

CONVERGENCE OF LAPLACIANS ON SMOOTH SPACES TOWARDS THE FRACTAL SIERPIŃSKI GASKET

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Abstract. The purpose of this article is to prove that – under reasonable assumptions – the canonical energy form on a graph-like manifold is quasi-unitarily equivalent with the energy form on the underlying discrete graph. Then we will apply this to approximate the standard energy form on the Sierpiński gasket by a family of energy forms on suitable graph-like manifolds.

1. Introduction

In [9] the first author introduced the notion of quasi-unitary equivalence, which generalises the concept of norm resolvent convergence to the case of energy forms resp. their associated linear operators defined in different Hilbert spaces. The consequences (such as convergence of spectra, of operator functions etc.) are basically the same as in the case of the classical norm resolvent convergence. It turns out in many applications (see e.g. [7, 11, 1, 14]) that the setting is tailor-made for these kind of linear approximation problems on varying spaces and might even be easier to apply than the weaker notion of strong (or Mosco-)convergence. We briefly introduce the notion of quasi-unitary equivalence in Appendix A and refer the interested reader to [9, 10, 13] for more details.

The aim of the article is to apply the aforementioned convergence schema to approximate the standard energy form on the Sierpiński gasket by the (scaled) canonical energy forms on a sequence of shrinking graph-like manifolds. We first introduce the notion of a graph-like manifold in Subsection 2.1. Then in Subsection 2.2 we link the data of a graph-like manifold with the weights of the discrete graph, called *uniform compatibility* here (see Definitions 2.4 and 2.6). In Subsection 2.3 we prove that a properly rescaled energy form on a graph-like manifold and the energy form on a suitable underlying discrete weighted graph are quasi-unitarily equivalent provided the data are uniformly compatible. In order to prove this result for the transversally scaled graph-like manifold (Corollary 2.10) it is enough to deal with the transversally unscaled manifold (Theorem 2.9).

The results presented here appeared already in [12] in a more general way. Here, in contrast, we would like to give a concrete and straightforward presentation, taking the prototype of a post-critically finite fractal, the Sierpiński gasket, as an example. Moreover, we also simplified several steps in the proofs.

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The idea for the proof in our application to the Sierpiński gasket in Section 3 is to use the transitivity of the notion of quasi-unitary equivalence (Proposition A.3) and the main result from [11] which states that the standard energy form on the Sierpiński gasket and the energy forms on the underlying family of discrete weighted graphs are quasi-unitarily equivalent.

Moreover, we are confident that with the improvements done — in particular in Corollary 2.10 — it is possible to treat more complex examples than just symmetric post-critically finite self-similar fractals (in the sense of [11, Sec. 5.2]) in the future.

Related works

Our analysis can be seen in some sense as an analytic and rigorous confirmation of numerical results found by Berry, Heilman and Strichartz [2]. There, the authors approximate the eigenvalues of the usual Laplacian on the Sierpiński gasket K (and other related spaces) numerically by the eigenvalues of a sequence of Neumann Laplacians on domains Ω_m , where $\Omega_{m+1} = F(\Omega_m)$ is obtained via the iterated function system (IFS) from an open neighbourhood Ω_0 of K . Such a sequence is called *outer approximation* as $K \subset \Omega_m$ for all $m \in \mathbb{N}_0$. Another closely related approach can be found in [3], where the authors also consider an outer approximation $\{\Omega'_m\}_m$. There, Ω'_{m+1} is obtained from Ω'_m by subtracting some triangles with small balls removed around the vertices. Again, they obtain similar results as in [2], but numerically better approximations.

Our sequence of manifolds $\{X_m\}_m$ is not an outer approximation of the Sierpiński gasket K , even in the embedded case where X_m is a subset of \mathbb{R}^2 . Moreover, $\{X_m\}_m$ is not obtained from the IFS, as we need a faster decay in the transversal direction (namely $\varepsilon_m = \varepsilon_0 E^m \ll \ell_{m,e} = \ell_{0,e} \Lambda^m$, see Subsection 3.2 for the notation). As an example, we let X_m be the $\varepsilon_m/2$ -neighbourhood of the graph G_m , where G_m is the m -th approximation graph of K . Here, we consider G_m as a metric graph embedded in \mathbb{R}^2 . In this case, X_m is a *graph-like manifold* (in the sense of Definition 2.1) with transversal manifolds $Y_{e,m} = [-\varepsilon_m/2, \varepsilon_m/2]$ and length function ℓ_m of order Λ^m with $\Lambda = 1/2$. We obtain this embedded concrete case from the abstract setting in Corollary 3.5. Our analysis confirms the energy rescaling factors $\tau_m = (5/4)^m$ in front of the standard energy form on X_m already numerically found in [2, 3].

In this context we should also mention the article [8] (see also the references therein) where the authors construct a sequence of *weighted* energy forms on open domains that Mosco-converge to an energy form with singular potential supported on a nested fractal such as the Koch curve or the Sierpiński triangle. For further references, especially on approximations of energy forms on manifolds by discrete graphs, we refer to the introduction of [12].

2. Approximation of discrete graphs by graph-like manifolds

2.1. Graph-like manifolds and their energy forms

In this section, we briefly introduce the notion of *graph-like manifolds*. For more details we refer to the monograph [10] and references therein.

Let $G = (V, E, \partial)$ be a *discrete graph*, i.e., V and E are at most countable sets, called the *vertex* and *edge* set of G . Moreover, $\partial : E \rightarrow V \times V, e \mapsto (\partial_-e, \partial_+e)$, assigns to each edge e an *initial vertex* ∂_-e and a *terminal vertex* ∂_+e , and hence determines an *orientation*. We define

$$E_v := E_v^+ \cup E_v^- \quad \text{where} \quad E_v^\pm := \{e \in E \mid \partial_\pm e = v\}.$$

We call $\deg v := |E_v|$ the *degree* of the vertex $v \in V$. We assume that the graph is *simple*, i.e., that there are no loops (i.e., $\partial_+e \neq \partial_-e$ for all $e \in E$) and no multiple edges (i.e., an edge e is uniquely determined by its adjacent vertices $\{\partial_-e, \partial_+e\}$).

Moreover, let $\mu : V \rightarrow (0, \infty)$ be a vertex weight function and $\gamma : E \rightarrow (0, \infty)$ an edge weight function. Then there is a natural Hilbert space structure on G given by

$$\ell_2(V, \mu) := \left\{ f \mid f : V \rightarrow \mathbb{C} \text{ such that } \sum_{v \in V} \mu(v) |f(v)|^2 < \infty \right\} \tag{2.1}$$

with norm $\|f\|_{\ell_2(G, \mu)}^2 := \sum_{v \in V} \mu(v) |f(v)|^2$ and a non-negative quadratic form

$$\mathcal{E}_G(f) := \sum_{e \in E} \gamma_e |(df)_e|^2 \quad \text{where} \quad (df)_e = f(\partial_+e) - f(\partial_-e).$$

The associated non-negative self-adjoint operator is given by

$$\Delta_G f(v) = \frac{1}{\mu(v)} \sum_{e \in E_v} \gamma_e (f(v) - f(v_e)),$$

where v_e denotes the vertex on e opposite to v . The graph energy form \mathcal{E}_G fulfills

$$\mathcal{E}_G(f) \leq \left(\sup_{v \in V} \frac{2}{\mu(v)} \sum_{e \in E_v} \gamma_e \right) \|f\|_{\ell_2(V, \mu)}^2.$$

In what follows we assume that the above supremum is *finite* (i.e. \mathcal{E}_G is bounded), or, equivalently, we assume the boundedness of the *relative weights*

$$\rho(v) := \frac{1}{\mu(v)} \sum_{e \in E_v} \gamma_e.$$

A *graph-like manifold* associated with the discrete graph $G = (V, E, \partial)$ is a Riemannian manifold of dimension $d \geq 2$ which decomposes into *vertex neighbourhoods* and *edge neighbourhoods* respecting the structure of the underlying graph. The following definition makes this precise (see also Figure 1).

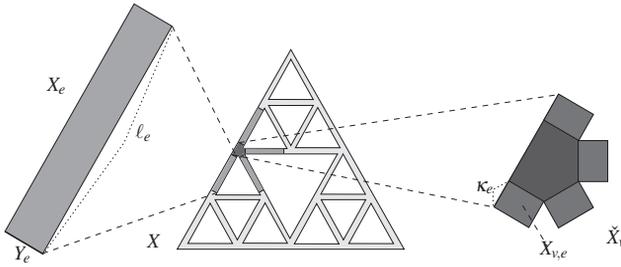


Figure 1: A graph-like manifold X with transversal manifold Y_e being an interval; together with an edge neighbourhood X_e and a core vertex neighbourhood X_v . The enlarged vertex neighbourhood X_v here consists of the core vertex neighbourhood \check{X}_v (dark grey) together with the four edge neighbourhoods (medium grey). The underlying graph G (see also Figure 2) here is the graph approximating the Sierpiński gasket in generation $m = 2$ called G_2 in Subsection 3.1.

DEFINITION 2.1. (Graph-like manifold) We say that a d -dimensional Riemannian manifold X is a *graph-like manifold* with associated discrete graph $G = (V, E)$, if the following properties hold:

1. We can decompose X into compact and connected subsets \check{X}_v and X_e , i.e.,

$$X = \bigcup_{v \in V} \check{X}_v \cup \bigcup_{e \in E} X_e,$$

such that $\check{X}_v \cap X_e \neq \emptyset$ if and only if $e \in E_v$; all other sets \check{X}_v and X_e are pairwise disjoint. We call \check{X}_v the *core vertex neighbourhood* of $v \in V$ and X_e the *edge neighbourhood* of $e \in E$.

If $\check{X}_v \cap X_e \neq \emptyset$ we assume that $\check{X}_v \cap X_e$ is isometric with a $(d - 1)$ -dimensional Riemannian manifold Y_e , called *transversal manifold* of e .

2. There is a so-called *edge length function* $\ell : E \rightarrow (0, \infty)$, $e \mapsto \ell_e > 0$, such that each edge neighbourhood X_e is isometric with $M_e \times Y_e$, where $M_e = [0, \ell_e]$ and where Y_e is the transversal manifold from (1).
3. There exists a function $\kappa : E \rightarrow (0, \infty)$, $e \mapsto \kappa_e$, such that $\partial_e \check{X}_v := \check{X}_v \cap X_e$ (isometric with Y_e by (2)) has a κ_e -collar neighbourhood $X_{v,e}$ inside \check{X}_v , i.e., $X_{v,e}$ is isometric with $[0, \kappa_e] \times Y_e$; we assume that $(X_{v,e})_{e \in E_v}$ are pairwise disjoint.

In what follows, we will identify points in the manifold with its coordinates, e.g., we write $x = (t, y) \in M_e \times Y_e$ if we mean a point $x \in X_e$, and similarly for $x \in X_{v,e}$. Moreover, we choose the isometry $X_e \cong M_e \times Y_e$ in such a way that points in $\partial_e \check{X}_v$ correspond to $\{0\} \times Y_e$ or $\{\ell_e\} \times Y_e$, depending whether $v = \partial_- e$ or $v = \partial_+ e$.

We also need the so-called (*enlarged*) *vertex neighbourhood* of v defined by

$$X_v := \check{X}_v \cup \bigcup_{e \in E_v} X_e.$$

We sometimes also refer to the data $(X, \ell, (\check{X}_v)_{v \in V}, \kappa, (Y_e)_{e \in E})$ as a graph-like manifold.

REMARKS 2.2.

1. If Y_e is the compact interval $[-1/2, 1/2]$, then X is diffeomorphic with a closed neighbourhood of the graph embedded in \mathbb{R}^2 (provided the graph has an embedding into \mathbb{R}^2). We would like to stress that X is not necessarily *isometrically* embeddable in \mathbb{R}^2 . The above definition is more general than the embedded case, as it refers to an *abstract* manifold. Nevertheless, we see in Corollary 3.5 that one can treat the embedded case as a perturbation of the abstract case.
2. Other cases are also included, e.g. if Y_e is a circle (a one-dimensional compact manifold), then X is diffeomorphic with the surface of a tubular neighbourhood of the underlying graph.
3. The decomposition into vertex neighbourhoods \check{X}_v and edge neighbourhoods X_e is not unique. Moreover, in the special case of a compact graph-like manifold, the third condition in the above definition follows from the first and the second one: For each edge $e \in E_V$, we take away a little piece from X_e and add it to \check{X}_v , as a result, ℓ_e will be smaller.
4. If we start with a metric graph instead of a discrete graph, we can construct a graph-like manifold by using the same length function ℓ and X is defined as an abstract space (cf. [12, Example 4.2]).

We mainly use not only the graph-like manifold, but a *scaled* version X_ε of it for some $\varepsilon > 0$. We use the following handy notation for a Riemannian manifold M with metric g : we denote by εM the Riemannian manifold M with metric $\varepsilon^2 g$. Technically, the ε -dependence only enters in the metric, hence εM and M have the same underlying manifold, allowing e.g. to define functions on εM without referring to ε , see e.g. (2.4).

DEFINITION 2.3. Let $\varepsilon > 0$. We say that X_ε is an (*transversally*) ε -scaled graph-like manifold if X_ε is a graph-like manifold with data $(X_\varepsilon, \ell, (\varepsilon \check{X}_v)_{v \in V}, \varepsilon \kappa, (\varepsilon Y_e)_{e \in E})$.

The idea of the above definition is to scale the transversal and core vertex manifolds Y_e and \check{X}_v , respectively, by a length factor ε . Note that the 1-scaled graph-like manifold X_1 is X itself.

The Hilbert space we consider here is the usual space of square integrable functions $L_2(X, \nu)$ with respect to the Riemannian measure ν and with the usual norm denoted by $\|\cdot\|_{L_2(X, \nu)}$. The energy form on X is given by

$$\mathcal{E}_X(u) = \int_X |\nabla u(x)|_x^2 \, d\nu(x) \tag{2.2}$$

for each $u \in H^1(X)$, i.e., the domain of \mathcal{E}_X is the closure of Lipschitz continuous functions with compact support in X with respect to the energy norm defined by

$$\|u\|_{H^1(X)}^2 = \|u\|_{L_2(X)}^2 + \mathcal{E}_X(u). \tag{2.3}$$

Here, ∇ denotes the gradient and $|\cdot|_x$ is the norm induced by the Riemannian metric at $x \in X$.

2.2. Compatibility of graph-like manifolds and weighted graphs

Let $G = (V, E)$ be a discrete graph. Let X be the associated graph-like manifold with Riemannian measure ν . We first define a partition of unity $\{\varphi_v\}_{v \in V}$ on X that respects the structure of the associated discrete graph. The idea in this section is quite similar to the approach we followed in [11] where we compared a finitely ramified fractal with its associated discrete graph. Let $\varphi_v : X \rightarrow [0, 1]$ be given by

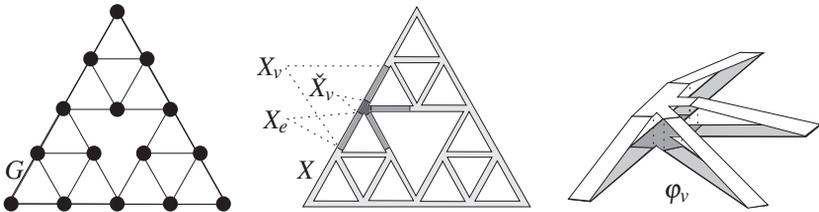


Figure 2: A discrete graph G together with an associated graph-like manifold X and the function φ_v supported on the (enlarged) vertex neighbourhood X_v .

$$\varphi_v(x) = 1 \quad \text{if } x \in \check{X}_v, \quad \varphi_v(x) = \frac{t}{\ell_e} \quad \text{if } x = (t, y) \in M_e \times Y_e \quad (2.4)$$

and $\varphi_v(x) = 0$ if $x \in X \setminus X_v$. Here we assume for simplicity that v is the terminal vertex for all edges $e \in E_v$, i.e., $v = \partial_+ e$ and the corresponding coordinate is $t = \ell_e$. Note that $\varphi_v \in H^1(X)$ since these functions are continuous and (piecewise) harmonic, hence Lipschitz continuous on X . In particular, φ_v is constant on the core vertex neighbourhoods \check{X}_v , affine linear in longitudinal and constant in transversal direction on the edge neighbourhoods X_e . Similarly, we can define a partition of unity $\{\varphi_v\}_{v \in V}$ for an ε -scaled graph-like manifold X_ε (without referring to ε , as the underlying manifold is X , the parameter ε enters only via the metric, see the remark before Definition 2.3).

The partition of unity allows us to define a vertex measure $\{v(v)\}_{v \in V}$ inherited by the Riemannian measure of X on the underlying discrete graph: We define $v : V \rightarrow (0, \infty)$ by

$$v(v) := \int_X \varphi_v(x) d\nu(x) = \text{vol} \check{X}_v + \frac{1}{2} \sum_{e \in E_v} \text{vol} X_e = \text{vol} \check{X}_v + \frac{1}{2} \sum_{e \in E_v} \ell_e \text{vol} Y_e. \quad (2.5)$$

Note that $\{v(v)\}_{v \in V}$ would be the natural choice for a vertex measure on the discrete graph G , which we want to compare with the graph-like manifold X . But keep in mind that it is our goal to compare the family of discrete graphs associated with the Sierpiński gasket with its associated graph-like manifolds. In that case, we have already a given vertex and edge weight function on G . Hence, we need to make sure that the weights induced by a post-critically finite fractal and the once induced by the graph-like manifold and other objects fit to each other, called *compatibility* here:

DEFINITION 2.4. (Compatibility of weights and manifold geometry) Let X be a graph-like manifold defined by the data $(X, \ell, (\check{X}_v)_{v \in V}, \kappa, (Y_e)_{e \in E})$ as in Definition 2.1 and let G be its underlying discrete graph. Let $\mu : V \rightarrow [0, \infty)$ and $\gamma : E \rightarrow [0, \infty)$ be a vertex weight and edge weight respectively.

1. We say that X and (G, μ, γ) are *compatible* if there exist two constants $c > 0$ and $\tau > 0$ such that the graph weights μ and γ , the edge length function ℓ and the transversal volumes $(\text{vol}Y_e)_{e \in E}$ fulfil

$$\frac{1}{2\mu(v)} \sum_{e \in E_v} \ell_e \text{vol}Y_e = \frac{1}{c^2} \quad \text{and} \quad \frac{\gamma_e \ell_e}{\text{vol}Y_e} = c^2 \tau \tag{2.6a}$$

for all vertices $v \in V$ and edges $e \in E$.

2. We say that X has *uniformly small (core) vertex neighbourhoods* if there are constants $\alpha_0 \in (0, 1]$ and $\alpha_\infty > 0$ such that

$$\alpha_0 \leq \alpha(v) := \frac{2 \text{vol}\check{X}_v}{\sum_{e \in E_v} \text{vol}X_e} \leq \alpha_\infty, \tag{2.6b}$$

for all $v \in V$ and, in addition, if

$$K_\infty := \sup_{v \in V} \max_{e \in E_v} \left(\kappa_e + \frac{2}{\kappa_e \lambda_2(\check{X}_v)} \right) < \infty, \tag{2.6c}$$

where $\lambda_2(\check{X}_v) > 0$ is the second (first non-zero) Neumann eigenvalue on \check{X}_v .

3. We say that X has *uniform transversal volume*, if¹

$$0 < \text{vol}_0 := \inf_{e \in E} \text{vol}Y_e \leq \text{vol}_\infty := \sup_{e \in E} \text{vol}Y_e < \infty \tag{2.6d}$$

REMARK 2.5. Let us comment on the meaning of the assumptions (2.6a)–(2.6d):

1. The first equality in (2.6a) means that the *leg volume*

$$\text{vol}(X_v \setminus \check{X}_v) = \sum_{e \in E_v} \text{vol}X_e = \sum_{e \in E_v} \ell_e \text{vol}Y_e \tag{2.7}$$

is proportional to the vertex measure $\mu(v)$ (where “proportional” here means that the constant — here $2/c^2$ — is independent of $v \in V$).

2. The second equality in (2.6a) means that the edge weight (or discrete conductance) γ_e is proportional to the “conductance” $\text{vol}Y_e/\ell_e$ of the graph-like manifold: recall that ℓ_e is the length of the cylinder; a large ℓ_e will lead to a bad conductance; while a “thick” cylinder (i.e., a large value of $\text{vol}Y_e$) leads to a good conductance.

¹Here, $\text{vol}Y_e$ is the volume of the $(d - 1)$ -dimensional manifold Y_e , where d is the dimension of X . We will not distinguish here between volume measures in different dimensions in the notation.

3. The estimate in (2.6b) assures that the core vertex volume \check{X}_v is comparable with the leg volume (2.7).
4. The estimate in (2.6c) measures in some sense the uniform “connectedness” of \check{X}_v (as family in v): recall that $\lambda_2(\check{X}_v)$ can be controlled by an isoperimetric constant, the Cheeger constant: the smaller $\lambda_2(\check{X}_v)$ (and hence the smaller the Cheeger constant), the less connected \check{X}_v is (and hence the larger K_∞ is). The extreme case would be $\lambda_2(\check{X}_v) = 0$ and hence $K_\infty = \infty$; this would happen if \check{X}_v was disconnected (which is excluded in Definition 2.1).
5. Finally, (2.6d) assures that the transversal volume can be controlled by some lower and upper constant.

Let us stress that we meticulously mention all the above constants as we deal with an *infinite family* of finite graphs in our main application in Section 3. Moreover, Theorem 2.9 and Corollary 2.10 are also valid for *infinite* graphs under the given assumptions. In addition, our results in Section 3 also apply to so-called non-compact *fractafolds* like the tessellation of the plane with Sierpiński gaskets touching in the boundary points and a suitable (non-compact) graph-like manifold.

Note that there is still some freedom in the choice of the parameters τ and c (adopting the other data), as they only need to satisfy the two equations (2.6a). We conclude

$$c = \left(\frac{1}{2\mu(v)} \sum_{e \in E_v} \ell_e \text{vol} Y_e \right)^{-1/2} \quad \text{and} \quad \tau = \frac{1}{2\mu(v)} \sum_{e \in E_v} \ell_e^2 \gamma_e.$$

Moreover, the compatibility conditions in (2.6a) gives the following necessary condition on the vertex weights μ and edge weights γ of the weighted graph G and the transversal volume $\text{vol} Y_e$; namely, we need that

$$\frac{1}{2\mu(v)} \sum_{e \in E_v} \frac{(\text{vol} Y_e)^2}{\gamma_e} = \frac{1}{c^4 \tau}$$

is independent of the vertices $v \in V$.

Summarising the above three conditions of Definition 2.4, we require:

DEFINITION 2.6. (Uniform compatibility of a graph-like manifold and a discrete graph) We say that a graph-like manifold X and a discrete weighted graph (G, μ, γ) are *uniformly compatible*, if they are compatible and if X has uniformly small vertex neighbourhoods and uniform transversal volume, i.e., if the conditions of Definition 2.4 (1)–(3) hold.

We now need to know how the various constants in Definition 2.4 depend on ε for an ε -scaled graph-like manifold. Recall that X_ε is obtained from X by scaling the

transversal and core vertex manifolds by a length scaling factor $\varepsilon > 0$. Note that we have the scaling behaviour

$$\text{vol}(\varepsilon M) = \varepsilon^d M \quad \text{and} \quad \lambda_k(\varepsilon M) = \frac{1}{\varepsilon^2} \lambda_k(M)$$

for a d -dimensional (compact) Riemannian manifold M , where k denotes the k -th eigenvalue of its Laplacian (without boundary, or Neumann boundary conditions). It follows immediately from Definition 2.4 that an ε -scaled graph-like manifold X_ε is uniformly compatible with (G, μ, γ) if the constants in Definition 2.4 fulfil the following scaling behaviour: τ is ε -independent and

$$\begin{aligned} c_\varepsilon &= \varepsilon^{-(d-1)/2} c, & K_{\infty, \varepsilon} &= \varepsilon K_\infty, \\ \alpha_{0, \varepsilon} &= \varepsilon \alpha_0, & \alpha_{\infty, \varepsilon} &= \varepsilon \alpha_\infty, \\ \text{vol}_{0, \varepsilon} &= \varepsilon^{d-1} \text{vol}_0, & \text{vol}_{\infty, \varepsilon} &= \varepsilon^{d-1} \text{vol}_\infty, \end{aligned}$$

where d is the dimension of X . Here, the quantities with subscript ε refer to X_ε whereas those without refer to the unscaled graph-like manifold $X = X_1$.

On each vertex neighbourhood X_v , we define a quadratic form by

$$\mathcal{E}_{X_v}(u) = \int_{X_v} |\nabla u(x)|_x^2 \, d\nu(x)$$

for all $u \in \text{dom } \mathcal{E}_{X_v} := \{u|_{X_v} \mid u \in H^1(X)\}$. This allows us to decompose the energy form with respect to the building blocks, and we have

$$\mathcal{E}_X(u) \leq \sum_{v \in V} \mathcal{E}_{X_v}(u|_{X_v}) \leq 2\mathcal{E}_X(u) \tag{2.8}$$

for each $u \in H^1(X)$, and the same is true for an ε -scaled graph-like manifold X_ε .

We need some facts about an auxiliary weighted eigenvalue problem on X_v , their proof can be found in [12, Lem. 2.6]):

PROPOSITION 2.7. *The quadratic form $(\mathcal{E}_{X_v}, \text{dom } \mathcal{E}_{X_v})$ is closable in the weighted Hilbert space*

$$\mathbb{L}_2(X_v, \varphi_v) := \left\{ u \mid \|u\|_{\mathbb{L}_2(X_v, \varphi_v)}^2 := \int_{X_v} |u(x)|^2 \varphi_v(x) \, d\nu(x) < \infty \right\}.$$

Moreover, the first eigenvalue of the associated operator is $\lambda_1(X_v, \varphi_v) = 0$ with constant eigenfunction $\mathbb{1}_{X_v}$; and its second eigenvalue $\lambda_2(X_v, \varphi_v)$ fulfils

$$\lambda_2(X_v) \leq \lambda_2(X_v, \varphi_v). \tag{2.9}$$

Here, $\lambda_2(X_v)$ denotes the second (first non-zero) eigenvalue of \mathcal{E}_{X_v} in the unweighted Hilbert space $\mathbb{L}_2(X_v, \nu)$.

For an ε -scaled graph-like manifold, we can estimate the weighted eigenvalue on $X_{v, \varepsilon}$ (the core vertex neighbourhood εX_v together with $|E_v|$ many cylindrical ends $[0, \ell_e] \times \varepsilon Y_e$) by a scaling argument (see [12, Prop. B.3]):

PROPOSITION 2.8. Assume that X_ε is an ε -scaled graph-like manifold uniformly compatible with (G, μ, γ) such that there are constants $\ell_0, \ell_\infty \in (0, \infty)$, $\check{\lambda}_2 > 0$ and $\lambda_2^{\uparrow} > 0$ with

$$\ell_0 \leq \ell_e \leq \ell_\infty, \quad \lambda_2(Y_e) \geq \lambda_2^{\uparrow} \quad \text{and} \quad \lambda_2(\check{X}_v) \geq \check{\lambda}_2 \quad \text{for all } e \in E \text{ and } v \in V.$$

Then there exists a constant $C > 0$ depending only on ℓ_∞/ℓ_0 , λ_2^{\uparrow} and $\check{\lambda}_2$ (from the unscaled graph-like manifold $X = X_1$) such that

$$\lambda_2(X_{v,\varepsilon}) \geq \frac{1}{\ell_\infty^2} \quad \text{for all } 0 < \varepsilon \leq \varepsilon_0 := \frac{\ell_0}{C^2}$$

for all $v \in V$.

Combining Propositions 2.7 and 2.8 we conclude that the weighted eigenvalue problem on $X_{v,\varepsilon}$ of an ε -scaled graph-like manifold X_ε satisfies

$$\frac{1}{\ell_\infty^2} \leq \lambda_2(X_{v,\varepsilon}, \varphi_v) \tag{2.10}$$

for all $v \in V$ and $\varepsilon \in (0, \varepsilon_0]$.

2.3. Quasi-unitary equivalence of the energy forms on a graph-like manifold and its related discrete graph

We are now able to state our first main result: the quasi-unitary equivalence of the (rescaled) natural energy forms on the graph-like manifold X and the energy form of the underlying discrete weighted graph (G, μ, γ) . We first state the result for an unscaled graph-like manifold and later introduce the transversal scaling parameter ε in Corollary 2.10.

THEOREM 2.9. Assume that (G, μ, γ) is a weighted discrete graph with weights fulfilling

$$\mu_\infty := \sup_{v \in V} \mu(v) < \infty \quad \text{and} \quad 0 < \gamma_0 := \inf_{e \in E} \gamma_e \leq \gamma_\infty := \sup_{e \in E} \gamma_e < \infty. \tag{2.11}$$

Assume in addition that X is a graph-like manifold with underlying graph G and that X and (G, μ, γ) are uniformly compatible. Then there exist $\ell_0, \ell_\infty \in (0, \infty)$ such that $\ell_0 \leq \ell_e \leq \ell_\infty$ for all $e \in E$. If finally

$$\frac{1}{\ell_\infty^2} \leq \lambda_2(X_v, \varphi_v), \tag{2.12}$$

then the discrete energy form \mathcal{E}_G on the weighted graph and the rescaled energy form $\tilde{\mathcal{E}}_X = \tau \mathcal{E}_X$ with domain $\text{dom } \tilde{\mathcal{E}}_X = H^1(X)$ in $L_2(X, \nu)$ are δ -quasi-unitarily equivalent where

$$\delta^2 = \max \left\{ \alpha_\infty^2, \frac{4}{\alpha_0} \left(\frac{\text{vol}_\infty}{\text{vol}_0} \right)^2 \frac{\gamma_\infty}{\gamma_0} \cdot \frac{1}{d_0} \cdot \frac{\mu_\infty}{\gamma_0}, \frac{4K_\infty}{\ell_0} \right\}, \tag{2.13}$$

where $d_0 := \min_{v \in V} \deg v$ is the minimal degree, and α_∞ , K_∞ and vol_0 , vol_∞ are defined in Definition 2.4.

The lower bound (2.12) on the second eigenvalue can be verified e. g. by estimating an isoperimetric constant $h(X_v)$ from below, using Cheeger’s inequality $\lambda_2(\tilde{X}_v) \geq h(X_v)^2/4$ (and Proposition 2.7). We will not use this approach here. Instead, we use Proposition 2.8 (and also Proposition 2.7) and an ε -scaled graph-like manifold here (together with the scaling behaviour of the constants stated above):

COROLLARY 2.10. *Assume that as in Theorem 2.9, (G, μ, γ) is a weighted graph fulfilling (2.11), and that X_ε is an ε -scaled graph-like manifold uniformly compatible with (G, μ, γ) . Moreover, assume that $\lambda_2^{(h)} = \inf_e \lambda_2(Y_e) > 0$ and $\check{\lambda}_2 = \inf_v \lambda_2(\tilde{X}_v) > 0$. Then there is $\varepsilon_0 = \ell_0/C > 0$ with C depending only on ℓ_∞/ℓ_0 , $\lambda_2^{(h)}$ and $\check{\lambda}_2$ such that the discrete energy form \mathcal{E}_G on the weighted graph and the rescaled energy form $\tilde{\mathcal{E}}_{X_\varepsilon} = \tau \mathcal{E}_{X_\varepsilon}$ on X_ε are δ_ε -quasi-unitarily equivalent with*

$$\delta_\varepsilon^2 = \max \left\{ \varepsilon^2 \alpha_\infty^2, \frac{4}{\varepsilon \alpha_0} \left(\frac{\text{vol}_\infty}{\text{vol}_0} \right)^2 \frac{\gamma_\infty}{\gamma_0} \cdot \frac{1}{d_0} \cdot \frac{\mu_\infty}{\gamma_0}, \frac{4\varepsilon K_\infty}{\ell_0} \right\} \tag{2.14}$$

for $\varepsilon \in (0, \varepsilon_0]$.

Note that δ_ε becomes small as $\varepsilon \rightarrow 0$ only if the graph weights fulfil $\mu_\infty/(\gamma_0\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. This will be fulfilled in Section 3 where we consider not only a fixed weighted graph, but a sequence $(G_m)_{m \in \mathbb{N}_0}$ of weighted graphs with associated ε_m -scaled graph-like manifold X_{m, ε_m} for some $\varepsilon_m \rightarrow 0$. The weights μ_m and γ_m (now also depending on m) are chosen in such a way that $\delta_m \rightarrow 0$ as $m \rightarrow \infty$. In particular, this implies that the length function $e \mapsto \ell_{m,e}$ decays slower to 0 than ε_m as $m \rightarrow \infty$.

The proof of the above theorem is similar to the one that states the quasi-unitary equivalence of the energy form on a post-critically finite fractal and the energy forms on their associated sequence of discrete weighted graphs in [11]. Again, we begin by defining the identification operators needed in Definition A.2. Let

$$J: \ell_2(V, \mu) \rightarrow \mathbb{L}_2(X, \nu), \quad Jf = c \cdot \sum_{v \in V} f(v) \varphi_v$$

and let J' be the adjoint of J , that is,

$$J': \mathbb{L}_2(X, \nu) \rightarrow \ell_2(V, \mu), \quad J'u(v) = \frac{1}{c} \cdot \frac{1}{\nu(v)} \langle u, \varphi_v \rangle_{\mathbb{L}_2(X, \nu)},$$

for each $v \in V$. Moreover, let $J^1: \text{dom } \mathcal{E}_G = \ell_2(V, \mu) \rightarrow \text{dom } H^1(X)$, $J^1 := J|_{H^1(X)}$ and define

$$J'^1: H^1(X) \rightarrow \text{dom } \mathcal{E}_G = \ell_2(V, \mu), \quad J'^1 u(v) = \frac{1}{c} \cdot \frac{1}{\text{vol } \tilde{X}_v} \int_{\tilde{X}_v} u \, d\nu$$

for each $v \in V$. The operators J and J' are also called *smoothing* and *discretisation* operator in [4, Sec. VI.5].

PROPOSITION 2.11. *Let X be a graph-like manifold with uniformly spectrally small vertex neighbourhoods and compatible discrete graph (G, μ, γ) . Then the operators J and J' fulfil (A.2a) with δ_a and (A.2b) with δ_b , where*

$$\delta_a = \alpha_\infty \quad \text{and} \quad \delta_b^2 = \max \left\{ \frac{2\mu_\infty}{\gamma_0}, \frac{2}{\tau\lambda_2} \right\}.$$

Here, the constant α_∞ is given in (2.6b); moreover, μ_∞ and γ_0 are defined in (2.11) and $\lambda_2 := \inf_{v \in V} \lambda_2(X_v, \varphi_v)$.

Proof. First, we verify (A.2a) and we begin with the boundedness of J and J' . For all $f \in \ell_2(V, \mu)$ we estimate, applying the Cauchy-Young inequality,

$$\begin{aligned} \|Jf\|_{L_2(X, \nu)}^2 &= \sum_{v \in V} \sum_{v' \in V} c^2 f(v) \overline{f(v')} \langle \varphi_v, \varphi_{v'} \rangle_{L_2(X, \nu)} \\ &\leq c^2 \sum_{v \in V} |f(v)|^2 \sum_{v' \in V} \langle \varphi_v, \varphi_{v'} \rangle_{L_2(X, \nu)} \leq \sup_{v \in V} \frac{c^2 \nu(v)}{\mu(v)} \|f\|_{\ell_2(V, \mu)}^2. \end{aligned}$$

Using (2.6a) and (2.6b) we see that J is bounded by $1 + \alpha_\infty$. Let $u \in L_2(X, \nu)$. Then, by the Cauchy-Schwarz inequality and since $\int |\varphi_v|^2 d\nu \leq \int \varphi_v = \nu(v)$ for all $v \in V$, we have

$$\begin{aligned} \|J'u\|_{\ell_2(V, \mu)}^2 &= \sum_{v \in V} \frac{\mu(v)}{c^2 \nu(v)^2} |\langle u, \varphi_v \rangle_{L_2(X, \nu)}|^2 \leq \sum_{v \in V} \frac{\mu(v)}{c^2 \nu(v)} \|u\|_{L_2(X, \varphi_v)}^2 \\ &\leq \sum_{v \in V} \frac{1}{1 + \alpha(v)} \|u\|_{L_2(X, \varphi_v)}^2 \leq \|u\|_{L_2(X, \nu)}^2. \end{aligned}$$

Since $\alpha(v)$ is non-negative, J' is bounded by 1. For the second condition in (A.2a), we define the function

$$\Xi(\xi) := \left| \sqrt{\xi} - \frac{1}{\sqrt{\xi}} \right| = 2 \sinh \left| \frac{1}{2} \log \xi \right|. \tag{2.15}$$

Clearly, $\Xi(1) = 0$, $\Xi(1/\xi) = \Xi(\xi)$ and $0 \leq \Xi(\xi) \leq \xi - 1$ for $\xi \geq 1$. Hence,

$$\begin{aligned} \left| \langle Jf, u \rangle_{L_2(X, \nu)} - \langle f, J'u \rangle_{\ell_2(V, \mu)} \right| &= \left| \sum_{v \in V} \left(c - \frac{\mu(v)}{c\nu(v)} \right) f(v) \langle \varphi_v, u \rangle_{L_2(X, \nu)} \right| \\ &\leq \sup_{v \in V} \Xi \left(\frac{c^2 \nu(v)}{\mu(v)} \right) \sum_{v \in V} \left| \sqrt{\mu(v)} f(v) \frac{\langle \varphi_v, u \rangle_{L_2(X, \nu)}}{\sqrt{\nu(v)}} \right| \\ &\leq \sup_{v \in V} \Xi \left(\frac{c^2 \nu(v)}{\mu(v)} \right) \|f\|_{\ell_2(V, \mu)} \|u\|_{L_2(X, \nu)}, \end{aligned}$$

where we used the Cauchy-Schwarz inequality in the last estimate. We can further estimate the supremum by

$$\sup_{v \in V} \Xi \left(\frac{c^2 \nu(v)}{\mu(v)} \right) = \sup_{v \in V} \Xi(1 + \alpha(v)) \leq \Xi(1 + \alpha_\infty),$$

Since $\sqrt{1 + \alpha_\infty} - 1 \leq \alpha_\infty$ and $\Xi(1 + \alpha_\infty) \leq \alpha_\infty$, we conclude that (A.2a) holds with $\delta_a = \alpha_\infty$.

Next, we check the estimates in (A.2b). For the first one, let $f \in \text{dom } \mathcal{E}_G = \ell_2(V, \mu)$. Then,

$$f(v) - J'Jf(v) = \frac{1}{v(v)} \sum_{v' \in V} (f(v) - f(v')) \langle \varphi_{v'}, \varphi_v \rangle_{L_2(X, v)},$$

for each $v \in V$. Thus, estimating in norm, we get

$$\begin{aligned} \|f - J'Jf\|_{\ell_2(V, \mu)}^2 &= \sum_{v \in V} \frac{\mu(v)}{v(v)^2} \left| \sum_{v' \in V} (f(v) - f(v')) \langle \varphi_{v'}, \varphi_v \rangle_{L_2(X, v)} \right|^2 \\ &\leq \sum_{v \in V} \frac{\mu(v)}{v(v)^2} \sum_{e \in E_v} \gamma_e^{-1} |\langle \varphi_{v_e}, \varphi_v \rangle_{L_2(X, v)}|^2 \sum_{e \in E_v} \gamma_e |f(v) - f(v_e)|^2, \end{aligned}$$

where we used the Cauchy-Schwartz inequality. Note that φ_v is non-negative, and by the definition of these functions, we have that $\gamma_e > 0$ if and only if $\langle \varphi_{v_e}, \varphi_v \rangle_{L_2(X, v)} > 0$. Let us further estimate the sum in the middle as follows

$$\sum_{e \in E_v} \gamma_e^{-1} |\langle \varphi_{v_e}, \varphi_v \rangle_{L_2(X, v)}|^2 \leq \sum_{e \in E_v} \gamma_e^{-1} \langle \varphi_{v_e}, \varphi_v \rangle_{L_2(X, v)} \cdot v(v) \leq \frac{v(v)^2}{\gamma_0}$$

and we conclude

$$\|f - J'Jf\|_{\ell_2(V, \mu)}^2 \leq \sum_{v \in V} \frac{\mu(v)}{\gamma_0} \sum_{e \in E_v} \gamma_e |f(v) - f(v_e)|^2 \leq \frac{2\mu_\infty}{\gamma_0} \cdot \mathcal{E}_G(f).$$

Hence, the first inequality in (A.2b) holds with $\delta_b^2 = 2\mu_\infty/\gamma_0$.

For the second inequality in (A.2b), we first compute

$$u - JJ'u = \sum_{v \in V} u \varphi_v - \sum_{v \in V} \frac{1}{v(v)} \langle u, \varphi_v \rangle_{L_2(X, v)} \cdot \varphi_v,$$

where we used that the family $\{\varphi_v\}_{v \in V}$ is a partition of unity. Hence,

$$\begin{aligned} \|u - JJ'u\|_{L_2(X, v)}^2 &= \int_X \left| \sum_{v \in V} \left(u(x) - \frac{1}{v(v)} \langle u, \varphi_v \rangle_{L_2(X, v)} \right) \cdot \varphi_v(x) \right|^2 dv(x) \\ &\leq \int_X \sum_{v \in V} \left| u(x) - \frac{1}{v(v)} \langle u, \varphi_v \rangle_{L_2(X, v)} \right|^2 \cdot \varphi_v(x) \sum_{v \in V} \varphi_v(x) dv(x) \\ &= \sum_{v \in V} \int_X \left| u(x) - \frac{1}{v(v)} \langle u, \varphi_v \rangle_{L_2(X, v)} \right|^2 \cdot \varphi_v(x) dv(x), \end{aligned}$$

where we again used the Cauchy-Schwarz inequality. Since $\|\mathbb{1}_{X_v}\|_{L_2(X_v, \varphi_v)}^2 = v(v)$,

$$u - \frac{1}{v(v)} \langle u, \mathbb{1}_{X_v} \rangle_{L_2(X_v, \varphi_v)} \mathbb{1}_{X_v}$$

is the projection onto the orthogonal complement of the first eigenspace $\mathbb{C}\mathbb{1}_{X_v}$ in the weighted Hilbert space $L_2(X_v, \varphi_v)$. The min-max characterisation of eigenvalues implies that

$$\int_X \left| u(x) - \frac{1}{v(v)} \langle u, \varphi_v \rangle_{L_2(X,v)} \right|^2 \cdot \varphi_v(x) \, dv(x) \leq \frac{1}{\lambda_2(X_v, \varphi_v)} \mathcal{E}_{X_v}(u). \tag{2.16}$$

Thus, we can further estimate

$$\|u - JJ'u\|_{L_2(X,v)}^2 \leq \sum_{v \in V} \frac{1}{\lambda_2(X_v, \varphi_v)} \mathcal{E}_{X_v}(u|_{X_v}) \leq \frac{2}{\lambda_2} \mathcal{E}_X(u) \leq \frac{2}{\tau \lambda_2} \tilde{\mathcal{E}}_X(u),$$

where $\lambda_2 := \inf_{v \in V} \lambda_2(X_v, \varphi_v)$ and where we applied (2.8) for the last inequality. Hence, the second inequality in A.2b is fulfilled with $\delta_b^2 = 2/(\tau \lambda_2)$.

The following proposition has already been stated in [12] but here we give an easier proof which leads to a more convenient error estimate.

PROPOSITION 2.12. *Let X be a graph-like manifold with uniformly spectrally small vertex neighbourhoods and compatible discrete graph (G, μ, γ) . Then the operators J^1 and J^1 fulfil the inequalities in (A.2c) with*

$$\delta_c^2 = \frac{2}{\alpha_0 \tau \lambda_2},$$

where α_0 is given in (2.6b) and $\lambda_2 := \inf_{v \in V} \lambda_2(X_v, \varphi_v)$.

Proof. Since $J^1 = J$, the first estimate in (A.2c) is fulfilled with $\delta_c = 0$. For the second one, let $u \in H^1(X)$. Then we have

$$J^1 u(v) - J'u(v) = \frac{1}{c} \left(\frac{1}{\text{vol}\check{X}_v} \langle u, \mathbb{1}_{\check{X}_v} \rangle_{L_2(X,v)} \right) - \frac{1}{v(v)} \langle u, \varphi_v \rangle_{L_2(X,v)}$$

for all $v \in V$ and thus, we can estimate in norm

$$\begin{aligned} \|J^1 u - J'u\|_{\ell_2(V,\mu)}^2 &= \sum_{v \in V} \frac{\mu(v)}{c^2} \left| \frac{1}{\text{vol}\check{X}_v} \langle u, \mathbb{1}_{\check{X}_v} \rangle_{L_2(X,v)} - \frac{1}{v(v)} \langle u, \varphi_v \rangle_{L_2(X,v)} \right|^2 \\ &= \sum_{v \in V} \frac{\mu(v)}{c^2} \left| \frac{1}{\text{vol}\check{X}_v} \int_{\check{X}_v} \left(u(x) - \frac{1}{v(v)} \langle u, \varphi_v \rangle_{L_2(X,v)} \right) dv(x) \right|^2 \\ &\leq \sum_{v \in V} \frac{\mu(v)}{(c \cdot \text{vol}\check{X}_v)^2} \int_{\check{X}_v} \left| u(x) - \frac{1}{v(v)} \langle u, \varphi_v \rangle_{L_2(X,v)} \right|^2 dv(x) \cdot \text{vol}\check{X}_v \\ &\leq \sum_{v \in V} \frac{\mu(v)}{c^2 \text{vol}\check{X}_v} \int_X \left| u(x) - \frac{1}{v(v)} \langle u, \varphi_v \rangle_{L_2(X,v)} \right|^2 \cdot \varphi_v(x) dv(x) \\ &\leq \sum_{v \in V} \frac{1}{\alpha(v)} \cdot \frac{1}{\lambda_2(X_v, \varphi_v)} \mathcal{E}_{X_v}(u) \leq \frac{2}{\alpha_0 \tau \lambda_2} \tilde{\mathcal{E}}_X(u), \end{aligned}$$

where we used the Cauchy-Schwarz inequality in the first estimate, the definition of φ_v in the second (i.e., that φ_v is equal to $\mathbb{1}_{\check{X}_v}$ on the core vertex neighbourhood \check{X}_v and non-negative on X) and (2.16) and (2.6a)–(2.6b) in the third estimate, and finally, (2.8) in the last estimate.

PROPOSITION 2.13. *Let X be a graph-like manifold with compatible discrete graph (G, μ, γ) . The discrete energy form \mathcal{E}_G and the rescaled energy form on the graph-like manifold $\tilde{\mathcal{E}}_X := \tau \mathcal{E}_X$ fulfil (A.2d) with*

$$\delta_d^2 = \frac{4K_\infty}{\ell_0},$$

where K_∞ is defined in (2.6c) and where $\ell_0 := \inf_{e \in E} \ell_e$.

Proof. Let $f \in \text{dom } \mathcal{E}_G = \ell_2(V, \mu)$ and $u \in H^1(X)$. Then, on the discrete graph, we have

$$\mathcal{E}_G(f, J^1 u) = \frac{1}{c} \sum_{e \in E} \gamma_e (f(\partial_+ e) - f(\partial_- e)) \left(\frac{1}{\text{vol} \check{X}_{\partial_+ e}} \int_{\check{X}_{\partial_+ e}} \bar{u} \, d\nu - \frac{1}{\text{vol} \check{X}_{\partial_- e}} \int_{\check{X}_{\partial_- e}} \bar{u} \, d\nu \right)$$

and for the rescaled energy form $\tilde{\mathcal{E}}_X := \tau \mathcal{E}_X$ form on the graph-like manifold,

$$\begin{aligned} \tilde{\mathcal{E}}_X(Jf, u) &= c\tau \sum_{v \in V} f(v) \int_X \langle \nabla \varphi_v(x), \nabla u(x) \rangle_x \, d\nu(x) & (2.17) \\ &= c\tau \sum_{v \in V} f(v) \sum_{e \in E_v} \frac{1}{\ell_e} \int_0^{\ell_e} \int_{Y_e} \bar{u}'_e(t, y) \, dt \, dy \\ &= c\tau \sum_{v \in V} f(v) \sum_{e \in E_v} \frac{1}{\ell_e} \int_{Y_e} (\bar{u}_e(\ell_e, y) - \bar{u}_e(0_e, y)) \, dy \\ &= \frac{1}{c} \sum_{e \in E} \gamma_e (f(\partial_+ e) - f(\partial_- e)) \left(\frac{1}{\text{vol} Y_e} \int_{\partial_e \check{X}_{\partial_+ e}} \bar{u} \, dy - \frac{1}{\text{vol} Y_e} \int_{\partial_e \check{X}_{\partial_- e}} \bar{u} \, dy \right) \end{aligned}$$

where $(\cdot)'$ denotes the derivative with respect to the first (i.e., longitudinal) variable, and where we used that φ_v is supported on X_v , constant on \check{X}_v and $\varphi_v(x) = t/\ell_e$ on each edge $e \in E_v$, for $x = (t, y) \in [0, \ell_e] \times Y_e$. In the last equation, we rearranged the sum via $\sum_{v \in V} \sum_{e \in E_v} = \sum_{e \in E} \sum_{v = \partial_{\pm e}}$ and used the choice of the edge weight in (2.6a) to replace $c\tau/\ell_e$ by $\gamma_e/c \text{vol} Y_e$. Combining the above equations and rearranging the integral terms we get by applying the Cauchy-Schwarz inequality,

$$\left| \mathcal{E}_G(f, J^1 u) - \mathcal{E}_X(Jf, u) \right|^2 \leq \frac{2}{c^2} \cdot \mathcal{E}_G(f) \sum_{e \in E} \sum_{v = \partial_{\pm e}} \gamma_e \left| \frac{1}{\text{vol} \check{X}_v} \int_{\check{X}_v} u \, d\nu - \frac{1}{\text{vol} Y_e} \int_{\partial_e \check{X}_v} u \, dy \right|^2.$$

In order to estimate the sum in the above equation, we need the following standard estimates (a min-max and a Sobolev trace estimate)

$$\left\| u - \frac{1}{\text{vol} \check{X}_v} \int_{\check{X}_v} u \, d\nu \right\|_{L_2(X_{v,e})}^2 \leq \frac{1}{\lambda_2(\check{X}_v)} \|du\|_{L_2(\check{X}_v)}^2 \tag{2.18}$$

$$\int_{\partial_e \check{X}_v} |u(y)|^2 dy \leq \kappa_e \|du\|_{L_2(X_{v,e})}^2 + \frac{2}{\kappa_e} \|u\|_{L_2(X_{v,e})}^2, \tag{2.19}$$

see e.g. [10, Proposition 5.1.1 and Corollary A.2.12]; recall that κ_e and $X_{v,e}$ are defined Definition 2.1 (3). Then

$$\begin{aligned} & \sum_{e \in E_v} \gamma_e \left| \frac{1}{\text{vol} \check{X}_v} \int_{\check{X}_v} u dv - \frac{1}{\text{vol} Y_e} \int_{\partial_e \check{X}_v} u dy \right|^2 \\ &= \sum_{e \in E_v} \gamma_e \left| \frac{1}{\text{vol} Y_e} \int_{\partial_e \check{X}_v} \left(u(y) - \frac{1}{\text{vol} \check{X}_v} \int_{\check{X}_v} u dv \right) dy \right|^2 \\ &\leq \sum_{e \in E_v} \frac{\gamma_e}{\text{vol} Y_e} \int_{\partial_e \check{X}_v} \left| u(y) - \frac{1}{\text{vol} \check{X}_v} \int_{\check{X}_v} u dv \right|^2 dy \\ &\leq \sum_{e \in E_v} \frac{\gamma_e}{\text{vol} Y_e} \left(\kappa_e \|du\|_{L_2(X_{v,e})}^2 + \frac{2}{\kappa_e} \left\| u - \frac{1}{\text{vol} \check{X}_v} \int_{\check{X}_v} u dv \right\|_{L_2(X_{v,e})}^2 \right) \\ &\leq c^2 \tau \max_{e \in E_v} \frac{1}{\ell_e} \left(\kappa_e + \frac{2}{\kappa_e^2 \lambda_2(\check{X}_v)} \right) \cdot \|du\|_{L_2(\check{X}_v)}^2, \end{aligned}$$

where we used the Cauchy-Schwarz inequality in the first estimate, (2.19) in the second, and (2.18) and (2.6a) for the last one. Hence, by the above and (2.8), we conclude

$$\left| \mathcal{E}_G(f, J^1 u) - \mathcal{E}_X(Jf, u) \right|^2 \leq \frac{4\tau K_\infty}{\ell_0} \cdot \mathcal{E}_G(f) \cdot \mathcal{E}_X(u)$$

using the definition of K_∞ in (2.6c).

We are now prepared to prove our first main result Theorem 2.9.

Proof of Theorem 2.9. By the second equation of (2.6a), we have

$$\ell_e = \frac{c^2 \tau \text{vol} Y_e}{\gamma_e} \begin{cases} \leq \frac{c^2 \tau \text{vol}_\infty}{\gamma_0} =: \ell_\infty < \infty \\ \geq \frac{c^2 \tau \text{vol}_0}{\gamma_\infty} =: \ell_0 > 0. \end{cases}$$

For the error estimate, we have to collect the individual terms from Propositions 2.11, 2.12 and 2.13. We need to estimate the term $2/(\tau \lambda_2)$ appearing in δ_b and δ_c :

$$\frac{2}{\tau \lambda_2} \leq \frac{2\ell_\infty^2}{\tau} = \frac{2c^2 \text{vol}_\infty}{\gamma_0} \cdot \ell_\infty = \frac{2 \text{vol}_\infty}{\text{vol}_0} \cdot \frac{\ell_\infty}{\ell_0} \cdot \frac{d_0 c^2 \ell_0 \text{vol}_0}{d_0 \gamma_0} \leq \frac{4 \text{vol}_\infty}{\text{vol}_0} \cdot \frac{\ell_\infty}{\ell_0} \cdot \frac{\mu_\infty}{d_0 \gamma_0}.$$

In the first inequality above, we use (2.12), and for the first equality the definition of ℓ_∞ . The last estimate follows from the first equation in (2.6a) because

$$2\mu_\infty \geq 2\mu(v) = c^2 \sum_{e \in E_v} \ell_e \text{vol} Y_e \geq d_0 c^2 \ell_0 \text{vol}_0,$$

for each $v \in V$. Finally using the definition of ℓ_0 and ℓ_∞ we end up with the desired term (note that $\alpha_0 \leq 1$).

Similarly as in the proof of Proposition 2.11, we can extract the following proposition, which allows us to compare two different weights on the same discrete graph. This result will later be used to treat the embedded case in Corollary 3.5.

PROPOSITION 2.14. *Let $G = (V, E, \partial)$ be a discrete graph. Let $v, \tilde{v}: V \rightarrow (0, \infty)$ be two vertex weight functions and moreover, let $\gamma, \tilde{\gamma}: E \rightarrow (0, \infty)$ be two edge weight functions. Then the energy forms \mathcal{E}_G associated with the edge weight γ in $\ell_2(V, v)$ and $\tilde{\mathcal{E}}_G$ associated with $\tilde{\gamma}$ in $\ell_2(V, \tilde{v})$ are δ -quasi-unitarily equivalent with*

$$\delta = \max \left\{ \sup_{v \in V} \Xi \left(\frac{\tilde{v}(v)}{v(v)} \right), \sup_{e \in E} \Xi \left(\frac{\tilde{\gamma}_e}{\gamma_e} \right) \right\}. \tag{2.20}$$

where Ξ is defined in (2.15); recall that $\Xi(1 + \eta) = O(\eta)$.

Proof. We choose the identification operators J, J', J^1 and J'^1 to be the corresponding identity operators. Then (A.2b) and (A.2c) are trivially fulfilled with $\delta_b = \delta_c = 0$. The two inequalities in (A.2a) are satisfied with $\delta_a = \sup_{v \in V} \Xi(\tilde{v}(v)/v(v))$ which follows as in the first part of the proof of Proposition 2.11. Let us now check condition (A.2d). Using the same arguments as before, we obtain

$$\left| \tilde{\mathcal{E}}_G(Jf, g) - \mathcal{E}_G(f, J'g) \right| = \left| \sum_{e \in E} (\tilde{\gamma}_e - \gamma_e) (df)_e (\overline{dg})_e \right| \leq \sup_{e \in E} \Xi \left(\frac{\tilde{\gamma}_e}{\gamma_e} \right) \mathcal{E}_G(f) \tilde{\mathcal{E}}_G(g),$$

for $f, g: V \rightarrow \mathbb{C}$.

We end this section with the following lemma, that states that the identification operators J^1 and J'^1 , acting on the form domains, are also bounded in the corresponding energy norms. This is later needed if we want to use the transitivity Proposition A.3 of the notion of quasi-unitary equivalence. The proof uses similar arguments as before.

LEMMA 2.15. ([12, Proposition 2.12]) *Assume the situation from Theorem 2.9. Then the operators J^1 and J'^1 fulfil*

$$\|J^1 f\|_{\tilde{\mathcal{E}}_X} \leq (1 + \delta) \|f\|_{\mathcal{E}_G} \quad \text{and} \quad \|J'^1 u\|_{\mathcal{E}_G} \leq (1 + \delta) \|u\|_{\tilde{\mathcal{E}}_X},$$

for $f \in \ell_2(V, \mu)$ and $u \in H^1(X)$. Recall that $\tilde{\mathcal{E}}_X := \tau \mathcal{E}_X$.

3. Approximating the Sierpiński gasket by graph-like manifolds

3.1. The Sierpiński gasket and its canonical energy form

In this section, we briefly introduce the Sierpiński gasket interpreted as a self-similar set together with its canonical energy form. For more details we refer to the monographs [6, 16].

Let $V_0 := \{p_1, p_2, p_3\}$ be the vertices of an equilateral triangle in the plain and let $F := \{F_j \mid j = 1, 2, 3\}$ be given by

$$F_j: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad F_j(x) = \frac{1}{2}(x - p_j) + p_j.$$

The family F is called an *iterated function system (IFS)* and each F_j is a contraction with ratio $\theta_j = 1/2$ and fixed point p_j . Then, there exists a unique non-empty compact subset K of \mathbb{R}^2 such that

$$K = F(K) := F_1(K) \cup F_2(K) \cup F_3(K) \tag{3.1}$$

and this set K is called the *Sierpiński gasket*. The set V_0 is called the *boundary* of K . Moreover, the contractions describe a cell structure on the Sierpiński gasket K via the map

$$w := w_1 w_2 \dots w_m \mapsto F_w(K) := F_{w_1} \circ F_{w_2} \circ \dots \circ F_{w_m}(K)$$

where $w \in \{1, 2, 3\}^m$ is a word of length $|w| = m$. We call $F_w(K)$ an m -cell of K whenever $w \in W_m$ and we write W_\star for the collection of all the words over the alphabet $\{1, 2, 3\}$.

We define a sequence of discrete graphs by applying the IFS inductively to the boundary vertices V_0 as follows: Let G_0 be the complete graph with three vertices V_0 . For each $m \in \mathbb{N}$, we set $G_m := (V_m, E_m)$, where

$$V_m := \bigcup_{w \in W_m} F_w(V_0) \quad \text{and} \quad E_m := \{e \mid e = \{x, y\} \subset V_m \text{ and } x \sim_m y\}.$$

Here, $x \sim_m y$ if and only if x and y are two distinct vertices in V_m and there exists a word $w \in W_m$ such that $x, y \in F_w(K)$.

On each graph G_m there is a canonical energy form \mathcal{E}_m , given by

$$\mathcal{E}_m(f) := \gamma_m \sum_{\{x,y\} \in E_m} |f(x) - f(y)|^2 \quad \text{with} \quad \gamma_m := \left(\frac{5}{3}\right)^m \tag{3.2}$$

for each $f: V_m \rightarrow \mathbb{R}$. Here the edge weight (also called *conductance*) $\gamma_m = (5/3)^m$ is independent of the edges $e \in E_m$ and is chosen in order to guarantee that the sequence of discrete energy forms $\{\mathcal{E}_m\}_{m \in \mathbb{N}_0}$ is *compatible*. Compatibility here means that the minimisation problem

$$\mathcal{E}_m(\rho) = \min\{\mathcal{E}_{m+1}(f) \mid f: V_{m+1} \rightarrow \mathbb{R} \text{ and } f|_{V_m} = \rho\}$$

has a unique solution for each $\rho: V_m \rightarrow \mathbb{R}$ and each $m \in \mathbb{N}_0$. As a consequence, $\{\mathcal{E}_m(u|_{V_m})\}_{m \in \mathbb{N}_0}$ is non-decreasing for all continuous $u: K \rightarrow \mathbb{C}$, and hence, the limit exists. We therefore define an energy form on K as the limit

$$\mathcal{E}(u) = \lim_{m \rightarrow \infty} \mathcal{E}_m(u|_{V_m}) \quad \text{where} \quad u \in \text{dom } \mathcal{E} := \{u \in C(K) \mid \lim_{m \rightarrow \infty} \mathcal{E}_m(u|_{V_m}) < \infty\}$$

in the Hilbert space $L_2(K, \mu)$. The measure μ we choose is the so-called (*homogeneous*) *self-similar measure*, i.e., μ is the unique Hausdorff measure on K of dimension $\log 3 / \log 2$ and such that any m -cell has measure $1/3^m$.

This measure μ also induces a vertex measure $\{\mu_m(v)\}_{v \in V_m}$ on the graph G_m . Let $\psi_{v,m}: K \rightarrow \mathbb{R}$ be the unique solution of

$$\mathcal{E}_m(\mathbb{1}_{\{v\}}) = \min\{\mathcal{E}(u) \mid u \in \text{dom } \mathcal{E} \text{ and } u|_{V_m} = \mathbb{1}_{\{v\}}\}.$$

Using the symmetry of the Sierpiński gasket and the fact that $\{\psi_{v,m}\}_{v \in V_m}$ is a partition of unity, we have

$$\mu_m(v) := \int_K \psi_{v,m}(x) d\mu(x) = \begin{cases} \frac{1}{3^{m+1}} & v \in V_0, \\ \frac{2}{3^{m+1}} & v \in V_m \setminus V_0. \end{cases} \tag{3.3}$$

The Hilbert space structure on the discrete weighted graph G is given by $\ell_2(V_m, \mu_m)$, see (2.1).

In [11] we have shown that an energy form on a self-similar fractal such as the Sierpiński gasket and its associated sequence of discrete graph energies are quasi-unitarily equivalent with an explicitly given error converging to 0. In our situation this precisely means the following:

PROPOSITION 3.1. ([11]) *The standard energy form on the Sierpiński gasket $\hat{\mathcal{E}}$ in the Hilbert space $L_2(K, \mu)$ and the discrete energy form \mathcal{E}_m in $\ell_2(V_m, \mu_m)$ are $\hat{\delta}_m$ -quasi-unitarily equivalent with error*

$$\hat{\delta}_m = \frac{(1 + \sqrt{3}) \cdot \sqrt{2}}{\sqrt{3}} \cdot \frac{1}{5^{m/2}}.$$

3.2. Quasi-unitary equivalence of the Sierpiński gasket and its associated graph-like manifold

Let (G_m, μ_m, γ_m) be the m -th approximation of the energy form on the Sierpiński gasket K . In this section, we apply Corollary 2.10 to the energy form on the discrete weighted graph (G_m, μ_m, γ_m) and an energy form on a compatible graph-like manifold X_m associated with (G_m, μ_m, γ_m) . The situation can easily be generalised to certain other post-critically finite fractals, see [12]. Let us now fix the different data for the family of scaled graph-like manifolds $\{X_m\}_{m \in \mathbb{N}_0}$: Recall that the scaling parameter ε_m scale the transversal and core vertex manifold as

$$Y_{m,e} = \varepsilon_m Y_e \quad \text{and} \quad \check{X}_{m,v} = \varepsilon_m \check{X}_v \tag{3.4}$$

for all $v \in V$ and $e \in E$, respectively (see the text before Definition 2.3). We fix the scaling parameter ε_m and the length function (here constant, i.e., $\ell_{m,e} = \ell_m$ for all $e \in E_m$) as

$$\ell_m = \ell_0 \Lambda^m \quad \text{and} \quad \varepsilon_m = \varepsilon_0 E^m,$$

where $\ell_0 \in (0, \infty)$ and $\varepsilon_0 \in (0, \infty)$ are some given parameters and $\Lambda, E \in (0, \infty)$ are specified later. Probably the most interesting case is $\Lambda = 1/2$, obtained from the IFS as in Subsection 3.1.

Next, we assume that Y_e is isometric with a fixed manifold Y_0 with $\text{vol} Y_e = 1$, e.g. $Y_e = [-1/2, 1/2]$ or Y_e is a circle of circumference 1 if $d = 2$. As the graphs approximating a Sierpiński gasket have only vertices of degree 2 and 4, we can work with two properly scaled building blocks \check{X}_2 and \check{X}_4 for the core vertex neighbourhoods

\check{X}_v in all generations, one with two and one with four boundary components, each isometric with Y_0 . We assume for simplicity that the one with four ends has larger volume than the one with only two ends. In particular, we have

$$\alpha_m(v) = \frac{\text{vol} \check{X}_v}{(\text{deg } v)\ell_0} \cdot \Lambda^{-m},$$

for the unscaled graph-like manifold associated with G_m . Here, $\text{deg } v \in \{2, 4\}$ is the degree of v .

Recall that the unscaled graph-like manifold as well as the ε_m -scaled graph-like manifold X_m are both defined as abstract manifolds and are not necessarily embedded in \mathbb{R}^2 or some other space. In particular, the edge lengths shrink as $\ell_m = \ell_0 \Lambda^m$ while the transversal manifolds and the core vertex neighbourhoods do not shrink if $m \rightarrow \infty$ for the *unscaled* manifold. We treat the embedded case in Corollary 3.5.

Moreover, the numbers $\lambda_2(Y_e) = \lambda_2(Y_0)$ and $\lambda_2(\check{X}_v)$ only achieve a finite set of numbers for $e \in E_m$ and $v \in V_m$ *independently* of $m \in \mathbb{N}_0$. In particular, the constant C in Proposition 2.8 is *independent* of m .

For κ_m in the (unscaled) graph-like manifold associated with G_m , we choose a fixed value κ_0 again *independent* of m (the shrinking later enters via the parameter ε_m). As before, this means that the constant K_∞ of the unscaled graph-like manifold associated with G_m is independent of m .

A straightforward calculation now shows that the constants c_m (unscaled) and τ_m are given by

$$c_m = (3\ell_0)^{-1/2} (3\Lambda)^{-m/2} \quad \text{and} \quad \tau_m = 3\ell_0^2 (5\Lambda^2)^m.$$

We can now state one of our main results:

THEOREM 3.2. *Let (G_m, μ_m, γ_m) be the m -th approximation graph of the Sierpiński gasket K . Moreover, let \mathcal{E}_m be the discrete energy form defined in (3.2) and let $\tilde{\mathcal{E}}_m := \tau_m \mathcal{E}_{X_m}$ be the rescaled energy form on the (transversally ε_m -scaled) graph-like manifold X_m associated with G_m with edge length $\ell_m = \ell_0 \Lambda^m$ and scaling factor $\varepsilon_m = \varepsilon_0 E^m$ as above. Moreover, we assume that*

$$\frac{1}{5} < \frac{E}{\Lambda} < 1.$$

Then \mathcal{E}_m and $\tilde{\mathcal{E}}_m$ are δ_m -quasi-unitarily equivalent with

$$\delta_m = \begin{cases} O\left(\left(\frac{E}{\Lambda}\right)^{m/2}\right), & \frac{1}{\sqrt{5}} \leq \frac{E}{\Lambda} < 1 \\ O\left(\left(\frac{\Lambda}{5E}\right)^{m/2}\right), & \frac{1}{5} < \frac{E}{\Lambda} \leq \frac{1}{\sqrt{5}} \end{cases}$$

(the precise term is given in (3.5)). In particular, if we choose $E/\Lambda = 1/\sqrt{5}$, then $\delta_m = O((1/5)^{m/4})$ which is the best possible choice.

REMARKS 3.3.

1. A natural choice in terms of geometry is to set $\Lambda = 1/2$ thus choosing the natural length scale of the Sierpiński gasket as subset of \mathbb{R}^2 given by the IFS. In this case we have (c_m unscaled)

$$c_m = (3\ell_0)^{-1/2} \left(\frac{2}{3}\right)^{m/2} \quad \text{and} \quad \tau_m = 3\ell_0^2 \left(\frac{5}{4}\right)^m.$$

Note that the energy rescaling factor $\tau_m = O((5/4)^m)$ was already found numerically in [2, 3]. Our analysis hence confirms their results.

2. One condition in Theorem 3.2 is $E < \Lambda$. In particular, this means that the transversal length parameter ε_m shrinks *faster* than the longitudinal length ℓ_m . If we are interested in a sequence of subsets X_m in \mathbb{R}^2 obtained via the m -th iteration of the IFS (i.e., $X_{m+1} = F(X_m)$ as in (3.1) for some starting set X_0 , a thickened equilateral triangle), then the scaling parameter $1/2$ of the IFS forces $\Lambda = E$. Unfortunately, we are not able to cover this case with our methods here.

Proof of Theorem 3.2. A careful analysis of δ_ε in Corollary 2.10 using the above considerations shows that

$$\delta_m^2 = \max \left\{ \left(\frac{\varepsilon_0 \text{vol} \check{X}_4}{2\ell_0} \right)^2 \cdot \left(\frac{E}{\Lambda} \right)^{2m}, \frac{4}{3} \left(\frac{\varepsilon_0 \text{vol} \check{X}_2}{4\ell_0} \right)^{-1} \cdot \left(\frac{\Lambda}{5E} \right)^m, \frac{4K_\infty \varepsilon_0}{\ell_0} \cdot \left(\frac{E}{\Lambda} \right)^m \right\} \tag{3.5}$$

where we have used that $\text{vol} \check{X}_2 \leq \text{vol} \check{X}_4$.

Now, we are prepared to state the quasi-unitary equivalence of the canonical energy form on the Sierpiński gasket and a properly rescaled energy form on a suitable graph-like manifold. The result is a direct consequence of the transitivity stated in Proposition A.3:

COROLLARY 3.4. *Assume the situation as in Theorem 3.2, then the rescaled energy form $\tilde{\mathcal{E}}_m = \tau_m \mathcal{E}_{X_m}$ on the ε_m -scaled graph-like manifold X_m and the energy form \mathcal{E}_K on the Sierpiński gasket are $\tilde{\delta}_m$ -unitarily equivalent with $\tilde{\delta}_m$ being of the same order as δ_m in Theorem 3.2. In particular, $\tilde{\mathcal{E}}_m$ converges to \mathcal{E}_K in the sense of Definition A.2.*

Proof. The $\hat{\delta}_m$ -quasi-unitary equivalence of $\hat{\mathcal{E}} = \mathcal{E}_K$ and \mathcal{E}_m is stated in Proposition 3.1; the δ_m -quasi-unitary equivalence of \mathcal{E}_m and $\tilde{\mathcal{E}}_m$ is just stated in Theorem 3.2. The boundedness of the corresponding operators respecting the form domains as in (A.3a) for both cases can be deduced from [12, Prp. 2.12], see also Lemma 2.15. Note that the error $\hat{\delta}_m = O((1/5)^{m/2})$ from Proposition 3.1 is not dominated by the error δ_m of Theorem 3.2, as $1/5 < E/\Lambda$, and hence $(E/\Lambda)^{m/2}$ dominates already $(1/5)^{m/2}$. In particular, $\tilde{\delta}_m = 14(\delta_m + \hat{\delta}_m)$ (from Proposition A.3) is of the same order as δ_m .

Let us now show that we can also find a sequence of open subsets $X_m \subset \mathbb{R}^2$ such that X_m approximates the Sierpiński gasket; moreover, the corresponding sequence

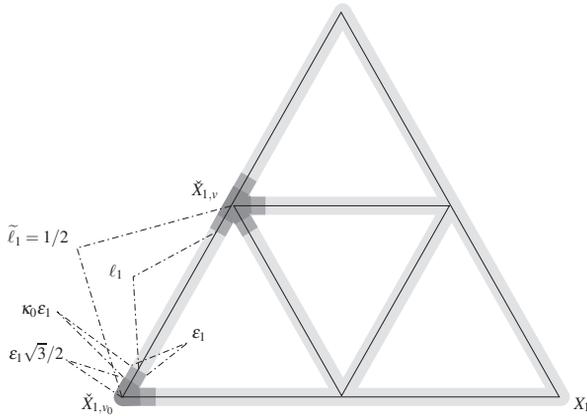


Figure 3: The $\varepsilon_m/2$ -neighbourhood of the equilateral metric graph associated with G_m with edge length $\tilde{\ell}_m = (1/2)^m$; here $m = 1$. Note that the edge neighbourhood has length $\ell_m = \tilde{\ell}_m - 2(\kappa_0 + \sqrt{3}/2)\varepsilon_m$. Moreover, the two building blocks for the core vertex neighbourhoods $\check{X}_{1,v}$ (upper vertex of degree 4) and \check{X}_{1,v_0} (lower left corner vertex of degree 2) are ε_1 -homothetic versions of the prototypes \check{X}_4 and \check{X}_2 .

of energy forms corresponding to the Neumann Laplacian converge to the canonical energy form on the Sierpiński gasket. We fix $\Lambda = 1/2$.

COROLLARY 3.5. *Let X_m be the $\varepsilon_m/2$ -neighbourhood of G_m considered as metric graph embedded in \mathbb{R}^2 with edge length $\tilde{\ell}_m = 1/2^m$. Then the corresponding rescaled energy form $\tilde{\mathcal{E}}_m$ given by*

$$\tilde{\mathcal{E}}_m(u) = \left(\frac{5}{4}\right)^m \int_{X_m} |\nabla u(x)|^2, \quad u \in \text{dom } \tilde{\mathcal{E}}_m := H^1(X_m),$$

is $\tilde{\delta}_m$ -quasi unitarily equivalent with the energy form \mathcal{E}_K on the Sierpiński gasket, where

$$\tilde{\delta}_m = \begin{cases} O((2E)^{m/2}), & \frac{1}{2\sqrt{5}} \leq E < \frac{1}{2} \\ O\left(\left(\frac{1}{10E}\right)^{m/2}\right), & \frac{1}{10} < E \leq \frac{1}{2\sqrt{5}}. \end{cases}$$

Proof. If we want the metric edge lengths in generation m to have length $\tilde{\ell}_m = 1/2^m$, then the length ℓ_m of the edge neighbourhoods in the embedded case (again independent of the edges $e \in E_m$) is related via

$$\tilde{\ell}_m = \ell_m + 2\left(\kappa_0 + \frac{\sqrt{3}}{2}\right)\varepsilon_m = \ell_0 \left(1 + k_0 \left(\frac{E}{\Lambda}\right)^m\right) \Lambda^m, \quad \text{where } k_0 = \frac{\varepsilon_0}{\ell_0} \left(2\kappa_0 + \sqrt{3}\right).$$

We now use the second equation in (2.6a) to calculate the corresponding edge weights $\tilde{\gamma}_m$: Note first that we have

$$c_m^2 \tau_m = \ell_0 \left(\frac{5\Lambda}{3} \right)^m,$$

and hence (recall $\text{vol} Y_e = \text{vol} Y_0 = 1$)

$$\tilde{\gamma}_m = \frac{c_m^2 \tau_m}{\tilde{\ell}_m} = \frac{1}{1 + k_0(\text{E}/\Lambda)^m} \left(\frac{5}{3} \right)^m.$$

We now use Proposition 2.14 to see that the energy forms on the weighted graphs (G_m, μ_m, γ_m) and $(G_m, \mu_m, \tilde{\gamma}_m)$ are $\bar{\delta}_m$ -quasi unitarily equivalent with

$$\bar{\delta}_m = \Xi \left(\frac{\tilde{\gamma}_m}{\gamma_m} \right) = \Xi \left(\frac{\gamma_m}{\tilde{\gamma}_m} \right) = \Xi \left(1 + k_0 \left(\frac{\text{E}}{\Lambda} \right)^m \right) = \mathcal{O} \left(\left(\frac{\text{E}}{\Lambda} \right)^m \right).$$

This error term is again not dominant. The boundedness of the identification operators (here the identities) with respect to the energy norms can also be seen easily. We now use the transitivity of quasi-unitary equivalence, Theorem 3.2, Proposition 3.1 and the above considerations to conclude the result fixing $\Lambda = 1/2$.

A. Quasi-unitary equivalence

In this appendix we briefly introduce the notion of *quasi-unitary equivalence*. The concept appeared first in [9] and was outlined in greater details in [10, Ch. 4]. Roughly speaking, we define a sort of “distance” between two energy forms \mathcal{E} and $\tilde{\mathcal{E}}$ acting in Hilbert spaces \mathcal{H} resp. $\tilde{\mathcal{H}}$. The distance is expressed as a parameter $\delta \geq 0$, and appears in the concept of δ -quasi-unitary equivalence.

Let \mathcal{H} and $\tilde{\mathcal{H}}$ be two separable complex Hilbert spaces. We call \mathcal{E} an *energy form in \mathcal{H}* if \mathcal{E} is a closed, non-negative quadratic form in \mathcal{H} , i.e., if $\mathcal{E}(f) := \mathcal{E}(f, f)$ for some sesquilinear form $\mathcal{E}: \mathcal{H}^1 \times \mathcal{H}^1 \rightarrow \mathbb{C}$ (denoted by the same symbol), if $\mathcal{E}(f) \geq 0$ and if $\mathcal{H}^1 := \text{dom } \mathcal{E}$, endowed with the norm

$$\|f\|_{\mathcal{E}}^2 := \|f\|_{\mathcal{H}}^2 + \mathcal{E}(f), \tag{A.1}$$

is itself a Hilbert space and dense (as a set) in \mathcal{H} . We call the corresponding non-negative, self adjoint operator Δ (see e.g. [5, Sec. VI.2]) the *Laplacian* associated with \mathcal{E} . Similarly, let $\tilde{\mathcal{E}}$ be an energy form in $\tilde{\mathcal{H}}$ with Laplacian $\tilde{\Delta}$.

DEFINITION A.1. (Quasi-unitary equivalence for energy forms) Let $\delta \geq 0$. Let $J: \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ and $J': \tilde{\mathcal{H}} \rightarrow \mathcal{H}$, resp. $J^1: \mathcal{H}^1 \rightarrow \tilde{\mathcal{H}}^1$ and $J'^1: \tilde{\mathcal{H}}^1 \rightarrow \mathcal{H}^1$ be linear operators on the Hilbert spaces and energy form domains. Then \mathcal{E} and $\tilde{\mathcal{E}}$ are δ -quasi-unitarily equivalent, if for all $f \in \mathcal{H}$ and $u \in \tilde{\mathcal{H}}$,

$$\|Jf\|_{\tilde{\mathcal{H}}} \leq (1 + \delta) \|f\|_{\mathcal{H}} \quad \left| \langle Jf, u \rangle - \langle f, J'u \rangle \right| \leq \delta \|f\|_{\mathcal{H}} \|u\|_{\tilde{\mathcal{H}}} \tag{A.2a}$$

and if for all $f \in \mathcal{H}^1$ and $u \in \widetilde{\mathcal{H}}^1$,

$$\|f - J'Jf\|_{\mathcal{H}} \leq \delta \|f\|_{\mathcal{E}} \qquad \|u - JJ'u\|_{\widetilde{\mathcal{H}}} \leq \delta \|u\|_{\widetilde{\mathcal{E}}} \quad (\text{A.2b})$$

$$\|J^1f - Jf\|_{\widetilde{\mathcal{H}}} \leq \delta \|f\|_{\mathcal{E}} \qquad \|J^1u - J'u\|_{\mathcal{H}} \leq \delta \|u\|_{\widetilde{\mathcal{E}}} \quad (\text{A.2c})$$

$$|\widetilde{\mathcal{E}}(J^1f, u) - \mathcal{E}(f, J^1u)| \leq \delta \|f\|_{\mathcal{E}} \|u\|_{\widetilde{\mathcal{E}}}. \quad (\text{A.2d})$$

DEFINITION A.2. Let $\{\mathcal{E}_m\}_{m \in \mathbb{N}}$ be a sequence of energy forms acting in the Hilbert spaces \mathcal{H}_m and \mathcal{E}_∞ be an energy form in \mathcal{H}_∞ . Moreover, assume that \mathcal{E}_m and \mathcal{E}_∞ are δ_m -quasi-unitarily equivalent and the $\delta_m \rightarrow 0$ as $m \rightarrow \infty$. Then we say that $\{\mathcal{E}_m\}_{m \in \mathbb{N}}$ converges to \mathcal{E}_∞ (with error $\{\delta_m\}_{m \in \mathbb{N}}$).

One essential ingredient to prove our main result is the following proposition which states the transitivity of the notion of quasi-unitary equivalence.

PROPOSITION A.3. ([13, Prop. 1.6]) Let $\delta, \widetilde{\delta} \in [0, 1]$. Assume that \mathcal{E} and $\widetilde{\mathcal{E}}$ are δ -quasi-unitarily equivalent with identification operators J, J^1, J' and J'^1 . Moreover, assume that $\widehat{\mathcal{E}}$ and \mathcal{E} are $\widehat{\delta}$ -quasi-unitarily equivalent with identification operators $\widehat{J}, \widehat{J}^1, \widehat{J}'$ and \widehat{J}'^1 . Assume in addition that, for all $u \in \widetilde{\mathcal{H}}^1$ and $w \in \widehat{\mathcal{H}}^1$,

$$\|\widehat{J}^1w\|_{\mathcal{E}} \leq (1 + \widehat{\delta})\|w\|_{\widehat{\mathcal{E}}} \quad \text{and} \quad \|J^1u\|_{\mathcal{E}} \leq (1 + \delta)\|u\|_{\widetilde{\mathcal{E}}}. \quad (\text{A.3a})$$

Then $\widehat{\mathcal{E}}$ and $\widetilde{\mathcal{E}}$ are $\widetilde{\delta}$ -quasi-unitarily equivalent with $\widetilde{\delta} = 14(\delta + \widehat{\delta})$.

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