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SEMIGROUP GENERATIONS OF UNBOUNDED BLOCK OPERATOR MATRICES BASED ON THE SPACE DECOMPOSITION

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Abstract. This paper deals with the problem for unbounded block operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with natural domain to generate C_0 semigroups, based on the space decomposition. By describing the spectral inclusion relations between the numerical range of M and its inner entries, using the quadratic complements of M, some necessary and sufficient conditions for M to generate C_0 semigroups are given.

1. Introduction

The research of operator matrices is motivated by systems of linear evolution equations. It is well known that such systems are well-posed if and only if the corresponding operator matrix is the infinitesimal generator of a C_0 semigroup on underlying spaces[1, 2]. One usually concerns with conditions for the operator matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ to generate C_0 semigroups, and obtain some conclusions[3, 4]. However, most of the results are discussed in the diagonal domain $\mathscr{D}(M) = \mathscr{D}(A) \oplus \mathscr{D}(D)$, by using the standard perturbation theorems. The change on the domain of the infinitesimal generator M has a great influence on its semigroup generation property. How can an unbounded operator matrix M with natural domain generates a semigroup? The problem is need to be discussed in other methods. In this paper, we consider the semigroup generation properties of the operator matrix M with natural domain $\mathscr{D}(M) = (\mathscr{D}(A) \cap \mathscr{D}(C)) \oplus (\mathscr{D}(B) \cap \mathscr{D}(D))$ in the different way.

As we know, the main obstacle for unbounded operators to generate semigroups is the unboundedness of their numerical range and the non-emptiness of their residual spectrum. Hence, we characterize the right boundedness of M with the quadratic numerical range of M, and consider the residual spectrum based on the space decomposition and quadratic complements.

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2. Preliminaries

Let T be a linear operator between Hilbert spaces, and let $\mathscr{Y}$ be a linear subspace of a Hilbert space. Then, the closure and orthogonal complement of $\mathscr{Y}$ are denoted by $\overline{\mathscr{Y}}$ and $\mathscr{Y}^{\perp}$, respectively. Write $P_{\mathscr{Y}}$ for the orthogonal projection onto $\mathscr{Y}$ along $\mathscr{Y}^{\perp}$ (when $\mathscr{Y}$ is closed) and $T|_{\mathscr{Y}}$ for the restriction of T to $\mathscr{Y}$. Also, we use $\mathscr{D}(T)$, $\mathscr{N}(T)$ and $\mathscr{R}(T)$ to denote the domain, nullspace and range of T, respectively. $n(T)$ is the dimension of $\mathscr{N}(T)$. Recall that T is said to be *dissipative*[5], if the numerical range of T, i.e., $W(T) = \{(Tv,v) : v \in \mathscr{D}(T), \|v\| = 1\}$ is contained in the closed left half plane.

Throughout this paper, $\mathscr{X}_1, \mathscr{X}_2$ are always Hilbert spaces. In the product space $\mathscr{X}_1 \oplus \mathscr{X}_2$, we consider the unbounded closed block operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.1}$$

where $A : \mathscr{D}(A) \subset \mathscr{X}_1 \to \mathscr{X}_1$, $B : \mathscr{D}(B) \subset \mathscr{X}_2 \to \mathscr{X}_1$, $C : \mathscr{D}(C) \subset \mathscr{X}_1 \to \mathscr{X}_2$ and $D : \mathscr{D}(D) \subset \mathscr{X}_2 \to \mathscr{X}_2$ are densely defined closable operators. Write $\mathscr{D}_1 = \mathscr{D}(A) \cap \mathscr{D}(C), \mathscr{D}_2 = \mathscr{D}(B) \cap \mathscr{D}(D)$. We suppose that the natural domain of M defined in (2.1), i.e.,

$$\mathscr{D}(M) = \mathscr{D}_1 \oplus \mathscr{D}_2 \tag{2.2}$$

is also dense in $\mathscr{X}_1 \oplus \mathscr{X}_2$. The followings are some constants

$$\begin{aligned}
&\alpha_0 = \inf\{\mathrm{Re}\lambda : \lambda = (-Af,f), \|f\| = 1, f \in \mathscr{D}_1\}, \\
&\beta_0 = \inf\{\mathrm{Re}\lambda : \lambda = (-Dg,g), \|g\| = 1, g \in \mathscr{D}_2\}, \\
&\delta_0 = \min\{\alpha_0, \beta_0\}, \\
&\gamma_0 = \sup\Big\{\mathrm{Re}\lambda : \lambda = \frac{(Cf,g)+(Bg,f)}{\|f\|^2+\|g\|^2}, (f\ g)^t \in \mathscr{D}(M)\Big\},
\end{aligned} \tag{2.3}$$

where $\mathrm{Re}\lambda$ is the real part of the complex number λ.

DEFINITION 2.1. [6] Let M be the block operator matrix defined in (2.1)(2.2). For $f \in \mathscr{D}_1, g \in \mathscr{D}_2$ with $\|f\| = \|g\| = 1$, define the 2×2 matrix

$$M_{f,g} = \begin{pmatrix} (Af,f) & (Bg,f) \\ (Cf,g) & (Dg,g) \end{pmatrix}.$$

Then the set

$$W^2(M) = \bigcup_{\substack{f \in \mathscr{D}_1, g \in \mathscr{D}_2, \\ \|f\| = \|g\| = 1}} \sigma_p(M_{f,g})$$

is called the *quadratic numerical range* of M.

DEFINITION 2.2. [7] Let M be the block operator matrix defined in (2.1)(2.2). Suppose that either C or B is boundedly invertible. Then the quadratic operator polynomials T_1 and T_2 defined by

$$T_1(\lambda) = C - (D-\lambda)B^{-1}(A-\lambda) \text{ if } B \text{ is boundedly invertible},$$

$$T_2(\lambda) = B - (A-\lambda)C^{-1}(D-\lambda) \text{ if } C \text{ is boundedly invertible,}$$

for $\lambda \in \mathbb{C}$ are called *quadratic complements* of M.

LEMMA 2.1. [8] *Let M be the block operator matrix defined in (2.1)(2.2). Then:*

(i) $W^2(M) \subset W(M)$,

(ii) $\dim \mathscr{X}_2 \geqslant 2 \Longrightarrow W(\tilde{A}) \subset W^2(M)$,

(iii) $\dim \mathscr{X}_1 \geqslant 2 \Longrightarrow W(\tilde{D}) \subset W^2(M)$,

where $\tilde{A} = A|_{\mathscr{D}_1}, \tilde{D} = D|_{\mathscr{D}_2}$.

LEMMA 2.2. [5] *Let T be a closed linear operator in Hilbert space $\mathscr{X}$. Then for any $\lambda \notin \overline{W(T)}$, $\mathscr{N}(T-\lambda) = \{0\}$ and $\mathscr{R}(T-\lambda)$ is closed.*

LEMMA 2.3. [9] *Let T be a densely defined closed linear operator in a Hilbert space $\mathscr{X}$. Then:*

(i) $\sigma(T) = \sigma_{app}(T) \cup \sigma_{r,1}(T)$,

(ii) $\sigma_{app}(T) \subset \overline{W(T)}$,

where

$$\sigma_{r,1}(T) = \{\lambda \in \mathbb{C} : T-\lambda \textit{ is injective}, \overline{\mathscr{R}(T-\lambda)} \neq \mathscr{X} \textit{ and } \mathscr{R}(T-\lambda) \textit{ is closed}\},$$
$$\sigma_{app}(T) = \{\lambda \in \mathbb{C} : (T-\lambda)v_n \to 0, \{v_n\}_{n=1}^{+\infty} \subset \mathscr{D}(T), \|v_n\| = 1, n = 1, 2, \cdots\}.$$

PROPOSITION 2.1. *Let $\mathscr{X}_1, \mathscr{X}_2$ be infinite dimensional Hilbert spaces, and let M be the block operator matrix defined in (2.1)(2.2). Then the boundedness to the right with bound β $(\in \mathbb{R})$[10] of M, i.e., $\mathrm{Re}(Mv, v) \leqslant \beta(v,v), v \in \mathscr{D}(M)$ implies those of $\tilde{A}$ and $\tilde{D}$, where $\tilde{A} = A|_{\mathscr{D}_1}, \tilde{D} = D|_{\mathscr{D}_2}$.*

Proof. According to Lemma 2.1, it is easy to obtain that $W(\tilde{A}) \subset W(M)$ and $W(\tilde{D}) \subset W(M)$, since $\mathscr{X}_1, \mathscr{X}_2$ are infinite dimensional Hilbert spaces.

3. Main results

In what follows, we assume that the spaces $\mathscr{X}_1, \mathscr{X}_2$ defined in (2.1)(2.2) are infinite dimensional Hilbert spaces.

THEOREM 3.1. *Let M be the operator matrix defined in (2.1)(2.2), and let $\gamma_0 \leqslant \delta_0$ with γ_0, δ_0 defined as in (2.3). Write $\tilde{A} = A|_{\mathscr{D}_1}$, $\tilde{B} = B|_{\mathscr{D}_2}$, $\tilde{C} = C|_{\mathscr{D}_1}$, $\tilde{D} = D|_{\mathscr{D}_2}$. Suppose that $\tilde{D}$ is a closed and $0 < n(\tilde{C}) < \infty$. Then M generates a C_0 semigroup $(T(t))_{t \geqslant 0}$ with $\|T(t)\| \leqslant e^{\beta t}$ for some $\beta \geqslant 0$ on $\mathscr{X}_1 \oplus \mathscr{X}_2$ if and only if the following statements hold:*

(a) $\tilde{A}$ *and* $\tilde{D}$ *are bounded to the right with bound* β,

(b) $\mathscr{R}\big((A_1-\lambda \ \ T_2(\lambda))\big)=\mathscr{X}_1$ *and* $\mathscr{R}(C_{11})=\mathscr{R}(\tilde{D}-\lambda)^{\perp}$ *for* $\lambda>\beta$, *where* $(A_1-\lambda \ \ T_2(\lambda))$ *is a line operator,* $A_1=A|_{\mathscr{N}(\tilde{C})}, T_2(\lambda)=A_{22}-\lambda-\tilde{B}(\tilde{D}-\lambda)^{-1}C_{22}$, $A_{22}=A|_{\mathscr{N}(C_1)}$, $C_1=P_{\mathscr{R}(\tilde{D}-\lambda)^{\perp}}C|_{\mathscr{N}(\tilde{C})^{\perp}\cap\mathscr{D}_1}$, $C_{11}=P_{\mathscr{R}(\tilde{D}-\lambda)^{\perp}}C|_{(\mathscr{N}(\tilde{C})^{\perp}\ominus\mathscr{N}(C_1))\cap\mathscr{D}_1}$ *and* $C_{22}=P_{\mathscr{R}(\tilde{D}-\lambda)}C|_{\mathscr{N}(C_1)}$.

Proof. The assertion (a) implies $W(-\tilde{A})\subset\{z\in\mathbb{C}:\mathrm{Re}z\geqslant-\beta\}$ and $W(-\tilde{D})\subset\{z\in\mathbb{C}:\mathrm{Re}z\geqslant-\beta\}$, so α_0,β_0 are well defined. For each $v=(f\ g)^t\in\mathscr{D}(M)$ with $f\neq 0$ and $g\neq 0$, let $\tau=\dfrac{\|f\|^2}{\|f\|^2+\|g\|^2}$. It follows from $\gamma_0\leqslant\delta_0$ that

$$\begin{aligned}\frac{\mathrm{Re}(Cf,g)+\mathrm{Re}(Bg,f)}{\|f\|^2+\|g\|^2}&\leqslant\gamma_0\leqslant\delta_0\leqslant\tau\alpha_0+(1-\tau)\beta_0\\&\leqslant\tau(\alpha_0+\beta)+(1-\tau)(\beta_0+\beta)\\&\leqslant\tau\cdot\frac{\mathrm{Re}(-Af,f)+\beta(f,f)}{\|f\|^2}+(1-\tau)\cdot\frac{\mathrm{Re}(-Dg,g)+\beta(g,g)}{\|g\|^2}\\&=\frac{\mathrm{Re}(-Af,f)+\beta(f,f)}{\|f\|^2+\|g\|^2}+\frac{\mathrm{Re}(-Dg,g)+\beta(g,g)}{\|f\|^2+\|g\|^2}.\end{aligned}\tag{3.1}$$

Hence

$$\mathrm{Re}(Cf,g)+\mathrm{Re}(Bg,f)+\mathrm{Re}(Af,f)+\mathrm{Re}(Dg,g)\leqslant\beta(f,f)+\beta(g,g),$$

and

$$\begin{aligned}\mathrm{Re}(Mv,v)&=\mathrm{Re}\left(\begin{pmatrix}A&B\\C&D\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix},\begin{pmatrix}f\\g\end{pmatrix}\right)\\&=\mathrm{Re}(Cf,g)+\mathrm{Re}(Bg,f)+\mathrm{Re}(Af,f)+\mathrm{Re}(Dg,g)\\&\leqslant\beta(v,v).\end{aligned}$$

If $f=0$ or $g=0$, it is easy to prove $\mathrm{Re}(Mv,v)\leqslant\beta(v,v)$. Thus, M is bounded to the right with bound β.

To complete the proof, it suffices to show that $M-\lambda\ (\lambda>\beta)$ is surjective.

Since $0<n(\tilde{C})<\infty$, $\mathscr{N}(\tilde{C})$ is a closed subspace of $\mathscr{X}_1$. From $\mathscr{N}(\tilde{C})\subset\mathscr{D}(\tilde{C})=\mathscr{D}_1$, we know

$$\mathscr{D}(\tilde{C})=\mathscr{N}(\tilde{C})\oplus(\mathscr{N}(\tilde{C})^{\perp}\cap\mathscr{D}_1).$$

On the other hand, from assertion (i) follows that $\lambda\notin\overline{W(\tilde{A})}\cup\overline{W(\tilde{D})}$, and $\mathscr{R}(\tilde{D}-\lambda)$ is a closed subspace of $\mathscr{X}_2$ by Lemma 2.2. Then, as an operator from $\mathscr{N}(\tilde{C})\oplus(\mathscr{N}(\tilde{C})^{\perp}\cap\mathscr{D}_1)\oplus\mathscr{D}_2\subset\mathscr{N}(\tilde{C})\oplus\mathscr{N}(\tilde{C})^{\perp}\oplus\mathscr{X}_2$ to $\mathscr{X}_1\oplus\mathscr{R}(\tilde{D}-\lambda)^{\perp}\oplus\mathscr{R}(\tilde{D}-\lambda)$, $M-\lambda$ admits the following block representation

$$M-\lambda=\begin{pmatrix}A_1-\lambda&A_2-\lambda&\tilde{B}\\0&C_1&0\\0&C_2&\tilde{D}-\lambda\end{pmatrix}.\tag{3.2}$$

Here $A_2 = A|_{\mathscr{N}(\tilde{C})^{\perp}\cap\mathscr{D}_1}$ and $C_2 = P_{\mathscr{R}(\tilde{D}-\lambda)}C|_{\mathscr{N}(\tilde{C})^{\perp}\cap\mathscr{D}_1}$. From assertion (i) follows $\lambda \notin \overline{W(\tilde{A})} \cup \overline{W(\tilde{D})}$, and hence $A_1 - \lambda : \mathscr{N}(\tilde{C}) \to \mathscr{X}_1$ is injective and $\tilde{D} - \lambda : \mathscr{D}_2 \to \mathscr{R}(\tilde{D}-\lambda)$ is boundedly invertible by Lemma 2.2. It follows from $0 < n(\tilde{C}) < \infty$ that $0 < n(C_1) < \infty$, hence

$$\mathscr{D}(C_1) = \mathscr{N}(\tilde{C})^{\perp} \cap \mathscr{D}_1 = \big((\mathscr{N}(\tilde{C})^{\perp} \ominus \mathscr{N}(C_1)) \cap \mathscr{D}_1\big) \oplus \mathscr{N}(C_1).$$

As an operator from $\mathscr{N}(\tilde{C}) \oplus \big((\mathscr{N}(\tilde{C})^{\perp} \ominus \mathscr{N}(C_1)) \cap \mathscr{D}_1\big) \oplus \mathscr{N}(C_1) \oplus \mathscr{D}_2 \subset \mathscr{N}(\tilde{C}) \oplus (\mathscr{N}(\tilde{C})^{\perp} \ominus \mathscr{N}(C_1)) \oplus \mathscr{N}(C_1) \oplus \mathscr{X}_2$ to $\mathscr{X}_1 \oplus \mathscr{R}(\tilde{D}-\lambda)^{\perp} \oplus \mathscr{R}(\tilde{D}-\lambda)$, $M-\lambda$ has the following representation

$$M - \lambda = \begin{pmatrix} A_1 - \lambda & A_{21} - \lambda & A_{22} - \lambda & \tilde{B} \\ 0 & C_{11} & 0 & 0 \\ 0 & C_{21} & C_{22} & \tilde{D} - \lambda \end{pmatrix}.$$

Clearly, $C_{11} : (\mathscr{N}(\tilde{C})^{\perp} \ominus \mathscr{N}(C_1)) \cap \mathscr{D}_1 \to \mathscr{R}(\tilde{D}-\lambda)^{\perp}$ is injective, and hence C_{11} is left invertible, i.e., there exist C_{11}^{-1} such that $C_{11}^{-1}C_{11} = I$ holds. Hence, $M - \lambda (\lambda > 0)$ has the following transformation

$$E_1(M-\lambda) = N,$$

where

$$E_1 = \begin{pmatrix} I & \tilde{B}(\tilde{D}-\lambda)^{-1}C_{21}C_{11}^{-1} - (A_{21}-\lambda)C_{11}^{-1} & -\tilde{B}(\tilde{D}-\lambda)^{-1} \\ 0 & I & 0 \\ 0 & -C_{21}C_{11}^{-1} & I \end{pmatrix},$$

$$N = \begin{pmatrix} A_1 - \lambda & 0 & A_{22} - \lambda - \tilde{B}(\tilde{D}-\lambda)^{-1}C_{22} & 0 \\ 0 & C_{11} & 0 & 0 \\ 0 & 0 & C_{22} & \tilde{D} - \lambda \end{pmatrix}.$$

According to the relative boundedness of unbounded operators([7, P92]), operator $\tilde{B}(\tilde{D}-\lambda)^{-1}$, $C_{21}C_{11}^{-1}$ and $(A_{21}-\lambda)C_{11}^{-1}$ are bounded operator, hence E_1 is bijective. Thus, $M-\lambda$ is surjective if and only if so is N. Since $\tilde{D}-\lambda$ is boundedly invertible, N has a further transformation that

$$NE_2 = L,$$

where

$$E_2 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & -(\tilde{D}-\lambda)^{-1}C_{22} & I \end{pmatrix},$$

$$L = \begin{pmatrix} A_1 - \lambda & 0 & A_{22} - \lambda - \tilde{B}(\tilde{D}-\lambda)^{-1}C_{22} & 0 \\ 0 & C_{11} & 0 & 0 \\ 0 & 0 & 0 & \tilde{D} - \lambda \end{pmatrix}.$$

Thus, N is surjective if and only if so is L. It is clear that the condition (b) implies L is surjective, so $M-\lambda$ is surjective. Therefore, M generates a C_0 semigroup $(T(t))_{t\geqslant 0}$ with $\|T(t)\|\leqslant e^{\beta t}$ on $\mathscr{X}_1\oplus\mathscr{X}_2$.

Conversely, if M is the infinitesimal generator of a C_0 semigroup $(T(t))_{t\geqslant 0}$ with $\|T(t)\|\leqslant e^{\beta t}$ for some $\beta\geqslant 0$ on $\mathscr{X}_1\oplus\mathscr{X}_2$, then M is bounded to the right with bound β and $\lambda\in\rho(M)$ for $\lambda>\beta$. By Proposition 2.1, $\tilde{A}$ and $\tilde{D}$ are both bounded to the right with bound β, i.e., the assertion (a) holds. According to the proof in sufficiency, $M-\lambda$, as an operator from $\mathscr{N}(\tilde{C})\oplus\big((\mathscr{N}(\tilde{C})^{\perp}\ominus\mathscr{N}(C_1))\cap\mathscr{D}_1\big)\oplus\mathscr{N}(C_1)\oplus\mathscr{D}_2\subset\mathscr{N}(\tilde{C})\oplus(\mathscr{N}(\tilde{C})^{\perp}\ominus\mathscr{N}(C_1))\oplus\mathscr{N}(C_1)\oplus\mathscr{X}_2$ to $\mathscr{X}_1\oplus\mathscr{R}(\tilde{D}-\lambda)^{\perp}\oplus\mathscr{R}(\tilde{D}-\lambda)$, is surjective. Hence, the assertion (b) holds immediately.

THEOREM 3.2. *Let M be the operator matrix defined in (2.1)(2.2), and let $\gamma_0\leqslant\delta_0$ with γ_0,δ_0 defined as in (2.3). Write $\tilde{A}=A|_{\mathscr{D}_1}$, $\tilde{B}=B|_{\mathscr{D}_2}$, $\tilde{C}=C|_{\mathscr{D}_1}$, $\tilde{D}=D|_{\mathscr{D}_2}$. Suppose that $\tilde{A}$ is a closed and $0<n(\tilde{B})<\infty$, then M generates a C_0 semigroup $(T(t))_{t\geqslant 0}$ with $\|T(t)\|\leqslant e^{\beta t}$ for some $\beta\geqslant 0$ on $\mathscr{X}_1\oplus\mathscr{X}_2$ if and only if the following statements hold:*

(i) $\tilde{A}$ and $\tilde{D}$ are bounded to the right with bound β,

(ii) $\mathscr{R}\big((D_1-\lambda\ \ T_1(\lambda))\big)=\mathscr{X}_2$ and $\mathscr{R}(B_{11})=\mathscr{R}(\tilde{A}-\lambda)^{\perp}$ for $\lambda>\beta$, where $(D_1-\lambda\ \ T_1(\lambda))$ is a line operator, $D_1=D|_{\mathscr{N}(\tilde{B})}$, $T_1(\lambda)=D_{22}-\lambda-\tilde{C}(\tilde{A}-\lambda)^{-1}B_{22}$, $D_{22}=D|_{\mathscr{N}(B_1)}$, $B_1=P_{\mathscr{R}(\tilde{A}-\lambda)^{\perp}}B|_{\mathscr{N}(\tilde{B})^{\perp}\cap\mathscr{D}_2}$ and $B_{22}=P_{\mathscr{R}(\tilde{A}-\lambda)}B|_{\mathscr{N}(B_1)}$.

Proof. The proof is similar to that of Theorem 3.1. We only need to note that the operator $M-\lambda$ $(\lambda>\beta)$, discussed from $\mathscr{X}_1\oplus\mathscr{N}(\tilde{B})\oplus(\mathscr{N}(\tilde{B})^{\perp}\ominus\mathscr{N}(B_1))\oplus\mathscr{N}(B_1)$ to $\mathscr{R}(\tilde{A}-\lambda)^{\perp}\oplus\mathscr{R}(\tilde{A}-\lambda)\oplus\mathscr{X}_2$ has the representation

$$M-\lambda=\begin{pmatrix}0&0&B_{11}&0\\ \tilde{A}-\lambda&0&B_{21}&B_{22}\\ \tilde{C}&D_1-\lambda&D_{21}-\lambda&D_{22}-\lambda\end{pmatrix}.$$

Under more finer subdivision of the domain space, we may have the following results.

THEOREM 3.3. *Let M be the operator matrix defined in (2.1)(2.2), and let $\gamma_0\leqslant\delta_0$ with γ_0,δ_0 defined as in (2.3). Write $\tilde{A}=A|_{\mathscr{D}_1}$, $\tilde{B}=B|_{\mathscr{D}_2}$, $\tilde{C}=C|_{\mathscr{D}_1}$, $\tilde{D}=D|_{\mathscr{D}_2}$. Suppose that $\tilde{A}$ and $\tilde{D}$ are closed, $0<n(\tilde{B})<\infty$ and $0<n(\tilde{C})<\infty$. Then M generates a C_0 semigroup $(T(t))_{t\geqslant 0}$ with $\|T(t)\|\leqslant e^{\beta t}$ for some $\beta\geqslant 0$ on $\mathscr{X}_1\oplus\mathscr{X}_2$ if and only if the following statements hold:*

(i) $\tilde{A}$ and $\tilde{D}$ are bounded to the right with bound β,

(ii) $\mathscr{R}\big((A_{22}-\lambda\ \ B_{22})\big)=\mathscr{R}(A_2-\lambda)$, $\mathscr{R}\big((C_{22}\ \ D_{22}-\lambda)\big)=\mathscr{R}(D_2-\lambda)$, $\mathscr{R}(B_{11})=\mathscr{R}(\tilde{A}-\lambda)^{\perp}$, $\mathscr{R}(C_{11})=\mathscr{R}(\tilde{D}-\lambda)^{\perp}$ for $\lambda>\beta$, where $A_2=A|_{\mathscr{N}(\tilde{C})^{\perp}\cap\mathscr{D}_1}$, $D_2=D|_{\mathscr{N}(\tilde{B})^{\perp}\cap\mathscr{D}_2}$, $C_1=P_{\mathscr{R}(\tilde{D}-\lambda)^{\perp}}C|_{\mathscr{N}(\tilde{C})^{\perp}\cap\mathscr{D}_1}$, $A_{22}=A|_{\mathscr{N}(C_1)}$, $D_{22}=D|_{\mathscr{N}(B_1)}$, $B_1=P_{\mathscr{R}(\tilde{A}-\lambda)^{\perp}}B|_{\mathscr{N}(\tilde{B})^{\perp}\cap\mathscr{D}_2}$, $B_{22}=P_{\mathscr{R}(\tilde{A}-\lambda)}B|_{\mathscr{N}(B_1)}$ and $C_{22}=P_{\mathscr{R}(\tilde{D}-\lambda)}C|_{\mathscr{N}(C_1)}$.

Proof. The proof is similar to that of Theorem 3.1. We only need to note that the operator $M-\lambda$ $(\lambda>\beta)$, discussed from $\mathscr{N}(\tilde{C})\oplus\big(\mathscr{N}(\tilde{C})^{\perp}\ominus\mathscr{N}(C_1)\big)\oplus\mathscr{N}(C_1)\oplus\mathscr{N}(\tilde{B})\oplus\big(\mathscr{N}(\tilde{B})^{\perp}\ominus\mathscr{N}(B_1)\big)\oplus\mathscr{N}(B_1)$ to $\mathscr{R}(\tilde{A}-\lambda)\oplus\mathscr{R}(\tilde{D}-\lambda)\oplus\mathscr{R}(\tilde{D}-\lambda)^{\perp}\oplus\mathscr{R}(\tilde{A}-\lambda)^{\perp}$ has the representation

$$M-\lambda=\begin{pmatrix} A_1-\lambda & A_{21}-\lambda & A_{22}-\lambda & 0 & B_{21} & B_{22}\\ 0 & C_{21} & C_{22} & D_1-\lambda & D_{21}-\lambda & D_{22}-\lambda\\ 0 & C_{11} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & B_{11} & 0 \end{pmatrix}.$$

THEOREM 3.4. *Let M be the operator matrix defined in* (2.1) *with the dense domain*

$$\mathscr{D}(M)=\mathscr{D}(C)\oplus(\mathscr{D}(B)\cap\mathscr{D}(D)):=\mathscr{D}(C)\oplus\mathscr{D}_2,$$

and let $\gamma_0\leqslant\delta_0$ with γ_0,δ_0 defined as in (2.3)*. Write $\tilde{A}=A|_{\mathscr{D}(C)},\tilde{D}=D|_{\mathscr{D}_2}$. If C is boundedly invertible and $C^{-1}D$ is bounded on $\mathscr{D}(D)$, then M generates a C_0 semigroup $(T(t))_{t\geqslant 0}$ with $\|T(t)\|\leqslant e^{\beta t}(t\geqslant 0)$ for some $\beta\geqslant 0$ on $\mathscr{X}_1\oplus\mathscr{X}_2$ if and only if the following statements hold:*

(i) $\tilde{A}$ and $\tilde{D}$ are bounded to the right with bound β,

(ii) $\sigma_{r,1}(T_2)\cap\{z\in\mathbb{C}:\mathrm{Re}z>\beta\}=\varnothing$, where $T_2(\lambda)=B-(A-\lambda)C^{-1}(D-\lambda)$ for $\lambda\in\mathbb{C}$.

Proof. Combining (i) and assumption $\gamma_0\leqslant\delta_0$, we see that M is bounded to the right with bound β. Since C is boundedly invertible, $M-\lambda$ $(\lambda\in\mathbb{C})$ has the following factorization

$$M-\lambda=\begin{pmatrix} I & (A-\lambda)C^{-1}\\ 0 & I\end{pmatrix}\begin{pmatrix} 0 & T_2(\lambda)\\ C & 0\end{pmatrix}\begin{pmatrix} I & C^{-1}(D-\lambda)\\ 0 & I\end{pmatrix}.$$

As $\mathscr{D}(C)\subset\mathscr{D}(A)$ and $C^{-1}D$ is bounded on $\mathscr{D}(D)$, $\begin{pmatrix} I & (A-\lambda)C^{-1}\\ 0 & I\end{pmatrix}$ and $\begin{pmatrix} I & C^{-1}(D-\lambda)\\ 0 & I\end{pmatrix}$ are both bounded and invertible, and hence

$$\lambda\in\sigma_{r,1}(M)\iff\lambda\in\sigma_{r,1}(T_2). \tag{3.3}$$

By Lemma 2.3, we have

$$\sigma_{app}(M)\subset\overline{W(M)}\subset\{z\in\mathbb{C}:\mathrm{Re}z\leqslant\beta\}$$

since M is bounded to the right with bound β. Hence, it follows from assertion (ii) that

$$\sigma(M)\subset\{z\in\mathbb{C}:\mathrm{Re}z\leqslant\beta\},$$

and hence

$$\{z\in\mathbb{C}:\mathrm{Re}z>\beta\}\subset\rho(M). \tag{3.4}$$

Thus, M generates a C_0 semigroup $(T(t))_{t\geqslant 0}$ with $\|T(t)\| \leqslant e^{\beta t} (t \geqslant 0)$ for some $\beta \geqslant 0$ on $\mathscr{X}_1 \oplus \mathscr{X}_2$.

Conversely, suppose that M is the infinitesimal generator of a C_0 semigroup $(T(t))_{t\geqslant 0}$ with $\|T(t)\| \leqslant e^{\beta t} (t \geqslant 0)$ for some $\beta \geqslant 0$ on $\mathscr{X}_1 \oplus \mathscr{X}_2$. Then, we may have that (3.4) is valid, and that $\tilde{A}$ and $\tilde{D}$ are both bounded to the right with bound β. Since C is boundedly invertible and $C^{-1}D$ is bounded on $\mathscr{D}(D)$, (3.3) holds. Therefore, the assertion (ii) follows.

In the same way, we have the following symmetric result.

THEOREM 3.5. *Let M be the operator matrix defined in (2.1) with dense domain*

$$\mathscr{D}(M) = (\mathscr{D}(A) \cap \mathscr{D}(C)) \oplus \mathscr{D}(B) := \mathscr{D}_1 \oplus \mathscr{D}(B),$$

and let $\gamma_0 \leqslant \delta_0$ with γ_0, δ_0 defined as in (2.3). Write $\tilde{A} = A|_{\mathscr{D}_1}, \tilde{D} = D|_{\mathscr{D}(B)}$. If B is boundedly invertible and $B^{-1}A$ is bounded on $\mathscr{D}(A)$, then M generates a C_0 semigroup $(T(t))_{t\geqslant 0}$ with $\|T(t)\| \leqslant e^{\beta t} (t \geqslant 0)$ for some $\beta \geqslant 0$ on $\mathscr{X}_1 \oplus \mathscr{X}_2$ if and only if the following statements hold:

(i) $\tilde{A}$ and $\tilde{D}$ are bounded to the right with bound β,

(ii) $\sigma_{r,1}(T_1) \cap \{z \in \mathbb{C} : \mathrm{Re} z > \beta\} = \varnothing$, where $T_1(\lambda) = C - (D-\lambda)B^{-1}(A-\lambda)$ for $\lambda \in \mathbb{C}$.

Proof. The proof is similar to that of Theorem 3.4. Note that $M - \lambda$ $(\lambda \in \mathbb{C})$ has the following factorization

$$M - \lambda = \begin{pmatrix} I & 0 \\ (D-\lambda)B^{-1} & I \end{pmatrix} \begin{pmatrix} 0 & B \\ T_1(\lambda) & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ B^{-1}(A-\lambda) & I \end{pmatrix}$$

as B is boundedly invertible. It follows from $\mathscr{D}(B) \subset \mathscr{D}(D)$ and $B^{-1}A$ is bounded on $\mathscr{D}(A)$ that $\begin{pmatrix} I & 0 \\ (D-\lambda)B^{-1} & I \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ B^{-1}(A-\lambda) & I \end{pmatrix}$ are both bounded and invertible, and hence

$$\lambda \in \sigma_{r,1}(M) \iff \lambda \in \sigma_{r,1}(T_1).$$

4. Example

Consider the mixed problem of partial differential equation

$$\begin{cases} \dfrac{\partial^4 u(x,y)}{\partial x^4} + \dfrac{\partial^2 u(x,y)}{\partial y^2} + 2\dfrac{\partial u(x,y)}{\partial y} + u = 0, & 0 < x < 1, 0 < y < h, \\ u(0,y) = u(1,y) = 0, u''_x(0,y) = u''_x(1,y) = 0, & 0 \leqslant y \leqslant h, \\ u(x,0) = \varphi(x), u'_y(x,0) = \psi(x), & 0 \leqslant x \leqslant 1. \end{cases} \tag{4.1}$$

Let $\mathscr{X} = L^2[0,1]$, $p = \frac{\partial^2 u}{\partial x^2}$, $q = -\frac{\partial u}{\partial y} - u$, then problem (4.1) can be transformed into abstract Cauchy problem in $\mathscr{X} \oplus \mathscr{X}$ as follows

$$\frac{\partial}{\partial y}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -1 & -\frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial x^2} & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}.$$

The corresponding block operator matrix is

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -I & -\frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial x^2} & -I \end{pmatrix} : \mathscr{D}(M) = \mathscr{D}(C) \oplus \mathscr{D}(B) \to \mathscr{X} \oplus \mathscr{X},$$

and

$$\mathscr{D}(B) = \mathscr{D}(C) = \{v(x) \in \mathscr{X} : v(x)', v(x)'' \in \mathscr{X}, v(0) = v(1) = 0\}.$$

It is easy to see that $\gamma_0 = 0 \leqslant \delta_0$ and $0 \in \rho(B)$. Clearly, A, D are dissipative and

$$T_1(\lambda) := C - (D - \lambda)B^{-1}(A - \lambda) = \frac{\partial^2}{\partial x^2} + (\lambda + 1)^2 (\frac{\partial^2}{\partial x^2})^{-1}, \lambda \in \mathbb{C}.$$

It follows from $0 \in \rho(B) \cap \rho(C)$ that $\mathscr{R}(T_1(\lambda)) = \mathscr{X}$, and hence $\sigma_r(T_1) = \varnothing$. Thus, M satisfies the assertions of Theorem 3.5 and hence M generates a contraction semigroup on $\mathscr{X} \oplus \mathscr{X}$.

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