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n–FOLD JORDAN PRODUCT COMMUTING MAPS WITH A λ–ALUTHGE TRANSFORM

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Abstract. Let $\mathscr{B}(H)$ be the set of all bounded linear operators from H to H, where H is a complex Hilbert space. In this paper, we study the properties of T when the λ-Aluthge transform of T^n is T. Also we prove that the bijective map $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ commutes with a λ-Aluthge transform under the n-fold jordan product if and only if there exists an unitary operator $U : H \to K$ such that $\Phi(T) = UTU^*$ for every T in $\mathscr{B}(H)$.

1. Introduction

Let H and K be complex Hilbert spaces and let $\mathscr{B}(H,K)$ denote the set of all bounded linear operators from H to K. If $H = K$, we write $\mathscr{B}(H)$ in place of $\mathscr{B}(H,K)$. Let $\mathscr{R}(T)$, $\mathscr{N}(T)$, and T^* be the range, the null space, and the operator adjoint of T, respectively. An operator $T \in \mathscr{B}(H,K)$ is an *isometry* if $T^*T = I_H$. In addition, an operator $T \in \mathscr{B}(H,K)$ is called *unitary* if T is a surjective isometry. An operator $T \in \mathscr{B}(H,K)$ is called a *partial isometry* if T^*T is an orthogonal projection, which means $TT^*T = T$. We denote the module of T by $|T| = (T^*T)^{1/2}$.

Let $T \in \mathscr{B}(H)$. An operator T is called *normal* if $T^*T = TT^*$. An operator T is called *quasinormal* if $T^*TT = TT^*T$. If T has a normal extension, it is called a *subnormal* operator. In addition, T is called p*-hyponormal* if $(T^*T)^p \geqslant (TT^*)^p$, where $p > 0$. If $p = 1$, T is said to be *hyponormal*. If $p = \frac{1}{2}$, the operator T is called *semi-hyponormal*. The following inclusion relations are well known: {normal} $\subset$ {quasinormal} $\subset$ {subnormal} $\subset$ {hyponormal} $\subset$ {semi-hyponormal}.

An operator $T \in \mathscr{B}(H)$ has a unique polar decomposition $T = U|T|$, where U is the appropriate partial isometry satisfying $\ker U = \ker |T| = \ker T$ and $\ker U^* = \ker T^*$. Then the Aluthge transform of $T \in \mathscr{B}(H)$ is defined by

$$\tilde{T} = |T|^{1/2}U|T|^{1/2},$$

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which is introduced by Aluthge [1].

The Aluthge transform of $T \in B(H)$ satisfying p-hyponormality becomes a $(p+\frac{1}{2})$-hyponormal operator preserving its spectrum when $0 \leqslant p < \frac{1}{2}$. Otherwise, it transforms a p-hyponormal operator into a hyponormal operator preserving its spectrum when $\frac{1}{2} \leqslant p < 1$. In addition, the sequence of consecutive iterations of the Aluthge trasform, which is denoted by $\{\tilde{T}^{(n)}\}$, is convergent to a normal operator under some conditions. Note that the Aluthge transform does not preserve hyponormality in the unbounded case.

In 2003, K. Okubo [6] introduced a more general notion called λ-Aluthge transform. For operator T with the polar decomposition $T = |U|T$, where U is the appropriate partial isometry satisfying $\ker U = \ker |T| = \ker T$ and $\ker U^* = \ker T^*$, the λ-Aluthge transform of T is defined by

$$\Delta_\lambda(T) = |T|^\lambda U |T|^{1-\lambda}$$

for any $0 < \lambda \leqslant 1$. Note that $\Delta_1(T) = |T|U$ is known as the Duggal transform of T.

In 2000, I. Jung, E. Ko, and C. Pearcy [5] showed that quasinormal operators T are exactly the fixed points of Δ_λ, which means $\Delta_\lambda(T) = T$. In 2016, F. Chabbabi and M. Mbekhta [4] obtained that the property $T = I$ is equivalent to the property $\Delta_\lambda(T^2) = T$, where T and T^* are one-to-one. In this paper, we show that the properties $T^{(n-1)^2} = I$ and $(T^{n-1})^* = T^{n-1}$ are equivalent to that the property $\Delta_\lambda(T^n) = T$, where T and T^* are one-to-one. This result plays an important role in the proof of Theorem 3.4.

In 2016, F. Botelho, L. Molnar, and G. Nagy [2] described the linear bijective mapping on von Neumann factors which commutes with the λ-Aluthge transform. Later, F. Chabbabi [3] described the bijective map $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ satisfying the following condition, which is not necessarily linear,

$$\Delta_\lambda(\Phi(A)\Phi(B)) = \Phi(\Delta_\lambda(AB)) \textit{ for every } A, B \in \mathscr{B}(H)$$

for some $\lambda \in [0,1]$.

DEFINITION 1.1. The *Jordan product* of A_1, A_2 is defined by

$$A_1 \circ A_2 = \frac{1}{2}(A_1A_2 + A_2A_1)$$

for $A_1, A_2 \in \mathscr{B}(H)$.

DEFINITION 1.2. For an integer $n > 1$, the *n-fold Jordan product* of $A_1, A_2, \cdots, A_n$ is defined by

$$A_1 \circ A_2 \circ \cdots \circ A_n = (A_1 \circ A_2 \circ \cdots \circ A_{n-1}) \circ A_n$$

for $A_1, A_2, \cdots, A_n \in \mathscr{B}(H)$.

This definition can be applied recursively to the $(n-1)$-fold. Therefore the n-fold Jordan product of $A_1, A_2, \cdots, A_n$ can be written as

$$A_1 \circ A_2 \circ \cdots \circ A_n = (((A_1 \circ A_2) \circ A_3) \cdots \circ A_{n-1}) \circ A_n.$$

This includes the usual Jordan product $A_1 \circ A_2 = \frac{A_1A_2+A_2A_1}{2}$, and the 3-fold Jordan product $A_1 \circ A_2 \circ A_3 = (A_1 \circ A_2) \circ A_3 = \frac{A_1A_2A_3+A_2A_1A_3+A_3A_1A_2+A_3A_2A_1}{4}$. Furthermore, if $A_1 = A_2 = \cdots = A_n$ then we have $A_1 \circ A_2 \circ \cdots \circ A_n = A^n$.

In 2017, F. Chabbabi and M. Mbekhta [4] gave a description of the bijective maps which consider the Jordan product commuting maps with the λ-Aluthge transform. In this paper, we show that the bijective map $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ commutes with a λ-Aluthge transform under the n-fold jordan product if and only if there exists a unitary operator $U : H \to K$ such that $\Phi(T) = UTU^*$ for every T in $\mathscr{B}(H)$.

2. Properties of the λ-Aluthge transform

In this section, we show that the properties $T^{(n-1)^2} = I$ and $(T^{n-1})^* = T^{n-1}$ are equivalent to that the property $\Delta_\lambda(T^n) = T$, that is the λ-Aluthge transform of the nth power of T is T, where T and T^* are one-to-one. We begin with the following lemma.

LEMMA 2.1. *Let $T \in \mathscr{B}(H)$ and let $T^n = U|T^n|$ be the polar decomposition of T^n for $n \in \mathbb{N}$. Suppose that T and T^* are one-to-one. If $\Delta_\lambda(T^n) = T$ for $n \in \mathbb{N}$, then the following statements hold:*

(i) $(T^n)^{n-1} = (T^*)^{n-1}$.

(ii) T^{n-1} *is quasinormal.*

Proof. (i) Let $T^n = U|T^n|$ be the polar decomposition of T^n. Suppose that $\Delta_\lambda(T^n) = T$. It means that $|T^n|^\lambda U|T^n|^{1-\lambda} = T$. From $|T^n|^\lambda U|T^n| = |T^n|^\lambda U|T^n|^{1-\lambda}|T^n|^\lambda = T|T^n|^\lambda$ and $|T^n|^\lambda U|T^n| = |T^n|^\lambda T^n$, we obtain

$$T|T^n|^\lambda = |T^n|^\lambda T^n. \tag{1}$$

Furthermore, we deduce that

$$T^{n-1}|T^n|^\lambda = T^{n-2}T|T^n|^\lambda = T^{n-2}|T^n|^\lambda T^n = T^{n-3}|T^n|^\lambda T^{2n} = \cdots = |T^n|^\lambda T^{(n-1)n}. \tag{2}$$

Using $\Delta_\lambda(T^n) = T$, we obtain that

$$T^{n-1}|T^n|^\lambda U|T^n|^{1-\lambda} = T^{n-1}\Delta_\lambda(T^n) = T^n = U|T^n| = U|T^n|^\lambda|T^n|^{1-\lambda}. \tag{3}$$

Since T^n is injective, it follows that

$$T^{n-1}|T^n|^\lambda U = U|T^n|^\lambda. \tag{4}$$

Using (2) and (4), we obtain

$$|T^n|^\lambda T^{n(n-1)}U = T^{n-1}|T^n|^\lambda U = U|T^n|^\lambda.$$

Since $|T^n|^\lambda T^{n(n-1)} = U|T^n|^\lambda U^* \geqslant 0$, it follows that

$$|T^n|^\lambda T^{n(n-1)} = (|T^n|^\lambda T^{n(n-1)})^* = T^{*n(n-1)}|T^n|^\lambda$$

$$\begin{aligned}
&= (|T^n|U^*)^{n-1}|T^n|^\lambda = (|T^n|U^*)^{n-2}|T^n|U^*|T^n|^\lambda \\
&= (|T^n|U^*)^{n-2}|T^n|^\lambda(|T^n|)^{1-\lambda}U^*|T^n|^\lambda \\
&= (|T^n|U^*)^{n-2}|T^n|^\lambda\Delta_\lambda(T^n)^* = (|T^n|U^*)^{n-2}|T^n|^\lambda T^* \\
&= (|T^n|U^*)^{n-3}|T^n|^\lambda\Delta_\lambda(T^n)^*T^* \\
&= (|T^n|U^*)^{n-3}|T^n|^\lambda(T^*)^2 = \cdots = |T^n|^\lambda(T^*)^{n-1}.
\end{aligned}$$

Since T^n is injective, we thus obtain $T^{n(n-1)} = (T^*)^{n-1}$.
(ii) Let $T^n = U|T^n|$ be a polar decomposition of T^n. Using $T^{n(n-1)} = (T^*)^{n-1}$ from (i), we also derive that

$$((T^{n-1})^*T^{n-1})T^{n-1} = (T^{n(n-1)}T^{n-1})T^{n-1} = T^{n-1}(T^{n(n-1)}T^{n-1}) = T^{n-1}((T^{n-1})^*T^{n-1}).$$

Hence, we conclude that the operator T^{n-1} is quasinormal.

THEOREM 2.2. *Let $T \in \mathscr{B}(H)$ and let T and T^* be one-to-one. If $\Delta_\lambda(T^n) = T$ for $n \in \mathbb{N}$, then the following statements hold:*

(i) T^{n-1} *is self-adjoint.*

(ii) $\Delta_\lambda(T^{n-1}) = T^{n-1}$.

(iii) $T^{(n-1)^2} = I$.

Proof. (i) Let $T^n = U|T^n|$ be the polar decomposition of T^n for $n \in \mathbb{N}$. By Lemma 2.1, we deduce that

$$\Delta_\lambda((T^{n-1})^*) = \Delta_\lambda(T^{n(n-1)}) = \Delta_\lambda((T^{n-1})^n) = T^{n-1}. \tag{5}$$

Using Lemma 2.3 in [4], we obtain

$$T^{n-1} = (T^{n-1})^*$$

since T^{n-1} is quasinormal from Lemma 2.1. Hence we conclude that T^{n-1} is self-adjoint.
(ii) Since T^{n-1} is self-adjoint, we have $\Delta_\lambda((T^{n-1})) = \Delta_\lambda((T^{n-1})^*)$.
By (5), for all $n \in \mathbb{N}$

$$\Delta_\lambda((T^{n-1})) = \Delta_\lambda((T^{n-1})^*) = T^{n-1}.$$

(iii) Since T^{n-1} is self-adjoint, we thus obtain

$$T^{n-1} = (T^{n-1})^* = T^{n(n-1)}$$

from Lemma 2.1.
This yields for all $n \in \mathbb{N}$

$$T^{n-1}\left(T^{n^2-2n+1} - I\right) = 0.$$

Since T^{n-1} is injective, we deduce

$$T^{n^2-2n+1}-I=0.$$

Hence we conclude that

$$T^{(n-1)^2}=I.$$

COROLLARY 2.3. *Let $T\in\mathscr{B}(H)$ and let T and T^* be one-to-one. If $\Delta_\lambda(T^n)=T$ for $n\in\mathbb{N}$, then T is an algebraic operator of order $(n-1)^2$ and the spectrum $\sigma(T)$ of T consists of $(n-1)^2$th roots of unity.*

Proof. Using $T^{(n-1)^2}=I$ from Theorem 2.2, we obtain $p(T)=0$ for some polynomial $p(z)=z^{(n-1)^2}-1$. Hence T is an algebraic operator of order $(n-1)^2$ and $\sigma(T)$ consists of roots of $p(z)=0$.
As some applications of Lemma 2.1 and Theorem 2.2, we get the following theorem.

THEOREM 2.4. *Let $T\in\mathscr{B}(H)$ and let T and T^* be one-to-one. Then the following statements are equivalent:*

(i) $\Delta_\lambda(T^n)=T$.

(ii) $T\in\left\{T\in\mathscr{B}(H):T^{(n-1)^2}=I \text{ and } (T^{n-1})^*=T^{n-1}\right\}$.

Proof. Let $T^n=U|T^n|$ be the polar decomposition of T^n for $n\in\mathbb{N}$. Suppose that $\Delta_\lambda(T^n)=T$. Then $T^{(n-1)^2}=I$ and $(T^{n-1})^*=T^{n-1}$ hold from Theorem 2.2. Assume that (ii) holds. Using $I=T^{n^2-2n+1}=T^{n(n-1)}T^{-(n-1)}$ and $(T^{n-1})^*=T^{n-1}$, we obtain

$$T^{n(n-1)}=T^{n-1}=(T^{n-1})^*.$$

From the given conditions, we derive that

$$\begin{aligned}|T^{n(n-1)}|^\lambda &= \left((T^*)^{n(n-1)}T^{n(n-1)}\right)^{\lambda/2}\\ &= \left((T^*)^{(n-1)}(T^*)^{(n-1)^2}T^{(n-1)^2}T^{(n-1)}\right)^{\lambda/2}=\left((T^*)^{(n-1)}IT^{(n-1)}\right)^{\lambda/2}\\ &= \left((T^*)^{(n-1)}T^{(n-1)}\right)^{\lambda/2}=|T^{n-1}|^\lambda\end{aligned}$$

for any $0<\lambda\leqslant 1$.
Similarly, we have $|T^{n(n-1)}|^{1-\lambda}=|T^{n-1}|^{1-\lambda}$ for any $0<\lambda\leqslant 1$. Hence, we conclude that

$$\Delta_\lambda((T^{(n-1)})^n)=|T^{n(n-1)}|^\lambda U|T^{n(n-1)}|^{1-\lambda}=|T^{n-1}|^\lambda U|T^{n-1}|^{1-\lambda}=\Delta_\lambda(T^{(n-1)}). \quad (6)$$

Let $T^{n-1}=V|T^{n-1}|$ be the polar decomposition of T^{n-1}. By assumption we obtain

$$V|T^{n-1}|=T^{n-1}=T^{(n-1)^2}T^{n-1}=T^{n(n-1)}=U|T^{n(n-1)}|=U|T^{n-1}|.$$

Since T is one to one and $(V-U)|T^{n-1}|=0$, we have $V=U$.
From this and (6), we have

$$\Delta_\lambda((T^{n-1})^n)=\Delta_\lambda(T^{n-1})=|T^{n-1}|^\lambda U|T^{n-1}|^{1-\lambda}$$
$$=|T^{n-1}|^\lambda V|T^{n-1}|^{1-\lambda}=V|T^{n-1}|=T^{n-1}$$

since the operator T^{n-1} is self-adjoint. Hence we conclude that $\Delta_\lambda(T^n)=T$ for all n.

3. Bijective maps commuting with λ-Aluthge transforms

Let H and K be two complex Hilbert spaces with $dim(H)\geqslant 2$. Let $\Phi:\mathscr{B}(H)\to\mathscr{B}(K)$ be a bijective map such that:

$$\Delta_\lambda(\Phi(A_1)\circ\Phi(A_2)\circ\cdots\circ\Phi(A_n))=\Phi(\Delta_\lambda(A_1\circ A_2\circ\cdots\circ A_n)) \tag{7}$$

for some $n\in\mathbb{N}$, where $A_1,\cdots,A_n\in\mathscr{B}(H)$.
We now consider some properties of a bijective map Φ which commutes with a λ-Aluthge transform under the n-fold jordan product.

LEMMA 3.1. *Let T be a bounded operator defined on H. Let $\Phi:\mathscr{B}(H)\to\mathscr{B}(K)$ be a bijective map satisfying (7) for some $n\in\mathbb{N}$. Then the following properties hold:*

(i) $\Phi(0)=0$ *and* $\Phi(I)\in\left\{T\in\mathscr{B}(K):T^{(n-1)^2}=I \text{ and } (T^{n-1})^*=T^{n-1}\right\}$.

(ii) The map Φ preserves the following:

$$A\in\left\{T\in\mathscr{B}(H):T^{(n-1)^2}=I \text{ and } (T^{n-1})^*=T^{n-1}\right\}$$
$$\Longleftrightarrow\Phi(A)\in\left\{T\in\mathscr{B}(K):T^{(n-1)^2}=I \text{ and } (T^{n-1})^*=T^{n-1}\right\}.$$

(iii) For all $A\in\{T\in\mathscr{B}(H):T^{(n-1)^2}=I \text{ and } (T^{n-1})^*=T^{n-1}\}$, we have $\Phi(A^{n-1})=(\Phi(A))^{n-1}$.

(iv) $\Phi(I)=I$.

Proof. (i) Since the bijective map Φ is onto, there exists $A\in\mathscr{B}(H)$ such that $\Phi(A)=0$. From (7), we have $\Delta_\lambda(\Phi(0)\circ\Phi(A)\circ\cdots\circ\Phi(A))=\Phi(\Delta_\lambda(0\circ A\circ\cdots\circ A))=\Phi(\Delta_\lambda(0))=\Phi(0)$ and $\Delta_\lambda(\Phi(0)\circ\Phi(A)\circ\cdots\circ\Phi(A))=\Delta_\lambda(\Phi(0)\circ 0\circ\cdots\circ 0)=\Delta_\lambda(0)=0$. It follows that $\Phi(0)=0$. From (7), we thus obtain

$$\Delta_\lambda((\Phi(I))^n)=\Delta_\lambda(\Phi(I)\circ\Phi(I)\circ\cdots\circ\Phi(I))=\Phi(\Delta_\lambda(I\circ I\circ\cdots\circ I))=\Phi(I).$$

By Theorem 2.4, we conclude that

$$\Phi(I)\in\{T\in\mathscr{B}(K):T^{(n-1)^2}=I \text{ and } (T^{n-1})^*=T^{n-1}\}.$$

(ii) Suppose that $A \in \left\{T \in \mathscr{B}(H) : T^{(n-1)^2} = I \textit{ and } (T^{n-1})^* = T^{n-1}\right\}$. By Theorem 2.4, $\Delta_\lambda(A^n) = A$. From (7), we thus obtain $\Delta_\lambda(\Phi(A)^n) = \Delta_\lambda(\Phi(A) \circ \Phi(A) \circ \cdots \circ \Phi(A)) = \Phi(\Delta_\lambda(A \circ A \circ \cdots \circ A)) = \Phi(\Delta_\lambda(A^n)) = \Phi(A)$. It means that $\Delta_\lambda(\Phi(A)^n) = \Phi(A)$. By Theorem 2.4, $\Phi(A)$ satisfies $\Phi(A)^{(n-1)^2} = I$ and $(\Phi(A)^{n-1})^* = \Phi(A)^{n-1}$. Conversely, we can prove in the same way.

(iii) By assumption, A^{n-1} is self-adjoint. Hence $(\Phi(A))^{n-1}$ is also self-adjoint from (ii). From (7), we obtain that $\Delta_\lambda(\Phi(A) \circ \cdots \Phi(A)) = \Delta_\lambda(\Phi(A)^{n-1}) = \Phi(A)^{n-1}$. Since A^{n-1} is self-adjoint, we thus obtain $\Phi(\Delta_\lambda(A \circ A \circ \cdots A)) = \Phi(\Delta_\lambda(A^{n-1})) = \Phi(A^{n-1})$. Hence we conclude that $\Phi(A)^{n-1} = \Phi(A^{n-1})$.

(iv) From (i) and (iii), we obtain that $\Phi(I)^{n-1} = \Phi(I^{n-1}) = \Phi(I)$. Since the bijective map Φ is injective and $\Phi(0) = 0$ from (i), we deduce that $\Phi(I)^{n-2} = I$. Since $\Phi(I)^{(n-1)^2} = I$ from (ii), we deduce that $I = \Phi(I)^{(n-1)^2} = (\Phi(I)^{n-2})^n \Phi(I)$. Because of $\Phi(I)^{n-2} = I$, we obtain $\Phi(I) = I$.

COROLLARY 3.2. *Let $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ be a bijective map satisfying (7). Then $\Delta_\lambda(\Phi(A)) = \Phi(\Delta_\lambda(A))$ holds for any $A \in \mathscr{B}(H)$. Furthermore, we have $\Delta_\lambda(\Phi(A)^k) = \Phi(\Delta_\lambda(A^k))$ for all $k \in \mathbb{N}$.*

Proof. Since $\Phi(I) = I$ from Lemma 3.1, we have

$$\Delta_\lambda(\Phi(A)) = \Delta_\lambda(\Phi(A) \circ \Phi(I) \circ \cdots \circ \Phi(I)) = \Phi(\Delta_\lambda(A \circ I \circ \cdots \circ I)) = \Phi(\Delta_\lambda(A))$$

for any $A \in \mathscr{B}(H)$. Similarly we also obtain

$$\Delta_\lambda(\Phi(A)^k) = \Delta_\lambda(\Phi(A)^k \circ \Phi(I) \cdots \circ \Phi(I)) = \Phi(\Delta_\lambda(A^k \circ I \cdots \circ I)) = \Phi(\Delta_\lambda(A^k))$$

for all $k \in \mathbb{N}$.

The next lemma characterizes the properties of the bijective map for the rank one orthogonal projections.

LEMMA 3.3. *Let $P = x \otimes x$ and $P' = x' \otimes x'$ be two rank one orthogonal projections such that $\langle x, x' \rangle = 0$. Let $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ be a bijective map satisfying (7). Then there exists a bijective function $h : \mathbb{C} \to \mathbb{C}$ such that:*

(i)

$$\Phi(\alpha P + \beta P') = h(\alpha)\Phi(P) + h(\beta)\Phi(P')$$

for every $\alpha, \beta \in \mathbb{C}$. In addition, the function h is multiplicative.

(ii) For every $A \in \mathscr{B}(H)$, we deduce that $\langle \Phi(A)y, y \rangle = h(\langle Ax, x \rangle)$ for all unit vectors x, y such that $\Phi(x \otimes x) = y \otimes y$. In addition, the function h is additive.

Proof. (i) Since $P = x \otimes x$ and $P' = x' \otimes x'$ are rank one orthogonal projections such that $\langle x, x' \rangle = 0$, we have $(\alpha P + \beta P') \circ (P + P') \circ \cdots \circ (P + P') = \alpha P + \beta P'$. Since

P is self-adjoint, we have $P^{n-1} = P^{n(n-1)}$. Therefore we have $P^{(n-1)^2} = I$. By Lemma 3.1 (iii), we obtain $\Phi(P^{n-1}) = \Phi(P)^{n-1}$. Therefore we deduce that

$$\Phi(P) = \Phi(P)^{n-1} = (\Phi(P)^{n-1})^{n-1} = \Phi(P)^{(n-1)^2} = I$$

from Lemma 3.1 (ii). Then we conclude that $\Phi(P)^2 = \Phi(P) = \Phi(P^2)$, and thus the bijective map Φ preserves the set of orthogonal projections. We denote a partial ordering between orthogonal projections by $P \leqslant Q \ if \ PQ = QP = P$. Since the bijective map Φ preserves the set of orthogonal projections, we also obtain $\Delta_\lambda(\Phi(Q)\Phi(P)) = \Phi(Q) = \Phi(\Delta_\lambda(Q)) = \Delta_\lambda(\Phi(Q))$. Since $\Phi(P)$ is an orthogonal projection, we have $\Phi(Q)\Phi(P) = \Phi(P)\Phi(Q) = \Phi(Q)$. It follows that $\Phi(Q) \leqslant \Phi(P)$. It means that the bijective map Φ preserves the order relation on the set of orthogonal projections in the both direction. Since P,Q are orthogonal, we have $P \leqslant P+Q$ and $Q \leqslant P+Q$. We also obtain $\Phi(P) \leqslant \Phi(P+Q)$ and $\Phi(Q) \leqslant \Phi(P+Q)$. Since $\Phi(P) \perp \Phi(Q)$, it follows that $\Phi(P)+\Phi(Q) \leqslant \Phi(P+Q)$. Since $\Phi(P+Q) = \Phi(\Phi^{-1}(\Phi(P)) + \Phi^{-1}(\Phi(Q))) \leqslant \Phi(\Phi^{-1}(\Phi(P)+\Phi(Q))) = \Phi(P)+\Phi(Q)$, we conclude that $\Phi(P+Q) = \Phi(P)+\Phi(Q)$ for all orthogonal projections P,Q such that $P \perp Q$. Since $\alpha P + \beta Q$ is normal for all orthogonal projections P,Q such that $P \perp Q$, the bijective map $\Phi(\alpha P + \beta Q)$ is quasinormal. By the result of I. Jung, E. Ko, and C. Pearcy [5] that quasinormal operators are exactly the fixed points of Δ_λ, we have $\Phi(\alpha P + \beta Q) = \Delta_\lambda(\Phi(\alpha P + \beta Q))$.
By Corollary 3.2, we obtain that

$$\begin{aligned}
&\Phi(\alpha P + \beta P') = \Delta_\lambda(\Phi(\alpha P + \beta P')) = \Phi(\Delta_\lambda(\alpha P + \beta P'))\\
=&\Phi(\Delta_\lambda((\alpha P + \beta P') \circ (P+P') \circ (P+P') \circ \cdots \circ (P+P')))\\
=&\Delta_\lambda(\Phi(\alpha P + \beta P') \circ \Phi(P+P') \circ \cdots \circ \Phi(P+P'))\\
=&\Delta_\lambda(\Phi(\alpha P + \beta P') \circ (\Phi(P)+\Phi(P')) \circ \cdots \circ (\Phi(P)+\Phi(P')))\\
=&\Delta_\lambda\left(\Phi(\alpha P + \beta P') \circ \Phi(P) \circ \cdots \circ \Phi(P) + \Phi(\alpha P + \beta P') \circ \Phi(P') \circ \cdots \circ \Phi(P')\right).
\end{aligned}$$

Since $(\alpha P + \beta P') \circ P \circ P \circ \cdots \circ P = \alpha P$, we deduce that

$$\begin{aligned}
\Delta_\lambda\left(\Phi(\alpha P + \beta P') \circ \Phi(P) \circ \cdots \circ \Phi(P)\right) &= \Phi(\Delta_\lambda((\alpha P + \beta P') \circ P \circ P \circ \cdots \circ P)))\\
&= \Phi(\Delta_\lambda(\alpha P)) = \Phi(\Delta_\lambda(x \otimes \overline{\alpha} x))\\
&= \Phi\left(\frac{\langle x, \overline{\alpha} x\rangle}{\|\overline{\alpha} x\|^2}(\overline{\alpha} x \otimes \overline{\alpha} x)\right) = \Phi(\alpha(x \otimes x))\\
&= \Phi(\alpha P).
\end{aligned}$$

Similarly, we also obtain $\Delta_\lambda(\Phi(\alpha P + \beta P') \circ \Phi(P') \circ \cdots \circ \Phi(P'))) = \Phi(\beta P')$.
Since $\Phi(I) = I$ from Lemma 3.1 and $\Phi(P)^2$ is quasinormal, we obtain that

$$\begin{aligned}
\Phi(P)^2 = \Delta_\lambda(\Phi(P)^2) &= \Delta_\lambda(\Phi(P) \circ \Phi(P) \circ \Phi(I) \circ \Phi(I) \circ \cdots \Phi(I)\\
&= \Phi(\Delta_\lambda(P \circ P \circ I \circ I \circ \cdots \circ I).
\end{aligned}$$

Since $\Phi(I) = I$ from Lemma 3.1, we let $x, x' \in H$ and $y, y' \in K$ be unit vectors such that $\Phi(P) = \Phi(x \otimes x) = y \otimes y$ and $\Phi(P') = \Phi(x' \otimes x') = y' \otimes y'$. Since $P = x \otimes x$ and

$P' = x' \otimes x'$ are two rank one orthogonal projections such that $\langle x, x' \rangle = 0$, then y and y' are orthogonal too. Therefore $\Phi(P)$ is rank one orthogonal projection. Therefore we have

$$\begin{aligned}\Delta_\lambda(\Phi(\alpha P) \circ \Phi(P)) &= \Delta_\lambda(\Phi(\alpha P) \circ \Phi(P) \circ \cdots \circ \Phi(P)) = \Phi(\Delta_\lambda(\alpha P \circ P \circ \cdots \circ P)) \\ &= \Phi(\Delta_\lambda(\alpha P)) = \Phi(\Delta_\lambda(x \otimes \overline{\alpha} x)) = \Phi\left(\frac{\langle x, \overline{\alpha} x \rangle}{\|\overline{\alpha} x\|^2}(\overline{\alpha} x \otimes \overline{\alpha} x)\right) \\ &= \Phi(\alpha(x \otimes x)) = \Phi(\alpha P).\end{aligned}$$

By applying the proposition 2.3. in [4], there exists $h(\alpha) \in \mathbb{C}$ such that $\Phi(\alpha P) = h(\alpha)\Phi(P)$. In the same way, there exists $h(\beta) \in \mathbb{C}$ such that $\Phi(\beta P') = h(\beta)\Phi(P')$. Finally, we obtain that

$$\Phi(\alpha P + \beta P') = h(\alpha)\Phi(P) + h(\beta)\Phi(P')$$

for every $\alpha, \beta \in \mathbb{C}$.
Since $\Phi(0) = h(0)\Phi(P)$ and $\Phi(P) = h(1)\Phi(P)$, we have $h(0) = 0$ and $h(1) = 1$. By using (iv) in Lemma 3.1, we obtain that

$$\begin{aligned}&h(\alpha)h(\beta)h(1)\cdots h(1)I = \Delta_\lambda(\Phi(\alpha I) \circ \Phi(\beta I) \circ \Phi(I) \circ \cdots \circ \Phi(I)) \\ =&\Phi(\Delta_\lambda(\alpha I \circ \beta I \circ I \circ \cdots \circ I)) = h(\alpha\beta)I.\end{aligned}$$

Therefore we conclude that the function h is multiplicative.

(ii) Since $\Phi(I) = I$ from Lemma 3.1, we let $x \in H$ and $y \in K$ be unit vectors such that $\Phi(x \otimes x) = y \otimes y$. Then we obtain that

$$\begin{aligned}&\Delta_\lambda(\Phi(A)(y \otimes y)) = \Delta_\lambda(\Phi(A) \circ (y \otimes y)) \\ =&\Delta_\lambda(\Phi(A) \circ (y \otimes y) \circ (y \otimes y) \cdots \circ (y \otimes y)) \\ =&\Delta_\lambda(\Phi(A) \circ (\Phi(x \otimes x)) \circ (\Phi(x \otimes x)) \cdots \circ (\Phi(x \otimes x)) \\ =&\Phi(\Delta_\lambda(A \circ (x \otimes x) \circ \cdots \circ (x \otimes x))) \\ =&\Phi(\Delta_\lambda(A \circ (x \otimes x)) = \Phi(\Delta_\lambda(A(x \otimes x)) = \Phi(\Delta_\lambda(Ax \otimes x))\end{aligned}$$

where $A \in \mathscr{B}(H)$. By using Proposition 2.1 in [4], we deduce that

$$\langle \Phi(A)y, y \rangle y \otimes y = \Phi(\langle Ax, x \rangle x \otimes x) = h(\langle Ax, x \rangle) y \otimes y.$$

Finally, we have

$$\langle \Phi(A)y, y \rangle = h(\langle Ax, x \rangle)$$

for all unit vectors x, y such that $\Phi(x \otimes x) = y \otimes y$ for every $A \in \mathscr{B}(H)$.
Let $z = \frac{1}{\sqrt{2}}(x + x')$. Then there exists an unit vector $t \in K$ such that $\Phi(z \otimes z) = t \otimes t$. Then we obtain that

$$\begin{aligned}h(\langle \alpha P z + \beta P' z, z \rangle) &= \langle \Phi(\alpha P + \beta P')t, t \rangle \\ &= \langle (h(\alpha)\Phi(P) + h(\beta)\Phi(P'))t, t \rangle\end{aligned}$$

$$= h(\alpha)\langle \Phi(P)t,t\rangle + h(\beta)\langle \Phi(P')t,t\rangle$$
$$= h(\alpha)h(\langle Pz,z\rangle) + h(\beta)h(\langle P'z,z\rangle)$$
$$= h(\alpha)h\left(\frac{1}{2}\right) + h(\beta)h\left(\frac{1}{2}\right)$$

using $\langle Pz,z\rangle = \langle\langle z,x\rangle x,z\rangle = \left\langle \frac{1}{\sqrt{2}}x,z\right\rangle = \frac{1}{2}$ and $\langle P'z,z\rangle = \langle\langle z,x'\rangle x',z\rangle = \left\langle \frac{1}{\sqrt{2}}x',z\right\rangle = \frac{1}{2}$. Since the function h is multiplicative by Lemma 3.3 (i), we obtain that $h(\langle \alpha Pz + \beta P'z,z\rangle) = h(\alpha||Pz||^2 + \beta||P'z||^2) = h\left(\frac{1}{2}\right)h(\alpha+\beta)$.
Finally, we deduce that

$$h\left(\frac{1}{2}\right)h(\alpha+\beta) = h\left(\frac{1}{2}\right)(h(\alpha)+h(\beta)).$$

Because of $h\left(\frac{1}{2}\right) \neq 0$, we conclude that the function h is additive.
Recall that if T is a bounded linear operator acting on a complex Hilbert space H, then the *numerical range* of T is defined by

$$W(T) = \{(Tx,x) : x \in H, ||x|| = 1\}.$$

The next theorems give a description of the bijective maps between Banach spaces which commute with a λ-Aluthge transform under the n-fold jordan product.

THEOREM 3.4. *Let H and K be two complex Hilbert spaces with $dim(H) \geqslant 2$. Let $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ be a bijective map. Then the following statements are equivalent:*

(α) *For every $A_1, \cdots, A_n \in \mathscr{B}(H)$, the bijective map Φ satisfies*

$$\Delta_\lambda(\Phi(A_1)\circ\Phi(A_2)\circ\cdots\circ\Phi(A_n)) = \Phi(\Delta_\lambda(A_1\circ A_2\circ\cdots\circ A_n)).$$

for some $n \in \mathbb{N}$.

(β) *There exists a unitary operator $U : H \to K$ such that*

$$\Phi(T) = UTU^* \text{ for every } T \in \mathscr{B}(H).$$

COROLLARY 3.5. *Let H and K be two complex Hilbert spaces with $dim(H) \geqslant 2$. Let $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ be a bijective map. Then the following statements are equivalent:*

(α) *For every $A_1, \cdots, A_n \in \mathscr{B}(H)$, the bijective map Φ satisfies*

$$\Delta_\lambda(\Phi(A_1)\Phi(A_2)\cdots\Phi(A_n)) = \Phi(\Delta_\lambda(A_1A_2\cdots A_n)).$$

(β) *There exists a unitary operator $U : H \to K$ such that*

$$\Phi(T) = UTU^* \text{ for every } T \in \mathscr{B}(H).$$

Now we are in a position to present the proof of Theorem 3.4. Our proof follows the similar steps to verify for Theorem 1.1 in [4].

Proof of Theorem 3.4. Let $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ be a bijective map satisfying the condition (α) in Theorem 3.4. Since $\Phi(I) = I$ from Lemma 3.1, we let $x \in H$ and $y \in K$ be unit vectors such that $\Phi(x \otimes x) = y \otimes y$. Since $\langle \Phi(A+B)y, y\rangle = h(\langle (A+B)x, x\rangle) = h(\langle Ax, x\rangle) + h(\langle Bx, x\rangle) = \langle \Phi(A)y, y\rangle + \langle \Phi(B)y, y\rangle = \langle (\Phi(A)+\Phi(B))y, y\rangle$ and $\langle \Phi(\alpha A)y, y\rangle = h(\langle \alpha Ax, x\rangle) = h(\alpha)h(\langle Ax, x\rangle) = h(\alpha)\langle \Phi(A)y, y\rangle$ from Lemma 3.3, we deduce that

$$\Phi(A+B) = \Phi(A) + \Phi(B) \;\; and \;\; \Phi(\alpha A) = h(\alpha)\Phi(A)$$

for every $A, B \in \mathscr{B}(H)$, where $\alpha \in \mathbb{C}$.
Let $\mathscr{E}$ be a bounded subset in $\mathbb{C}$ such that $\mathscr{E} \subset W(A)$ for $A \in \mathscr{B}(H)$. By using (ii) in Lemma 3.3, we obtain that

$$h(\mathscr{E}) \subset h(W(A)) = W(\Phi(A)).$$

Since $W(\Phi(A))$ is bounded, we claim that the function h is bounded on the bounded set. Since the function h is multiplicative and additive by Lemma 3.3, we conclude that the function h is continuous. It follows that the function h is the identity or the complex conjugation map. Therefore the bijective map Φ is linear or antilinear.
By Corollary 3.2, the bijective map Φ commutes with the λ-Aluthge transform. Using the Theorem 1 in [2], there exists a linear and bijective operator $V : H \to K$ such that Φ takes one of the following either

$$\Phi(A) = VAV^* \quad \text{for all } A \in \mathscr{B}(H), \tag{8}$$

or

$$\Phi(A) = VA^*V^* \quad \text{for all } A \in \mathscr{B}(H). \tag{9}$$

Suppose that the bijective map Φ takes the form (9). Since V is unitary, we deduce that $V^*\Phi(A)V = V^*VA^*V^*V = A^*$. Then we obtain the following equation

$$\Delta_\lambda(A^*) = \Delta_\lambda(V^*\Phi(A)V) = V^*\Delta_\lambda(\Phi(A))V = V^*\Phi(\Delta_\lambda(A))V = (\Delta_\lambda(A))^* \tag{10}$$

for all bounded linear operators $A \in \mathscr{B}(H)$.
If we consider $A = x \otimes x'$ such that x, x' are unit independent vectors in H, we obtain that

$$\Delta_\lambda(A) = \Delta_\lambda(x \otimes x) = \frac{\langle x, x'\rangle}{\|x'\|^2}(x' \otimes x') = \langle x, x'\rangle(x' \otimes x')$$
$$\Delta_\lambda(A^*) = \Delta_\lambda(x' \otimes x) = \frac{\langle x', x\rangle}{\|x\|^2}(x \otimes x) = \langle x', x\rangle(x \otimes x).$$

Then we conclude that

$$(\Delta_\lambda(A))^* = \langle x', x\rangle (x' \otimes x') \neq \Delta_\lambda(A^*).$$

It is a contradiction to (10).
Therefore there exists a unitary operator $V : H \to K$ such that

$$\Phi(A) = VAV^* \quad \text{for all } A \in \mathscr{B}(H).$$

REFERENCES

[1] A. ALUTHGE, *On p-hyponormal operators for $0 < p < 1$*, Integral Equations Operator Theory **13** (1990) 307–315.
[2] F. BOTELHO, L. MOLNAR, AND G. NAGY, *Linear bijections on von Neumann factors commuting with λ-Aluthge transform*, Bull. London Math. Soc. **48** (2016) 74–84.
[3] F. CHABBABI, *Product commuting maps with the λ-Aluthge transform*, J. Math. Anal. Appl. **449** (2017) 589–600.
[4] F. CHABBABI AND M. MBEKHTA, *Jordan-product commuting nonlinear maps with λ-Aluthge transform*, J. Math. Anal. Appl. **450** (2017) 293–313.
[5] I. JUNG, E. KO AND C. PEARCY, *Aluthge transform of operators*, Integral Equations Operator Theory **37** (2000) 437–448.
[6] K. OKUBO, *On weakly unitarily invariant norm and the Aluthge transformation*, Linear Algebra Appl. **371** (2003) 369–375.

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