

LEFT AND RIGHT GENERALIZED DRAZIN INVERTIBLE OPERATORS ON BANACH SPACES AND APPLICATIONS

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Abstract. In this paper, left and right generalized Drazin invertible operators on Banach spaces are defined and characterized by means of the generalized Kato decomposition. Then, new binary relations associated with these operators are presented and studied. In addition, a new characterization of the generalized Drazin pre-order and a sufficient condition for that to be a partial order are given by using a matrix operator technique.

1. Introduction and background

Generalized inverses of matrices and operators have received an increasing interest in the last decade [2, 9, 11]. Specially, they are used to introduce new pre-orders and partial orders on matrix spaces [7, 8, 14], on Hilbert or Banach spaces or Banach algebras [4, 12, 13], and even on rings [6, 10]. In these papers, authors introduced new generalized inverses and new pre-orders and partial orders, most of them extending some known results existing for matrices in the literature.

Let X be a complex Banach space and let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on X . An element $A \in \mathcal{L}(X)$ is quasi-nilpotent if $\lim_{n \rightarrow \infty} \|A^n\|^{1/n} = 0$.

An operator $A \in \mathcal{L}(X)$ is g -Drazin invertible if there exists a unique $B \in \mathcal{L}(X)$ such that

$$AB = BA, \quad BAB = B, \quad \text{and} \quad A - ABA \text{ is quasi-nilpotent.}$$

The operator B is called the generalized Drazin inverse of A and it is denoted by A^d (see for instance [9]).

For each $A \in \mathcal{L}(X)$, we denote by $N(A)$ and $R(A)$ the kernel and the range of A , respectively. Let $A \in \mathcal{L}(X)$ be a g -Drazin invertible operator, then the matrix forms of operators A and A^d with respect to the decomposition $X = N(I - AA^d) \oplus R(I - AA^d)$ are given by

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad A^d = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

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where A_1 is invertible and A_2 is quasi-nilpotent [9, Theorem 7.1]. If we write

$$A_C = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_Q = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix},$$

then $A = A_C + A_Q$ is known as the core quasi-nilpotent decomposition of A . The operator A_C is called the core part of A and A_Q is called the quasi-nilpotent part of A (see [9] for more details).

Let $A \in \mathcal{L}(X)$. An element in the set $\{B \in \mathcal{L}(X) : BA = I\}$ is denoted by $(A)_{\text{left}}^{-1}$.

Recall that a binary relation on a nonempty set which is reflexive and transitive is called a pre-order. A partial order is a pre-order that also satisfies the antisymmetric property. The relation \preceq^d on the set of all g -Drazin invertible operators of $\mathcal{L}(X)$ is known as the Drazin pre-order (see [13]). Recall that $A \preceq^d B$ if and only if $A^d A = A^d B$ and $AA^d = BA^d$.

In [12, 13], the authors defined and investigated new pre-orders on the set of all bounded linear operators on Banach spaces generalizing the definitions of the binary relations given in [7]. Two generalized Drazin pre-orders and an extension of the generalized Drazin pre-order to a partial order was considered in [12].

Throughout this paper, $X_1 \oplus X_2$ will denote the topological direct sum of X_1 and X_2 , both of them closed subspaces of X . If $X = X_1 \oplus X_2$, it is said that X_1 is complemented in X with X_2 .

Let $A \in \mathcal{L}(X)$. A subspace M of X is A -stable if $A(M) \subseteq M$, in this case the restriction of A to M is denoted by A_M , with $A_M \in \mathcal{L}(M)$. A subspace $X_1 \subset X$ reduces A if there exists another subspace $X_2 \subset X$ such that $X = X_1 \oplus X_2$ with X_1 and X_2 being A -stable subspaces. When a pair of subspaces (X_1, X_2) satisfies this property we write $(X_1, X_2) \in \text{Red}(A)$. In this case, the restrictions $A_1 := A_{X_1}$ and $A_2 := A_{X_2}$ act on X_1 and X_2 , respectively, and $A = A_1 \oplus A_2$ in the sense that any $x \in X$ has a unique decomposition $x = x_1 + x_2$, $x_i \in X_i$, with $Ax = A_1 x_1 + A_2 x_2$, where $A_i \in \mathcal{L}(X_i)$, for $i = 1, 2$.

This paper is organized as follows. In Section 2, we describe the relationship between the subspaces of a generalized Kato decomposition and the Mbekhta subspaces for operators weaker than pseudo-Fredholm operators. In Section 3, we define and characterize left and right generalized Drazin invertible operators in terms of the generalized Kato decomposition. Then, we give new binary relations by means of left and right generalized Drazin invertible operators that become pre-orders. As it is shown, these pre-orders extend the pre-orders given by [13]. As a consequence, the well-known Drazin pre-order can be obtained by combining these new left and right pre-orders. Section 4 develops a new technique to define new partial orders by using left and right generalized Drazin invertible operators on a Banach space.

2. On the Kato decompositon and the Mbekhta subspaces

There exist two subspaces of X that play an important role in the development of the generalized Drazin inverse of an operator $A \in \mathcal{L}(X)$, namely, the quasi-nilpotent

part $H_0(A)$ of A :

$$H_0(A) = \left\{ x \in X : \lim_{n \rightarrow \infty} \|A^n x\|^{1/n} = 0 \right\}$$

and the analytical core part $K(A)$ of A :

$$K(A) = \{x \in X : \text{there exist a sequence } (x_n) \in X \text{ and a constant } \delta > 0 \text{ such that } Ax_1 = x, Ax_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n \in \mathbb{N}\}.$$

From the above definitions (see [1, Theorem 1.41 and Lemma 1.67]), it follows that

$$A(K(A)) = K(A) \quad \text{and} \quad A(H_0(A)) \subseteq H_0(A). \tag{1}$$

DEFINITION 2.1. [1, Definition 1.11] Let $A \in \mathcal{L}(X)$. If $R(A)$ is closed and $N(A^n) \subseteq R(A)$, for every $n \in \mathbb{N}$, then A is said to be semi-regular.

DEFINITION 2.2. Let $A \in \mathcal{L}(X)$. If there exists a pair of subspaces (X_1, X_2) in X such that $(X_1, X_2) \in \text{Red}(A)$ with A_1 semi-regular and A_2 quasi-nilpotent, then A is said to be a pseudo-Fredholm operator. In this case, the pair (X_1, X_2) is called a generalized Kato decomposition for A and it is denoted by $\text{GKD}(X_1, X_2)$.

The following results provide us with some useful and important properties.

THEOREM 2.3. [3, 5, 15] Let $A \in \mathcal{L}(X)$. Then the following statements are equivalent:

- (a) A is g -Drazin invertible;
- (b) $X = K(A) \oplus H_0(A)$;
- (c) $A = A_1 \oplus A_2$ where $A_1 = A_{K(A)} \in \mathcal{L}(K(A))$ is invertible and $A_2 = A_{H_0(A)} \in \mathcal{L}(H_0(A))$ is quasi-nilpotent.

THEOREM 2.4. [1, Theorem 1.41 and Corollary 1.69] If $A \in \mathcal{L}(X)$ is a pseudo-Fredholm operator with $\text{GKD}(X_1, X_2)$ then:

- (a) $K(A) = K(A_1)$ and $K(A)$ is closed;
- (b) $H_0(A) = H_0(A_1) \oplus H_0(A_2) = H_0(A_1) \oplus X_2$;
- (c) $N(A_1) = K(A) \cap N(A)$.

THEOREM 2.5. [1, Theorem 1.22] Let F be a closed subspace of X and let $A \in \mathcal{L}(X)$. If $A(F) = F$ then $F \subseteq K(A)$.

Next result will be crucial for the subsequent proofs. It basically describes the relationship between the subspaces of a generalized Kato decomposition and the Mbekhta subspaces when the operator $A \in \mathcal{L}(X)$ admits a left and/or right invertible part plus a quasi-nilpotent part.

LEMMA 2.6. Let $A \in \mathcal{L}(X)$ be a pseudo-Fredholm operator with $\text{GKD}(X_1, X_2)$.

(a) If A_1 is left invertible then $H_0(A)$ is closed in X and $X_2 = H_0(A)$.

(b) If A_1 is right invertible then $K(A)$ is closed in X and $X_1 = K(A)$.

Proof.

(a) Assume that A_1 is left invertible and let $x \in H_0(A_1)$. Then there exists $M > 0$ such that

$$\|x\|M^n \leq \|A_1^n x\|, \quad n \in \mathbb{N}.$$

Hence, $\limsup_{n \rightarrow \infty} \|x\|^{1/n} = 0$, i.e., $H_0(A_1) = \{0\}$. Thus, by Theorem 2.4 (b) it follows that $X_2 = H_0(A)$.

(b) Suppose now that A_1 is right invertible. Since $A_1(X_1) = X_1$, it then results $X_1 = K(A_1) = K(A)$ by Theorem 2.5 and Theorem 2.4 (a). \square

COROLLARY 2.7. Let $A \in \mathcal{L}(X)$ be a pseudo-Fredholm operator with $\text{GKD}(X_1, X_2)$. If A_1 is invertible then $X_1 = K(A)$ and $X_2 = H_0(A)$.

3. Left and right pre-orders on $\mathcal{L}(X)$

We start with the following definitions of left and right generalized Drazin invertible operators.

DEFINITION 3.1. An operator $A \in \mathcal{L}(X)$ is said to be left g -Drazin invertible if $H_0(A)$ is complemented in X with a subspace L^A such that $A(L^A)$ is complemented in L^A .

The set of all subspaces associated with an operator A satisfying Definition 3.1 is denoted by $\{L^A\}$.

DEFINITION 3.2. An operator $A \in \mathcal{L}(X)$ is said to be right g -Drazin invertible if $K(A)$ is complemented in X with a subspace R^A such that $A(R^A) \subseteq R^A \subseteq H_0(A)$ and $N(A) \cap K(A)$ is complemented in $K(A)$.

The set of all subspaces associated with an operator A satisfying Definition 3.2 is denoted by $\{R^A\}$.

Now, we present a characterization of left generalized Drazin invertible operators by using a generalized Kato decomposition.

THEOREM 3.3. Let $A \in \mathcal{L}(X)$. Then the following statements are equivalent:

(a) A is left g -Drazin invertible with $L^A \in \{L^A\}$;

(b) A is a pseudo-Fredholm operator with $\text{GKD}(L^A, H_0(A))$, where A_1 is left invertible;

(c) *there exists a projection $P \in \mathcal{L}(X)$ such that $AP = PA$, $(A + P)_{N(P)}$ is left invertible and AP is quasi-nilpotent.*

Proof. (a) \Rightarrow (b) If A is left g -Drazin invertible with $L^A \in \{L^A\}$, then there exist subspaces $L^A \in \{L^A\}$ and $B \subseteq L^A$ such that

$$L^A \oplus H_0(A) = X \quad \text{and} \quad A(L^A) \oplus B = L^A. \tag{2}$$

By (1), $(L^A, H_0(A)) \in \text{Red}(A)$ holds. It is clear that $A_2 = A_{H_0(A)}$ is quasi-nilpotent. Furthermore, if $A_1 = A_{L^A}$ then we have $N(A_1) = L^A \cap N(A) \subseteq L^A \cap H_0(A) = \{0\}$. So A_1 is injective, and by (2), A_1 is left invertible. Consequently, A is a pseudo-Fredholm operator with $\text{GKD}(L^A, H_0(A))$, where A_1 is left invertible.

(b) \Rightarrow (c) Assume that A is a pseudo-Fredholm operator with $\text{GKD}(L^A, H_0(A))$, where A_1 is left invertible. Let $P \in \mathcal{L}(X)$ be the projection onto $R(P) = H_0(A)$ along $N(P) = L^A$. Thus, $X = N(P) \oplus R(P)$. By hypothesis, we have that $A(R(P)) \subseteq R(P)$ and $A(N(P)) \subseteq N(P)$.

Let $x \in X$. It is clear that $x = x_1 + x_2$ is uniquely represented in this form with $x_1 \in N(P)$ and $x_2 \in R(P)$. Note that $APx = A_2x_2 = PAx$, i.e., $AP = PA$. So, $(AP)^n x = A^n P^n x = A_2^n x_2$, for any $n \in \mathbb{N}$, and consequently

$$\|(AP)^n x\|^{\frac{1}{n}} = \|A_2^n x_2\|^{\frac{1}{n}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Therefore, $H_0(AP) = X$. Hence, AP is quasi-nilpotent.

Now, by hypothesis and from $(A + P)_{N(P)} = A_1$ item (c) holds.

(c) \Rightarrow (a) Suppose that there exists a projection $P \in \mathcal{L}(X)$ such that $AP = PA$, $(A + P)_{N(P)}$ is left invertible, and AP is quasi-nilpotent. We know that $X = N(P) \oplus R(P)$ since P is a continuous projection. As A and P commute, $N(P)$ and $R(P)$ are A -stable subspaces, and so $(N(P), R(P)) \in \text{Red}(A)$. For $x \in R(P)$, we have $A_{R(P)}^n x = A^n P^n x = (AP)^n x$, for any $n \in \mathbb{N}$. Thus, $H_0(A_{R(P)}) = R(P)$ since AP is quasi-nilpotent. This implies that $A_{R(P)}$ is quasi-nilpotent.

On the other hand, clearly $A_{N(P)}$ is left invertible since $A_{N(P)} = (A + P)_{N(P)}$. Thus, $H_0(A_{N(P)}) = \{0\}$. Hence, A is a pseudo-Fredholm operator with $\text{GKD}(N(P), R(P))$. Consequently, $H_0(A) = H_0(A_{N(P)}) \oplus R(P) = R(P)$ by Theorem 2.4. Summarizing this reasoning we have that $H_0(A) \oplus N(P) = X$, where $A(N(P))$ is complemented in $N(P)$ because $A_{N(P)}$ is left invertible. It follows that A is left g -Drazin invertible with $N(P) \in \{L^A\}$. The proof is complete. \square

We get a similar result for right generalized Drazin invertible operators.

THEOREM 3.4. *Let $A \in \mathcal{L}(X)$. Then the following statements are equivalent:*

- (a) *A is right g -Drazin invertible with $R^A \in \{R^A\}$;*
- (b) *A is a pseudo-Fredholm operator with $\text{GKD}(K(A), R^A)$, where A_1 is right invertible;*
- (c) *there exists a projection $P \in \mathcal{L}(X)$ such that $AP = PA$, $(A + P)_{N(P)}$ is right invertible and AP is quasi-nilpotent.*

Proof. (a) \Rightarrow (b) If A is right g -Drazin invertible with $R^A \in \{R^A\}$, then there exist subspaces $R^A \in \{R^A\}$ and $B \subseteq K(A)$ such that

$$K(A) \oplus R^A = X, \quad A(R^A) \subseteq R^A \subseteq H_0(A), \quad \text{and} \quad (N(A) \cap K(A)) \oplus B = K(A). \quad (3)$$

By (1), $(K(A), R^A) \in \text{Red}(A)$. From (1) and (3) it follows that $A_1 = A_{K(A)}$ is surjective and $N(A_1) = N(A) \cap K(A)$ is complemented in $K(A)$. Thus, A_1 is right invertible. Moreover, $A_2 = A_{R^A}$ is quasi-nilpotent since $H_0(A_2) = H_0(A) \cap R^A = R^A$. Hence, A is a pseudo-Fredholm operator with $\text{GKD}(L^A, H_0(A))$, where A_1 is right invertible.

(b) \Rightarrow (c) Assume that A is a pseudo-Fredholm operator with $\text{GKD}(K(A), R^A)$, where A_1 is right invertible. Let $P \in \mathcal{L}(X)$ be the projection onto $R(P) = R^A$ along $N(P) = K(A)$, and so $X = N(P) \oplus R(P)$. A similar reasoning to that in the proof of Theorem 3.3 allows us to obtain that $AP = PA$, AP is quasi-nilpotent, and $(A + P)_{N(P)} = A_1$ is right invertible.

(c) \Rightarrow (a) Suppose that there exists a projection $P \in \mathcal{L}(X)$ such that $AP = PA$, $(A + P)_{N(P)}$ is right invertible, and AP is quasi-nilpotent. As in the proof of Theorem 3.3 we have that $(N(P), R(P)) \in \text{Red}(A)$ and $A_{R(P)}$ is quasi-nilpotent.

On the other hand, clearly, $A_{N(P)}$ is right invertible since $A_{N(P)} = (A + P)_{N(P)}$. Thus, $N(A_{N(P)})$ is complemented in $N(P)$. As A is a pseudo-Fredholm operator with $\text{GKD}(N(P), R(P))$, Theorem 2.4 implies that $N(A_{N(P)}) = N(A) \cap K(A)$ and $R(P) \subseteq H_0(A)$ hold. Also, from Lemma 2.6 we get $N(P) = K(A)$. Summarizing this reasoning we have that $K(A) \oplus R(P) = X$, $A(R(P)) \subseteq R(P) \subseteq H_0(A)$, and $N(A) \cap K(A)$ is complemented in $K(A)$. Hence, A is right g -Drazin invertible with $R(P) \in \{R^A\}$. \square

COROLLARY 3.5. *Let $A \in \mathcal{L}(X)$. Then A is g -Drazin invertible if and only if A is left and right g -Drazin invertible.*

In what follows we define new binary relations based on the left and right g -Drazin inverses of a bounded linear operator on Banach space.

DEFINITION 3.6. Let $A, B \in \mathcal{L}(X)$ be left g -Drazin invertible operators. It is said that $A \preceq^{LD} B$ if L^A is complemented in L^B with a B -stable subspace L of X such that $B_{L^A} = A_{L^A}$, for some $L^A \in \{L^A\}$ and $L^B \in \{L^B\}$.

REMARK 3.7. The subspace L in Definition 3.6 is closed in X .

Now, we are able to establish a characterization of the binary relation \preceq^{LD} .

THEOREM 3.8. *Let $A, B \in \mathcal{L}(X)$ be left g -Drazin invertible operators. Then the following statements are equivalent:*

(a) $A \preceq^{LD} B$;

(b) *there exist topological direct sums $X = X_1 \oplus X_2$ and $X_2 = X_1^2 \oplus X_2^2$ such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1^2 \oplus B_2^2 \end{bmatrix},$$

where $A_i \in \mathcal{L}(X_i)$, $B_i^2 \in \mathcal{L}(X_i^2)$, for $i = 1, 2$, with A_1, B_1^2 being left invertible and A_2, B_2^2 quasi-nilpotent.

Proof. (a) \Rightarrow (b) Since $A \preceq^{LD} B$, there exist subspaces $L \subseteq X$, $L^A \in \{L^A\}$ and $L^B \in \{L^B\}$ such that

$$B(L) \subseteq L, \quad L^A \oplus L = L^B, \quad \text{and} \quad B_{L^A} = A_{L^A}.$$

By Theorem 3.3, we have that A and B are pseudo-Fredholm operators with $\text{GKD}(L^A, H_0(A))$ and $\text{GKD}(L^B, H_0(B))$, respectively, and with A_1 and B_1 being left invertible. Then

$$B_1 = B_{L^B} = B_{L^A} \oplus B_L = A_{L^A} \oplus B_L = A_1 \oplus B_L.$$

Let us denote $X_1^2 = L$, $X_2^2 = H_0(B)$, $X_1 = L^A$, $X_2 = X_1^2 \oplus X_2^2$, $B_1^2 = B_L$, and $B_2^2 = B_2$. Since $X = L^B \oplus X_2^2$ and $L^B = X_1 \oplus X_1^2$, it is easy to see that $X = X_1 \oplus (X_1^2 \oplus X_2^2) = X_1 \oplus X_2$. Thus,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1^2 \oplus B_2^2 \end{bmatrix},$$

where $A_i \in \mathcal{L}(X_i)$, $B_i^2 \in \mathcal{L}(X_i^2)$, for $i = 1, 2$, with A_1, B_1^2 being left invertible and A_2, B_2^2 quasi-nilpotent.

(b) \Rightarrow (a) Suppose that A and B are given as in (b). Clearly, the pair (X_1, X_2) is a generalized Kato decomposition for A , because A_1 is left invertible and A_2 is quasi-nilpotent. Thus, from Lemma 2.6 (a) we obtain $X_2 = H_0(A)$ and then $X = X_1 \oplus X_2 = X_1 \oplus H_0(A)$. Moreover, since $A_1 : X_1 \rightarrow X_1$ is left invertible, then $A(X_1) = A_1(X_1)$ is complemented in X_1 . Hence, $X_1 \in \{L^A\}$.

As $X = X_1 \oplus X_2$ and $X_2 = X_1^2 \oplus X_2^2$, we arrive at

$$X = (X_1 \oplus X_1^2) \oplus X_2^2. \tag{4}$$

Then, the pair $(X_1 \oplus X_1^2, X_2^2) \in \text{Red}(B)$. Moreover, $B_1 = A_1 \oplus B_1^2 : X_1 \oplus X_1^2 \rightarrow X_1 \oplus X_1^2$ is left invertible because

$$\begin{bmatrix} (A_1)_{\text{left}}^{-1} & 0 \\ 0 & (B_1^2)_{\text{left}}^{-1} \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & B_1^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Therefore, since $B_2^2 : X_2^2 \rightarrow X_2^2$ is quasi-nilpotent, the pair $(X_1 \oplus X_1^2, X_2^2)$ is a generalized Kato decomposition of B . Thus, by applying Lemma 2.6 (a) we obtain

$$X_2^2 = H_0(B). \tag{5}$$

Also, the left invertibility of B_1 implies that $B(X_1 \oplus X_1^2)$ is complemented in $X_1 \oplus X_1^2$. So, from (4) and (5) we have $X_1 \oplus X_1^2 \in \{L^B\}$.

Finally, if we take $L^A = X_1$, $L = X_1^2$, and $L^B = X_1 \oplus X_1^2$ then there exists a B -stable subspace L such that $L^A \oplus L = L^B$ and $B_{L^A} = A_1 = A_{X_1} = A_{L^A}$, which completes the proof. \square

As a consequence of the proof the above theorem, we obtain the following remark.

REMARK 3.9. If $A \preceq^{LD} B$ then $H_0(A) = L \oplus H_0(B)$, with L satisfying Definition 3.6.

In general, the binary relation \preceq^{LD} is not antisymmetric as it can be derived from the following result.

COROLLARY 3.10. *Let $A, B \in \mathcal{L}(X)$ be left g -Drazin invertible operators. If $A \preceq^{LD} B$ and $B \preceq^{LD} A$ then there exists a topological direct sum $X = X_1 \oplus X_2$ such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

where $A_i \in \mathcal{L}(X_i)$, for $i = 1, 2$, $B_2 \in \mathcal{L}(X_2)$, with A_1 being left invertible and A_2, B_2 quasi-nilpotent.

Proof. Since $A \preceq^{LD} B$, by Remark 3.9 we obtain $H_0(A) = L \oplus H_0(B)$. Analogously, from $B \preceq^{LD} A$ we have $H_0(B) = L' \oplus H_0(A)$, for some subspace L' . Thus, $H_0(A) = H_0(B)$, $L = L' = \{0\}$, and $L^A = L^B$. The conclusion follows immediately from the proof of Theorem 3.8. \square

Notice that the set of all left g -Drazin invertible operators in $\mathcal{L}(X)$ is nonempty (for instance, $I \in \mathcal{L}(X)$). An important consequence of the characterization given in Theorem 3.8 is the following.

THEOREM 3.11. *The relation \preceq^{LD} is a pre-order on the set of all left g -Drazin invertible operators in $\mathcal{L}(X)$.*

Proof. It is immediate to see that reflexive property holds. Let $A, B, C \in \mathcal{L}(X)$ be left g -Drazin invertible operators such that $A \preceq^{LD} B$ and $B \preceq^{LD} C$. Since $A \preceq^{LD} B$, by Theorem 3.8 there exist topological direct sums $X = X_1 \oplus X_2$ and $X_2 = X_1^2 \oplus X_2^2$ such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1^2 \oplus B_2^2 \end{bmatrix},$$

where $A_i \in \mathcal{L}(X_i)$, $B_i^2 \in \mathcal{L}(X_i^2)$, for $i = 1, 2$, with A_1, B_1^2 being left invertible and A_2, B_2^2 quasi-nilpotent. As it was proved in Theorem 3.8, $X = (X_1 \oplus X_1^2) \oplus X_2^2$ and the operator B can be represented as

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2^2 \end{bmatrix},$$

where $B_1 = A_1 \oplus B_1^2 \in \mathcal{L}(X_1 \oplus X_1^2)$ is left invertible. From $B \preceq^{LD} C$, it is analogously proved that $X_2^2 = Y_1^2 \oplus Y_2^2$ and

$$C = \begin{bmatrix} B_1 & 0 \\ 0 & C_1^2 \oplus C_2^2 \end{bmatrix},$$

where $C_i^2 \in \mathcal{L}(Y_i^2)$, for $i = 1, 2$, with C_1^2 being left invertible, and $C_2^2 \in \mathcal{L}(Y_2^2)$ quasi-nilpotent. Clearly, $X = X_1 \oplus (X_1^2 \oplus (Y_1^2 \oplus Y_2^2)) = X_1 \oplus ((X_1^2 \oplus Y_1^2) \oplus Y_2^2)$. Therefore, the operator C can be written as

$$C = \begin{bmatrix} A_1 & 0 \\ 0 & (B_1^2 \oplus C_1^2) \oplus C_2^2 \end{bmatrix},$$

where $B_1^2 \oplus C_1^2 \in \mathcal{L}(X_1^2 \oplus Y_1^2)$ is left invertible. Hence, $A \preceq^{LD} C$. \square

By using the characterization given in Theorem 3.8, we next provide an example to show that the generalized Drazin pre-order \preceq^d defined in [13] is a proper subset of \preceq^{LD} . This means that our pre-order \preceq^{LD} is an extension of \preceq^d .

EXAMPLE 3.12. Let $X_1 = \{x = (x_n)_{n=1}^\infty \in l^2 : x_{2n-1} = 0\}$, $X_2 = \{x = (x_n)_{n=1}^\infty \in l^2 : x_{2n} = 0\}$ and let $A_i : X_i \rightarrow X_i$, $i = 1, 2$, be the bounded linear operators defined by

$$A_1x = (0, x_2, 0, x_4, \dots) \quad \text{and} \quad A_2x = \left(\frac{x_3}{3}, 0, \frac{x_5}{5}, 0, \frac{x_7}{7}, \dots\right).$$

Clearly, $l^2 = X_1 \oplus X_2$, A_1 is invertible, and

$$A_2^k x = \left(\frac{x_{2k+1}}{3 \cdot 5 \cdot 7 \dots (2k+1)}, 0, \frac{x_{2k+3}}{5 \cdot 7 \cdot 9 \dots (2k+3)}, 0, \frac{x_{2k+5}}{7 \cdot 9 \cdot 11 \dots (2k+5)}, \dots\right),$$

for $k \geq 1$. Note that if $n \in \mathbb{N}$ is odd then $\prod_{s=0}^{k-1} \frac{1}{2(s+1)+n} \leq \frac{1}{k!}$. Thus, $\|A_2^k x\| \leq$

$\frac{1}{k!} \left(\sum_{n=k}^\infty x_{2n+1}^2\right)^{\frac{1}{2}} \leq \frac{\|x\|}{k!}$ and then $\|A_2^k\|^{\frac{1}{k}} \leq \frac{1}{\sqrt[k]{k!}}$. Therefore, A_2 is quasi-nilpotent and then $H_0(A_2) = X_2$.

Hence, the operator $A : l^2 \rightarrow l^2$ defined by

$$Ax = \left(\frac{x_3}{3}, x_2, \frac{x_5}{5}, x_4, \frac{x_7}{7}, \dots\right), \quad x = (x_n)_{n=1}^\infty$$

is a bounded linear operator that satisfies $A = A_1 \oplus A_2$. Thus, A is left g -Drazin invertible with $L^A = X_1$ and $H_0(A) = X_2$.

Let $B : l^2 \rightarrow l^2$ the injective bounded linear operator defined by

$$Bx = (0, x_2, x_1, x_4, x_3, \dots), \quad x = (x_n)_{n=1}^\infty.$$

It is easy to see that B is left invertible and then it is left g -Drazin invertible with $L^B = l^2$ and $H_0(B) = \{0\}$. If we take $L = X_2$, it then follows that $A \preceq^{LD} B$. However, B is not g -Drazin invertible. In addition, by Theorem 2.3 we have $K(B) = l^2$ and $B_{K(B)}$ is invertible, which is impossible since it is not surjective in l^2 . Therefore, $A \preceq^d B$ does not hold.

DEFINITION 3.13. Let $A, B \in \mathcal{L}(X)$ be right g -Drazin invertible operators. It is said that $A \preceq^{RD} B$ if $K(A)$ is complemented in $K(B)$ with a B -stable subspace L of X such that $B_{K(A)} = A_{K(A)}$.

REMARK 3.14. The subspace L in Definition 3.13 is closed in X .

THEOREM 3.15. *Let $A, B \in \mathcal{L}(X)$ be right g -Drazin invertible operators. Then the following statements are equivalent:*

(a) $A \preceq^{RD} B$;

(b) *there exist topological direct sums $X = X_1 \oplus X_2$ and $X_2 = X_1^2 \oplus X_2^2$ such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1^2 \oplus B_2^2 \end{bmatrix},$$

where $A_i \in \mathcal{L}(X_i)$, $B_i^2 \in \mathcal{L}(X_i^2)$, for $i = 1, 2$, with A_1, B_1^2 being right invertible, and A_2, B_2^2 quasi-nilpotent.

Proof. (a) \Rightarrow (b) Since $A \preceq^{RD} B$, there exists a subspace $L \subseteq X$ such that

$$B(L) \subseteq L, \quad K(A) \oplus L = K(B), \quad \text{and} \quad B_{K(A)} = A_{K(A)}.$$

By Theorem 3.4, there exist $R^A \in \{R^A\}$ and $R^B \in \{R^B\}$ such that A and B are pseudo-Fredholm operators with $\text{GKD}(K(A), R^A)$ and $\text{GKD}(K(B), R^B)$, respectively, with A_1 and B_1 being right invertible.

Let us denote $X_1^2 = L$, $X_2^2 = R^B$, $X_1 = K(A)$, $X_2 = X_1^2 \oplus X_2^2$, $B_1^2 = B_L$, and $B_2^2 = B_2$. Since $X = K(B) \oplus X_2^2$ and $K(B) = X_1 \oplus X_1^2$, we have $X = X_1 \oplus (X_1^2 \oplus X_2^2) = X_1 \oplus X_2$. Hence,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1^2 \oplus B_2^2 \end{bmatrix},$$

where $A_i \in \mathcal{L}(X_i)$, $B_i^2 \in \mathcal{L}(X_i^2)$, for $i = 1, 2$, with A_1, B_1^2 being right invertible and A_2, B_2^2 quasi-nilpotent.

(b) \Rightarrow (a) Suppose that A and B are given as in (b). Clearly, the pair (X_1, X_2) is a generalized Kato decomposition for A , because A_1 is right invertible and A_2 is quasi-nilpotent. Thus, by Lemma 2.6 (b) we have $X_1 = K(A)$ and then

$$X = K(A) \oplus X_2. \tag{6}$$

In a similar way to the proof of Theorem 3.8, it is proved that the pair $(X_1 \oplus X_1^2, X_2^2)$ is a generalized Kato decomposition of B , where $B_1 = A_1 \oplus B_1^2 : X_1 \oplus X_1^2 \rightarrow X_1 \oplus X_1^2$ is right invertible and $B_2^2 : X_2^2 \rightarrow X_2^2$ is quasi-nilpotent. Hence, from Lemma 2.6 (b) we have

$$K(A) \oplus X_1^2 = X_1 \oplus X_1^2 = K(B).$$

Since $B(X_1^2) = B_1^2(X_1^2) \subseteq X_1^2$ and $B_{K(A)} = A_1 = A_{K(A)}$, the proof is complete. \square

In general, the binary relation \preceq^{RD} is not antisymmetric as it can be derived from the following result.

COROLLARY 3.16. *Let $A, B \in \mathcal{L}(X)$ be right g -Drazin invertible operators. If $A \preceq^{RD} B$ and $B \preceq^{RD} A$ then there exists a topological direct sum $X = X_1 \oplus X_2$ such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

where $A_i \in \mathcal{L}(X_i)$, for $i = 1, 2$, $B_2 \in \mathcal{L}(X_2)$, with A_1 being right invertible and A_2, B_2 quasi-nilpotent.

Proof. Since $A \preceq^{RD} B$, by Definition 3.13 we have $K(B) = L \oplus K(A)$. Analogously, from $B \preceq^{RD} A$ we obtain $K(A) = L' \oplus K(B)$, for some subspace L' . Hence, $K(A) = K(B)$ and $L = L' = \{0\}$. The conclusion follows directly from the proof of Theorem 3.15. \square

Similar results to Theorem 3.11 can be obtained as a consequence of Theorem 3.15.

THEOREM 3.17. *The relation \preceq^{RD} is a pre-order on the set of all right g -Drazin invertible operators in $\mathcal{L}(X)$.*

The next example shows that the generalized Drazin pre-order \preceq^d defined in [13] is a proper subset of \preceq^{RD} .

EXAMPLE 3.18. Let $X_i \subset l^2$, $i = 1, 2$, and let $A : l^2 \rightarrow l^2$ be as in Example 3.12. Let $B : l^2 \rightarrow l^2$ be the bounded linear operator defined by

$$Bx = (x_3, x_2, x_5, x_4, x_7, \dots), \quad x = (x_n)_{n=1}^\infty.$$

It is easy to see that B is surjective and then right g -Drazin invertible, with $K(B) = l^2$ and $R^B = \{0\}$. If we take $L = X_2$, then $A \preceq^{RD} B$. However, B is not g -Drazin invertible. In addition, by Theorem 2.3 we have $B_{K(B)}$ is invertible, which is impossible since it is not injective in l^2 . Therefore, $A \preceq^d B$ does not hold.

Our next result establishes an important characterization of the generalized Drazin pre-order \preceq^d in terms of the new pre-orders \preceq^{LD} and \preceq^{RD} . Before that, we quote the following needed result.

THEOREM 3.19. [13, Corollary 2.1] *Let $A, B \in \mathcal{L}(X)$ be g -Drazin invertible operators. Then the following statements are equivalent:*

- (a) $A \preceq^d B$;
- (b) *there exist topological direct sums $X = X_1 \oplus X_2$ and $X_2 = X_1^2 \oplus X_2^2$ such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1^2 \oplus B_2^2 \end{bmatrix},$$

where $A_i \in \mathcal{L}(X_i)$, $B_i^2 \in \mathcal{L}(X_i^2)$, for $i = 1, 2$, with A_1, B_1^2 being invertible, and A_2, B_2^2 quasi-nilpotent.

Now, we are able to prove the following result.

THEOREM 3.20. *Let $A, B \in \mathcal{L}(X)$. Then the following statements are equivalent:*

- (a) $A \preceq^d B$;
- (b) $A \preceq^{LD} B$ and $A \preceq^{RD} B$.

Proof. (a) \Rightarrow (b) It follows immediately from Corollary 3.5 and Theorems 3.8, 3.15, and 3.19.

(b) \Rightarrow (a) Since A and B are left and right g -Drazin invertible operators, it is clear from Corollary 3.5 that A and B are g -Drazin invertible. By Theorem 2.3 we have that A and B are pseudo-Fredholm operators with $\text{GKD}(K(A), H_0(A))$ and $\text{GKD}(K(B), H_0(B))$, respectively, where $A_1 = A_{K(A)}$ and $B_1 = B_{K(B)}$ are invertible. Item (b) of Theorem 3.19 can be obtained in a similar way to the proof of Theorem 3.15. Hence, $A \preceq^d B$. \square

REMARK 3.21. The relations \preceq^d , \preceq^{LD} , and \preceq^{RD} are equivalent on the set of all g -Drazin invertible operators.

4. New partial orders on $\mathcal{L}(X)$

It is well known that the generalized Drazin binary relation \preceq^d is only a pre-order, because the quasi-nilpotente part of operators is not considered. As an application of the previous results, the aim of this section is to give new binary relations on the set $\mathcal{L}(X)$. In order to obtain partial orders on $\mathcal{L}(X)$, it will be used the one-sided pre-orders \preceq^{LD} or \preceq^{RD} on the analytical core part and any partial order on the quasi-nilpotent part of the operators. Up to now, the natural way to define a partial order is to consider two operators in the core quasi-nilpotent form and to order both, the core part and the quasi-nilpotent part. For that, it was used some known partial orders such as the sharp order and the minus order, among others. In this section one of these conditions is weakened. It is enough to consider a partial order on the quasi-nilpotent part and a pre-order on the core part. In this way, a new sufficient condition for the generalized Drazin pre-order to be a partial order is obtained.

By the symbol \leq^P we denote any partial order on the set of quasi-nilpotent operators in $\mathcal{L}(X)$.

REMARK 4.1. By Theorem 3.8 and Theorem 3.15 it is easy to see that the following statements hold:

- (a) $A \preceq^{LD} B$ if and only if $A_C \preceq^{LD} B_C$.
- (b) $A \preceq^{RD} B$ if and only if $A_C \preceq^{RD} B_C$.

DEFINITION 4.2. Let $A, B \in \mathcal{L}(X)$ be left g -Drazin invertible operators and let \leq^P be any partial order (fixed but arbitrary) on the set of quasi-nilpotent operators in $\mathcal{L}(X)$. It is said that $A \leq^{LD,P} B$ if $A_C \preceq^{LD} B_C$ and $A_Q \leq^P B_Q$.

Since \preceq^{LD} is a pre-order and \leq^P is a partial order we have that $\leq^{LD,P}$ is a pre-order on $\mathcal{L}(X)$.

THEOREM 4.3. *The relation $\leq^{LD,P}$ is a partial order on the set of left g -Drazin invertible operators of $\mathcal{L}(X)$.*

Proof. It is enough to check antisymmetric property. If $A_C \preceq^{LD} B_C$ and $B_C \preceq^{LD} A_C$, then by Corollary 3.10 $A_C = B_C$,

$$A_Q = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \text{and} \quad B_Q = \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix},$$

where $X = X_1 \oplus X_2$ and $A_2, B_2 \in \mathcal{L}(X_2)$ are quasi-nilpotent. Moreover, if $A_Q \leq^P B_Q$ and $B_Q \leq^P A_Q$ it follows that $A_2 = B_2$ and then $A = B$. \square

An operator $A \in \mathcal{L}(X)$ is said to be *regular* (or *relatively regular*) if there exists $B \in \mathcal{L}(X)$ such that $ABA = A$. The operator B is called an inner generalized inverse of A . Let $\mathcal{L}_{reg}(X)$ be the class of all regular operators in $\mathcal{L}(X)$.

The minus partial order was defined in [6] for regular elements in rings. Let $A, B \in \mathcal{L}_{reg}(X)$. Then A is said to be below B under the minus order if there exists an inner generalized inverse A^- of A such that $AA^- = BA^-$ and $A^-A = A^-B$. We write $A \leq^- B$.

The following characterization is obtained when the relation \leq^P is particularized to \leq^- in Definition 4.2.

THEOREM 4.4. *Let $A, B \in \mathcal{L}(X)$ be left g -Drazin invertible operators and let \leq^- be the minus order on $\mathcal{L}_{reg}(X)$. Then the following statements are equivalent:*

- (a) $A_Q, B_Q \in \mathcal{L}_{reg}(X)$ and $A \leq^{LD,-} B$;
- (b) there exist topological direct sums $X = X_1 \oplus X_2$ and $X_2 = X_1^2 \oplus X_2^2$ such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \oplus A_2^2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1^2 \oplus B_2^2 \end{bmatrix},$$

where $A_1 \in \mathcal{L}(X_1)$, $A_2^2 \in \mathcal{L}(X_2^2)$, $B_i^2 \in \mathcal{L}(X_i^2)$, for $i = 1, 2$, the operators A_1, B_1^2 being left invertible, $A_2^2 \leq^- B_2^2$ and A_2^2, B_2^2 being relatively regular and quasi-nilpotent.

Proof. (a) \Rightarrow (b) Since $A \leq^{LD,-} B$, by Theorem 3.8 there exist topological direct sums $X = X_1 \oplus X_2$ and $X_2 = X_1^2 \oplus X_2^2$ such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1^2 \oplus B_2^2 \end{bmatrix},$$

where $A_i \in \mathcal{L}(X_i)$, $B_i^2 \in \mathcal{L}(X_i^2)$, for $i = 1, 2$, the operators A_1, B_1^2 being left invertible and A_2, B_2^2 being quasi-nilpotent. Hence,

$$A_Q = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B_Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \oplus B_2^2 \end{bmatrix}.$$

Let us suppose that

$$T = \begin{bmatrix} 0 & 0 \\ 0 & B_2^2 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} A_1^2 & A_3^2 \\ A_4^2 & A_2^2 \end{bmatrix} : \begin{bmatrix} X_1^2 \\ X_2^2 \end{bmatrix} \rightarrow \begin{bmatrix} X_1^2 \\ X_2^2 \end{bmatrix}.$$

By hypotheses $A_Q \leq^- B_Q$, thus $A_2 \leq^- T$ which implies $R(A_2) \subseteq R(T)$ and $N(T) \subseteq N(A_2)$. Hence, $A_1^2 = 0$, $A_3^2 = 0$, and $A_4^2 = 0$, i.e., $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & A_2^2 \end{bmatrix}$. Then $A_2^2 \in \mathcal{L}(X_2^2)$ and $A_2^2 \leq^- B_2^2$. Finally, from $A_Q, B_Q \in \mathcal{L}_{reg}(X)$ we get that A_2^2, B_2^2 are relatively regular. This completes the proof.

(b) \Rightarrow (a) It is trivial. \square

Analogously, we can define the following pre-order on $\mathcal{L}(X)$.

DEFINITION 4.5. Let $A, B \in \mathcal{L}(X)$ be right g -Drazin invertible operators and let \leq^P any partial order (fixed but arbitrary) on the set of quasi-nilpotent operators in $\mathcal{L}(X)$. It is said that $A \leq^{RD,P} B$ if $A_C \leq^{RD} B_C$ and $A_Q \leq^P B_Q$.

In a similar way to left pre-order we can prove the following results.

THEOREM 4.6. *The relation $\leq^{RD,P}$ is a partial order on the set of right g -Drazin invertible operators in $\mathcal{L}(X)$.*

THEOREM 4.7. *Let $A, B \in \mathcal{L}(X)$ be right g -Drazin invertible operators and let \leq^- be the minus order on $\mathcal{L}_{reg}(X)$. Then the following statements are equivalent:*

(a) $A_Q, B_Q \in \mathcal{L}_{reg}(X)$ and $A \leq^{RD,-} B$;

(b) *there exist topological direct sums $X = X_1 \oplus X_2$ and $X_2 = X_1^2 \oplus X_2^2$ such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \oplus A_2^2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1^2 \oplus B_2^2 \end{bmatrix},$$

where $A_1 \in \mathcal{L}(X_1)$, $A_2^2 \in \mathcal{L}(X_2^2)$, $B_i^2 \in \mathcal{L}(X_i^2)$, for $i = 1, 2$, with the operators A_1, B_1^2 being are right invertible, $A_2^2 \leq^- B_2^2$ and A_2^2 and B_2^2 being relatively regular and quasi-nilpotent.

Notice that Definitions 4.2-4.5 and Theorems 4.4-4.7 extend [12, Definition 3.2] and [12, Corollary 3.1], respectively.

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REFERENCES

- [1] P. AIENA, *Fredholm and Local Spectral Theory, with Applications to Multipliers*, Kluwer Academic Publishers, Dordrecht 2004.
- [2] A. DAIJĆ, J. J. KOLIHA, *The weighted g-Drazin inverse for operators*, J. Aust. Math. Soc., **81** (2006), 405–423.
- [3] A. DAIJĆ, J. J. KOLIHA, *The σ -Drazin inverse and the generalized Mbekhta decomposition*, Integr. equ. oper. theory, **57** (2007), 309–326.
- [4] G. DOLINAR, J. MAROVT, *Star partial order on $B(H)$* , Linear Algebra Appl., **434** (2011), 319–326.
- [5] M. GONZÁLEZ, M. MBEKHTA, M. OUDGHIRI, *On the isolated points of the surjective spectrum of a bounded operator*, Proc. Amer. Math. Soc., (10) **136** (2008), 3521–3528.
- [6] R. E. HARTWIG, *How to partially order regular elements*, Math. Japon., **25** (1980), 1–13.
- [7] A. HERNÁNDEZ, M. LATTANZI, N. THOME, *Weighted binary relations involving the Drazin inverse*, Appl. Math. Comput., **253** (2015), 215–223.
- [8] A. HERNÁNDEZ, M. LATTANZI, N. THOME, *On some new pre-orders defined by weighted Drazin inverses*, Appl. Math. Comput., **282** (2016), 108–116.
- [9] J. J. KOLIHA, *A generalized Drazin inverse*, Glasg. Math. J., **38** (1996), 367–381.
- [10] L. LEBTAHI, P. PATRÍCIO, N. THOME, *The diamond partial order in rings*, Linear Multilinear Algebra (3) **62** (2014), 386–395.
- [11] S. MALIK, N. THOME, *A note on a revisited Moore-Penrose inverse of a linear operator on Hilbert spaces*, Filomat (31) **7** (2017) 1927–1931.
- [12] D. MOSIĆ, *Weighted binary relations for operators on Banach spaces*, Aequat. Math., (4) **90** (2016), 787–798.
- [13] D. MOSIĆ, D. S. DJORDJEVIĆ, *Weighted pre-orders involving the generalized Drazin inverse*, Appl. Math. Comput. **270** (2015), 496–504.
- [14] L. RUEDA, S. MALIK, N. THOME, *Further properties on the core partial order and other matrix partial orders*, Linear Multilinear Algebra., (12) **62** (2014), 1629–1648.
- [15] C. SCHMOEGER, *On Isolated Points of the Spectrum of a Bounded Linear Operator*, Proc. Amer. Math. Soc., (3) **117** (1993), 715–719.

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