (\dot{\theta}(\theta_+^{-1}(\theta(x_1) - \pi)))^2 |u|^2 dx_1.$$

Summarising our conclusions in these two case we finally arrive at

$$\int_{\omega_x^+} \left| \frac{\partial u}{\partial x_1} \right|^2 dx_1 \geq \int_{\theta_+^{-1}(\theta(0) + 2\pi)}^\infty (\dot{\theta}(\theta_+^{-1}(\theta(x_1) - \pi)))^2 |u|^2 dx_1, \tag{3.19}$$

with the function  $f$  being defined by (2.10).

Using the same arguments we get similar estimate for  $\omega_x^-$ :

$$\int_{\omega_x^-} \left| \frac{\partial u}{\partial x_1} \right|^2 dx_1 \geq \int_{-\infty}^{\theta_-^{-1}(\theta(0) + 2\pi)} (\dot{\theta}(\theta_-^{-1}(\theta(x_1) - \pi)))^2 |u|^2 dx_1. \tag{3.20}$$

Taking into account that  $\int_{\Omega} g(x_1, x_2, x_3) dx_1 dx_2 dx_3 = \int_{\mathbb{R}^2} [\int_{\omega_x} g(x_1, x_2, x_3) dx_1] dx$  for each  $g \in L^1(\mathbb{R}^3)$  with  $\text{supp}(g) \subset \bar{\Omega}$ , we get (3.11) from (3.19)-(3.20) and the definition of the function  $f$ . The lemma is proved.  $\square$

We come back to the proof of the theorem. Let us fix  $0 < \varepsilon < 1$  and  $\Lambda \geq 0$ . Given a function  $u \in \mathcal{H}_0^1(\Omega)$  the quadratic form of the Dirichlet Laplacian  $-\Delta_\Omega^D$  can be represented as follows,

$$\int_{\Omega} |\nabla u|^2 dx_1 dx_2 dx_3 \\ = \varepsilon \int_{\Omega} \left| \frac{\partial u}{\partial x_1} \right|^2 dx_1 dx + \int_{\Omega} \left( (1-\varepsilon) \left| \frac{\partial u}{\partial x_1} \right|^2 + \left| \frac{\partial u}{\partial x_2} \right|^2 + \left| \frac{\partial u}{\partial x_3} \right|^2 \right) dx_1 dx.$$

This together with (3.11) yields

$$\int_{\Omega} (|\nabla u|^2 - \Lambda |u|^2) dx_1 dx \geq \varepsilon \int_{\Omega} f(x_1) |u|^2 dx_1 dx \\ + \int_{\Omega} \left( (1-\varepsilon) \left| \frac{\partial u}{\partial x_1} \right|^2 + \left| \frac{\partial u}{\partial x_2} \right|^2 + \left| \frac{\partial u}{\partial x_3} \right|^2 - \Lambda |u|^2 \right) dx_1 dx \\ \geq (1-\varepsilon) \int_{\Omega} \left( |\nabla u|^2 + \frac{1}{1-\varepsilon} (\varepsilon f(x_1) - \Lambda)_- |u|^2 \right) dx_1 dx. \quad (3.21)$$

We introduce the complement  $\widehat{\Omega} := \mathbb{R}^3 \setminus \overline{\Omega}$  and consider the functions of the form  $h = u + v$  with  $u \in \mathcal{H}_0^1(\Omega)$  and  $v \in \mathcal{H}_0^1(\widehat{\Omega})$  which we may regard as functions in  $\mathbb{R}^3$  extending them by zero to  $\widehat{\Omega}$  and  $\Omega$ , respectively. Next we extend by zero the potential  $\frac{1}{1-\varepsilon}(\varepsilon f(x_1) - \Lambda)_-$  to potential  $V$  defined on the whole space  $\mathbb{R}^3$ . Then (3.21) implies

$$\int_{\Omega} (|\nabla u|^2 - \Lambda |u|^2) dx_1 dx + \int_{\widehat{\Omega}} |\nabla v|^2 dx_1 dx \\ \geq (1-\varepsilon) \int_{\mathbb{R}^3} (|\nabla h|^2 + V |h|^2) dx_1 dx. \quad (3.22)$$

The left-hand side in (3.22) is the quadratic form corresponding to the operator  $(-\Delta_D^\Omega - \Lambda) \oplus (-\Delta_D^{\widehat{\Omega}})$ , while the right-hand one is the form associated with the operator  $(1-\varepsilon)(-\Delta + V)$  on  $L^2(\mathbb{R}^3)$ . Since  $-\Delta_D^{\widehat{\Omega}}$  is a positive operator, we conclude from (3.22), using the minimax principle, that for each  $\sigma \geq 0$

$$\text{tr} \left( -\Delta_D^\Omega - \Lambda \right)_-^\sigma \leq (1-\varepsilon)^\sigma \text{tr} (-\Delta + V)_-^\sigma. \quad (3.23)$$

Further we apply the Lieb-Thirring inequality for the operator  $-\Delta + V$ . Recall, that it reads as

$$\text{tr} (-\Delta + V)_-^\sigma \leq L_\sigma \int_{\mathbb{R}^3} V_-^{\sigma+3/2} dx_1 dx_2 dx_3, \quad \sigma \geq 0 \quad (3.24)$$

(for  $\sigma = 0$  (3.24) is known as Cwikel-Lieb-Rozenblum inequality). It was proved in [20] for  $\sigma > 0$ , and in [22] for  $\sigma = 0$ . Moreover, for  $\sigma \geq 3/2$  the best constant  $L_\sigma$  for which (3.24) holds coincides with  $L_{\sigma,3}^{\text{cl}}$  given by (1.5) [19].

Using (3.24) we obtain from (3.23):

$$\begin{aligned}
 & \operatorname{tr} \left( -\Delta_D^\Omega - \Lambda \right)_-^\sigma \leq (1 - \varepsilon)^\sigma L_\sigma \int_{\mathbb{R}^3} V_-^{\sigma+3/2} dx_1 dx \\
 & = \frac{L_\sigma}{(1 - \varepsilon)^{3/2}} \int_\Omega (\varepsilon f(x_1) - \Lambda)_-^{\sigma+3/2} dx_1 dx, \\
 & = \frac{L_\sigma}{(1 - \varepsilon)^{3/2}} \int_{\mathbb{R}} \int_{\omega(x_1)} (\varepsilon f(x_1) - \Lambda)_-^{\sigma+3/2} dx_1 dx \\
 & = \frac{L_\sigma}{(1 - \varepsilon)^{3/2}} \int_{\mathbb{R}} (\varepsilon f(x_1) - \Lambda)_-^{\sigma+3/2} |\omega(x_1)| dx_1, \tag{3.25}
 \end{aligned}$$

where  $\omega(x_1)$  is the image of  $\omega$  after the rotation. Since for every  $x \in \mathbb{R}$ ,  $|\omega(x_1)| = |\omega|$ , inequality (3.25) immediately implies (2.9).

Theorem 2.1 is proved.

### 4. Discussion

#### 4.1. Other choices of $\theta$

Theorem 2.1 (with accordingly modified function  $f$ ) remains valid for some other choices of  $\theta$ .

Assume, for example, that  $\theta(x_1)$  is a continuously differentiable function satisfying (1.3) and additionally

$$\dot{\theta}(x_1) \geq 0, \tag{4.26}$$

$$\dot{\theta}(x_1) \text{ is increasing on } \mathbb{R}_+, \quad \dot{\theta}(x_1) \text{ is decreasing on } \mathbb{R}_- \tag{4.27}$$

(for example, one can choose  $\theta(x_1) = \sum_{k=0}^m A_k x_1^{2k+1}$  with  $m \in \mathbb{N}$ ,  $A_k \geq 0$ ,  $A_m \neq 0$ ). Then Theorem 2.1 remains valid with  $f(x_1)$  being replaced by

$$\begin{aligned}
 \tilde{f}(x_1) & = (\dot{\theta}(\theta_+^{-1}(\theta(x_1) - \pi)))^2 \chi_{\{x_1 \geq \theta_+^{-1}(\theta(0) + 2\pi)\}}(x_1) \\
 & + (\dot{\theta}(\theta_-^{-1}(\theta(x_1) + \pi)))^2 \chi_{\{x_1 \leq \theta_-^{-1}(\theta(0) - 2\pi)\}}(x_1). \tag{4.28}
 \end{aligned}$$

Moreover, if  $\theta$  satisfies (1.3), (2.6), (2.7) (respectively, (1.3), (4.26), (4.27)) only for  $|x_1| \geq s_0 > 0$  then Theorem 2.1 holds with  $f(x_1)$  (2.10) being replaced by

$$f(x_1) \chi_{\max\{s_0, \theta_+^{-1}(2\pi + \theta(s_0))\}}(x_1) + f(x_1) \chi_{\min\{-s_0, \theta_-^{-1}(2\pi + \theta(-s_0))\}}(x_1)$$

(respectively, with  $\tilde{f}(x_1)$  (4.28) being replaced by

$$\tilde{f}(x_1) \chi_{\max\{s_0, \theta_+^{-1}(2\pi + \theta(s_0))\}}(x_1) + \tilde{f}(x_1) \chi_{\min\{-s_0, \theta_-^{-1}(-2\pi + \theta(-s_0))\}}(x_1).$$

The proof for the above cases is almost the same as in the case (1.3), (2.6), (2.7).

#### 4.2. Asymptotics of the obtained bound for large $\Lambda$

The right-hand side of (2.9) looks rather cumbersome. The situation becomes easier when  $\Lambda \rightarrow \infty$ . The following statement takes place.

PROPOSITION 4.1. *The right-hand side of (2.9) has the asymptotics*

$$(1 + o(1)) \frac{L\sigma}{(1 - \varepsilon)^{3/2}} |\omega| \int_{\mathbb{R}} (\varepsilon \dot{\theta}^2(x_1) - \Lambda)_-^{\sigma+3/2} dx_1 \text{ as } \Lambda \rightarrow \infty. \quad (4.29)$$

*Proof.* Let us prove that the right hand side of (2.9) can be estimated from above by the expression of the form (4.29).

Due to the monotonicity of  $\theta$  (see (2.7)) and  $\dot{\theta}$  (see (2.6)) there exists  $s_0 > 0$  such that  $\theta(x_1) > \theta(0) + \pi$  and  $\dot{\theta}(\theta(x_1) - \pi) \geq \alpha > 0$  as  $x_1 > s_0$ . On each finite interval being contained in  $[\theta_+^{-1}(\theta(0) + \pi), \infty)$  the function  $\theta_+^{-1}(\theta(x_1) - \pi) - x_1$  is bounded. Moreover, applying the mean value theorem for  $x_1 > s_0$  one gets

$$\theta_+^{-1}(\theta(x_1) - \pi) - x_1 = \theta_+^{-1}(\theta(x_1) - \pi) - \theta_+^{-1}(\theta(x_1)) = -\frac{\pi}{\dot{\theta}(c(x_1))},$$

where  $c(x_1) \in (\theta(x_1) - \pi, \theta(x_1))$ . Hence

$$K_1 := \sup_{\{x_1 \geq \theta_+^{-1}(\theta(0) + \pi)\}} |\theta_+^{-1}(\theta(x_1) - \pi) - x_1| < \infty$$

Similarly,

$$K_2 := \sup_{\{x_1 \leq \theta_-^{-1}(\theta(0) + \pi)\}} |\theta_-^{-1}(\theta(x_1) - \pi) - x_1| < \infty.$$

Let  $K = \max\{K_1, K_2\}$ . Then

$$\begin{aligned} & \int_{\mathbb{R}} (\varepsilon f(x_1) - \Lambda)_-^{\sigma+3/2} dx_1 \\ &= \int_{\theta_+^{-1}(\theta(0) + 2\pi)}^{\infty} (\varepsilon \dot{\theta}^2(\theta_+^{-1}(\theta(x_1) - \pi)) - \Lambda)_-^{\sigma+3/2} dx_1 \\ & \quad + \int_{-\infty}^{\theta_-^{-1}(\theta(0) + 2\pi)} (\varepsilon \dot{\theta}^2(\theta_-^{-1}(\theta(x_1) - \pi)) - \Lambda)_-^{\sigma+3/2} dx_1 \\ &\leq \int_{\theta_+^{-1}(\theta(0) + \pi)}^{\infty} (\varepsilon \dot{\theta}^2(x_1 - K) - \Lambda)_-^{\sigma+3/2} dx_1 + \int_{-\infty}^{\theta_-^{-1}(\theta(0) + 2\pi)} (\varepsilon \dot{\theta}^2(x_1 + K) - \Lambda)_-^{\sigma+3/2} dx_1 \\ &= \int_{\theta_+^{-1}(\theta(0) + 2\pi) - K}^{\infty} (\varepsilon \dot{\theta}^2(x_1) - \Lambda)_-^{\sigma+3/2} dx_1 + \int_{-\infty}^{\theta_-^{-1}(\theta(0) + 2\pi) + K} (\varepsilon \dot{\theta}^2(x_1) - \Lambda)_-^{\sigma+3/2} dx_1 \\ &\leq \int_{\mathbb{R}} (\varepsilon \dot{\theta}^2(x_1) - \Lambda)_-^{\sigma+3/2} dx_1 + \Lambda^{\sigma+3/2} (|\theta_+^{-1}(\theta(0) + 2\pi) - K| + |\theta_-^{-1}(\theta(0) + 2\pi) + K|). \end{aligned} \quad (4.30)$$

One has:

$$\begin{aligned}
 & \int_{\mathbb{R}} (\varepsilon \dot{\theta}^2(x_1) - \Lambda)_-^{\sigma+3/2} dx_1 \\
 &= \int_{|\dot{\theta}(x_1)| \leq \sqrt{\Lambda/\varepsilon}} (\varepsilon \dot{\theta}^2(x_1) - \Lambda)_-^{\sigma+3/2} dx_1 \\
 &\geq \int_{|\dot{\theta}(x_1)| \leq \sqrt{\Lambda/(2\varepsilon)}} (\varepsilon \dot{\theta}^2(x_1) - \Lambda)_-^{\sigma+3/2} dx_1 \\
 &\geq \frac{\Lambda^{\sigma+3/2}}{2\sigma+3/2} \text{meas} \left\{ x_1 \in \mathbb{R} : |\dot{\theta}(x_1)| \leq \sqrt{\Lambda/(2\varepsilon)} \right\}. \tag{4.31}
 \end{aligned}$$

Evidently, the measure standing at the right-hand side of (4.31) tends to infinity as  $\Lambda \rightarrow 0$ . Therefore (4.31) implies

$$\Lambda^{\sigma+3/2} = o \left( \int_{\mathbb{R}} (\varepsilon \dot{\theta}^2(x_1) - \Lambda)_-^{\sigma+3/2} dx_1 \right), \quad \Lambda \rightarrow \infty. \tag{4.32}$$

Combining (4.30) and (4.32) we get the desired estimate

$$\int_{\mathbb{R}} (\varepsilon f(x_1) - \Lambda)_-^{\sigma+3/2} dx_1 \leq (1 + o(1)) \int_{\mathbb{R}} (\varepsilon \dot{\theta}^2(x_1) - \Lambda)_-^{\sigma+3/2} dx_1.$$

Using the same arguments one can prove that the right hand side of (2.9) can be also estimated *from below* by the expression of the form (4.29). The proof is similar: the chain of estimates (2.9) remains valid if we replace all “ $\leq$ ” by “ $\geq$ ”, all “ $\pm K$ ” by “ $\mp K$ ” and “ $+\Lambda^{\sigma+3/2}$ ” in the last line by “ $-\Lambda^{\sigma+3/2}$ ”.

Proposition 4.1 is proved.  $\square$

### 4.3. Comparison with the classical Berezin bound

In this subsection we show that the obtained estimate (2.9) can be used to improve the classical Berezin bound for *bounded* twisted tubes with sufficiently large rotation velocity in the regime  $\Lambda \ll N$ , where  $N$  is the length of the tube.

Let  $\Omega$  be a twisted tube considered in Section 2. Additionally, we assume that its rotation velocity satisfies

$$|\dot{\theta}(x_1)| \geq |x_1|. \tag{4.33}$$

Combining (2.9) and (4.29) and taking into account (4.33) one gets for large  $\Lambda$ :

$$\begin{aligned}
 & \text{tr} \left( -\Delta_D^\Omega - \Lambda \right)_-^\sigma \leq (1 + \bar{o}(1)) \frac{L_\sigma}{(1 - \varepsilon)^{3/2}} |\omega| \int_{\mathbb{R}} (\varepsilon \dot{\theta}^2(x_1) - \Lambda)_-^{\sigma+3/2} dx_1 \\
 &\leq (1 + \bar{o}(1)) \frac{L_\sigma}{(1 - \varepsilon)^{3/2}} |\omega| \Lambda^{\sigma+3/2} (\dot{\theta}_+^{-1}(\sqrt{\Lambda/\varepsilon}) - \dot{\theta}_-^{-1}(\sqrt{\Lambda/\varepsilon})) \\
 &\leq 2(1 + \bar{o}(1)) \frac{L_\sigma}{(1 - \varepsilon)^{3/2} \sqrt{\varepsilon}} |\omega| \Lambda^{\sigma+2}. \tag{4.34}
 \end{aligned}$$

Now, we consider the bounded twisted tube

$$\Omega_N := \{(x_1, x_2, x_3) \in \Omega : 0 < x_1 < N\}.$$

For this tube the classical Berezin inequality (1.4) reads

$$\mathrm{tr} \left( -\Delta_D^{\Omega_1} - \Lambda \right)_-^\sigma \leq L_{\sigma,3}^{\mathrm{cl}} \Lambda^{\sigma+3/2} |\Omega_N| = L_{\sigma,3}^{\mathrm{cl}} \Lambda^{\sigma+3/2} |\omega| N. \quad (4.35)$$

On the other hand, applying the Dirichlet bracketing technique, we get

$$\left( -\Delta_D^{\Omega_N} \right) \oplus \left( -\Delta_D^{\Omega \setminus \overline{\Omega_N}} \right) \geq -\Delta_D^\Omega,$$

whence

$$\mathrm{tr} \left( -\Delta_D^{\Omega_N} - \Lambda \right)_-^\sigma \leq \mathrm{tr} \left( \left( -\Delta_D^{\Omega_N} \right) \oplus \left( -\Delta_D^{\Omega \setminus \overline{\Omega_N}} \right) - \Lambda \right)_-^\sigma \leq \mathrm{tr} \left( -\Delta_D^\Omega - \Lambda \right)_-^\sigma.$$

Thus the right-hand side of (4.34) is also an upper bound  $\mathrm{tr} \left( -\Delta_D^{\Omega_N} - \Lambda \right)_-^\sigma$ .

Finally, assume that  $N = N(\lambda)$  and  $\Lambda \ll N$  as  $\Lambda \rightarrow \infty$ . Then for large  $\Lambda$  the right-hand side of (4.34) gives much better estimate for  $\mathrm{tr} \left( -\Delta_D^{\Omega_N} - \Lambda \right)_-^\sigma$  than the classical Berezin inequality (4.35).

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