

ISOLATED EIGENVALUES, POLES AND COMPACT PERTURBATIONS OF BANACH SPACE OPERATORS

BHAGWATI PRASHAD DUGGAL

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Abstract. Given a Banach space operator A , the isolated eigenvalues $E(A)$ and the poles $\Pi(A)$ (resp., eigenvalues $E^a(A)$ which are isolated points of the approximate point spectrum and the left poles $\Pi^a(A)$) of the spectrum of A satisfy $\Pi(A) \subseteq E(A)$ (resp., $\Pi^a(A) \subseteq E^a(A)$), and the reverse inclusion holds if and only if $E(A)$ (resp., $E^a(A)$) has empty intersection with the B-Weyl spectrum (resp., upper B-Weyl spectrum) of A . Evidently $\Pi(A) \subseteq E^a(A)$, but no such inclusion exists for $E(A)$ and $\Pi^a(A)$. The study of identities $E(A) = \Pi^a(A)$ and $E^a(A) = \Pi(A)$, and their stability under perturbation by commuting Riesz operators, has been of some interest in the recent past. This paper studies the stability of these identities under perturbation by (non-commuting) compact operators. Examples of analytic Toeplitz operators and operators satisfying the abstract shift condition are considered.

1. Introduction

Let $B(\mathcal{X})$ (resp., $B(\mathcal{H})$) denote the algebra of operators, equivalently bounded linear transformations, on a complex infinite dimensional Banach (resp., Hilbert) space into itself. For an operator $A \in B(\mathcal{X})$, let $\sigma(A)$, $\text{iso}\sigma(A)$, $\sigma_p(A)$, $\sigma_a(A)$ and $\text{iso}\sigma_a(A)$ denote, respectively, the spectrum, the set of isolated points of $\sigma(A)$, the point spectrum, the approximate point spectrum and the set of isolated points of $\sigma_a(A)$. Let $\text{asc}(A)$ (resp., $\text{dsc}(A)$) denote the ascent (resp., descent) of A , $A - \lambda$ denote $A - \lambda I$, $\alpha(A - \lambda) = \dim(A - \lambda)^{-1}(0)$, and let $E(A)$, $E_0(A)$, $E^a(A)$, $E_0^a(A)$, $\Pi(A)$, $\Pi_0(A)$, $\Pi^a(A)$ and $\Pi_0^a(A)$ denote, respectively the sets $E(A) = \{\lambda \in \text{iso}\sigma(A) : \lambda \in \sigma_p(A)\}$, $E_0(A) = \{\lambda \in E(A) : \alpha(A - \lambda) < \infty\}$, $E^a(A) = \{\lambda \in \text{iso}\sigma_a(A) : \lambda \in \sigma_p(A)\}$, $E_0^a(A) = \{\lambda \in E^a(A) : \alpha(A - \lambda) < \infty\}$, $\Pi(A) = \{\lambda \in \sigma(A) : \text{asc}(A - \lambda) = \text{dsc}(A - \lambda) < \infty\}$, $\Pi_0(A) = \{\lambda \in \Pi(A) : \alpha(A - \lambda) < \infty\}$, $\Pi^a(A) = \{\lambda \in \sigma_a(A) : \text{asc}(A - \lambda) = d < \infty, (A - \lambda)^{d+1}(\mathcal{X}) \text{ is closed}\}$ and $\Pi_0^a(A) = \{\lambda \in \Pi^a(A) : \alpha(A - \lambda) < \infty\}$. The sets $\Pi(A)$, $\Pi^a(A)$, $E(A)$ and $E^a(A)$ satisfy the inclusions $\Pi(A) \subseteq \Pi^a(A) \subseteq E^a(A)$ and $\Pi(A) \subseteq E(A) \subseteq E^a(A)$. The reverse inclusions in general do not hold. The reverse inclusions, in particular the properties

$$(P1) : \quad E(A) = \Pi^a(A) \quad \text{and} \quad (P2) : \quad E^a(A) = \Pi(A)$$

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and their stability under perturbations by commuting Riesz operators, have been studied in a number of papers in the recent past, amongst them [2, 3, 4, 8, 9, 17, 18, 20, 21]. It is easily seen that $A \in B(\mathcal{X})$ satisfies property (P1), $A \in (P1)$, if and only if $E(A) = \Pi^a(A) = \Pi(A)$ and $A \in (P2)$ if and only if $E^a(A) = \Pi(A) = \Pi^a(A) = E(A)$ (thus: $(P2) \implies (P1)$). Letting $\sigma_{Bw}(A)$ and $\sigma_{uBw}(A)$ denote, respectively, the B-Weyl and the left (or, upper) B-Weyl spectrum of $A \in B(\mathcal{X})$, it is seen that $A \in (P1)$ if and only if $E(A) \cap \sigma_{uBw}(A) = \emptyset$ and $A \in (P2)$ if and only if $E^a(A) \cap \sigma_{Bw}(A) = \emptyset$: Left polaroid operators (i.e., operators A for which $\lambda \in \text{iso}\sigma_a(A)$ implies $\lambda \in \Pi^a(A)$) satisfy (P1) and a-polaroid operators (i.e., operators A for which $\lambda \in \text{iso}\sigma_a(A)$ implies $\lambda \in \Pi(A)$) satisfy (P2) [9]. The isolated points of (the Weyl spectrum $\sigma_w(A)$ and) the left (or, upper) Weyl spectrum $\sigma_{aw}(A)$ of A play an important role in determining the stability of properties (P1) and (P2) under perturbation by commuting Riesz operators $R \in B(\mathcal{X})$. Thus, if $\text{iso}\sigma_{aw}(A) = \emptyset$, and $\text{iso}\sigma_a(A + R) = \text{iso}\sigma_a(A)$, then $A \in (Pi) \iff A + R \in (Pi)$; $i = 1, 2$ [9, Theorem 8.5].

This paper considers the preservation of properties (P1) and (P2), and their finite dimensional kernel versions

$$(P1)': \quad E_0(A) = \Pi_0^a(A) \quad \text{and} \quad (P2)': \quad E_0^a(A) = \Pi_0(A),$$

under perturbation by (non-commuting) compact operators. We give a number of examples to show that neither of the properties (P1), (P1)', (P2) and (P2)' travels well from A to $A + K$ under perturbation by compact operators $K \in B(\mathcal{X})$. It is proved that if $\text{iso}\sigma_a(A + K) = \text{iso}\sigma_a(A)$ and either $\text{iso}\sigma_w(A) \cap \{\sigma(A) \setminus \sigma_{Bw}(A)\} = \emptyset$ or $\text{iso}\sigma_{aw}(A) \cap \{\sigma_a(A) \setminus \sigma_{uBw}(A)\} = \emptyset$, then $A \in (P1) \implies A \in (P1)'$, and $A \in (P1) \implies A + K \in (P1)'$, if and only if $E_0(A + K) \subseteq E_0(A)$; $A \in (P2) \implies A \in (P2)'$, and $A \in (P2) \implies A + K \in (P2)'$, if and only if $E_0^a(A + K) \subseteq E_0^a(A)$. For A, K such that $\text{iso}\sigma_a(A + K) = \text{iso}\sigma_a(A)$, $\sigma_{Bw}(A) \setminus \sigma_{uBw}(A) = \sigma_{Bw}(A + K) \setminus \sigma_{uBw}(A + K)$ and $\text{iso}\sigma_a(A) \cap \{\sigma_{Bw}(A + K) \setminus \sigma_{Bw}(A)\} = \emptyset$, a sufficient condition for $A \in (Pi)$ implies $A + K \in (Pi)$, $i = 1, 2$, is that $\text{iso}\sigma_a(A) \cap \sigma_{uBw}(A) = \emptyset$. Analytic Toeplitz operators $A \in B(H^2(\partial\mathcal{D}))$, and operators $A \in B(\mathcal{X})$ satisfying the abstract shift condition (such that A is non-quasinilpotent and non-invertible), satisfy properties (P1) and (P2). We prove that a sufficient condition for $A + K \in (P1) \vee (P2)$ is $E^a(A + K) \cap \sigma_w(A) = \emptyset$, and a necessary and sufficient condition for $A + K \in (P1) \vee (P2)$ is $\sigma_a(A + K) \cap \{\sigma_w(A) \setminus \sigma_{aw}(A)\} = \emptyset$.

The plan of this paper is as follows. After introducing (most of) our notation and terminology in Section 2, we prove some complementary results on polaroid type operators, and a functional calculus for such operators, in Section 3. Section 4 is devoted to proving our main results, and Section 5 considers examples of analytic Toeplitz operators and operators which satisfy the abstract shift condition.

2. Notation and terminology

In addition to the (explained) notation and terminology already introduced, we shall use the following further notation and terminology. We shall use \mathbb{C} to denote the complex plane, and S^c to denote the complement of the subset S of \mathbb{C} in \mathbb{C} . (Thus

$\sigma_w(A)^c = \mathbb{C} \setminus \sigma_w(A)$.) We use $\mathcal{D}(0, r)$ to denote the open disc (in \mathbb{C}) of radius r centered at 0, \mathcal{D} to denote (the open) unit disc, $\overline{\mathcal{D}}$ to denote the closure of \mathcal{D} and $\partial\mathcal{D}$ to denote the boundary of \mathcal{D} . An operator $A \in B(\mathcal{X})$ has SVEP, *the single-valued extension property*, at $\lambda_0 \in \mathbb{C}$ if for every open disc D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \rightarrow \mathcal{X}$ satisfying $(A - \lambda)f(\lambda) = 0$ is the function $f \equiv 0$. Every operator A has SVEP at points in its resolvent set $\rho(A) = \mathbb{C} \setminus \sigma(A)$ and on the boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$. We say that T has SVEP on a set S if it has SVEP at every $\lambda \in S$. The *ascent* of A , $\text{asc}(A)$ (resp. *descent* of A , $\text{dsc}(A)$), is the least non-negative integer n such that $A^{-n}(0) = A^{-(n+1)}(0)$ (resp., $A^n(\mathcal{X}) = A^{n+1}(\mathcal{X})$): If no such integer exists, then $\text{asc}(A)$, resp. $\text{dsc}(A)$, $= \infty$. It is well known that $\text{asc}(A) < \infty$ implies A has SVEP at 0, $\text{dsc}(A) < \infty$ implies A^* ($=$ the dual operator) has SVEP at 0, finite ascent and descent for an operator implies their equality, and that a point $\lambda \in \sigma(A)$ is a pole (of the resolvent) of A if and only if $\text{asc}(A - \lambda) = \text{dsc}(A - \lambda) < \infty$ (see [1, 12, 14, 19]).

The operator $A \in B(\mathcal{X})$ is: *left semi-Fredholm* at $\lambda \in \mathbb{C}$, denoted $\lambda \in \Phi_+(A)$ or $A - \lambda \in \Phi_+(\mathcal{X})$, if $(A - \lambda)\mathcal{X}$ is closed and the deficiency index $\alpha(A - \lambda) < \infty$; *right semi-Fredholm* at $\lambda \in \mathbb{C}$, denoted $\lambda \in \Phi_-(A)$ or $A - \lambda \in \Phi_-(\mathcal{X})$, if $\beta(A - \lambda) = \dim(\mathcal{X}/(A - \lambda)(\mathcal{X})) < \infty$. A is *semi-Fredholm*, $\lambda \in \Phi_{sf}(A)$ or $A - \lambda \in \Phi_{sf}(\mathcal{X})$, if $A - \lambda$ is either left or right semi-Fredholm, and A is *Fredholm*, $\lambda \in \Phi(A)$ or $A - \lambda \in \Phi(\mathcal{X})$, if $A - \lambda$ is both left and right semi-Fredholm. The index of a semi-Fredholm operator is the integer, possibly infinite, $\text{ind}(A) = \alpha(A) - \beta(A)$. Corresponding to these classes of one sided Fredholm operators, we have the following spectra: The *left Fredholm spectrum* $\sigma_{ae}(A)$ of A defined by $\sigma_{ae}(A) = \{\lambda \in \sigma(A) : A - \lambda \notin \Phi_+(\mathcal{X})\}$, and the *right Fredholm spectrum* $\sigma_{se}(A)$ of A defined by $\sigma_{se}(A) = \{\lambda \in \sigma(A) : A - \lambda \notin \Phi_-(\mathcal{X})\}$. The *Fredholm spectrum* $\sigma_e(A)$ of A is the set $\sigma_e(A) = \sigma_{ae}(A) \cup \sigma_{se}(A)$. $A \in B(\mathcal{X})$ is *Weyl* (resp. *a-Weyl*) if it is Fredholm with $\text{ind}(A) = 0$ (resp., if it is left Fredholm with $\text{ind}(A) \leq 0$). It is well known that a *semi-Fredholm operator* A (resp., its dual operator A^*) has SVEP at a point λ if and only if $\text{asc}(A - \lambda) < \infty$ (resp., $\text{dsc}(A - \lambda) < \infty$) [1, Theorems 3.16, 3.17]; furthermore, if $A - \lambda$ is *Weyl*, i.e., if $\lambda \in \Phi(A)$ and $\text{ind}(A - \lambda) = 0$, then A has SVEP at λ implies $\lambda \in \text{iso}\sigma(A)$ with $\text{asc}(A - \lambda) = \text{dsc}(A - \lambda) < \infty$ (resp., if $A - \lambda$ is *a-Weyl*, i.e., if $\lambda \in \Phi_+(A)$ and $\text{ind}(A - \lambda) \leq 0$, then A has SVEP at λ implies $\lambda \in \text{iso}\sigma_a(A)$ with $\text{asc}(A - \lambda) < \infty$). The *Weyl* (resp., the left or approximate *Weyl*) spectrum of A is the set

$$\begin{aligned} \sigma_w(A) &= \{\lambda \in \sigma(A) : \lambda \notin \Phi(A) \text{ or } \text{ind}(A - \lambda) \neq 0\} \\ (\sigma_{aw}(A) &= \{\lambda \in \sigma_a(A) : \lambda \notin \Phi_+(A) \text{ or } \text{ind}(A - \lambda) > 0\}). \end{aligned}$$

A generalization of Fredholm and Weyl spectra is obtained as follows. An operator $A \in B(\mathcal{X})$ is *semi B-Fredholm* if there exists an integer $n \geq 1$ such that $A^n(\mathcal{X})$ is closed and the induced operator $A_{[n]} = A|_{A^n(\mathcal{X})}$, $A_{[0]} = A$, is semi Fredholm (in the usual sense). It is seen that if $A_{[n]} \in \Phi_{\pm}(\mathcal{X})$ for an integer $n \geq 1$, then $A_{[m]} \in \Phi_{\pm}(\mathcal{X})$ for all integers $m \geq n$, and one may unambiguously define the index of A by $\text{ind}(A) = \alpha(A) - \beta(A)$ ($= \text{ind}(A_{[n]})$) (see [7] and [4] for relevant references). Upper (or, left) semi B-Fredholm, lower (or, right) semi B-Fredholm and B-Fredholm spectra of A are then the sets $\sigma_{uBf}(A) = \{\lambda \in \sigma(A) : A - \lambda \text{ is not upper semi B-Fredholm}\}$, $\sigma_{lBf}(A) = \{\lambda \in \sigma(A) : A - \lambda \text{ is not lower semi B-Fredholm}\}$, and $\sigma_{Bf}(A) = \sigma_{uBf}(A) \cup \sigma_{lBf}(A)$.

Letting $\sigma_{Bw}(A) = \{\lambda \in \sigma(A) : \lambda \in \sigma_{Be}(A) \text{ or } \text{ind}(A - \lambda) \neq 0\}$, $\sigma_{uBw}(A) = \{\lambda \in \sigma_a(A) : \lambda \in \sigma_{uBf}(A) \text{ or } \text{ind}(A - \lambda) \not\leq 0\}$, $\sigma_{lBw}(A) = \{\lambda \in \sigma_s(A) : \lambda \in \sigma_{lBf}(A) \text{ or } \text{ind}(A - \lambda) \not\geq 0\}$ denote, respectively, *the B-Weyl, the upper B-Weyl and the lower B-Weyl spectrum of A*, we have $\sigma_{Bw}(A) = \sigma_{uBw}(A) \cup \sigma_{lBw}(A)$, and $\sigma_{uBw}(A) = \sigma_{lBw}(A^*)$. Just as in the case of Weyl and a-Weyl operators, if A has SVEP at $\lambda \in \sigma(A)$ and $A - \lambda$ is B-Weyl, then $\text{asc}(A - \lambda) = \text{dsc}(A - \lambda) < \infty$ and $\lambda \in \text{iso}\sigma(A)$ (resp., if A has SVEP at $\lambda \in \sigma_a(A)$ and $A - \lambda$ is upper B-Weyl, then $\text{asc}(A - \lambda) < \infty$ and $\lambda \in \text{iso}\sigma_a(A)$) [7].

We say in the following that $A \in B(\mathcal{X})$ is *polaroid* (resp., *finitely polaroid*) if $\text{iso}\sigma(A) \subseteq \Pi(A)$ (resp., $\text{iso}\sigma(A) \subseteq \Pi_0(A)$), *left polaroid* (resp., *finitely left polaroid*) if $\text{iso}\sigma_a(A) \subseteq \Pi^a(A)$ (resp., $\text{iso}\sigma_a(A) \subseteq \Pi_0^a(A)$), *a-polaroid* (resp., *finitely a-polaroid*) if $\text{iso}\sigma_a(A) \subseteq \Pi(A)$ (resp., $\text{iso}\sigma_a(A) \subseteq \Pi_0(A)$). It is clear that a-polaroid operators are polaroid, $\Pi_0(A) \subseteq \Pi_0^a(A) \subseteq \Pi^a(A)$ and $\Pi_0(A) \subseteq \Pi(A) \subseteq \Pi^a(A)$ [4, 7, 15].

3. Polaroid operators and compact perturbations

Given operators $A, K \in B(\mathcal{X})$ with K compact, it is clear that

$$\begin{aligned} A + K \text{ is finitely polaroid} &\iff \text{iso}\sigma(A + K) \subseteq \Pi_0(A + K) \\ &\iff \text{iso}\sigma(A + K) \cap \sigma_w(A) = \emptyset; \\ A + K \text{ is finitely left polaroid} &\iff \text{iso}\sigma_a(A + K) \subseteq \Pi_0^a(A + K) \\ &\iff \text{iso}\sigma_a(A + K) \cap \sigma_{aw}(A) = \emptyset; \text{ and} \\ A + K \text{ is finitely a-polaroid} &\iff \text{iso}\sigma_a(A + K) \subseteq \Pi_0(A + K) \\ &\iff \text{iso}\sigma_a(A + K) \cap \sigma_w(A) = \emptyset. \end{aligned}$$

A version of these observations extends to polaroid, left polaroid and a-polaroid operators. Recall from [9, Section 3] that

$$\sigma_{Bw}(A) = \sigma_w(A) \setminus \Phi_{Bw}^{\text{iso}}(A) \text{ and } \sigma_{uBw}(A) = \sigma_{aw}(A) \setminus \Phi_{uBw}^{\text{iso}}(A),$$

where

$$\Phi_{Bw}^{\text{iso}}(A) = \text{iso}\sigma_w(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} \text{ and } \Phi_{uBw}^{\text{iso}}(A) = \text{iso}\sigma_{aw}(A) \cap \sigma_{uBw}(A)^{\mathcal{C}}.$$

Recall also that for every $\lambda \in \text{iso}\sigma(A + K)$, either $\lambda \in \sigma_w(A + K) = \sigma_w(A)$ or $\lambda \in \sigma_w(A + K)^{\mathcal{C}} = \sigma_w(A)^{\mathcal{C}}$ (similarly, for every $\lambda \in \text{iso}\sigma_a(A + K)$, either $\lambda \in \sigma_{aw}(A + K) = \sigma_{aw}(A)$ or $\lambda \in \sigma_{aw}(A + K)^{\mathcal{C}} = \sigma_{aw}(A)^{\mathcal{C}}$). Hence, since $\sigma_w(A + K)^{\mathcal{C}} \cap \sigma_{Bw}(A + K) = \emptyset = \sigma_{aw}(A + K)^{\mathcal{C}} \cap \sigma_{uBw}(A + K)$,

$$\begin{aligned} \text{iso}\sigma(A + K) \cap \sigma_{Bw}(A + K) &= \{\text{iso}\sigma(A + K) \cap \sigma_w(A + K)\} \cap \sigma_{Bw}(A + K) \\ &\subseteq \text{iso}\sigma_w(A + K) \cap \sigma_{Bw}(A + K) \\ &= \text{iso}\sigma_w(A) \cap \sigma_{Bw}(A + K) \end{aligned}$$

and

$$\text{iso}\sigma_a(A + K) \cap \sigma_{uBw}(A + K) = \{\text{iso}\sigma_a(A + K) \cap \sigma_{aw}(A + K)\} \cap \sigma_{uBw}(A + K)$$

$$\begin{aligned} &\subseteq \text{iso}\sigma_{aw}(A+K) \cap \sigma_{uBw}(A+K) \\ &= \text{iso}\sigma_{aw}(A) \cap \sigma_{uBw}(A+K). \end{aligned}$$

The following theorem, which gives a necessary and sufficient condition for the perturbation of an operator by a compact operator to be polaroid (left polaroid, a-polaroid), improves [6, Theorem 6.4]. Let $[\text{iso}\sigma_w(A)]_K$ and $[\text{iso}\sigma_{aw}(A)]_K$ denote, respectively, the sets

$$[\text{iso}\sigma_w(A)]_K = \{\lambda \in \text{iso}\sigma_w(A) = \text{iso}\sigma_w(A+K) : \lambda \in \text{iso}\sigma(A+K)\}$$

and

$$[\text{iso}\sigma_{aw}(A)]_K = \{\lambda \in \text{iso}\sigma_{aw}(A) = \text{iso}\sigma_{aw}(A+K) : \lambda \in \text{iso}\sigma_a(A+K)\}.$$

Clearly, $\text{iso}\sigma(A+K) \cap \sigma_{Bw}(A+K) = \emptyset \iff [\text{iso}\sigma_w(A)]_K \cap \sigma_{Bw}(A+K) = \emptyset$ and $\text{iso}\sigma_a(A+K) \cap \sigma_{uBw}(A+K) = \emptyset \iff [\text{iso}\sigma_{aw}(A)]_K \cap \sigma_{uBw}(A+K) = \emptyset$.

THEOREM 3.1. *If $A, K \in B(\mathcal{X})$, then:*

(i)

$$\begin{aligned} A+K \text{ is polaroid} &\iff \text{iso}\sigma(A+K) \cap \sigma_{Bw}(A+K) = \emptyset \\ &\iff [\text{iso}\sigma_w(A)]_K \cap \sigma_{Bw}(A+K) = \emptyset; \end{aligned}$$

(ii)

$$\begin{aligned} A+K \text{ is left polaroid} &\iff \text{iso}\sigma_a(A+K) \cap \sigma_{uBw}(A+K) = \emptyset \\ &\iff [\text{iso}\sigma_{aw}(A)]_K \cap \sigma_{uBw}(A+K) = \emptyset; \end{aligned}$$

(iii)

$$A+K \text{ is a-polaroid} \iff \text{iso}\sigma_a(A+K) \cap \sigma_{Bw}(A+K) = \emptyset.$$

Proof. Start by observing that

$$\Pi(A+K) = \Pi_0(A+K) \cup \Pi_\infty(A+K)$$

and

$$\Pi^a(A+K) = \Pi_0^a(A+K) \cup \Pi_\infty^a(A+K),$$

where

$$\Pi_\infty(A+K) = \text{iso}\sigma(A+K) \cap \{\sigma_w(A+K) \setminus \sigma_{Bw}(A+K)\}$$

and

$$\Pi_\infty^a(A+K) = \text{iso}\sigma_a(A+K) \cap \{\sigma_{aw}(A+K) \setminus \sigma_{uBw}(A+K)\}.$$

(i) We have:

$$\begin{aligned} \text{iso}\sigma(A+K) &= \{\text{iso}\sigma(A+K) \cap \sigma_w(A+K)^{\complement}\} \cup \{\text{iso}\sigma(A+K) \\ &\quad \cap (\sigma_w(A+K) \setminus \sigma_{Bw}(A+K))\} \cup \{\text{iso}\sigma(A+K) \cap \sigma_{Bw}(A+K)\} \\ &= \Pi_0(A+K) \cup \Pi_\infty(A+K) \cup \{\text{iso}\sigma(A+K) \cap \sigma_{Bw}(A+K)\} \\ &= \Pi(A+K) \cup \{\text{iso}\sigma(A+K) \cap \sigma_{Bw}(A+K)\} \\ &= \Pi(A+K) \cup \{[\text{iso}\sigma_w(A)]_K \cap \sigma_{Bw}(A+K)\}, \end{aligned}$$

which implies

$$\begin{aligned} \text{iso}\sigma(A+K) = \Pi(A+K) &\iff \text{iso}\sigma(A+K) \cap \sigma_{Bw}(A+K) = \emptyset \\ &\iff [\text{iso}\sigma_w(A)]_K \cap \sigma_{Bw}(A+K) = \emptyset. \end{aligned}$$

(ii) Again:

$$\begin{aligned} \text{iso}\sigma_a(A+K) &= \{\text{iso}\sigma_a(A+K) \cap \sigma_{aw}(A+K)^{\complement}\} \cup \{\text{iso}\sigma_a(A+K) \\ &\quad \cap (\sigma_{aw}(A+K) \setminus \sigma_{uBw}(A+K))\} \cup \{\text{iso}\sigma_a(A+K) \cap \sigma_{uBw}(A+K)\} \\ &= \Pi_0^a(A+K) \cup \Pi_\infty^a(A+K) \cup \{\text{iso}\sigma_a(A+K) \cap \sigma_{uBw}(A+K)\} \\ &= \Pi^a(A+K) \cup \{\text{iso}\sigma_a(A+K) \cap \sigma_{uBw}(A+K)\} \\ &= \Pi^a(A+K) \cup \{[\text{iso}\sigma_{aw}(A)]_K \cap \sigma_{uBw}(A+K)\}, \end{aligned}$$

which implies

$$\begin{aligned} \text{iso}\sigma_a(A+K) = \Pi^a(A+K) &\iff \text{iso}\sigma_a(A+K) \cap \sigma_{uBw}(A+K) = \emptyset \\ &\iff [\text{iso}\sigma_{aw}(A)]_K \cap \sigma_{uBw}(A+K) = \emptyset. \end{aligned}$$

(iii) Finally

$$\begin{aligned} \text{iso}\sigma_a(A+K) &= \{\text{iso}\sigma_a(A+K) \cap \sigma_w(A+K)^{\complement}\} \cup \{\text{iso}\sigma_a(A+K) \\ &\quad \cap (\sigma_w(A+K) \setminus \sigma_{Bw}(A+K))\} \cup \{\text{iso}\sigma_a(A+K) \cap \sigma_{Bw}(A+K)\} \\ &= \Pi_0(A+K) \cup \Pi_\infty(A+K) \cup \{\text{iso}\sigma_a(A+K) \cap \sigma_{Bw}(A+K)\} \\ &= \Pi(A+K) \cup \{\text{iso}\sigma_a(A+K) \cap \sigma_{Bw}(A+K)\}, \end{aligned}$$

which implies

$$\text{iso}\sigma_a(A+K) = \Pi(A+K) \iff \text{iso}\sigma_a(A+K) \cap \sigma_{Bw}(A+K) = \emptyset.$$

This completes the proof. \square

REMARK 3.2. Commuting Riesz operators. Translated to operators $A \in B(\mathcal{X})$ and Riesz operators $R \in B(\mathcal{X})$ such that $[A, R] = AR - RA = 0$ and $\sigma_{Bw}(A+R) = \sigma_{Bw}(A)$ (resp., $\sigma_{uBw}(A+R) = \sigma_{uBw}(A)$) Theorem 3.1 implies that: $A+R$ is polaroid if and only if $\text{iso}\sigma(A) \cap \sigma_{Bw}(A) = \emptyset$, equivalently if and only if A is polaroid (resp., $A+R$ is left polaroid if and only if $\text{iso}\sigma_a(A) \cap \sigma_{uBw}(A) = \emptyset$, equivalently if and only if A is left polaroid). An important example of a class of operators satisfying the above spectral hypotheses is that of operators $F \in B(\mathcal{X})$ satisfying $[A, F] = 0$ and F^n is finite rank for some natural number n [9, Proposition 3.3].

Functional calculus Given $A \in B(\mathcal{X})$, let $Holo(\sigma(A))$ denote the set of functions f which are holomorphic in a neighbourhood of $\sigma(A)$, and let $Holo_c(\sigma(A))$ denote those $f \in Holo(\sigma(A))$ which are non-constant on the connected components of $\sigma(A)$. If we let $\sigma_D(A)$ denote the Drazin spectrum of A ,

$$\sigma_D(A) = \{\lambda \in \sigma(A) : \text{asc}(A - \lambda) \neq \text{dsc}(A - \lambda)\},$$

then $\sigma_D(A)$ satisfies the spectral mapping theorem

$$\sigma_D(f(A)) = f(\sigma_D(A)), f \in Holo_c(\sigma(A));$$

the left Drazin spectrum $\sigma_{lD}(A)$ of A ,

$$\sigma_{lD}(A) = \{\lambda \in \sigma_a(A) : \text{there does not exist an integer } p \geq 1 \\ \text{such that } \text{asc}(A - \lambda) \leq p \text{ and } (A - \lambda)^{p+1}(\mathcal{X}) \text{ is closed}\},$$

also satisfies a similar spectral mapping theorem:

$$\sigma_{lD}(f(A)) = f(\sigma_{lD}(A)), f \in Holo_c(\sigma(A))$$

[16]. It is straightforward to see that: A is polaroid if and only if $\text{iso}\sigma(A) \cap \sigma_D(A) = \emptyset$; A is left polaroid if and only if $\text{iso}\sigma_a(A) \cap \sigma_{lD}(A) = \emptyset$, and A is a-polaroid if and only if $\text{iso}\sigma_a(A) \cap \sigma_D(A) = \emptyset$. It is well known (see, for example, [5, Lemma 4.1]) that if $f \in Holo_c(\sigma(A))$, then $\text{iso}\sigma(f(A)) = f(\text{iso}\sigma(A))$. Hence, for $f \in Holo_c(\sigma(A))$,

$$\begin{aligned} f(A) \text{ is polaroid} &\iff \text{iso}\sigma(f(A)) \cap \sigma_D(f(A)) = \emptyset \\ &\iff f(\text{iso}\sigma(A)) \cap f(\sigma_D(A)) = f(\text{iso}\sigma(A) \cap \sigma_D(A)) = \emptyset \\ &\iff A \text{ is polaroid.} \end{aligned}$$

(See [5] for other alternative arguments.) This argument does not extend to left polaroid operators (for the reason that the spectral mapping theorem fails for $\text{iso}\sigma_a(A)$). However, given a $\lambda \in \text{iso}\sigma_a(f(A))$ for an $f \in Holo_c(\sigma(A))$, there always exists a $\mu \in \text{iso}\sigma_a(A)$ such that $f(\mu) = \lambda$. Hence

$$\begin{aligned} A \text{ is left polaroid} &\implies f(\text{iso}\sigma_a(A) \cap \sigma_{lD}(A)) = \emptyset \\ &\iff \{f(\mu) : \mu \in (\text{iso}\sigma_a(A)) \cap \sigma_{lD}(A)\} = \emptyset \\ &\iff \{\lambda \in \text{iso}\sigma_a(f(A)) : \lambda = f(\mu), \mu \in \text{iso}\sigma_a(A)\} \cap \sigma_{lD}(f(A)) = \emptyset \\ &\implies \text{iso}\sigma_a(f(A)) \cap \sigma_{lD}(f(A)) = \emptyset \iff f(A) \text{ is left polaroid.} \end{aligned}$$

For the reverse implication, a hypothesis guaranteeing $f(\text{iso}\sigma_a(A)) = \text{iso}\sigma_a(f(A))$, such as f is injective or $\text{iso}\sigma_a(A) \subseteq \text{iso}\sigma(A)$, is required. It is clear that A is a-polaroid implies $\text{iso}\sigma_a(A) = \text{iso}\sigma(A)$. Hence

$$A \text{ is a-polaroid} \implies f(A) \text{ is polaroid} \implies A \text{ is polaroid.}$$

Combining with Theorem 3.1, we have:

COROLLARY 3.3. *Given operators $A, K \in B(\mathcal{X})$ with K compact, and an $f \in \text{Holo}_c(\sigma(A))$:*

- (i) $f(A + K)$ is polaroid if and only if $[\text{iso}\sigma_w(A)]_K \cap \sigma_D(A + K) = \emptyset$;
- (ii) if f is injective, then $f(A + K)$ is left polaroid if and only if $[\text{iso}\sigma_{aw}(A)]_K \cap \sigma_D(A + K) = \emptyset$;
- (iii) if f is injective, then $f(A + K)$ is a -polaroid if and only if $\text{iso}\sigma_a(A + K) \cap \sigma_D(A + K) = \emptyset$.

Proof. The proof is immediate from Theorem 3.1 once one observes that if an operator T has SVEP at a point λ , then $\lambda \in \sigma_{Bw}(T)$ (resp., $\lambda \in \sigma_{uBw}(T)$) if and only if $\lambda \in \sigma_D(T)$ (resp., $\lambda \in \sigma_{lD}(T)$). \square

4. Properties (P1), (P2) and compact perturbations

Neither of the properties (P1) and (P2), or their finite kernel versions

$$(P1)' \quad E_0(A) = \Pi_0^a(A) \quad \text{and} \quad (P2)' \quad E_0^a(A) = \Pi_0(A),$$

travels well from $A \in B(\mathcal{X})$ to its perturbation by a compact operator $K \in B(\mathcal{X})$.

EXAMPLE 4.1. *If we let $A = U \oplus Q \in B(\mathcal{H} \oplus \mathcal{H})$, where U is the forward unilateral shift and Q is an injective compact quasinilpotent operator, then*

$$\sigma_w(A) = \sigma_{Bw}(A) = \overline{\mathcal{D}}, \sigma_{aw}(A) = \partial\mathcal{D} \cup \{0\} = \sigma_{uBw}(A), \text{iso}\sigma_w(A) = \emptyset, \\ \text{iso}\sigma_{aw}(A) = \{0\} \quad \text{and} \quad E(A) = \Pi^a(A) = \emptyset = E^a(A) = \Pi(A).$$

Let $K \in B(H \oplus \mathcal{H})$ be the compact operator $K = 0 \oplus -Q$. Then the perturbed operator $A + K = A \oplus 0$ satisfies

$$\sigma_w(A + K) = \sigma_{Bw}(A + K) = \overline{\mathcal{D}}, \sigma_{aw}(A + K) = \partial\mathcal{D} \cup \{0\}, \\ \sigma_{uBw}(A + K) = \partial\mathcal{D} \quad \text{and} \quad \text{iso}\sigma_a(A + K) = \text{iso}\sigma_a(A);$$

hence

$$E(A + K) = \emptyset \neq \Pi^a(A + K), E^a(A + K) = \{0\} \neq \Pi(A + K) = \emptyset.$$

EXAMPLE 4.2. *Let $A = U \oplus I \in B(\ell^2 \oplus \ell^2)$ and $K = 0 \oplus F \in B(\ell^2 \oplus \ell^2)$, where $U \in B(\ell^2)$ is the forward unilateral shift and F is the compact operator $F(x_1, x_2, x_3, \dots) = (-\frac{x_1}{2}, 0, 0, \dots)$. Then*

$$\text{iso}\sigma_w(A) = \text{iso}\sigma_{aw}(A) = \emptyset, \text{iso}\sigma_a(A) = \emptyset \neq \left\{\frac{1}{2}\right\} = \text{iso}\sigma_a(A + K)$$

and

$$E_0(A) = \Pi_0^a(A) = \emptyset = E_0^a(A) = \Pi_0(A), \\ \Pi_0(A + K) = \Pi(A + K) = E(A + K) = E_0(A + K) = \emptyset, \\ \Pi_0^a(A + K) = \Pi^a(A + K) = E^a(A + K) = E_0^a(A + K) = \left\{\frac{1}{2}\right\}.$$

EXAMPLE 4.3. If we let $A = U \oplus 0 \in B(\ell^2 \oplus \ell^2)$, where (as before) U is the forward unilateral shift, then

$$\begin{aligned}\sigma_w(A) &= \sigma_{Bw}(A) = \overline{\mathcal{D}}, \sigma_{aw}(A) = \partial\mathcal{D} \cup \{0\} \neq \sigma_{uBw}(A) = \partial\mathcal{D}, \\ \text{iso}\sigma_w(A) &= \emptyset \neq \text{iso}\sigma_{aw}(A) = \{0\}\end{aligned}$$

and

$$E_0(A) = \Pi_0^a(A) = \emptyset = E_0^a(A) = \Pi_0(A).$$

Let $K \in B(\ell^2 \oplus \ell^2)$ be the compact operator $K = 0 \oplus Q$, where Q is the compact operator $Q(x_1, x_2, x_3, \dots) = (0, \frac{x_2}{2}, \frac{x_3}{3}, \dots)$. Then

$$\begin{aligned}\sigma_w(A+K) &= \sigma_{Bw}(A+K) = \overline{\mathcal{D}}, \sigma_{aw}(A+K) = \partial\mathcal{D} \cup \{0\} = \sigma_{uBw}(A+K), \\ E(A+K) &= E_0(A+K) = \Pi_0(A+K) = \Pi(A+K) = \emptyset, \\ \Pi_0^a(A+K) &= \Pi^a(A+K) = \left\{\frac{1}{2}, \frac{1}{3}, \dots\right\}, E_0^a(A+K) = E(A+K) = \left\{0, \frac{1}{2}, \frac{1}{3}, \dots\right\}.\end{aligned}$$

Evidently,

$$\begin{aligned}E_0(A+K) &\neq \Pi_0^a(A+K), E(A+K) \neq \Pi^a(A+K), \\ E_0^a(A+K) &\neq \Pi_0(A+K) \text{ and } E^a(A+K) \neq \Pi(A+K).\end{aligned}$$

The above examples show that neither of the hypotheses $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A+K)$, $\sigma_{Bw}(A) = \sigma_{Bw}(A+K)$, $\sigma_{uBw}(A) = \sigma_{uBw}(A+K)$, $\text{iso}\sigma_w(A) = \emptyset$, $\text{iso}\sigma_{aw}(A) = \emptyset$ and (even) $[A, K] = 0$ is sufficient to guarantee the transfer of either of the properties (P1), (P1)', (P2) and (P2)' from A to $A+K$. Recalling, [9], $\sigma_{Bw}(A) = \sigma_w(A) \setminus \Phi_{Bw}^{\text{iso}}(A)$ and $\sigma_{uBw}(A) = \sigma_{aw}(A) \setminus \Phi_{uBw}^{\text{iso}}(A)$, where $\Phi_{Bw}^{\text{iso}}(A) = \text{iso}\sigma_w(A) \cap \sigma_{Bw}(A)^{\mathcal{C}}$ and $\Phi_{uBw}^{\text{iso}}(A) = \text{iso}\sigma_{aw}(A) \cap \sigma_{uBw}(A)^{\mathcal{C}}$, we have

$$\begin{aligned}\Phi_{Bw}^{\text{iso}}(A) = \Phi_{Bw}^{\text{iso}}(A+K) &\implies \sigma_{Bw}(A) = \sigma_{Bw}(A+K), \text{ and} \\ \Phi_{uBw}^{\text{iso}}(A) = \Phi_{uBw}^{\text{iso}}(A+K) &\implies \sigma_{uBw}(A) = \sigma_{uBw}(A+K).\end{aligned}$$

Furthermore, if also $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A+K)$, then

$$\begin{aligned}\Pi(A) = \text{iso}\sigma(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} &= \text{iso}\sigma_a(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} = \text{iso}\sigma_a(A+K) \cap \sigma_{Bw}(A+K)^{\mathcal{C}} \\ &= \Pi(A+K),\end{aligned}$$

and

$$\Pi^a(A) = \text{iso}\sigma_a(A) \cap \sigma_{uBw}(A)^{\mathcal{C}} = \text{iso}\sigma_a(A+K) \cap \sigma_{uBw}(A+K)^{\mathcal{C}} = \Pi^a(A+K).$$

This, however, is not enough to warranty the passage of properties (P1) and (P2) from A to $A+K$.

EXAMPLE 4.4. Choose $A = Q_1 \oplus Q_2 \in B(\mathcal{H} \oplus \mathcal{H})$, where Q_1 is an injective compact quasinilpotent operator and Q_2 is an injective quasinilpotent such that $Q_2^n(\mathcal{H})$ is non-closed for all natural numbers n . Then

$$\sigma_w(A) = \sigma_{Bw}(A) = \sigma_{uBw}(A) = \sigma_{aw}(A) = \{0\}, \text{iso}\sigma_w(A) = \text{iso}\sigma_{aw}(A) = \emptyset$$

$$(\implies \Phi_{Bw}^{iso}(A) = \Phi_{uBw}^{iso}(A) = \emptyset), E(A) = E^a(A) = \Pi^a(A) = \Pi(A) = \emptyset.$$

Now let $K \in B(\mathcal{H} \oplus \mathcal{H})$ be the compact operator $K = -Q_1 \oplus 0$. Then $A + K = 0 \oplus Q_2$ satisfies

$$\begin{aligned} iso\sigma_a(A + K) &= iso\sigma_a(A), \sigma_w(A + K) = \sigma_{Bw}(A + K) = \sigma_{uBw}(A + K) \\ &= \sigma_{aw}(A + K) = \{0\}, iso\sigma_w(A) = iso\sigma_{aw}(A) = \{0\} (\implies \Phi_{Bw}^{iso}(A + K) = \Phi_{uBw}^{iso}(A + K) \\ &= \emptyset), E(A + K) = E^a(A + K) = \{0\} \neq \Pi^a(A + K) = \Pi(A + K) = \emptyset. \end{aligned}$$

REMARK 4.5. We note for future reference that the hypothesis $\Phi_{Bw}^{iso}(A) = \emptyset$ implies $\sigma_{Bw}(A) = \sigma_w(A)$ and the hypothesis $\Phi_{uBw}^{iso}(A) = \emptyset$ implies $\sigma_{uBw}(A) = \sigma_{aw}(A)$. Furthermore, the hypothesis $\Phi_{uBw}^{iso}(A) = \emptyset$ also implies $\sigma_{Bw}(A) = \sigma_w(A)$, as the following argument proves. Evidently

$$\Phi_{uBw}^{iso}(A) = \emptyset \implies iso\sigma_{aw}(A) \cap \sigma_{uBw}(A)^{\mathcal{C}} = \emptyset \implies iso\sigma_{aw}(A) \subseteq \sigma_{uBw}(A) \subseteq \sigma_{Bw}(A).$$

Take a $\lambda \notin \sigma_{Bw}(A)$. Then

$$\lambda \in \sigma_{Bw}(A)^{\mathcal{C}} \subseteq \sigma_{uBw}(A)^{\mathcal{C}} = \sigma_{aw}(A)^{\mathcal{C}} \implies \lambda \in \sigma_w(A)^{\mathcal{C}}$$

(since $\lambda \in \sigma_{Bw}(A)^{\mathcal{C}}$ implies $\text{ind}(A - \lambda) = 0$, hence $\lambda \notin \sigma_{aw}(A)$ implies $(A - \lambda)(\mathcal{X})$ is closed and $\text{ind}(A - \lambda) = 0$). Thus $\lambda \notin \sigma_w(A)$, consequently $\sigma_w(A) \subseteq \sigma_{Bw}(A)$ (implies $\sigma_w(A) = \sigma_{Bw}(A)$).

THEOREM 4.6. Given operators $A, K \in B(\mathcal{X})$, K compact, such that $iso\sigma_a(A) = iso\sigma_a(A + K)$, if either $\Phi_{Bw}^{iso}(A) = \emptyset = \Phi_{Bw}^{iso}(A + K)$, or $\Phi_{uBw}^{iso}(A) = \emptyset = \Phi_{uBw}^{iso}(A + K)$, then:

- (i) $A \in (P1) \implies A \in (P1)'$ and $A \in (P1) \implies A + K \in (P1)'$ if and only if $E_0(A + K) \subseteq E_0(A)$;
- (ii) $A \in (P2) \implies A \in (P2)'$ and $A \in (P2) \implies A + K \in (P2)'$ if and only if $E_0^a(A + K) \subseteq E_0^a(A)$.

Proof. The hypothesis $\Phi_{Bw}^{iso}(A) = \emptyset = \Phi_{Bw}^{iso}(A + K)$ implies

$$\sigma_w(A) = \sigma_{Bw}(A) = \sigma_{Bw}(A + K) = \sigma_w(A + K)$$

and hence

$$\Pi(A) = iso\sigma(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} = iso\sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = \Pi_0(A);$$

furthermore, if also $iso\sigma_a(A) = iso\sigma_a(A + K)$, then

$$\begin{aligned} \Pi(A + K) &= iso\sigma(A + K) \cap \sigma_{Bw}(A + K)^{\mathcal{C}} = iso\sigma(A + K) \cap \sigma_w(A + K)^{\mathcal{C}} = \Pi_0(A + K) \\ &= iso\sigma_a(A + K) \cap \sigma_{Bw}(A + K)^{\mathcal{C}} = iso\sigma_a(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} = \Pi(A) \\ &= iso\sigma_a(A) \cap \sigma_w(A)^{\mathcal{C}} = \Pi_0(A). \end{aligned}$$

(Thus $\Pi(A+K) = \Pi_0(A+K) = \Pi_0(A) = \Pi(A)$.) Similarly, if $\Phi_{uBw}^{\text{iso}}(A) = \emptyset = \Phi_{uBw}^{\text{iso}}(A+K)$ and $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A+K)$, then

$$\sigma_{aw}(A) = \sigma_{uBw}(A) = \sigma_{uBw}(A+K) = \sigma_{aw}(A+K)$$

and

$$\begin{aligned} \Pi^a(A+K) &= \text{iso}\sigma_a(A+K) \cap \sigma_{uBw}(A+K)^{\mathcal{C}} = \text{iso}\sigma_a(A+K) \cap \sigma_{aw}(A+K)^{\mathcal{C}} \\ &= \Pi_0^a(A+K) = \text{iso}\sigma_a(A) \cap \sigma_{uBw}(A)^{\mathcal{C}} = \Pi^a(A) = \text{iso}\sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} \\ &= \Pi_0^a(A). \end{aligned}$$

(Thus $\Pi^a(A+K) = \Pi_0^a(A+K) = \Pi_0^a(A) = \Pi^a(A)$.)

(i) If $\Phi_{Bw}^{\text{iso}}(A) = \emptyset = \Phi_{Bw}^{\text{iso}}(A+K)$, then

$$\begin{aligned} A \in (P1) &\iff E(A) = \Pi^a(A) = \Pi(A) \implies E_0(A) = \Pi_0^a(A) = \Pi_0(A) \\ &(\iff A \in (P1)') \iff E_0(A) = \Pi_0^a(A) = \Pi_0(A) = \Pi_0(A+K), \end{aligned}$$

and this since

$$\begin{aligned} \Pi_0^a(A+K) &= \text{iso}\sigma_a(A+K) \cap \sigma_{aw}(A+K)^{\mathcal{C}} = \text{iso}\sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} = \Pi_0^a(A) \\ &= \Pi_0(A) = \Pi_0(A+K) \end{aligned}$$

implies

$$A \in (P1) \implies A \in (P1)' \implies E_0(A) = \Pi_0^a(A+K) = \Pi_0(A+K) \subseteq E_0(A+K).$$

Again, if $\Phi_{uBw}^{\text{iso}}(A) = \emptyset = \Phi_{uBw}^{\text{iso}}(A+K)$, then $(\sigma_w(A) = \sigma_{Bw}(A) = \sigma_{Bw}(A+K) = \sigma_w(A+K))$, and

$$\begin{aligned} A \in (P1) &\iff E(A) = (\Pi^a(A) = \Pi(A) =) \Pi_0^a(A) \implies E_0(A) = \Pi_0^a(A) \\ &(\iff A \in (P1)') \iff E_0(A) = \Pi_0(A) = \Pi_0^a(A) = \Pi_0^a(A+K), \end{aligned}$$

and this since

$$\begin{aligned} \Pi_0^a(A+K) &= \Pi_0(A) = \text{iso}\sigma_a(A) \cap \sigma_w(A)^{\mathcal{C}} = \text{iso}\sigma_a(A+K) \cap \sigma_w(A+K)^{\mathcal{C}} \\ &= \Pi_0(A+K) \end{aligned}$$

implies

$$A \in (P1) \implies A \in (P1)' \implies E_0(A) = \Pi_0^a(A+K) = \Pi_0(A+K) \subseteq E_0(A+K).$$

Hence, in either case, $A \in (P1) \implies A \in (P1)'$ and $A \in (P1) \implies A+K \in (P1)'$ if and only if $E_0(A+K) \subseteq E_0(A)$.

(ii) If $\Phi_{Bw}^{iso}(A) = \emptyset = \Phi_{Bw}^{iso}(A + K)$, then (since $\Pi(A) = \Pi_0(A)$, $\Pi^a(A) = \Pi_0(A) \subseteq \Pi_0^a(A)$ and $\Pi_0^a(A) \subseteq E_0^a(A) \subseteq E^a(A)$)

$$A \in (P2) \iff E^a(A) = \Pi(A) = \Pi^a(A) \implies E_0^a(A) = \Pi_0(A) = \Pi_0^a(A) \iff A \in (P2)'$$

implies

$$E_0^a(A) = \Pi_0(A) = \Pi_0(A + K) \subseteq E_0^a(A + K).$$

If, instead, $\Phi_{uBw}^{iso}(A) = \emptyset = \Phi_{uBw}^{iso}(A + K)$, then $\sigma_{Bw}(A) = \sigma_w(A)$ implies $\Pi(A) = \Pi_0(A) = \Pi_0(A + K)$, $\Pi^a(A) = \Pi_0^a(A)$ and $\Pi_0^a(A) \subseteq E_0^a(A) \subseteq E^a(A)$. Hence

$$A \in (P2) \iff E^a(A) = \Pi(A) = \Pi^a(A) \iff E_0^a(A) = \Pi_0(A) = \Pi_0^a(A) (\iff A \in (P2)')$$

implies

$$E_0^a(A) = \Pi_0(A + K) = \Pi_0^a(A) = \Pi_0^a(A + K) \subseteq E_0^a(A + K).$$

In either case, $A \in (P2) \implies A \in (P2)'$ and $A \in (P2) \implies A + K \in (P2)'$ if and only if $E_0^a(A + K) \subseteq E_0^a(A)$. \square

The hypotheses of the theorem are not sufficient to guarantee $E(A + K) = E_0(A + K)$, or, $E^a(A + K) = E_0^a(A + K)$ (see Example 4.4). A sufficient condition ensuring $E_0(A + K) \subseteq E_0(A)$ in (i) and (ii) above is that the operator A is finitely a-polaroid. This follows since $\lambda \in E_0(A + K)$ or $\lambda \in E^a(A + K)$ implies $\lambda \in \text{iso}\sigma_a(A)$, and the hypothesis A is finitely a-polaroid implies $\lambda \in \Pi_0(A)$ ($= E_0(A)$ in case (i) and $= E_0^a(A)$ in case (ii)). Observe that the hypothesis $\text{iso}\sigma_a(A) \cap \sigma_w(A) = \emptyset$ guarantees both $\Phi_{Bw}^{iso}(A) = \emptyset$ and A is a-polaroid, and the hypothesis $\text{iso}\sigma_{aw}(A) = \emptyset$ guarantees both $\Phi_{uBw}^{iso}(A) = \emptyset$ and A is left polaroid.

Hypotheses $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A + K)$ and $\Phi_{uBw}^{iso}(A) = \Phi_{uBw}^{iso}(A + K)$, where $A, K \in B(\mathcal{X})$ and K is compact, imply

$$\sigma_{Bw}(A) \setminus \sigma_{uBw}(A) = \sigma_w(A) \setminus \sigma_{aw}(A) = \sigma_w(A + K) \setminus \sigma_{aw}(A + K) = \sigma_{Bw}(A + K) \setminus \sigma_{uBw}(A + K)$$

and

$$\text{iso}\sigma_a(A) \cap \{\sigma_{Bw}(A + K) \setminus \sigma_{Bw}(A)\} = \text{iso}\sigma_a(A) \cap \{\sigma_w(A + K) \setminus \sigma_w(A)\} = \emptyset.$$

Observe here that if $A \in (P2)$ and $E^a(A + K) \subseteq E^a(A)$, then $E^a(A) = \text{iso}\sigma_a(A) \cap \sigma_{uBw}(A)^\complement$ and hence $\lambda \in E^a(A + K)$ implies $\lambda \notin \sigma_{uBw}(A)$. The following theorem is an analogue of Theorem 4.6 for operators $A \in (P1)$ or $(P2)$ such that $A + K \in (P1)$ or (respectively) $(P2)$.

THEOREM 4.7. *Given operators $A, K \in B(\mathcal{X})$ with K compact, if $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A + K)$, $\sigma_{Bw}(A) \cap \sigma_{uBw}(A)^\complement = \sigma_{Bw}(A + K) \cap \sigma_{uBw}(A + K)^\complement$ and $\text{iso}\sigma_a(A) \cap \{\sigma_{Bw}(A + K) \setminus \sigma_{Bw}(A)\} = \emptyset$, then a sufficient condition for*

$$A \in (Pi) \implies A + K \in (Pi), \quad i = 1, 2,$$

is that $\text{iso}\sigma_a(A) \cap \sigma_{uBw}(A) = \emptyset$.

Proof. Since $A \in (P1)$ if and only if $E(A) = \Pi^a(A) = \Pi(A)$, and $A \in (P2)$ if and only if $E^a(A) = \Pi(A) = \Pi^a(A)$, the hypothesis $A \in (Pi)$, $i = 1, 2$, implies $\Pi^a(A) = \Pi(A)$. Hence, if $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A + K)$ and $\sigma_{Bw}(A) \cap \sigma_{uBw}(A)^\complement = \sigma_{Bw}(A + K) \cap \sigma_{uBw}(A + K)^\complement$, then

$$\begin{aligned} \emptyset &= \Pi^a(A) \setminus \Pi(A) = \{\text{iso}\sigma_a(A) \cap \sigma_{uBw}(A)^\complement\} \cap \{\text{iso}\sigma_a(A) \cap \sigma_{Bw}(A)^\complement\}^\complement \\ &= \{\text{iso}\sigma_a(A) \cap \sigma_{uBw}(A)^\complement \cap \text{iso}\sigma_a(A)^\complement\} \cup \{\text{iso}\sigma_a(A) \cap \sigma_{uBw}(A)^\complement \cap \sigma_{Bw}(A)\} \\ &= \text{iso}\sigma_a(A) \cap \{\sigma_{Bw}(A) \setminus \sigma_{uBw}(A)\} = \text{iso}\sigma_a(A + K) \cap \{\sigma_{Bw}(A + K) \setminus \sigma_{uBw}(A + K)\} \\ &= \{\text{iso}\sigma_a(A + K) \cap \sigma_{uBw}(A + K)^\complement\} \cap \{\text{iso}\sigma_a(A + K) \cap \sigma_{Bw}(A + K)^\complement\}^\complement \\ &= \Pi^a(A + K) \setminus \Pi(A + K), \end{aligned}$$

i.e., $\Pi^a(A + K) \subseteq \Pi(A + K)$. Since $\Pi(A + K) \subseteq \Pi^a(A + K)$ always,

$$\Pi(A + K) = \Pi^a(A + K).$$

Again, since

$$\begin{aligned} \Pi(A) \setminus \Pi(A + K) &= \Pi(A) \cap \{\text{iso}\sigma_a(A + K) \cap \sigma_{Bw}(A + K)^\complement\}^\complement \\ &= \{\text{iso}\sigma_a(A) \cap \sigma_{Bw}(A)^\complement\} \cap \{\text{iso}\sigma_a(A) \cap \sigma_{Bw}(A + K)^\complement\}^\complement \\ &= \text{iso}\sigma_a(A) \cap \{\sigma_{Bw}(A + K) \setminus \sigma_{Bw}(A)\} = \emptyset, \end{aligned}$$

we must have

$$\Pi(A) \subseteq \Pi(A + K).$$

Consider now a $\lambda \in E(A + K)$. If $\text{iso}\sigma_a(A) \cap \sigma_{uBw}(A) = \emptyset$, then A is left polaroid, hence $\lambda \in E(A + K)$ implies

$$\lambda \in \text{iso}\sigma_a(A) \cap \sigma_{uBw}(A)^\complement = \Pi^a(A) = \Pi(A) \implies E(A + K) \subseteq E(A)$$

and, since $\Pi(A + K) \subseteq E(A + K) \subseteq E(A) = \Pi(A) \subseteq \Pi(A + K)$,

$$E(A + K) = \Pi^a(A + K) \iff A + K \in (P1).$$

Considering, instead, a $\lambda \in E^a(A + K)$, the above argument implies

$$\lambda \in \Pi^a(A) = \Pi(A) \subseteq \Pi(A + K)$$

and hence, since $\Pi(A + K) = \Pi^a(A + K) \subseteq E^a(A + K)$,

$$E^a(A + K) = \Pi(A + K) \iff A + K \in (P2).$$

This completes the proof. \square

If $A \in (P1)$, then the hypotheses of Theorem 4.7 imply $E(A) = \Pi^a(A) = \Pi(A) \subseteq \Pi(A + K) = \Pi^a(A + K) \subseteq E(A + K)$; similarly, if $A \in (P2)$, then the hypotheses of Theorem 4.7 imply $E^a(A) = \Pi(A) = \Pi^a(A) \subseteq \Pi(A + K) = \Pi^a(A + K) \subseteq E^a(A + K)$. Hence a necessary and sufficient condition for $A \in (P1)$ implies $A + K \in (P1)$ (resp. $A \in (P2)$ implies $A + K \in (P2)$) in Theorem 4.7 is that $E(A + K) \subseteq \Pi(A)$ (resp., $E^a(A + K) \subseteq \Pi^a(A)$).

COROLLARY 4.8. *If $A, K \in B(\mathcal{X})$ satisfy the hypotheses of Theorem 4.7, then*

$$A \in (P1) \implies A + K \in (P1) \iff E(A + K) \cap \sigma_{Bw}(A) = \emptyset, \text{ and}$$

$$A \in (P2) \implies A + K \in (P2) \iff E^a(A + K) \cap \sigma_{uBw}(A) = \emptyset.$$

Proof. A straightforward consequence of the facts that $E(A + K) \subseteq \Pi(A)$ if and only if $E(A + K) \cap \sigma_{Bw}(A) = \emptyset$ and $E^a(A + K) \subseteq \Pi^a(A)$ if and only if $E^a(A + K) \cap \sigma_{uBw}(A) = \emptyset$. \square

We conclude this section with a remark on Hilbert space operators.

REMARK 4.9. Given a Hilbert space operator $A \in B(\mathcal{H})$, there always exists a compact operator $K \in B(\mathcal{H})$ such that $\sigma_p(A + K) = \Phi_{sf}^+(A) = \{\lambda \in \sigma(A) : A - \lambda \text{ is semi-Fredholm of } \text{ind}(A - \lambda) > 0\} = \Phi_{sf}^+(A + K)$ [11, Proposition 3.4]. Consider a $\lambda \in E(A + K) = \sigma_p(A + K) \cap \text{iso}\sigma(A + K)$, or, $\lambda \in E^a(A + K) = \sigma_p(A + K) \cap \text{iso}\sigma_a(A + K)$. Since $A + K$ has SVEP at $\lambda \in \Phi_{sf}(A + K)$ implies $\text{ind}(A + K - \lambda) \leq 0$ [1], we have $E(A + K) = E^a(A + K) = \emptyset$. Hence $E(A + K) = \Pi(A + K) = \Pi^a(A + K) = E^a(A + K) = \emptyset$, and $A + K \in (P1) \wedge (P2)$. Conclusion: Given a Hilbert space operator $A \in B(\mathcal{H})$, there always exists a compact operator $K \in B(\mathcal{H})$ such that $A + K \in (P1) \wedge (P2)$.

In the absence of similar results for perturbed Banach space operators, a corresponding remark does not seem possible for Banach space operators.

5. Examples: Analytic Toeplitz operators and operators satisfying the abstract shift condition

If we let Ω denote the normalized arc length measure on $\partial\mathcal{D}$ and let $H^2 = H^2(\partial\mathcal{D})$ denote the Hardy space of analytic square summable (with respect to Ω) functions, then the Toeplitz operator T_f with symbol f is the operator in $B(H^2)$ defined by

$$T_f(g) = \mathcal{P}(fg), \quad g \in H^2,$$

where \mathcal{P} is the orthogonal projection of $L^2(\partial\mathcal{D}, \Omega)$ onto H^2 . The operator T_f is analytic Toeplitz if $f \in H^\infty(\partial\mathcal{D})$. (We assume in the following that $f \neq$ a constant.)

If $A \in B(H^2)$ is an analytic Toeplitz operator, then $\sigma(A) = \sigma_w(A)$ is a connected set, A (satisfies Bishop’s property (β) and so) has SVEP [14], and A has no eigenvalues [10, Page 139]. Hence

$$E(A) = E^a(A) = \Pi^a(A) = \Pi(A) = \emptyset \implies A \in (P1) \wedge (P2).$$

The connected property of $\sigma_w(A)$ implies that $A + K$ is polaroid for all compact operators $K \in B(H^2)$ [6, Theorem 6.4]; the connected property of $\sigma_w(A)$ also implies that

$$\sigma_w(A) = \sigma_{Bw}(A) = \sigma_{Bw}(A + K) = \sigma_w(A + K)$$

(consequently, $\Pi(A + K) = \Pi_0(A + K)$) for all compact $K \in B(H^2)$.

For $A, K \in B(H^2)$, A analytic Toeplitz and K compact, assume that $E(A+K) \neq \emptyset$ and consider a $\lambda \in E(A+K)$. Since $A+K$ is polaroid, $\lambda \in \Pi(A+K) - \Pi_0(A+K)$ and hence (since $\Pi(A+K) \subset E(A+K)$ always) $E(A+K) = E_0(A+K) = \Pi_0(A+K) = \Pi(A+K)$. Recall that $\Pi_0^a(A+K) = \Pi_0(A+K)$ if and only if $\text{iso}\sigma_a(A+K) \cap \{\sigma_w(A+K) \setminus \sigma_{aw}(A+K)\} = \text{iso}\sigma_a(A+K) \cap \{\sigma_w(A) \setminus \sigma_{aw}(A)\} = \emptyset$. Hence, if we now assume that $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A+K)$, then

$$\lambda \in \text{iso}\sigma_a(A+K) \cap \{\sigma_w(A) \setminus \sigma_{aw}(A)\} \implies \lambda \in E_0^a(A) \cap \sigma_w(A),$$

a contradiction since A has no eigenvalues. Conclusion: *Given operators $A, K \in B(H^2)$, with A analytic Toeplitz and K compact, if $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A+K)$, then $A+K \in (P1)'$. We do not know if the hypothesis $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A+K)$ is sufficient to guarantee $A+K \in (P2)'$.*

{ Added 12.12.2018: The hypothesis $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A+K)$ guarantees $A+K \in (P2)'$ as the following argument shows. Evidently, $\Pi_0(A+K) \subseteq E_0^a(A+K)$. Since $\lambda \in E_0^a(A+K)$ implies $\lambda \in \text{iso}\sigma_a(A)$, and since $\text{iso}\sigma_{aw}(A) = \emptyset$ (see P. Aiena, Fredholm and Local Spectral Theory II, Lecture Notes in Mathematics 2235, Springer 2018; Theorem 4.99),

$$\begin{aligned} \lambda \in E_0^a(A+K) &\implies \lambda \in \text{iso}\sigma_a(A), \lambda \notin \sigma_{aw}(A) \implies \lambda \in \text{iso}\sigma_a(A) \cap \Phi_{aw}(A) \\ \implies \lambda \in \Pi_0^a(A) = \Pi_0(A), &\text{ since } A \in (P2)' \\ \implies \lambda \in \text{iso}\sigma_a(A+K) \cap \Phi_w(A+K) &= \Pi_0(A+K). \end{aligned}$$

A hypothesis guaranteeing $A+K \in (P1) \wedge (P2)$ is given by the following:

THEOREM 5.1. *If $A, K \in B(H^2)$, where A is analytic Toeplitz and K is compact, satisfy $E^a(A+K) \cap \sigma_w(A) = \emptyset$, then $A+K \in (P1) \wedge (P2)$.*

Proof. It is clear from the above that if A is analytic Toeplitz and K is compact, then $\sigma_{Bw}(A+K) = \sigma_w(A+K) = \sigma_w(A) = \sigma_{Bw}(A)$ and $E(A+K) = \Pi(A+K) = \Pi_0(A+K) = E_0(A+K)$. Since $\Pi_0(A+K) \subseteq \Pi_0^a(A+K) \subseteq E_0^a(A+K)$ and $\Pi_0(A+K) \subseteq E_0(A+K) \subseteq E_0^a(A+K)$, it follows that $E_0(A+K) = \Pi_0(A+K) = \Pi_0^a(A+K) = E_0^a(A+K) = E^a(A+K)$ if and only if $E^a(A+K) \setminus \Pi_0(A+K) = \emptyset$. We have:

$$E^a(A+K) \setminus \Pi_0(A+K) = E^a(A+K) \cap \{\text{iso}\sigma_a(A+K) \cap \sigma_w(A+K)\}^{\mathcal{C}} = E^a(A+K) \cap \sigma_w(A),$$

which implies

$$E^a(A+K) \setminus \Pi_0(A+K) = \emptyset \iff E^a(A+K) \cap \sigma_w(A) = \emptyset.$$

This completes the proof. \square

The sufficient condition of the theorem is necessary too: For if $A+K \in (P1) \wedge (P2)$, then $E^a(A+K) = \Pi(A+K) = \Pi_0(A+K) = E_0^a(A+K)$, and hence $E^a(A+K) \setminus \Pi_0(A+K) = E^a(A+K) \cap \{\text{iso}\sigma_a(A+K) \cap \sigma_w(A+K)\}^{\mathcal{C}} = E^a(A+K) \cap \sigma_w(A) = \emptyset$.

Operators satisfying the ‘‘abstract shift condition’’ $A \in B(\mathcal{X})$ satisfies the *abstract shift condition*, $A \in (ASC)$, if $A^\infty(\mathcal{X}) = \bigcap_{n=1}^\infty A^n(\mathcal{X}) = \{0\}$ [14]. Operators

$A \in (ASC)$ satisfy the properties that $\sigma(A)$ is connected (so that either $\text{iso}\sigma(A) = \emptyset$, or, $\sigma(A) = \{0\}$ in which case A is quasinilpotent), $\alpha(A - \lambda) = 0$ for all non-zero $\lambda \in \sigma(A)$ (so that A has SVEP) and $\sigma(A) = \sigma_w(A)$ [1, 14]. If we let $r(A)$ denote the spectral radius of A and define $i(A)$ by

$$i(A) = \lim_{n \rightarrow \infty} \{\kappa(A^n)\}^{\frac{1}{n}} = \sup_{n \rightarrow \infty} \{\kappa(A^n)\}^{\frac{1}{n}},$$

where

$$\kappa(A) = \inf\{\|Ax\| : x \in \mathcal{X}, \|x\| = 1\}$$

denotes the lower bound of A , then $\mathcal{D}(0, i(A)) \subseteq \sigma(A)$. We assume henceforth that A is not quasinilpotent and $i(A) = r(A)$ for operators $A \in (ASC)$. Given a compact operator $K \in B(\mathcal{X})$, we prove in the following that $A + K \in (P1) \vee (P2)$ (‘inclusive’ or) if and only if $\text{iso}\sigma_a(A + K) \cap \eta' \sigma_{aw}(A) = \emptyset$, where $\eta' \sigma_{aw}(A)$ denotes the bounded component of the complement of $\sigma_{aw}(A)$ in $\sigma_w(A)$.

An important subclass of (ASC) operators is that of *weighted right shift operators* A , $A \in (WRS)$, in $B(\ell^p)$; $\ell^p = \ell^p(\mathbb{N}), 1 \leq p < \infty$. It is well known (see [1, 14] and some of the argument above) that

$$\begin{aligned} \sigma(A) = \sigma_w(A) = \sigma_{Bw}(A) &= \overline{\mathcal{D}(0, r(A))}, E(A) = E^a(A) = \emptyset, \\ \sigma_a(A) = \sigma_{aw}(A) = \sigma_{uBw}(A) &= \partial \mathcal{D}(0, r(A)) \end{aligned}$$

for operators $A \in (ASC)$ (recall: A is non-quasinilpotent and $i(A) = r(A)$), and

$$\begin{aligned} \sigma(A) = \sigma_w(A) = \sigma_{Bw}(A) &= \overline{\mathcal{D}(0, r(A))}, E(A) = E^a(A) = \emptyset, \\ \sigma_a(A) = \sigma_{aw}(A) = \sigma_{uBw}(A) &= \{\lambda : i(A) \leq |\lambda| \leq r(A)\} \end{aligned}$$

for operators $A \in (WRS)$. It is clear that $A \in (P1) \wedge (P2)$ for operators $A \in (ASC) \vee (WRS)$.

THEOREM 5.2. *Given an operator $A \in (ASC) \vee (WRS)$, and a compact operator K such that $K \in B(\mathcal{X})$ if $A \in (ASC)$ and $K \in B(\ell^p)$ if $A \in (WRS)$, $A + K \in (P1) \vee (P2)$, inclusive or, if and only if $\text{iso}\sigma_a(A + K) \cap \{\lambda : 0 \leq |\lambda| < i(A)\} = \emptyset$.*

Proof. If $A \in (ASC) \vee (WRS)$, then $\text{iso}\sigma_w(A) = \text{iso}\sigma_{aw}(A) = \emptyset$ implies that $A + K$ is both polaroid and left-polaroid (see Theorem 3.1). Consequently,

$$\lambda \in E(A + K) \implies \lambda \in \Pi(A + K), \text{ hence } E(A + K) = \Pi(A + K)$$

and

$$\lambda \in E^a(A + K) \implies \lambda \in \Pi^a(A + K), \text{ hence } E^a(A + K) = \Pi^a(A + K).$$

Thus

$$A + K \in (P1) \vee (P2) \iff \Pi^a(A + K) = \Pi(A + K)$$

$$\begin{aligned}
&\iff \Pi^a(A+K) \subseteq \Pi(A+K) \iff \Pi^a(A+K) \setminus \Pi(A+K) = \emptyset \\
&\iff \text{iso}\sigma_a(A+K) \cap \{\sigma_{Bw}(A+K) \setminus \sigma_{uBw}(A+K)\} = \emptyset \\
&\iff \text{iso}\sigma_a(A+K) \cap \{\sigma_w(A+K) \cap \sigma_{aw}(A+K)\}^c \\
&\quad = \text{iso}\sigma_a(A+K) \cap \{\lambda : 0 \leq |\lambda| < i(A)\} = \emptyset
\end{aligned}$$

(where $i(A) = r(A)$ if $A \in (ASC)$). \square

Operators $f(A)$ Let $f \in \text{Holo}_c(\sigma(A))$, where $A \in (ASC)$ or (WRS) or A is an analytic Toeplitz operator. (Recall: If $A \in (ASC)$, then $i(A) = r(A)$ and A is not quasinilpotent.) Since A has SVEP (everywhere) and $\sigma_w(A) = \sigma(A)$,

$$\sigma(f(A)) = f(\sigma(A)) = f(\sigma_w(A)) = \sigma_w(f(A)); f(A) \text{ is polaroid and } E(f(A)) = \Pi(f(A)).$$

Recall that $\sigma_p(A) = \emptyset$: We claim that $\sigma_p(f(A)) = \emptyset$. For suppose there exists a $\lambda \in \sigma_p(f(A))$. Then there exists a $\mu \in \sigma(A)$ such that

$$f(A) - \lambda = f(A) - f(\mu) = (A - \mu)^\alpha p(A)g(A),$$

for some integer $\alpha > 0$, a polynomial $p(z)$ such that $p(\mu) \neq 0$ and an analytic function $g(z)$ which does not vanish on $\sigma(A)$. But then $(f(A) - \lambda)x = 0$, $x \neq 0$, implies $\mu \in \sigma_p(A)$ – a contradiction. This proves our claim. The fact that $\sigma_p(f(A)) = \emptyset$ implies $E^a(f(A)) = \emptyset$ ensures (since $\Pi(f(A)) \subseteq \Pi^a(f(A)) \subseteq E^a(f(A))$) that

$$E(f(A)) = \Pi(f(A)) = \Pi^a(f(A)) = E^a(f(A)) \iff f(A) \in (P1) \wedge (P2).$$

Consider now operators $A, K \in B(H^2)$ such that A is analytic Toeplitz and K is compact. Given $f \in \text{Holo}_c(\sigma(A))$, $\text{iso}\sigma(f(A)) = \text{isof}(\sigma(A))$, $f(A+K)$ is polaroid and hence

$$E(f(A+K)) = \Pi(f(A+K)).$$

Assume further that f is injective and $\text{iso}\sigma_a(A+K) = \text{iso}\sigma_a(A)$. Then $\text{iso}\sigma_a(f(A+K)) = \text{iso}\sigma_a(f(A))$, hence (since A has no eigenvalues)

$$\begin{aligned}
&\text{iso}\sigma_a(f(A+K)) \cap \{\sigma_w(f(A+K)) \setminus \sigma_{aw}(f(A+K))\} \\
&= f(\text{iso}\sigma_a(A+K) \cap \{\sigma_w(A+K) \setminus \sigma_{aw}(A+K)\}) \\
&= f(\text{iso}\sigma_a(A) \cap \{\sigma_w(A) \setminus \sigma_{aw}(A)\}) = f(\Pi_0^a(A) \cap \sigma_w(A)) = \emptyset
\end{aligned}$$

(since $\Pi_0^g(A) = \Pi_0(A)$). Thus:

PROPOSITION 5.3. *If $f \in \text{Holo}_c(\sigma(A))$ is injective, then $f(A+K) \in (P1)$ for analytic Toeplitz operators $A \in B(H^2)$ perturbed by a compact operator $K \in B(H^2)$ such that $\text{iso}\sigma_a(A+K) = \text{iso}\sigma_a(A)$.*

The following theorem, an analogue of Theorem 5.1, gives a necessary and sufficient condition for $f(A+K) \in (P1) \wedge (P2)$.

THEOREM 5.4. *Given operators $A, K \in B(H^2)$, where A is analytic Toeplitz and K is compact, and an injective function $f \in \text{Holo}_c(\sigma(A))$, $f(A+K) \in (P1) \wedge (P2)$ if and only if $E^a(A+K) \cap \sigma_w(A) = \emptyset$.*

Proof. If the operators A, K and the function f are as in the statement of the theorem, then $\text{iso}\sigma_x(f(A + K)) = \text{iso}\sigma_x(f(A))$, $\sigma_x = \sigma$ or σ_a , $f(A + K)$ is polaroid (hence $E(f(A + K)) = \Pi(f(A + K)) = f(\Pi(A + K))$) and $f(A + K)$ is left polaroid (so that $\Pi^a(f(A + K)) = f(\Pi^a(A + K))$). Consequently, $f(A + K) \in (P1) \wedge (P2)$ if and only if

$$\begin{aligned} E(f(A + K)) &= \Pi(f(A + K)) = \Pi^a(f(A + K)) = E^a(f(A + K)) \\ \iff E^a(f(A + K)) \setminus \Pi(f(A + K)) &= \Pi^a(f(A + K)) \setminus \Pi(f(A + K)) = \emptyset. \end{aligned}$$

Recalling that $\sigma_{Bw}(A + K) = \sigma_w(A + K) = \sigma_w(A)$, we have

$$\begin{aligned} f(A + K) \in (P1) \wedge (P2) &\iff \Pi^a(f(A + K)) \setminus \Pi(f(A + K)) = \emptyset \\ &\iff f(\Pi^a(A + K) \setminus \Pi(A + K)) = \emptyset \\ &\iff f(\Pi^a(A + K) \cap \sigma_{Bw}(A + K)) = \emptyset \\ &\iff f(E^a(A + K) \cap \sigma_w(A)) = \emptyset \iff E^a(A + K) \cap \sigma_w(A) = \emptyset. \end{aligned}$$

This completes the proof. \square

For operators $A \in (ASC) \vee (WRS)$, $\text{iso}\sigma_a(A + K) = \text{iso}\sigma_a(A)$ ($= \emptyset$) implies $f(A + K) \in (P1) \wedge (P2)$ for all injective $f \in \text{Holo}_c(\sigma(A))$: The hypothesis $\text{iso}\sigma_a(A + K) = \emptyset$ may be relaxed.

THEOREM 5.5. *Given operators A and K , where $A \in (ASC) \vee (WRS)$ and K is compact, and an injective $f \in \text{Holo}_c(\sigma(A))$, $f(A + K) \in (P1) \vee (P2)$ if and only if $\text{iso}\sigma_a(A + K) \cap \{\lambda : 0 \leq |\lambda| < i(A)\} = \emptyset$.*

Proof. The injective hypothesis on $f \in \text{Holo}_c(\sigma(A))$ implies

$$\text{iso}\sigma_x(f(A + K)) = f(\text{iso}\sigma_x(A + K)), \sigma_x = \sigma \text{ or } \sigma_a,$$

and $A + K$ (alongwith being polaroid) is left polaroid. Since

$$\sigma_{Bw}(f(A + K)) = \sigma_w(f(A + K)) = f(\sigma_w(A + K)) = f(\sigma_{Bw}(A + K))$$

and

$$\sigma_{uBw}(f(A + K)) = \sigma_{aw}(f(A + K)) = f(\sigma_{aw}(A + K)) = f(\sigma_{uBw}(A + K)),$$

we have

$$E(f(A + K)) = \Pi(f(A + K)) \text{ and } E^a(f(A + K)) = \Pi^a(f(A + K)).$$

Thus

$$\begin{aligned} f(A + K) \in (P1) \vee (P2) &\iff \Pi^a(f(A + K)) \subseteq \Pi(f(A + K)) \\ \iff \text{iso}\sigma_a(f(A + K)) \cap f\{\sigma_{Bw}(A + K) \setminus \sigma_{uBw}(A + K)\} &= \emptyset \\ \iff f(\text{iso}\sigma_a(A + K) \cap \{\sigma_w(A + K) \setminus \sigma_{aw}(A + K)\}) &= \emptyset \\ \iff \text{iso}\sigma_a(A + K) \cap \{\sigma_w(A) \cap \sigma_{aw}(A)\}^c &= \emptyset \\ \iff \text{iso}\sigma_a(A + K) \cap \{\lambda : 0 \leq |\lambda| < i(A)\} &= \emptyset. \end{aligned}$$

(Recall: $i(A) = r(A)$ for $A \in (ASC)$.) \square

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Bhagwati Prashad Duggal
 8 Redwood Grove, London W5 4SZ, United Kingdom
 e-mail: bpduggal@yahoo.co.uk