

THE REFLEXIVITY OF HYPEREXPANSIONS AND THEIR CAUCHY DUAL OPERATORS

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Abstract. We discuss the reflexivity of hyperexpansions and their Cauchy dual operators. In particular, we show that any cyclic completely hyperexpansive operator is reflexive. We also establish the reflexivity of the Cauchy dual of an arbitrary 2-hyperexpansive operator. As a consequence, we deduce the reflexivity of the so-called Bergman-type operator, that is, a left-invertible operator T satisfying the inequality $TT^* + (T^*T)^{-1} \leq 2I_{\mathcal{H}}$.

1. Introduction

Completely hyperexpansive operators were introduced independently by Aleman [2] and Athavale [5]. It has been extensively studied by several authors (see, for example, [38], [26], [27], [3], [4]). It is worth mentioning that the class of completely hyperexpansive weighted shifts is antithetical to that of contractive subnormal weighted shifts, in the sense that its Cauchy duals are contractive subnormal weighted shifts (see [5, Remark 4]). The present paper investigates the class of hyperexpansions with a focus on reflexivity. It is to be noted that by a result of Olin and Thompson [29, Theorem 3], any subnormal operator is reflexive. Although the Cauchy dual of a completely hyperexpansive operator is not necessarily subnormal (refer to [4, Examples 6.6 and 7.10]), surprisingly, Proposition 3.1 ensures the reflexivity of the Cauchy dual of any 2-hyperexpansive operator.

We set below the notations used throughout this text. Let \mathbb{N} denote the set of positive integers. Let \mathbb{C} be the complex plane, while \mathbb{D} stand for the open unit disk in \mathbb{C} centered at the origin. All the Hilbert spaces to occur below are complex and separable. For a Hilbert space \mathcal{H} , we use $B(\mathcal{H})$ to denote the algebra of bounded linear operators on \mathcal{H} . Unless stated otherwise, the Hilbert spaces are infinite-dimensional. The kernel, range, adjoint and spectrum of an operator $T \in B(\mathcal{H})$ are denoted by $\ker T$,

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$T(\mathcal{H})$, T^* and $\sigma(T)$, respectively. The symbol $I_{\mathcal{H}}$ is reserved for the identity operator of $B(\mathcal{H})$. If F is a subset of \mathcal{H} , the closure of F is denoted by \bar{F} , while the closed linear span of F is denoted by $\vee\{x : x \in F\}$. If \mathcal{N} is a finite-dimensional subspace of \mathcal{H} , then $\dim \mathcal{N}$ denotes the vector space dimension of \mathcal{N} .

The concept of reflexivity first appeared in the work of Sarason [35] in 1966. In 1969 Radjavi and Rosenthal [31] formally defined the concept of reflexivity for algebras of operators. Following the terminology used by them, we now explain that concept. Let \mathcal{W} be a subalgebra of $B(\mathcal{H})$ containing $I_{\mathcal{H}}$, and let $\text{Lat } \mathcal{W}$ be the set of all closed linear subspaces of \mathcal{H} that are invariant under every operator in \mathcal{W} . For $T \in B(\mathcal{H})$, let $\text{Lat } T$ denote the set of all closed linear subspaces of \mathcal{H} that are invariant under T . The set

$$\text{AlgLat } \mathcal{W} := \{T \in B(\mathcal{H}) : \text{Lat } \mathcal{W} \subseteq \text{Lat } T\}$$

is a WOT-closed subalgebra of $B(\mathcal{H})$ which contains \mathcal{W} . We say that \mathcal{W} is *reflexive* if

$$\mathcal{W} = \text{AlgLat } \mathcal{W}.$$

For $T \in B(\mathcal{H})$, let \mathcal{W}_T (resp. \mathcal{A}_T) stand for the WOT-closed (resp. weak*-closed) subalgebra of $B(\mathcal{H})$ generated by T and $I_{\mathcal{H}}$. We say T is *reflexive* if \mathcal{W}_T is reflexive. It is easy to see that \mathcal{W}_T is reflexive if \mathcal{A}_T is reflexive, and in that case \mathcal{W}_T equals \mathcal{A}_T . An algebra \mathcal{W} is said to be *super-reflexive* if any unital WOT-closed subalgebra of \mathcal{W} is reflexive and we call an operator T *super-reflexive* if \mathcal{W}_T is super-reflexive. The concept of super-reflexivity first appeared in [25].

Let \mathcal{S} be a weak*-closed subspace of $B(\mathcal{H})$. We say that \mathcal{S} admits the *property* (\mathbb{A}_1) if for any weak*-continuous linear functional ϕ on \mathcal{S} there exist vectors f, g in \mathcal{H} such that $\phi(S) = \langle Sf, g \rangle$ for all $S \in \mathcal{S}$. Further, given $r \geq 1$, if for every weak*-continuous linear functional ϕ on \mathcal{S} and $s > r$ there exist vectors f, g in \mathcal{H} such that $\phi(S) = \langle Sf, g \rangle$ for all $S \in \mathcal{S}$ and $\|f\| \|g\| \leq s \|\phi\|$, then we say that \mathcal{S} has the *property* $(\mathbb{A}_1(r))$. For $r \geq 1$, we say $T \in B(\mathcal{H})$ admits the *property* $(\mathbb{A}_1(r))$ (resp. (\mathbb{A}_1)) if \mathcal{A}_T satisfies the *property* $(\mathbb{A}_1(r))$ (resp. (\mathbb{A}_1)). A comprehensive account of these properties and related topics can be found in [7].

DEFINITION 1. Let \mathcal{H} be a Hilbert space and $T \in B(\mathcal{H})$. Define $B_n(T)$ as the following

$$B_n(T) := \sum_{p=0}^n (-1)^p \binom{n}{p} T^{*p} T^p.$$

- (i) T is said to be *completely hyperexpansive* if $B_n(T) \leq 0$ for all $n \in \mathbb{N}$.
- (ii) For $m \in \mathbb{N}$, T is said to be *m-hyperexpansive* if $B_n(T) \leq 0$ for $n = 1, \dots, m$.
- (iii) For $m \in \mathbb{N}$, T is said to be *m-isometric* if $B_m(T) = 0$.

In case $m = 1$, an 1-hyperexpansive (resp. 1-isometry) operator T is said to be *expansive* (resp. *isometry*). The notion of 2-hyperexpansive operators first appeared in the paper [32] of Richter. He proved in [32, Lemma 1] that for any T if $B_2(T) \leq 0$,

then $B_1(T) \leq 0$. Hence any 2-isometric operator T is indeed 2-hyperexpansive. Also, it is well known that any 2-isometric operator is completely hyperexpansive (see [38, Remark 1.3]). In [36], Shimorin referred to 2-hyperexpansive operators as concave operators.

Following [36], we say that $T \in B(\mathcal{H})$ is *analytic* if $\bigcap_{n \in \mathbb{N}} T^n(\mathcal{H}) = \{0\}$. An operator T is called *completely non-unitary* if there is no nonzero reducing subspace \mathcal{N} of T such that $T|_{\mathcal{N}}$ is a unitary. Note that any analytic operator is completely non-unitary. We say T is *finitely multicyclic* if there is a finite, linearly independent subset W of \mathcal{H} such that

$$\mathcal{H} = \bigvee \{p(T)h : p \in \mathbb{C}[z], h \in W\}.$$

In particular, if cardinality of W is 1, then we say that the operator T is *cyclic*.

We now briefly recall the notion of Cauchy dual of a left-invertible operator, which was first investigated by Shimorin in [36]. An operator $T \in B(\mathcal{H})$ is said to be *left-invertible* if there exists $S \in B(\mathcal{H})$ such that $ST = I_{\mathcal{H}}$. Note that $T \in B(\mathcal{H})$ is left-invertible if and only if T^*T is invertible, which holds if and only if T is bounded from below. This allows us to define the *Cauchy dual* T' of a left-invertible operator T by

$$T' := T(T^*T)^{-1}.$$

For a left-invertible operator T , the following can be seen easily:

$$\begin{cases} T'^*T = I_{\mathcal{H}}, T^*T' = I_{\mathcal{H}}, \\ (T')' = T, T'^*T' = (T^*T)^{-1}. \end{cases} \tag{1.1}$$

A left-invertible operator T is said to be *Bergman-type operator* if

$$TT^* + (T^*T)^{-1} \leq 2I_{\mathcal{H}}.$$

We now give an outline of this paper. In Section 2, we establish the reflexivity of cyclic completely hyperexpansive operators (Proposition 2.3). As an application, we deduce that any cyclic 2-isometry is reflexive. This does not hold in general for cyclic m -isometries, $m > 2$. We also discuss some instances in which finitely multicyclic completely hyperexpansive operators are reflexive. In Section 3, we show that the Cauchy dual of 2-hyperexpansions are reflexive (Proposition 3.1). Among various applications, we prove the reflexivity of the Bergman-type operators. Also, we deduce the fact that the Cauchy dual of an m -isometry is reflexive for $m = 1, 2$; but the result is not true in general for $m > 2$. However, in case of a contractive or expansive m -isometry, its Cauchy dual is always reflexive. Further, we establish the reflexivity of certain power bounded left-invertible operators and study some cases where left-invertible operators and their Cauchy duals are reflexive. We end this paper with some possible directions for further research.

2. Completely hyperexpansive operators

In this section, we address the question of the reflexivity for a completely hyperexpansive operator. We focus our attention to the class of cyclic completely hyperexpansive operators primarily due to a result of Aleman, which asserts that any cyclic analytic completely hyperexpansive operator is unitarily equivalent to the multiplication operator M_z on a Dirichlet-type space D_μ for some finite positive measure μ supported on $\overline{\mathbb{D}}$ (see [2, IV, Theorem 2.5]). We now recall the definition of D_μ . Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be an analytic function of the form $f(z) = \sum_{n=0}^\infty \hat{f}(n)z^n$. For a finite positive measure μ on $\overline{\mathbb{D}}$, define

$$\|f\|_\mu^2 := \sum_{n=0}^\infty |\hat{f}(n)|^2 + \int_{\mathbb{D}} |f'(\zeta)|^2 U_\mu(\zeta) dm(\zeta),$$

where dm is the normalized area measure on \mathbb{D} , and

$$U_\mu(\zeta) := \int_{\mathbb{D}} \log \left| \frac{1 - \bar{\zeta}\zeta}{\zeta - z} \right| \frac{d\mu(z)}{1 - |z|^2} + \int_{\partial\mathbb{D}} \frac{1 - |\zeta|^2}{|z - \zeta|^2} d\mu(z), \quad \zeta \in \mathbb{D}.$$

The Dirichlet-type space D_μ is defined as

$$D_\mu := \{f : \mathbb{D} \rightarrow \mathbb{C} \mid f \text{ is analytic function such that } \|f\|_\mu < \infty\}.$$

One of the ingredients in obtaining reflexivity of cyclic completely hyperexpansions is a fact related to the multiplier algebra of scalar-valued reproducing kernel Hilbert spaces. We briefly discuss reproducing kernel Hilbert spaces and some related topics. Most of the facts mentioned below pertaining to reproducing kernel Hilbert spaces can be found in [30].

Let X be a set and \mathcal{E} be a Hilbert space. Let \mathcal{H} be a Hilbert space of \mathcal{E} -valued functions on X . We say \mathcal{H} is an \mathcal{E} -valued reproducing kernel Hilbert space (RKHS) on X provided for every $x \in X$, the evaluation map $E_x : \mathcal{H} \rightarrow \mathcal{E}$ given by $E_x(f) = f(x)$ is bounded. The map $k : X \times X \rightarrow B(\mathcal{E})$ defined by $k(x, y) := E_x E_y^*$ is called the $B(\mathcal{E})$ -valued reproducing kernel. In this case, for any $\eta \in \mathcal{E}$ and $x \in X$, we have $k(\cdot, x)\eta = E_x^*(\eta)$. The space \mathcal{H} admits the reproducing property:

$$\langle f(x), \eta \rangle_{\mathcal{E}} = \langle f, k(\cdot, x)\eta \rangle_{\mathcal{H}} \text{ for } f \in \mathcal{H}, x \in X \text{ and } \eta \in \mathcal{E}. \tag{2.1}$$

We omit the suffix of the inner product in future usages, whenever the context is clear. In case $\mathcal{E} = \mathbb{C}$, we say \mathcal{H} is a scalar-valued RKHS and k to be a scalar-valued reproducing kernel.

Let \mathcal{H} be a \mathcal{E} -valued RKHS on a set X . A map $\phi : X \rightarrow B(\mathcal{E})$ is said to be a multiplier of \mathcal{H} if $\phi f \in \mathcal{H}$ for every $f \in \mathcal{H}$, where $\phi f(x) := \phi(x)f(x)$ for $x \in X$. Let $\mathcal{M}(\mathcal{H})$ denote the set of all multipliers of \mathcal{H} . As an application of the closed graph theorem, every multiplier ϕ induces a bounded linear operator $M_\phi : \mathcal{H} \rightarrow \mathcal{H}$ given by $M_\phi f = \phi f$, $f \in \mathcal{H}$. It is well known that for any scalar-valued RKHS \mathcal{H} , the algebra $\mathcal{M}(\mathcal{H}) := \{M_\phi : \phi \in \mathcal{M}(\mathcal{H})\}$ is reflexive. Also, in case of an \mathcal{E} -valued RKHS, it follows from [6, Corollary 2.2] that $\mathcal{M}(\mathcal{H})$ is reflexive provided the

evaluation map $E_x : \mathcal{H} \rightarrow \mathcal{E}$ is either onto or zero for every $x \in X$. It turns out that if \mathcal{E} is finite-dimensional, then $\mathcal{M}(\mathcal{H})$ is reflexive irrespective of the aforementioned assumptions. The argument is routine and we include a proof for completeness sake.

LEMMA 2.1. *Let X be a set and \mathcal{E} be a finite-dimensional Hilbert space. Let \mathcal{H} be an \mathcal{E} -valued reproducing kernel Hilbert space on X associated with a $B(\mathcal{E})$ -valued reproducing kernel k . Then $\mathcal{M}(\mathcal{H})$ is reflexive.*

Proof. Given a multiplier $\phi \in \mathcal{M}(\mathcal{H})$, it is easy to verify that

$$M_\phi^* k(\cdot, x) \eta = k(\cdot, x) \phi(x)^* \eta, \quad x \in X, \eta \in \mathcal{E}.$$

In particular, for each $x \in X$, $V_x := \{k(\cdot, x) \eta : \eta \in \mathcal{E}\} \in \text{Lat } M_\phi^*$ for all $\phi \in \mathcal{M}(\mathcal{H})$. Let $T \in \text{Alg Lat } \mathcal{M}(\mathcal{H})$, then $\text{Lat } M_\phi^* \subseteq \text{Lat } T^*$ for every $\phi \in \mathcal{M}(\mathcal{H})$ and hence $T^*(V_x) \subset V_x$ for all $x \in X$. Given $x \in X$, since \mathcal{E} is finite-dimensional, for every $\eta \in \mathcal{E}$ there exist a unique $\tilde{\eta} \in E_x(\mathcal{H}) \subseteq \mathcal{E}$ such that $T^* k(\cdot, x) \eta = k(\cdot, x) \tilde{\eta}$. Hence for each $x \in X$, we define $\tilde{\psi}(x) : \mathcal{E} \rightarrow \mathcal{E}$ by $\tilde{\psi}(x)(\eta) := \tilde{\eta}$. Now we verify that $\tilde{\psi}(x)$ is linear. Let $\eta_1, \eta_2 \in \mathcal{E}$ and α be a scalar. Then by our choice there exist unique $\tilde{\eta}_1, \tilde{\eta}_2, \widetilde{\alpha\eta_1 + \eta_2} \in E_x(\mathcal{H})$ such that $T^* k(\cdot, x) \eta_1 = k(\cdot, x) \tilde{\eta}_1$, $T^* k(\cdot, x) \eta_2 = k(\cdot, x) \tilde{\eta}_2$ and $T^* k(\cdot, x) (\alpha\eta_1 + \eta_2) = k(\cdot, x) (\widetilde{\alpha\eta_1 + \eta_2})$. Now

$$\begin{aligned} T^* k(\cdot, x) (\alpha\eta_1 + \eta_2) &= \alpha T^* k(\cdot, x) \eta_1 + T^* k(\cdot, x) \eta_2 \\ &= \alpha k(\cdot, x) \tilde{\eta}_1 + k(\cdot, x) \tilde{\eta}_2 \\ &= k(\cdot, x) (\alpha \tilde{\eta}_1 + \tilde{\eta}_2). \end{aligned}$$

Hence by uniqueness $\widetilde{\alpha\eta_1 + \eta_2} = \alpha \tilde{\eta}_1 + \tilde{\eta}_2$ and thus $\tilde{\psi}(x)(\alpha\eta_1 + \eta_2) = \widetilde{\alpha\eta_1 + \eta_2} = \alpha \tilde{\eta}_1 + \tilde{\eta}_2 = \alpha \tilde{\psi}(x)(\eta_1) + \tilde{\psi}(x)(\eta_2)$. Since \mathcal{E} is finite-dimensional $\tilde{\psi}(x)$ is also bounded. Let $\psi(x) := \tilde{\psi}(x)^*$, then for any $f \in \mathcal{H}$, $\eta \in \mathcal{E}$ and $x \in X$,

$$\begin{aligned} \langle (Tf)(x), \eta \rangle &\stackrel{(2.1)}{=} \langle Tf, k(\cdot, x) \eta \rangle \\ &= \langle f, T^* k(\cdot, x) \eta \rangle \\ &= \langle f, k(\cdot, x) \psi(x)^* \eta \rangle \\ &\stackrel{(2.1)}{=} \langle f(x), \psi(x)^* \eta \rangle \\ &= \langle \psi(x) f(x), \eta \rangle. \end{aligned}$$

Hence $T = M_\psi \in \mathcal{M}(\mathcal{H})$, which completes the proof. \square

We recall the definition of complete Nevanlinna-Pick kernel from [37]. Let \mathcal{H} be a scalar-valued RKHS with a scalar-valued reproducing kernel k on a set X . We say that k has *complete Nevanlinna-Pick property* if

$$k(x, y) - \frac{k(x, w)k(w, y)}{k(w, w)} = B_w(x, y)k(x, y), \quad x, y \in X,$$

for some $w \in X$ such that $k(w, w) \neq 0$ and some positive semidefinite function $B_w(x, y)$ with $|B_w(x, y)| < 1$ for all $x, y \in X$.

In the following proposition we record the known facts [25, Proposition 2.5(5)], [7, Proposition 2.04, 2.055, 2.09], [18, Corollary 5.3], pertaining to the property (\mathbb{A}_1) for ready reference.

PROPOSITION 2.2. *The following statements hold.*

- (i) *If \mathcal{W} is a unital WOT-closed subalgebra and $\text{AlgLat } \mathcal{W}$ has property (\mathbb{A}_1) , then \mathcal{W} is super-reflexive.*
- (ii) *If \mathcal{S} is any weak*-closed subspace with property $(\mathbb{A}_1(r))$ for some $r \geq 1$, and \mathcal{T} is a weak*-closed subspace of \mathcal{S} , then \mathcal{T} has property $(\mathbb{A}_1(r))$.*
- (iii) *For some $r \geq 1$, a direct sum of two unital weak*-closed subalgebras has property $(\mathbb{A}_1(r))$ if each summand has property $(\mathbb{A}_1(r))$.*
- (iv) *If \mathcal{A} is a unital weak*-closed subalgebra of $B(\mathcal{H})$ that has property $(\mathbb{A}_1(r))$ for some $r \geq 1$, then \mathcal{A} is WOT-closed; and the weak* and weak operator topologies coincide on \mathcal{A} . Moreover, if L is an invertible operator in $B(\mathcal{H})$, then $L^{-1}\mathcal{A}L := \{L^{-1}AL : A \in \mathcal{A}\}$ is a unital weak*-closed subalgebra that has property $(\mathbb{A}_1(r'))$ for some $r' \geq 1$.*
- (v) *The multiplier algebra $\mathcal{M}(\mathcal{K})$ of every complete Nevanlinna-Pick kernel has property $(\mathbb{A}_1(1))$.*

It is known that any abelian von Neumann algebra has property $(\mathbb{A}_1(1))$ (see [17, Proposition 60.1]). Hence, by Proposition 2.2(ii), for any normal operator T , \mathcal{A}_T has property $(\mathbb{A}_1(1))$. The following is the main result of this section.

PROPOSITION 2.3. *Let $T \in B(\mathcal{H})$ be a completely hyperexpansive operator such that $\ker T^*$ is one-dimensional. Then T is reflexive and admits property $(\mathbb{A}_1(1))$.*

Proof. Since T is completely hyperexpansive operator, hence by [36, Theorem 3.6], T admits the following Wold-type decomposition

$$T = T_1 \oplus T_2 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2),$$

where $\mathcal{H}_1 = \bigcap_{n \in \mathbb{N}} T^n(\mathcal{H})$ and $\mathcal{H}_2 = \bigvee \{T^n x : x \in \mathcal{N}, n \in \mathbb{N} \cup \{0\}\}$, with $\mathcal{N} := \ker T^*$. Further, $T_1 = T|_{\mathcal{H}_1}$ is unitary and $T_2 = T|_{\mathcal{H}_2}$ is analytic. Note that, by [32, Theorem 1], T_2 is cyclic as $\dim \mathcal{N} = 1$ and hence by a theorem of Aleman [2, IV, Theorem 2.5], T_2 is unitarily equivalent to M_z on D_μ for some finite positive measure μ on $\overline{\mathbb{D}}$. Furthermore, from [37, Theorem 1.1], it follows that the scalar-valued reproducing kernel k_μ associated with D_μ is complete Nevanlinna-Pick. Thus, by Proposition 2.2(v), the algebra $\mathcal{M}(D_\mu)$ has property $(\mathbb{A}_1(1))$ and $\mathcal{M}(D_\mu)$ is reflexive by Lemma 2.1. Therefore, by Proposition 2.2(i), $\mathcal{M}(D_\mu)$ is super-reflexive, that is, every unital WOT-closed subalgebra of $\mathcal{M}(D_\mu)$ is reflexive. In particular \mathcal{W}_{M_z} , and hence T_2 , is reflexive. Also, \mathcal{A}_{T_2} has property $(\mathbb{A}_1(1))$ by Proposition 2.2(ii). On the other hand, T_1 , being a unitary operator, is reflexive and \mathcal{A}_{T_1} has property $(\mathbb{A}_1(1))$ (see the discussion prior to Proposition 2.3). Since T_1 and T_2 are reflexive and both \mathcal{A}_{T_1} , \mathcal{A}_{T_2} have

property $(\mathbb{A}_1(1))$, one may now imitate the proof of [16, VII, Theorem 8.5, Case 2] to conclude the reflexivity of \mathcal{A}_T , and hence of T . Moreover, \mathcal{A}_T has property $(\mathbb{A}_1(1))$ since each \mathcal{A}_{T_i} ($i = 1, 2$) has property $(\mathbb{A}_1(1))$ (see Proposition 2.2(iii)). \square

One part of the proof of reflexivity of a cyclic analytic 2-isometry $T \in B(\mathcal{H})$ is implicit in [34]. Indeed, any such T is unitarily equivalent to M_z acting on D_μ for some finite positive measure μ supported on $\partial\mathbb{D}$ (see [33, Theorem 5.1]). It follows from [34, Lemma 5.4] that for any multiplier $\phi \in \mathcal{M}(D_\mu)$ the multiplication operator $M_\phi \in \mathcal{W}_{M_z}$. One can also verify that $\text{AlgLat } \mathcal{W}_{M_z} \subseteq \mathcal{M}(D_\mu)$ (see, for instance, [14, Proof of Theorem 4.1]) and thus M_z is reflexive. The novelty of our result is that it removes the analyticity assumption and at the same time ensures the property $(\mathbb{A}_1(1))$ (cf. [23, Theorem 4]).

COROLLARY 2.4. *If T is a cyclic 2-isometry in $B(\mathcal{H})$, then T is reflexive and has property $(\mathbb{A}_1(1))$.*

We discuss here one instance in which the reflexivity of (not necessarily cyclic) completely hyperexpansive operators can be ensured. Let $T \in B(\mathcal{H})$ be a finitely multicyclic completely non-unitary 2-isometry satisfying the kernel condition, that is, $T^*T(\ker T^*) \subseteq \ker T^*$. It follows from [3, Corollary 3.7] that T is unitarily equivalent to an orthogonal sum of n unilateral weighted shifts T_i , where n is the order of multicyclivity of T . Since T_i 's are cyclic 2-isometries, each T_i is reflexive and has property $(\mathbb{A}_1(1))$ (Corollary 2.4). Then applying [36, Theorem 3.6] and using the fact that any analytic operator is completely non-unitary, we can show that any finitely multicyclic 2-isometry satisfying the kernel condition is reflexive.

The preceding discussion together with Proposition 2.3 motivates us to the following question:

QUESTION 1. Is every completely hyperexpansive operator in $B(\mathcal{H})$ reflexive?

3. Cauchy dual operators

The main result of this section establishes the reflexivity of the Cauchy dual of certain expansive operators. Before proceeding to the main result, we recall the following notion. Let $H^\infty(\mathbb{D})$ be the algebra of all bounded holomorphic functions on \mathbb{D} . A subset Ω of \mathbb{C} is *dominating for the algebra* $H^\infty(\mathbb{D})$ if

$$\sup_{\lambda \in \mathbb{D}} |f(\lambda)| = \sup_{\lambda \in \Omega \cap \mathbb{D}} |f(\lambda)|, \quad f \in H^\infty(\mathbb{D}).$$

PROPOSITION 3.1. *Let T be a 2-hyperexpansive operator in $B(\mathcal{H})$ and T' be the Cauchy dual of T . Then T' is reflexive and has property $(\mathbb{A}_1(1))$.*

Proof. If T is invertible, then T is a unitary [38, Remark 3.4]. In that case, $T' = T$ and the result follows. So, we assume that T is not invertible. Since T is expansive, T' is a contraction. Let us recall from [8, p. 1] that T' admits canonical decomposition

$$T' = T_1 \oplus T_2 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2),$$

where $T_1 = T'|_{\mathcal{H}_1}$ is an absolutely continuous contraction and $T_2 = T'|_{\mathcal{H}_2}$ is a singular unitary operator. It is well known that a contraction is reflexive if and only if its absolutely continuous contractive part is reflexive (see, for example, [15, Lemma 7.1]). We now establish the reflexivity of T_1 . As T_1 is an absolutely continuous contraction, T_1 has (Sz.-Nagy-Foias) H^∞ -functional calculus $\Phi_{T_1} : H^\infty(\mathbb{D}) \rightarrow \mathcal{A}_{T_1}$ (see [7, Theorem 4.1]). Since T is not invertible, $\sigma(T') = \overline{\mathbb{D}}$ by [12, Lemma 2.14(ii)] and thus $\sigma(T_1) = \overline{\mathbb{D}}$. Therefore, $\sigma(T_1) \cap \mathbb{D}$ is dominating for $H^\infty(\mathbb{D})$. It follows then from [7, Proposition 4.6] that the H^∞ -functional calculus Φ_{T_1} is an isometric isomorphism and a weak* homeomorphism. Hence T_1 is reflexive by [10] (see also [24, Theorem 1]).

By [9, Theorem 1.2], \mathcal{A}_{T_1} satisfies property $(\mathbb{A}_1(1))$. Since T_2 is a unitary operator, \mathcal{A}_{T_2} has property $(\mathbb{A}_1(1))$ (see the discussion prior to Proposition 2.3). Therefore, by Proposition 2.2(iii), $\mathcal{A}_{T'}$ has property $(\mathbb{A}_1(1))$. This completes the proof. \square

REMARK 1. The reflexivity of T_1 can also be obtained from [23, Theorem 5]. The proof there is based on the fact that every von Neumann operator is reflexive.

Bergman-type operators were studied by Shimorin [36] in the context of wandering subspace problem. As an immediate application of the above proposition, we now establish the reflexivity of such operators.

COROLLARY 3.2. *Let $T \in B(\mathcal{H})$ be a Bergman-type operator. Then T is reflexive and has property $(\mathbb{A}_1(1))$.*

Proof. Suppose $T \in B(\mathcal{H})$ is a Bergman-type operator. It was noted in the proof of [36, Theorem 3.6] that T' is 2-hyperexpansive. Now by applying Proposition 3.1, we deduce the reflexivity of $(T')' = T$ and property $(\mathbb{A}_1(1))$. \square

REMARK 2. Let T be a 2-hyperexpansive operator in $B(\mathcal{H})$ and let T' be the Cauchy dual of T . It was observed in the proof of [12, Theorem 2.9] that $T'T$ is an expansion, TT' is a contraction and TT' is similar to an isometry. Since T is expansive, T' is a contraction. Now for any $x \in \mathcal{H}$ and $n \in \mathbb{N}$, one has

$$\|x\| \leq \|(T'T)^n x\| = \|T'(TT')^{n-1}Tx\| \leq \|(TT')^{n-1}Tx\| \leq \|Tx\| \leq \|T\| \|x\|.$$

We now deduce from Proposition 1.15 and the discussion following Corollary 1.16 in [28], $T'T$ is similar to an isometry. Since any isometry is reflexive (see [19]) and reflexivity is invariant under similarity (see [17, Section 57, pp. 323]), it follows that TT' and $T'T$ are reflexive. Moreover, since an isometry has property $(\mathbb{A}_1(1))$, so by applying Proposition 2.2(iv) we deduce that TT' (resp. $T'T$) has property $(\mathbb{A}_1(r))$ (resp. $(\mathbb{A}_1(r'))$) for some $r \geq 1$ (resp. $r' \geq 1$).

As discussed earlier, any 2-isometry is 2-hyperexpansive. Therefore by Proposition 3.1, Cauchy-dual of any 2-isometry is reflexive. It turns out that any m -isometry is left-invertible (see [1, I. Lemma 1.21]). This gives rise to a natural question, whether the Cauchy dual of any m -isometry is reflexive or not. This has a negative answer for $m = 3$. In this regard, we discuss below an example of a cyclic 3-isometry T from [1, III. pp 406], for which neither T nor T' is reflexive.

EXAMPLE 1. Let us consider the following operator

$$T := \bigoplus_{n=1}^{\infty} \begin{bmatrix} \alpha_n & c \\ 0 & \alpha_n \end{bmatrix} \text{ on } \mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n,$$

where each $\mathcal{H}_n = \mathbb{C}^2$, $c > 0$ and $(\alpha_n)_{n \in \mathbb{N}}$ is a sequence of unimodular complex numbers which do not contain any of its accumulation points. It is easy to verify that

$$T^*T = \bigoplus_{n=1}^{\infty} \begin{bmatrix} 1 & \overline{\alpha_n}c \\ \alpha_n c & 1 + c^2 \end{bmatrix} \text{ and } T' = \bigoplus_{n=1}^{\infty} \begin{bmatrix} \alpha_n & 0 \\ -\alpha_n^2 c & \alpha_n \end{bmatrix}.$$

Let $\{e_1, e_2\}$ be the standard basis of \mathbb{C}^2 . For $j = 1, 2$, let

$$x_j^n = (0, \dots, 0, e_j, 0, \dots),$$

where e_j is in the n^{th} position. Note that $\{x_j^n : j = 1, 2, \text{ and } n \in \mathbb{N}\}$ is an orthonormal basis for \mathcal{H} . Let $\mathcal{M} \in \text{Lat } T$, then $\mathcal{M} = \bigoplus_{n=1}^{\infty} \mathcal{M}_n$, where each \mathcal{M}_n is T -invariant (note that, each \mathcal{H}_n is reducing for T). One readily sees that a non-trivial T -invariant subspace $\mathcal{M}_n \subset \mathcal{H}_n$ has to be of the form $\vee \{x_j^n\}$. Hence, a routine verification shows that

$$\text{Lat } T = \left\{ \bigoplus_{n=1}^{\infty} \mathcal{M}_n : \mathcal{M}_i = \{0\}, \text{ or } \vee \{x_j^i\}, \text{ or } \mathcal{H}_i, i \in \mathbb{N} \right\},$$

$$\text{AlgLat } \mathcal{W}_T = \left\{ \bigoplus_{n=1}^{\infty} \begin{bmatrix} p_n & q_n \\ 0 & r_n \end{bmatrix} : p_n, q_n, r_n \in \mathbb{C} \right\}.$$

It is easy to show that any $A \in \mathcal{W}_T$ is of the form

$$\bigoplus_{n=1}^{\infty} \begin{bmatrix} t_n & q \\ 0 & t_n \end{bmatrix} \text{ for some } t_n, q \in \mathbb{C}.$$

Thus \mathcal{W}_T is a proper subset of $\text{AlgLat } \mathcal{W}_T$ and hence T is not reflexive. A similar computation shows that T' is not reflexive.

Observe that the 3-isometry considered in Example 1 is neither expansive nor contractive. In the following proposition we show that the Cauchy dual T' of an m -isometry T is reflexive provided T is either expansive or contractive.

PROPOSITION 3.3. *Let $T \in B(\mathcal{H})$ be an m -isometry and let T' be the Cauchy dual of T . If T is either contractive or expansive, then T' is reflexive and has property $(\mathbb{A}_1(1))$.*

Proof. If $m = 1$, then T is an isometry. In that case, $T' = T$ and the result follows as any isometry is reflexive (see [19]) and has property $(\mathbb{A}_1(1))$. Let $m > 1$.

Case I. Suppose T is a contraction. It follows from [13, Lemma 2.4] that T is expansive and hence T is an isometry. Thus, T as well as T' is reflexive and both of them have property $(\mathbb{A}_1(1))$.

Case II. Suppose T is an expansion. It is well known for any m -isometry T , $\sigma(T) \subseteq \overline{\mathbb{D}}$ (see [1, I. Lemma 1.21]). Note that for any $\lambda \in \mathbb{D} \setminus \{0\}$,

$$T'^* - \lambda I_{\mathcal{H}} \stackrel{(1.1)}{=} T'^*(I_{\mathcal{H}} - \lambda T) = -\lambda T'^*(T - (1/\lambda)I_{\mathcal{H}}).$$

If T is not invertible, then T' is not invertible. In that case, $\overline{\mathbb{D}} \subseteq \sigma(T')$. In addition, the contractivity of T' implies $\sigma(T') = \overline{\mathbb{D}}$. Now it follows from the argument of the proof of Proposition 3.1 that for any non-invertible expansive m -isometry T , the Cauchy dual T' is reflexive and has property $(\mathbb{A}_1(1))$. On the other hand, if T is invertible, then T^{-1} is a contractive m -isometry (see the discussion above [13, Lemma 2.4]). Therefore, by Case I, T^{-1} is an isometry and hence T is a unitary. This completes the proof. \square

Although T' in Proposition 3.3 is reflexive and every contractive or invertible expansive, m -isometry T is reflexive; we do not have any conclusive evidence about the reflexivity of T in general. This situation does not occur in case of finite-dimensional Hilbert spaces.

PROPOSITION 3.4. *Let T be an invertible operator acting on a finite dimensional Hilbert space \mathcal{H} . Then T is reflexive if and only if T' is reflexive.*

Proof. Note that $T' = (T^{-1})^*$. If λ is an eigenvalue of T corresponding to a generalized eigenvector v , then $1/\lambda$ is the eigenvalue of T^{-1} corresponding to the generalized eigenvector v . It is easy to verify that numbers and sizes of Jordan blocks corresponding to λ and $1/\lambda$ are same for T and T^{-1} in their respective Jordan canonical forms. Now applying the result of Deddens and Fillmore [20] (see also [17, Theorem 57.2]), T is reflexive if and only if T^{-1} is reflexive. Since reflexivity is preserved under adjoint operation, therefore T is reflexive if and only if T' is reflexive. \square

In the following proposition we give a class of left-invertible operators for which both T and T' are reflexive (cf. [14, Corollary 4.5], [21, Corollary 30]). Recall that an operator $T \in B(\mathcal{H})$ is said to be *power bounded* if $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$.

PROPOSITION 3.5. *Let $T \in B(\mathcal{H})$ be a left-invertible operator. If both T and T' are power bounded, then T and T' are reflexive. Moreover, T (resp. T') has property $(\mathbb{A}_1(r))$ (resp. $(\mathbb{A}_1(r'))$) for some $r \geq 1$ (resp. $r' \geq 1$).*

Proof. Suppose that T and T' are power bounded operators. Then there exist positive numbers K_1 and K_2 such that $\|T^n\| \leq K_1$ and $\|T'^n\| \leq K_2$ for all $n \in \mathbb{N}$. Now for any $x \in \mathcal{H}$, $\|T^n x\| \leq K_1 \|x\|$ and

$$\|x\| \stackrel{(1.1)}{=} \|T'^{*n} T^n x\| \leq \|T'^{*n}\| \|T^n x\| \leq K_2 \|T^n x\|.$$

Thus $(1/K_2)\|x\| \leq \|T^n x\| \leq K_1 \|x\|$ for $x \in \mathcal{H}$. We now deduce from Proposition 1.15 and the discussion following Corollary 1.16 in [28], T is similar to an isometry. Then

from the reflexivity of isometry (see [19]) and invariance of reflexivity under similarity, we conclude that T is reflexive. Further, since an isometry has property $(\mathbb{A}_1(1))$, so by applying Proposition 2.2(iv) we deduce that T has property $(\mathbb{A}_1(r))$ for some $r \geq 1$. Similarly one can show that T' is reflexive and has property $(\mathbb{A}_1(r'))$ for some $r' \geq 1$. \square

For an expansive power bounded operator T , it is evident from (1.1) T' is also power bounded. Hence the following corollary is immediate from the above proposition.

COROLLARY 3.6. *Let $T \in B(\mathcal{H})$ be expansive. If T is power bounded, then both T and T' are reflexive. Moreover, T (resp. T') has property $(\mathbb{A}_1(r))$ (resp. $(\mathbb{A}_1(r'))$) for some $r \geq 1$ (resp. $r' \geq 1$).*

4. Concluding remarks

We conclude this note with some possible directions for further investigations. We would like to draw the reader’s attention to the recent work [22], where the notion of the Cauchy dual operator has been generalized to the closed range operators. It would be interesting to know counter-parts of the results in this paper for closed range operators. Although we have obtained reflexivity of any cyclic completely hyperexpansive operator (Proposition 2.3), we do not know whether an arbitrary completely hyperexpansive operator is reflexive. In view of the reflexivity of the Cauchy dual of a 2-hyperexpansion (Proposition 3.1), one may explore the possibility to relate $\text{Lat } T$ with $\text{Lat } T'$, \mathcal{W}_T with $\mathcal{W}_{T'}$ and $\text{AlgLat } \mathcal{W}_T$ with $\text{AlgLat } \mathcal{W}_{T'}$ for a left-invertible operator T . Since $(T')' = T$, these relations may play a decisive role in deriving reflexivity of an arbitrary completely hyperexpansive operator. Indeed, the inclusion of \mathcal{W}_T in $\mathcal{W}_{T'}$ (or vice-versa) implies the reflexivity of both T and T' .

PROPOSITION 4.1. *Let $T \in B(\mathcal{H})$ be left-invertible and let T' be the Cauchy dual of T . If $\mathcal{W}_T \subseteq \mathcal{W}_{T'}$, then both T and T' are reflexive.*

Proof. Assume that $\mathcal{W}_T \subseteq \mathcal{W}_{T'}$, then $TT' = T'T$. Now multiplying both sides from left by T'^* gives $T' \stackrel{(1.1)}{=} (T^*T)^{-1}T$. Hence $(T^*T)T = T(T^*T)$, that is, T is quasinormal and by [39, Theorem 2] T is reflexive. It is easy to verify that T is quasinormal if and only if T' is quasinormal. This completes the proof. \square

Note that the converse of the above proposition is not true in general. Indeed, such examples are ample in the class of weighted unilateral shifts operators (denoted by S_w) on $l^2(\mathbb{N})$. As noted in the proof if $\mathcal{W}_{S_w} \subseteq \mathcal{W}'_{S_w}$ then S_w is necessarily quasinormal. But it is easy to verify from the definition that a weighted shift is quasinormal if and only if it is a constant multiple of the (unweighted) unilateral shift. Moreover in this case $\mathcal{W}_{S_w} = \mathcal{W}'_{S_w}$. But the class of reflexive weighted shifts is strictly bigger than unilateral shifts (see [23] for a large class of examples).

Recently, reflexivity of certain classes of weighted shifts on directed trees was studied in [11, 21]. A result in [21] suggests another class of left-invertible operators

T , for which both T and T' are reflexive. Indeed, it follows from [21, Corollary 30], for a left-invertible weighted shift T on rooted directed tree, both T and T' are reflexive. In view of Propositions 3.4, 3.5 and the discussion above, one may ask whether the reflexivity of a left-invertible operator T implies the reflexivity of the Cauchy dual T' . Nevertheless, it is interesting to study the class of left-invertible operators for which reflexivity is preserved under Cauchy dual operation.

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