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NEW CHARACTERIZATIONS FOR DIFFERENCES OF INTEGRAL–TYPE OPERATORS FROM α–BLOCH SPACE TO β–BLOCH–ORLICZ SPACE

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Abstract. Several new equivalent characterizations for the boundedness of the differences of integral-type operators from α-Bloch space to β-Bloch-Orlicz space are presented in this paper. Especially, we estimate their essential norms in terms of the n-th power of the induced analytic self-maps on the unit disk, which can provide new and interesting compactness criteria and be seen as extensions of several existing results in the literature. Moreover, some applications are also exhibited in an example for readers' convenience.

1. Introduction and preliminaries

Let $H(\mathbb{D})$ denote the space of all holomorphic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the collection of all holomorphic self-maps on $\mathbb{D}$, where $\mathbb{D}$ is the unit disk in the complex plane $\mathbb{C}$. Given a continuous linear operator T on a Banach space X, its essential norm is the distance from the operator T to compact operators on X, that is, $\|T\|_e = \inf\{\|T-K\| : K \text{ is compact}\}$. It's trivial that $\|T\|_e = 0$ if and only if T is compact, see, e.g. [6] and their references therein. For $a \in \mathbb{D}$, let φ_a be the automorphism of $\mathbb{D}$ exchanging 0 for a, that is, $\varphi_a(z) = (a-z)/(1-\bar{a}z)$. For $z, w \in \mathbb{D}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z,w) = |\varphi_w(z)| = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

In the sequel, given $\phi, \psi \in S(\mathbb{D})$, we denote $\rho(z) = \rho(\phi(z), \psi(z))$ for simplicity.

For an analytic self-map $\phi : \mathbb{D} \to \mathbb{D}$, the composition operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ is defined by

$$C_\phi f = f \circ \phi, \; f \in H(\mathbb{D}).$$

The study of composition operators is a fairly active field. For general references on the theory of composition operators, see the two excellent books [3] by Cowen and MacCluer, and [20] by Shapiro.

In this paper, we mainly pay our attention to the boundedness and essential norms of the differences of integral-type operators defined below. As we all know, the operator theoretic properties of integral-type operators expressed in terms of function theoretic

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conditions on symbols have been a subject of high interest, which can be found in [9, 10, 13, 17] and their reference therein. Subsequently, we list some relevant integral-type operators in details.

(a) Given $g \in H(\mathbb{D})$, the operator L^g is defined by

$$L^g f(z) = \int_0^z f'(t)g(t)dt, \ f \in H(\mathbb{D}), \ z \in \mathbb{D}.$$

(b) Given $g \in H(\mathbb{D})$, the operator L_g is defined by

$$L_g f(z) = \int_0^z f'(t)g'(t)dt, \ f \in H(\mathbb{D}), \ z \in \mathbb{D}.$$

(c) Let $\phi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, the operator C_ϕ^g is defined by

$$C_\phi^g f(z) = \int_0^z f'(\phi(t))g(t)dt, \ f \in H(\mathbb{D}), \ z \in \mathbb{D}.$$

(d) Let $\phi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, the operator V_ϕ^g is defined by

$$V_\phi^g f(z) = \int_0^z f'(\phi(t))g'(t)dt, \ f \in H(\mathbb{D}), \ z \in \mathbb{D}.$$

Indeed, the above integral-type operators have close connections. On the one hand, let $\phi = id$ be the identity map in C_ϕ^g and V_ϕ^g, then

$$C_{id}^g = L^g \text{ and } V_{id}^g = L_g.$$

This means the operators L^g and L_g are special cases of C_ϕ^g and V_ϕ^g, respectively. On the other hand, if we replace $g \in H(\mathbb{D})$ with $g' \in H(\mathbb{D})$ in C_ϕ^g, then $C_\phi^{g'} = V_\phi^g$. Moreover, C_ϕ^g is called the generalized composition operator due to the fact $C_\phi^g - C_\phi$ is a constant if $g = \phi'$. Based on the above facts, we firstly provide interesting descriptions for $C_\phi^g - C_\psi^h$ acting from the α-Bloch space to β-Bloch-Orlicz space, then the analogous results for residual integral-type operators follow naturally. Our result can generalize the work in [11] to some extent. Next we present some Banach spaces we concentrated on.

Let μ be a weight; that is, μ is a positive continuous function on $\mathbb{D}$. We recall that the μ-Bloch space $\mathscr{B}_\mu := \mathscr{B}_\mu(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{B}_\mu} = |f(0)| + \|f'\|_\mu < \infty$$

with $\|f'\|_\mu := \sup_{z \in \mathbb{D}} \mu(z)|f'(z)|$. It is a well-known fact that the μ-Bloch space $\mathscr{B}_\mu$ is a Banach space under the norm $\|f\|_{\mathscr{B}_\mu}$. In particular, if $\mu(z) = (1-|z|^2)^\alpha$, we obtain the α-Bloch space $\mathscr{B}^\alpha$ [1, 18, 23]. For $\alpha = 1$, $\mathscr{B}^\alpha = \mathscr{B}$, the classical Bloch space; if $0 < \alpha < 1$, $\mathscr{B}^\alpha = Lip_{1-\alpha}$, the analytic Lipschitz space which consists of all $f \in H(\mathbb{D})$ satisfying

$$|f(z) - f(w)| \leqslant C|z-w|^{1-\alpha},$$

for some constant $C > 0$ and all $z, w \in \mathbb{D}$; when $\alpha > 1$, $\mathscr{B}^\alpha = H^\infty_{\alpha-1}$, the $\alpha - 1$ weighted-type space of analytic functions that contains all $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha-1}|f(z)| < \infty.$$

More generally, let v be a weight on $\mathbb{D}$. The weighted-type space H^∞_v is defined to be the collection of all functions $f \in H(\mathbb{D})$ that satisfy

$$\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty,$$

provided we identify that differs by a constant, and then H^∞_v is a Banach space under the norm $\|.\|_v$, see, e.g. [4, 8] and the references therein.

As another generalization of the classical Bloch space $\mathscr{B}$, Ramos Fernández [19] employed Young's functions to define the Bloch-Orlicz space $\mathscr{B}^\varphi$ in 2010. More precisely, let $\varphi : [0,+\infty) \to [0,+\infty)$ be an $\mathscr{N}$-function, that is, φ is a strictly increasing convex function such that $\varphi(0) = 0$, which implies that $\lim\limits_{t\to\infty}\varphi(t) = +\infty$. The Bloch-Orlicz space $\mathscr{B}^\varphi$, linked with the function φ, is the collection of all $f \in H(\mathbb{D})$ fulfilling

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(\lambda|f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f. We further suppose that φ^{-1} is continuously differentiable. If φ^{-1} is not differentiable everywhere, we can define the function

$$\psi(t) = \int_0^t \frac{\varphi(x)}{x}dx,\ t \geqslant 0,$$

then ψ is differentiable, whence ψ^{-1} is differentiable everywhere on $[0,\infty)$. Since φ is a strictly increasing, convex function satisfying $\varphi(0) = 0$, therefore the function $\varphi(t)/t$, $t > 0$, is increasing and

$$\varphi(t) \geqslant \psi(t) \geqslant \int_{t/2}^t \frac{\varphi(x)}{x}dx \geqslant \varphi\left(\frac{t}{2}\right) \text{ for all } t \geqslant 0.$$

As a consequence, $\mathscr{B}^\varphi = \mathscr{B}^\psi$. The convexity of φ can further imply the Minkowski's functional

$$\|f\|_\varphi = \inf\left\{k > 0 :\ S_\varphi\left(\frac{f'}{k}\right) \leqslant 1\right\},$$

defines a seminorm for $\mathscr{B}^\varphi$, where

$$S_\varphi(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(|f(z)|).$$

At the same time, $\mathscr{B}^\varphi$ is a Banach space endowed with the norm

$$\|f\|_{\mathscr{B}^\varphi} = |f(0)| + \|f\|_\varphi.$$

Observing from the fact

$$S_{\varphi}\left(\frac{f'}{\|f\|_{\mathscr{B}^{\varphi}}}\right) \leqslant 1,$$

it leads to the following lemma.

LEMMA 1.1. *The Bloch-Orlicz space is isometrically equal to* μ_1^{φ}*-Bloch space, where*

$$\mu_1^{\varphi}(z) = \frac{1}{\varphi^{-1}\left(\frac{1}{1-|z|^2}\right)}, \ z \in \mathbb{D}.$$

Whence for any $f \in \mathscr{B}^{\varphi}$,

$$\|f\|_{\mathscr{B}^{\varphi}} = |f(0)| + \sup_{z\in\mathbb{D}} \mu_1^{\varphi}(z)|f'(z)|.$$

As far as we know the readers can consult, e.g. [2, 22] and their reference therein for Bloch-Orlicz spaces. In [12], the β-Bloch-Orlicz space $\mathscr{B}_{\beta}^{\varphi} = \mathscr{B}_{\beta}^{\varphi}(\mathbb{D})$ is defined as the class of all $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\varphi(\lambda|f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f. And the β-Bloch-Orlicz space $\mathscr{B}_{\beta}^{\varphi}$ is also a Banach space under the norm

$$\|f\|_{\mathscr{B}_{\beta}^{\varphi}} = |f(0)| + \|f\|_{\varphi,\beta},$$

where

$$\|f\|_{\varphi,\beta} = \inf\left\{k>0 : \ S_{\varphi,\beta}\left(\frac{f'}{k}\right) \leqslant 1\right\}$$

and

$$S_{\varphi,\beta}(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\varphi(|f(z)|).$$

It holds that $\mathscr{B}_{\beta}^{\varphi} = \mathscr{B}^{\varphi}$ when $\beta = 1$. Furthermore, a standard fact is

$$S_{\varphi,\beta}\left(\frac{f'}{\|f\|_{\mathscr{B}_{\beta}^{\varphi}}}\right) \leqslant 1,$$

which yields a lemma linking with Lemma 1.1.

LEMMA 1.2. *The* β*-Bloch-Orlicz space is isometrically equal to* μ_{β}^{φ}*-Bloch space, where*

$$\mu_{\beta}^{\varphi}(z) = \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\beta}}\right)}, \ z \in \mathbb{D}.$$

Whence for any $f \in \mathscr{B}_\beta^\varphi$,

$$\|f\|_{\mathscr{B}_\beta^\varphi} = |f(0)| + \|f'\|_{\mu_\beta^\varphi}, \tag{1.1}$$

with $\|f'\|_{\mu_\beta^\varphi} := \sup_{z\in\mathbb{D}} \mu_\beta^\varphi(z)|f'(z)|$.

In the sequel, Lemma 1.2 will play an important role in showing our main results. And we always use $\mu_\beta^\varphi(z)$ to stand for $1/\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)$ for the convenience of writing. Meanwhile, we let $\mathbb{N}_0$ denote the set of all nonnegative integers and the notations $A \approx B$, $A \preceq B$, $A \succeq B$ mean that there maybe different positive constants C such that $B/C \leqslant A \leqslant CB$, $A \leqslant CB$, $CB \leqslant A$.

For a long time, huge interest has been in characterizing the properties of composition operator C_ϕ acting on Bloch-type spaces in terms of the n-th power of ϕ of $\mathbb{D}$. In particular, Wulan, Zheng and Zhu [21] obtained a new compactness criterion for C_ϕ on the Bloch space in term of the norm of ϕ^n, where ϕ^n means the n-th power of ϕ. That is, C_ϕ is compact on $\mathscr{B}$ if and only if $\lim_{n\to\infty} \|\phi^n\|_{\mathscr{B}} = 0$. As regards to α-Bloch spaces, Zhao [23] proved that $\|C_\phi\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta} \approx \limsup_{n\to\infty} n^{\alpha-1}\|\phi^n\|_\beta$ for $0<\alpha,\beta<\infty$. Since then, many work have contributed to the development of new characterizations for several kinds of operators, the interested readers can refer to [5, 6, 7, 14, 15, 16, 21]. The integration, differentiation and composition operators are well-studied objects because they are quite natural and provide information about spaces and functions in them. They are often related to some operations in algebras of functions and are useful in solving differential and integral equations. Especially, they may represent models of operators with particular properties and have important links with geometric function theory or dynamics. To the best of our knowledge, there has been no such new descriptions for differences of integral-type operators acting on β-Bloch-Orlicz spaces. Hence these characterizations are in desired need of response. Motivated by this, we will treat the differences of integral-type operators acting from α-Bloch space to β-Bloch-Orlicz space in this paper, which shows the interesting role of Bloch-Orlicz spaces. The outline of the paper is as follows, the boundedness of $C_\phi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$ is investigated in Section 2 and then its essential norm is estimated in Section 3. Finally, some corollaries and an example of application are presented in Section 4.

2. The boundedness of $C_\phi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$

In this section, we will present several equivalent characterizations for the boundedness of $C_\phi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$. For $a \in \mathbb{D}$, define two test functions as below:

$$f_a(z) = \int_0^z \frac{(1-|a|^2)^\alpha}{(1-\bar{a}t)^{2\alpha}}dt,$$

$$\hat{f}_a(z) = \int_0^z \frac{(1-|a|^2)^\alpha}{(1-\bar{a}t)^{2\alpha}} \cdot \frac{a-t}{1-\bar{a}t}dt.$$

It turns out that

$$\|\hat{f}_a\|_{\mathscr{B}^\alpha} \preceq \|f_a\|_{\mathscr{B}^\alpha} = 1.$$

Moreover, by the direct calculations, it yields that

$$f_a'(z) = \frac{(1-|a|^2)^\alpha}{(1-\bar{a}z)^{2\alpha}} \text{ and } \hat{f}_a'(z) = \frac{(1-|a|^2)^\alpha}{(1-\bar{a}z)^{2\alpha}} \cdot \frac{a-z}{1-\bar{a}z}.$$

For our further use, we denote two notations

$$\mathscr{T}_\alpha^\beta(g\phi)(z) = \frac{\mu_\beta^\varphi(z)g(z)}{(1-|\phi(z)|^2)^\alpha}, \ \mathscr{T}_\alpha^\beta(h\psi)(z) = \frac{\mu_\beta^\varphi(z)h(z)}{(1-|\psi(z)|^2)^\alpha}.$$

LEMMA 2.1. *Let $0<\alpha<\infty$, then for each $f \in \mathscr{B}^\alpha$, it holds that*

$$|(1-|z|^2)^\alpha f'(z) - (1-|w|^2)^\alpha f'(w)| \leqslant C\|f\|_{\mathscr{B}^\alpha}\rho(z,w)$$

for all $z,w \in \mathbb{D}$.

Proof. For $f \in \mathscr{B}^\alpha$, it follows that $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty$. That is to say $f' \in H_\alpha^\infty$, and $\|f'\|_{H_\alpha^\infty} \preceq \|f\|_{\mathscr{B}^\alpha}$. Using [4, Lemma 3.2], it yields that

$$\begin{aligned}&|(1-|z|^2)^\alpha f'(z) - (1-|w|^2)^\alpha f'(w)|\\ &\preceq \|f'\|_{H_\alpha^\infty}\rho(z,w) \preceq \|f\|_{\mathscr{B}^\alpha}\rho(z,w). \quad \square\end{aligned}$$

LEMMA 2.2. *Let $0<\alpha,\beta<\infty$ and $\varphi:[0,\infty)\to[0,\infty)$ be an $\mathscr{N}$-function. Suppose $\phi,\psi \in S(\mathbb{D})$ and $g,h \in H(\mathbb{D})$, then the following three inequalities hold,*

$$\begin{aligned}&(i)\ \sup_{z\in\mathbb{D}}\left|\mathscr{T}_\alpha^\beta(g\phi)(z)\right|\rho(z)\\ &\leqslant \sup_{w\in\mathbb{D}}\|(C_\phi^g - C_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi} + \sup_{w\in\mathbb{D}}\|(C_\phi^g - C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi}; \qquad (2.1)\\ &(ii)\ \sup_{z\in\mathbb{D}}\left|\mathscr{T}_\alpha^\beta(h\psi)(z)\right|\rho(z)\\ &\leqslant \sup_{w\in\mathbb{D}}\|(C_\phi^g - C_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi} + \sup_{w\in\mathbb{D}}\|(C_\phi^g - C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi}; \qquad (2.2)\\ &(iii)\ \sup_{z\in\mathbb{D}}\left|\mathscr{T}_\alpha^\beta(g\phi)(z) - \mathscr{T}_\alpha^\beta(h\psi)(z)\right|\\ &\leqslant \sup_{w\in\mathbb{D}}\|(C_\phi^g - C_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi} + \sup_{w\in\mathbb{D}}\|(C_\phi^g - C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi}. \qquad (2.3)\end{aligned}$$

Proof. From the consequences in Lemma 1.2, we use the norm given in (1.1) to show this lemma. For $a \in \mathbb{D},$ we arrive at

$$\begin{aligned}
&\|(C_\phi^g - C_\psi^h) f_{\phi(a)}\|_{\mathscr{B}_\beta^\varphi} \\
&= \sup_{z\in\mathbb{D}} \mu_\beta^\varphi(z) |f'_{\phi(a)}(\phi(z))g(z) - f'_{\phi(a)}(\psi(z))h(z)| \\
&\geqslant \mu_\beta^\varphi(a)|f'_{\phi(a)}(\phi(a))g(a) - f'_{\phi(a)}(\psi(a))h(a)| \qquad (2.4)\\
&\geqslant \frac{\mu_\beta^\varphi(a)|g(a)|}{(1-|\phi(a)|^2)^\alpha} - \frac{(1-|\phi(a)|^2)^\alpha(1-|\psi(a)|^2)^\alpha}{|1-\overline{\phi(a)}\psi(a)|^{2\alpha}} \frac{\mu_\beta^\varphi(a)|h(a)|}{(1-|\psi(a)|^2)^\alpha} \\
&= |\mathscr{T}_\alpha^\beta(g\phi)(a)| - \frac{(1-|\phi(a)|^2)^\alpha(1-|\psi(a)|^2)^\alpha}{|1-\overline{\phi(a)}\psi(a)|^{2\alpha}} |\mathscr{T}_\alpha^\beta(h\psi)(a)|. \qquad (2.5)
\end{aligned}$$

Similarly, it turns out that

$$\begin{aligned}
&\|(C_\phi^g - C_\psi^h) \hat{f}_{\phi(a)}\|_{\mathscr{B}_\beta^\varphi} \\
&\geqslant \mu_\beta^\varphi(a)|\hat{f}'_{\phi(a)}(\phi(a))g(a) - \hat{f}'_{\phi(a)}(\psi(a))h(a)| \\
&= \mu_\beta^\varphi(a)|h(a)| \frac{(1-|\phi(a)|^2)^\alpha}{|1-\overline{\phi(a)}\psi(a)|^{2\alpha}} \rho(a) \\
&= \frac{(1-|\phi(a)|^2)^\alpha(1-|\psi(a)|^2)^\alpha}{|1-\overline{\phi(a)}\psi(a)|^{2\alpha}} \frac{\mu_\beta^\varphi(a)|h(a)|}{(1-|\psi(a)|^2)^\alpha} \rho(a) \\
&= \frac{(1-|\phi(a)|^2)^\alpha(1-|\psi(a)|^2)^\alpha}{|1-\overline{\phi(a)}\psi(a)|^{2\alpha}} |\mathscr{T}_\alpha^\beta(h\psi)(a)|\rho(a). \qquad (2.6)
\end{aligned}$$

Putting (2.6) into (2.5) we deduce that

$$\begin{aligned}
&\left|\mathscr{T}_\alpha^\beta(g\phi)(a)\right|\rho(a) \\
&\leqslant \|(C_\phi^g - C_\psi^h) f_{\phi(a)}\|_{\mathscr{B}_\beta^\varphi}\rho(a) \\
&\quad + \frac{(1-|\phi(a)|^2)^\alpha(1-|\psi(a)|^2)^\alpha}{|1-\overline{\phi(a)}\psi(a)|^{2\alpha}} |\mathscr{T}_\alpha^\beta(h\psi)(a)|\rho(a) \\
&\leqslant \|(C_\phi^g - C_\psi^h) f_{\phi(a)}\|_{\mathscr{B}_\beta^\varphi} + \|(C_\phi^g - C_\psi^h) \hat{f}_{\phi(a)}\|_{\mathscr{B}_\beta^\varphi}, \qquad (2.7)
\end{aligned}$$

where the last inequality follows from $\rho(a) \leqslant 1.$ Analogously, we deduce that

$$\left|\mathscr{T}_\alpha^\beta(h\psi)(a)\right|\rho(a) \leqslant \|(C_\phi^g - C_\psi^h) f_{\psi(a)}\|_{\mathscr{B}_\beta^\varphi} + \|(C_\phi^g - C_\psi^h) \hat{f}_{\psi(a)}\|_{\mathscr{B}_\beta^\varphi}. \qquad (2.8)$$

Taking the supremum about $a \in \mathbb{D}$ in (2.7) and (2.8), we conclude that

$$\begin{aligned}
&(i)\ \sup_{a\in\mathbb{D}} \left|\mathscr{T}_\alpha^\beta(g\phi)(a)\right|\rho(a) \\
&\leqslant \sup_{a\in\mathbb{D}}\left(\|(C_\phi^g - C_\psi^h)f_{\phi(a)}\|_{\mathscr{B}_\beta^\varphi} + \|(C_\phi^g - C_\psi^h)\hat{f}_{\phi(a)}\|_{\mathscr{B}_\beta^\varphi}\right) \\
&\leqslant \sup_{w\in\mathbb{D}}\left(\|(C_\phi^g - C_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi} + \|(C_\phi^g - C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi}\right). \qquad (2.9) \\
&(ii)\ \sup_{a\in\mathbb{D}} \left|\mathscr{T}_\alpha^\beta(h\psi)(a)\right|\rho(a) \\
&\leqslant \sup_{w\in\mathbb{D}}\left(\|(C_\phi^g - C_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi} + \|(C_\phi^g - C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi}\right). \qquad (2.10)
\end{aligned}$$

On the other hand, we change (2.4) into

$$\begin{aligned}
&\|(C_\phi^g - C_\psi^h)f_{\phi(a)}\|_{\mathscr{B}_\beta^\varphi} \\
&\geqslant \mu_\beta^\varphi(a)\left|\frac{g(a)}{(1-|\phi(a)|^2)^\alpha} - \frac{h(a)(1-|\phi(a)|^2)^\alpha}{(1-\overline{\phi(a)}\psi(a))^{2\alpha}}\right| \\
&\geqslant \left|\mathscr{T}_\alpha^\beta(g\phi)(a) - \mathscr{T}_\alpha^\beta(h\psi)(a)\right| - \frac{\mu_\beta^\varphi(a)|h(a)|}{(1-|\psi(a)|^2)^\alpha} \\
&\quad \cdot\left|(1-|\phi(a)|^2)^\alpha f'_{\phi(a)}(\phi(a)) - (1-|\psi(a)|^2)^\alpha f'_{\phi(a)}(\psi(a))\right| \\
&= \left|\mathscr{T}_\alpha^\beta(g\phi)(a) - \mathscr{T}_\alpha^\beta(h\psi)(a)\right| - \left|\mathscr{T}_\alpha^\beta(h\psi)(a)\right| \\
&\quad \cdot\left|(1-|\phi(a)|^2)^\alpha f'_{\phi(a)}(\phi(a)) - (1-|\psi(a)|^2)^\alpha f'_{\phi(a)}(\psi(a))\right| \\
&\succeq \left|\mathscr{T}_\alpha^\beta(g\phi)(a) - \mathscr{T}_\alpha^\beta(h\psi)(a)\right| - C\left|\mathscr{T}_\alpha^\beta(h\psi)(a)\right|\rho(a), \qquad (2.11)
\end{aligned}$$

the last inequality in (2.11) is due to Lemma 2.1. The above inequalities yield

$$\begin{aligned}
&(iii)\ \sup_{a\in\mathbb{D}}\left|\mathscr{T}_\alpha^\beta(g\phi)(a) - \mathscr{T}_\alpha^\beta(h\psi)(a)\right| \\
&\preceq \sup_{a\in\mathbb{D}}\left(\|(C_\phi^g - C_\psi^h)f_{\phi(a)}\|_{\mathscr{B}_\beta^\varphi} + \left|\mathscr{T}_\alpha^\beta(h\psi)(a)\right|\rho(a)\right) \\
&\preceq \sup_{w\in\mathbb{D}}\left(\|(C_\phi^g - C_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi} + \|(C_\phi^g - C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi}\right). \qquad (2.12)
\end{aligned}$$

(2.9), (2.10) together with (2.12) verify the inequalities (2.1)–(2.3) are true. This completes the proof. $\square$

LEMMA 2.3. *Let $0<\alpha,\beta<\infty$ and $\varphi:[0,\infty)\to[0,\infty)$ be an $\mathscr{N}$-function. Suppose that $\phi,\psi\in S(\mathbb{D})$ and $g,h\in H(\mathbb{D})$, then the following statements hold,*

$$(i)\ \sup_{w\in\mathbb{D}}\|(C_\phi^g-C_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi}\preceq\sup_{n\in\mathbb{N}_0}n^\alpha\|g\phi^n-h\psi^n\|_{\mu_\beta^\varphi};$$
$$(ii)\ \sup_{w\in\mathbb{D}}\|(C_\phi^g-C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi}\preceq\sup_{n\in\mathbb{N}_0}n^\alpha\|g\phi^n-h\psi^n\|_{\mu_\beta^\varphi}.$$

Proof. Recall that

$$\frac{1}{(1-\bar{a}t)^{2\alpha}}=\sum_{k=0}^\infty\frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!}(\bar{a}t)^k.$$

Integrating the above formulae we express f_a into Maclaurin expansion as below

$$\begin{aligned}f_a(z)&=(1-|a|^2)^\alpha\int_0^z\sum_{k=0}^\infty\frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!}(\bar{a}t)^kdt\\&=(1-|a|^2)^\alpha\sum_{k=0}^\infty\frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)(k+1)!}\bar{a}^kz^{k+1},\ z\in\mathbb{D}.\end{aligned}\tag{2.13}$$

On the other hand, it turn out that

$$\begin{aligned}&\frac{(1-|a|^2)^\alpha}{(1-\bar{a}t)^{2\alpha}}\cdot\frac{a-t}{1-\bar{a}t}\\&=(1-|a|^2)^\alpha\left(\sum_{k=0}^\infty\frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!}\bar{a}^kt^k\right)\left(\frac{a(1-\bar{a}t)+|a|^2t-t}{1-\bar{a}t}\right)\\&=(1-|a|^2)^\alpha\left(\sum_{k=0}^\infty\frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!}\bar{a}^kt^k\right)\left(a-(1-|a|^2)\frac{t}{1-\bar{a}t}\right)\\&=(1-|a|^2)^\alpha\left(\sum_{k=0}^\infty\frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!}\bar{a}^kt^k\right)\left(a-(1-|a|^2)\sum_{k=0}^\infty\bar{a}^kt^{k+1}\right)\\&=a(1-|a|^2)^\alpha\left(\sum_{k=0}^\infty\frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!}\bar{a}^kt^k\right)\\&\quad-(1-|a|^2)^{\alpha+1}\left(\sum_{k=0}^\infty\frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!}\bar{a}^kt^k\right)\left(\sum_{k=0}^\infty\bar{a}^kt^{k+1}\right)\\&=a(1-|a|^2)^\alpha\left(\sum_{k=0}^\infty\frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!}\bar{a}^kt^k\right)\\&\quad-(1-|a|^2)^{\alpha+1}\sum_{k=1}^\infty\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+2\alpha)}{\Gamma(2\alpha)l!}\right)\bar{a}^{k-1}t^k.\end{aligned}$$

Integrating the above, we obtain the Maclaurin expansion of $\hat{f}_a$ as below

$$\begin{aligned}
\hat{f}_a(z) &= \int_0^z \frac{(1-|a|^2)^\alpha}{(1-\bar{a}t)^{2\alpha}} \cdot \frac{a-t}{1-\bar{a}t} dt \\
&= a f_a(z) - (1-|a|^2)^{\alpha+1} \int_0^z \sum_{k=1}^{\infty} \left(\sum_{l=0}^{k-1} \frac{\Gamma(l+2\alpha)}{\Gamma(2\alpha) l!} \right) \bar{a}^{k-1} t^k dt \\
&= a f_a(z) - (1-|a|^2)^{\alpha+1} \sum_{k=1}^{\infty} \frac{1}{k+1} \left(\sum_{l=0}^{k-1} \frac{\Gamma(l+2\alpha)}{\Gamma(2\alpha) l!} \right) \bar{a}^{k-1} z^{k+1}. \qquad (2.14)
\end{aligned}$$

By the Maclaurin expansion in (2.13), we formulate that

$$\begin{aligned}
&\|(C_\phi^g - C_\psi^h) f_a\|_{\mathscr{B}_\beta^\varphi} \\
&\leqslant (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)(k+1)!} |\bar{a}|^k \|(C_\phi^g - C_\psi^h) z^{k+1}\|_{\mathscr{B}_\beta^\varphi} \\
&= (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)(k+1)!} |\bar{a}|^k \sup_{z\in\mathbb{D}} \left[\mu_\beta^\varphi(z) |(k+1)\phi^k(z) g(z) - (k+1)\psi^k(z) h(z)| \right] \\
&= (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha) k!} |\bar{a}|^k \sup_{z\in\mathbb{D}} \left[\mu_\beta^\varphi(z) |\phi^k(z) g(z) - \psi^k(z) h(z)| \right] \\
&= (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha) k!} |\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \qquad (2.15) \\
&\preceq (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha) k!} |\bar{a}|^k \left(\frac{2k}{k+1} \right)^\alpha \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
&\preceq (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha) k!} |\bar{a}|^k (k+1)^{-\alpha} \cdot \sup_{n\in\mathbb{N}_0} n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}. \qquad (2.16)
\end{aligned}$$

By Stirling's formula, it follows that

$$\begin{aligned}
&\frac{\Gamma(k+\alpha)}{k!\Gamma(\alpha)} \approx (k+1)^{\alpha-1}, \text{ as } k \to \infty; \\
&\frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha) k!} (k+1)^{-\alpha} \approx (k+1)^{\alpha-1}, \text{ as } k \to \infty.
\end{aligned}$$

Therefore it yields that

$$\begin{aligned}
\frac{1}{(1-|a|)^\alpha} &= \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{k!\Gamma(\alpha)} |a|^k \\
&\approx \sum_{k=0}^{\infty} (k+1)^{\alpha-1} |a|^k \\
&\approx \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha) k!} |\bar{a}|^k (k+1)^{-\alpha}. \qquad (2.17)
\end{aligned}$$

Putting (2.17) into (2.16), we deduce that

$$\|(C_\phi^g - C_\psi^h) f_a\|_{\mathscr{B}_\beta^\varphi} \preceq \sup_{n\in\mathbb{N}_0} n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}. \tag{2.18}$$

Using the Maclaurin expansion in (2.14), it turns out that

$$\begin{aligned}
&\|(C_\phi^g - C_\psi^h)\hat{f}_a\|_{\mathscr{B}_\beta^\varphi} \\
\preceq\ &\|(C_\phi^g - C_\psi^h) f_a\|_{\mathscr{B}_\beta^\varphi} \\
&+(1-|a|^2)^{\alpha+1}\sum_{k=1}^{\infty}\frac{1}{k+1}\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+2\alpha)}{\Gamma(2\alpha)l!}\right)|\bar{a}|^{k-1}\|(C_\phi^g - C_\psi^h)z^{k+1}\|_{\mathscr{B}_\beta^\varphi} \\
=\ &\|(C_\phi^g - C_\psi^h) f_a\|_{\mathscr{B}_\beta^\varphi} + (1-|a|^2)^{\alpha+1}\sum_{k=1}^{\infty}\frac{1}{k+1}\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+2\alpha)}{\Gamma(2\alpha)l!}\right)|\bar{a}|^{k-1} \\
&\cdot\sup_{z\in\mathbb{D}}\left[(k+1)\mu_\beta^\varphi(z)|g(z)\phi^k(z) - h(z)\psi^k(z)|\right] \\
=\ &\|(C_\phi^g - C_\psi^h) f_a\|_{\mathscr{B}_\beta^\varphi} \\
&+(1-|a|^2)^{\alpha+1}\sum_{k=1}^{\infty}\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+2\alpha)}{\Gamma(2\alpha)l!}\right)|\bar{a}|^{k-1}\|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi}.
\end{aligned} \tag{2.19}$$

Furthermore by Stirling's formula again, we obtain

$$\sum_{l=0}^{k-1}\frac{\Gamma(l+2\alpha)}{\Gamma(2\alpha)l!} \approx \sum_{l=0}^{k-1}(l+1)^{2\alpha-1} \approx k^{2\alpha}, \text{ as } k\to\infty.$$

The reason for the second equivalent display is exhibited as below. For simplicity, we denote $a_k = \sum_{l=0}^{k-1}(l+1)^{2\alpha-1}$, and employ the Binomial Theorem to formulate

$$\begin{aligned}
&k^{2\alpha} - (k-1)^{2\alpha} \\
=\ &(k-1+1)^{2\alpha} - (k-1)^{2\alpha} \\
=\ &(k-1)^{2\alpha} + (2\alpha)(k-1)^{2\alpha-1} + \cdots + 1 - (k-1)^{2\alpha} \\
=\ &(2\alpha)(k-1)^{2\alpha-1} + \cdots + 1,
\end{aligned}$$

here the sum on the right-hand is not necessarily finite if α is not an integer, which does not essentially affect the following estimates. We deduce from Stole-Cesáro formulae that

$$\begin{aligned}
\lim_{k\to\infty}\frac{a_k}{k^{2\alpha}} &= \lim_{k\to\infty}\frac{a_k - a_{k-1}}{k^{2\alpha} - (k-1)^{2\alpha}} \\
&= \lim_{k\to\infty}\frac{k^{2\alpha-1}}{k^{2\alpha} - (k-1)^{2\alpha}} \\
&= \lim_{k\to\infty}\frac{k^{2\alpha-1}}{(2\alpha)(k-1)^{2\alpha-1} + \cdots + 1} \\
&= \frac{1}{2\alpha}.
\end{aligned}$$

The above facts entail (2.19) into

$$\begin{aligned}
&\|(C_\phi^g - C_\psi^h)\hat{f}_a\|_{\mathscr{B}_\beta^\varphi} \\
\preceq\ &\|(C_\phi^g - C_\psi^h)f_a\|_{\mathscr{B}_\beta^\varphi} + (1-|a|^2)^{\alpha+1}\sum_{k=1}^{\infty} k^{2\alpha}|\bar{a}|^{k-1}\|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \qquad (2.20)\\
\leqslant\ &\|(C_\phi^g - C_\psi^h)f_a\|_{\mathscr{B}_\beta^\varphi} \\
&+(1-|a|^2)^{\alpha+1}\sum_{k=1}^{\infty} k^{2\alpha}|\bar{a}|^{k-1}k^{-\alpha}\cdot \sup_{n\in\mathbb{N}_0} n^\alpha\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \\
\preceq\ &\|(C_\phi^g - C_\psi^h)f_a\|_{\mathscr{B}_\beta^\varphi} \\
&+(1-|a|^2)^{\alpha+1}\sum_{k=1}^{\infty} k^{\alpha}|\bar{a}|^{k-1}\cdot \sup_{n\in\mathbb{N}_0} n^\alpha\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \\
\preceq\ &\|(C_\phi^g - C_\psi^h)f_a\|_{\mathscr{B}_\beta^\varphi} \\
&+(1-|a|^2)^{\alpha+1}\frac{1}{(1-|a|)^{\alpha+1}}\cdot \sup_{n\in\mathbb{N}_0} n^\alpha\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \\
\preceq\ &\sup_{n\in\mathbb{N}_0} n^\alpha\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}. \qquad (2.21)
\end{aligned}$$

After that, we take the supremum about $a\in\mathbb{D}$ in (2.18) and (2.21), which completes the proof. $\square$

In this section, our main result is presented as below.

THEOREM 2.4. *Let $0<\alpha,\beta<\infty$ and $\varphi:[0,\infty)\to[0,\infty)$ be an $\mathscr{N}$-function. Suppose $\phi,\psi\in S(\mathbb{D})$ and $g,h\in H(\mathbb{D})$, then the following statements are equivalent,*

(i) $C_\phi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$ *is bounded;*

(ii)

$$\sup_{z\in\mathbb{D}}|\mathscr{T}_\alpha^\beta(g\phi)(z)|\rho(z) + \sup_{z\in\mathbb{D}}\left|\mathscr{T}_\alpha^\beta(g\phi)(z) - \mathscr{T}_\alpha^\beta(h\psi)(z)\right| < \infty,$$
$$\sup_{z\in\mathbb{D}}|\mathscr{T}_\alpha^\beta(h\psi)(z)|\rho(z) + \sup_{z\in\mathbb{D}}\left|\mathscr{T}_\alpha^\beta(g\phi)(z) - \mathscr{T}_\alpha^\beta(h\psi)(z)\right| < \infty;$$

(iii)

$$\sup_{w\in\mathbb{D}}\|(C_\phi^g - C_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi} + \sup_{w\in\mathbb{D}}\|(C_\phi^g - C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi} < \infty;$$

(iv)

$$\sup_{n\in\mathbb{N}_0} n^\alpha\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} < \infty.$$

Proof. The implications $(iv) \Rightarrow (iii) \Rightarrow (ii)$ follow from Lemma 2.3 and Lemma 2.2, respectively. We only need to prove $(i) \Rightarrow (iv)$ and $(ii) \Rightarrow (i)$.

$(i) \Rightarrow (iv)$. Suppose that $C_\phi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$ is bounded. Concerning the monomial function $z^{n+1} \in \mathscr{B}^\alpha$, we recall that $\|z^{n+1}\|_{\mathscr{B}^\alpha} \approx (n+1)^{1-\alpha}$ from [16, Section 2(6)]. And then we conclude that

$$\begin{aligned}
\infty &> \|C_\phi^g - C_\psi^h\|_{\mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi} \\
&\succeq \left\| (C_\phi^g - C_\psi^h) \frac{z^{n+1}}{\|z^{n+1}\|_{\mathscr{B}^\alpha}} \right\|_{\mathscr{B}_\beta^\varphi} \\
&\succeq (n+1)^{\alpha-1} \sup_{z\in\mathbb{D}} \mu_\beta^\varphi(z) |(n+1)g(z)\phi^n(z) - (n+1)h(z)\psi^n(z)| \\
&= (n+1)^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \\
&\approx n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}.
\end{aligned}$$

The above formulas imply

$$\sup_{n\in\mathbb{N}_0} n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \preceq \|C_\phi^g - C_\psi^h\|_{\mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi} < \infty,$$

which entails the statement $(i) \Rightarrow (iv)$.

$(ii) \Rightarrow (i)$. For any $f \in \mathscr{B}^\alpha$, we employ Lemma 2.1 to show that

$$\begin{aligned}
&\|(C_\phi^g - C_\psi^h) f\|_{\mathscr{B}_\beta^\varphi} \\
&= \sup_{z\in\mathbb{D}} \mu_\beta^\varphi(z) |g(z) f'(\phi(z)) - h(z) f'(\psi(z))| \\
&\preceq \sup_{z\in\mathbb{D}} \left| \mathscr{T}_\alpha^\beta (g\phi)(z) \right| \left| (1-|\phi(z)|^2)^\alpha f'(\phi(z)) - (1-|\psi(z)|^2)^\alpha f'(\psi(z)) \right| \\
&\quad + \sup_{z\in\mathbb{D}} (1-|\psi(z)|^2)^\alpha |f'(\psi(z))| \left| \mathscr{T}_\alpha^\beta (g\phi)(z) - \mathscr{T}_\alpha^\beta (h\psi)(z) \right| \\
&\preceq \sup_{z\in\mathbb{D}} |\mathscr{T}_\alpha^\beta (g\phi)(z)| \rho(z) + \sup_{z\in\mathbb{D}} \left| \mathscr{T}_\alpha^\beta (g\phi)(z) - \mathscr{T}_\alpha^\beta (h\psi)(z) \right| < \infty.
\end{aligned} \tag{2.22}$$

Analogously to (2.22), we can also obtain that

$$\begin{aligned}
&\|(C_\phi^g - C_\psi^h) f\|_{\mathscr{B}_\beta^\varphi} \\
&\preceq \sup_{z\in\mathbb{D}} |\mathscr{T}_\alpha^\beta (h\psi)(z)| \rho(z) + \sup_{z\in\mathbb{D}} \left| \mathscr{T}_\alpha^\beta (g\phi)(z) - \mathscr{T}_\alpha^\beta (h\psi)(z) \right| < \infty.
\end{aligned} \tag{2.23}$$

The two inequalities (2.22) and (2.23) imply that each one of conditions (ii) can ensure the boundedness of $C_\phi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$. This finishes the proof. $\square$

3. The essential norm of $C_\phi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$

In this section, we deduce several estimations for essential norm of $C_\phi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$. Firstly, we present some parallel results with Lemma 2.2 as follows.

LEMMA 3.1. *Let $0 < \alpha, \beta < \infty$ and $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathscr{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$, then the following inequalities hold,*

$$(i)\ \lim_{r\to 1}\sup_{|\phi(z)|>r}\left|\mathscr{T}_\alpha^\beta(g\phi)(z)\right|\rho(z) \preceq \limsup_{|w|\to 1}\|(C_\phi^g - C_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi} + \limsup_{|w|\to 1}\|(C_\phi^g - C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi};$$

$$(ii)\ \lim_{r\to 1}\sup_{|\psi(z)|>r}\left|\mathscr{T}_\alpha^\beta(h\psi)(z)\right|\rho(z) \preceq \limsup_{|w|\to 1}\|(C_\phi^g - C_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi} + \limsup_{|w|\to 1}\|(C_\phi^g - C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi};$$

$$(iii)\ \lim_{r\to 1}\sup_{\min\{|\phi(z)|,|\psi(z)|\}>r}\left|\mathscr{T}_\alpha^\beta(g\phi)(z) - \mathscr{T}_\alpha^\beta(h\psi)(z)\right| \preceq \limsup_{|w|\to 1}\|(C_\phi^g - C_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi} + \limsup_{|w|\to 1}\|(C_\phi^g - C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi}.$$

Proof. These results can be deduced directly from the inequalities (2.7), (2.8) and (2.11) in Lemma 2.2. □

LEMMA 3.2. *Let $0 < \alpha, \beta < \infty$ and $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathscr{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$ such that the operator $C_\phi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$ is bounded, then the following statements hold,*

$$(i)\ \limsup_{|w|\to 1}\|(C_\phi^g - C_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi} \preceq \limsup_{n\to\infty} n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}; \tag{3.1}$$

$$(ii)\ \limsup_{|w|\to 1}\|(C_\phi^g - C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi} \preceq \limsup_{n\to\infty} n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}. \tag{3.2}$$

Proof. For any $a \in \mathbb{D}$ and each positive integer N, employing (2.15) we obtain

$$\begin{aligned}
&\|(C_\phi^g - C_\psi^h)f_a\|_{\mathscr{B}_\beta^\varphi}\\
&\leqslant (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!}|\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi}\\
&\leqslant (1-|a|^2)^\alpha \sum_{k=0}^{N} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!}|\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi}\\
&\quad + (1-|a|^2)^\alpha \sum_{k=N+1}^{\infty} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!}|\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi}.
\end{aligned} \tag{3.3}$$

We denote two displays as below,

$$J_1 := (1-|a|^2)^\alpha \sum_{k=0}^{N} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!} |\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi},$$

$$J_2 := (1-|a|^2)^\alpha \sum_{k=N+1}^{\infty} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!} |\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi}.$$

For $k = 0,1,\cdots,N$, choosing $z^{k+1} \in \mathscr{B}^\alpha$ and using the boundedness of $C_\phi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$, it turns out that $\|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} < \infty$. Hence

$$\limsup_{|a|\to 1} J_1 = 0. \tag{3.4}$$

On the other hand, it follows from (2.17) that

$$\begin{aligned}
J_2 &= (1-|a|^2)^\alpha \sum_{k=N+1}^{\infty} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!} |\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
&\leqslant (1-|a|^2)^\alpha \sum_{k=N+1}^{\infty} \frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)k!} |\bar{a}|^k k^{-\alpha} \cdot \sup_{n\geqslant N+1} n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \\
&\preceq \sup_{n\geqslant N+1} n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}.
\end{aligned}$$

Furthermore, letting $|a| \to 1$ in the above inequality, it leads to

$$\limsup_{|a|\to 1} J_2 \preceq \sup_{n\geqslant N+1} n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}. \tag{3.5}$$

Putting (3.4) and (3.5) into (3.3) and letting $n \to \infty$, we arrive at (3.1). Similarly, by (2.20), we conclude that

$$\begin{aligned}
&\|(C_\phi^g - C_\psi^h)\hat{f}_a\|_{\mathscr{B}_\beta^\varphi} \\
&\preceq \|(C_\phi^g - C_\psi^h)f_a\|_{\mathscr{B}_\beta^\varphi} + (1-|a|^2)^{\alpha+1} \sum_{k=1}^{\infty} k^{2\alpha} |\bar{a}|^{k-1} \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
&\leqslant \|(C_\phi^g - C_\psi^h)f_a\|_{\mathscr{B}_\beta^\varphi} + (1-|a|^2)^{\alpha+1} \sum_{k=1}^{N} k^{2\alpha} |\bar{a}|^{k-1} \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
&\quad + (1-|a|^2)^{\alpha+1} \sum_{k=N+1}^{\infty} k^{2\alpha} |\bar{a}|^{k-1} \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
&\leqslant \|(C_\phi^g - C_\psi^h)f_a\|_{\mathscr{B}_\beta^\varphi} + (1-|a|^2)^{\alpha+1} \sum_{k=1}^{N} k^{2\alpha} |\bar{a}|^{k-1} \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
&\quad + (1-|a|^2)^{\alpha+1} \sum_{k=N+1}^{\infty} k^{2\alpha} |\bar{a}|^{k-1} k^{-\alpha} \cdot \sup_{n\geqslant N+1} n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \\
&\preceq \|(C_\phi^g - C_\psi^h)f_a\|_{\mathscr{B}_\beta^\varphi} + (1-|a|^2)^{\alpha+1} \sum_{k=1}^{N} k^{2\alpha} |\bar{a}|^{k-1} \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
&\quad + \sup_{n\geqslant N+1} n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}.
\end{aligned}$$

Letting $|a| \to 1$ in the above formulas, we can get that

$$\begin{aligned}&\limsup_{|a|\to 1} \|(C_\phi^g - C_\psi^h)\hat{f}_a\|_{\mathscr{B}_\beta^\varphi} \\ \preceq &\limsup_{|a|\to 1} \|(C_\phi^g - C_\psi^h) f_a\|_{\mathscr{B}_\beta^\varphi} + \sup_{n \geqslant N+1} n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}.\end{aligned}$$

The above inequality together with (3.1) verify (3.2). This ends the proof. $\square$

In the sequel, we employ the similar methods from [16, Section 4], we let $K_r f(z) = f(rz)$ for $r \in (0,1)$. And then K_r is a compact operator on α-Bloch space $\mathscr{B}^\alpha$ or the little α-Bloch space $\mathscr{B}_0^\alpha$ for any $\alpha > 0$, with $\|K_r\| \leqslant 1$. Here we cite an interesting lemma, which provides a sequence of compact operators acting on $\mathscr{B}^\alpha$ with $0 < \alpha < \infty$.

LEMMA 3.3. [23, Lemma 4.1–4.3] *Let $0 < \alpha < \infty$. Then there is a sequence $\{r_k\}$, with $0 < r_k < 1$ tending to 1, such that the compact operator*

$$L_n = \frac{1}{n} \sum_{k=1}^{n} K_{r_k}$$

on $\mathscr{B}_0^\alpha$ satisfies $\lim_{n\to\infty} \sup_{\|f\|_{\mathscr{B}^\alpha} \leqslant 1} \sup_{|z|\leqslant t} |((I - L_n)f)'(z)| = 0$ for any $t \in [0,1)$. Furthermore, this statement holds as well for the sequence of biadjoints L_n^{**} on $\mathscr{B}^\alpha$.

The following is our main theorem in this section.

THEOREM 3.4. *Let $0 < \alpha, \beta < \infty$ and $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathscr{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$ such that the operators C_ϕ^g, $C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$ are bounded, then the following equivalences hold,*

$$\begin{aligned}&\|C_\phi^g - C_\psi^h\|_{e, \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi} \\ \approx &\lim_{r\to 1} \sup_{|\phi(z)|>r} \left|\mathscr{T}_\alpha^\beta(g\phi)(z)\right| \rho(z) + \lim_{r\to 1} \sup_{|\psi(z)|>r} \left|\mathscr{T}_\alpha^\beta(h\psi)(z)\right| \rho(z) \\ &+ \lim_{r\to 1} \sup_{\min\{|\phi(z)|, |\psi(z)|\}>r} \left|\mathscr{T}_\alpha^\beta(g\phi)(z) - \mathscr{T}_\alpha^\beta(h\psi)(z)\right| \\ \approx &\limsup_{|w|\to 1} \|(C_\phi^g - C_\psi^h) f_w\|_{\mathscr{B}_\beta^\varphi} + \limsup_{|w|\to 1} \|(C_\phi^g - C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi} \\ \approx &\limsup_{n\to\infty} n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}.\end{aligned}$$

Proof. Firstly, the boundedness of $C_\phi^g : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$ and $C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$ yield that

$$M_g = \sup_{z\in\mathbb{D}} \mu_\beta^\varphi(z)|g(z)| < \infty \text{ and } M_h = \sup_{z\in\mathbb{D}} \mu_\beta^\varphi(z)|h(z)| < \infty.$$

Lemma 3.1 together with Lemma 3.2 verify that

$$\begin{aligned}
&\lim_{r\to1}\sup_{|\phi(z)|>r}\left|\mathscr{T}_\alpha^\beta(g\phi)(z)\right|\rho(z)+\lim_{r\to1}\sup_{|\psi(z)|>r}\left|\mathscr{T}_\alpha^\beta(h\psi)(z)\right|\rho(z)\\
&\quad+\lim_{r\to1}\sup_{\min\{|\phi(z)|,|\psi(z)|\}>r}\left|\mathscr{T}_\alpha^\beta(g\phi)(z)-\mathscr{T}_\alpha^\beta(h\psi)(z)\right|\\
&\preceq\limsup_{|w|\to1}\|(C_\phi^g-C_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi}+\limsup_{|w|\to1}\|(C_\phi^g-C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi}\\
&\preceq\limsup_{n\to\infty}n^\alpha\|g\phi^n-h\psi^n\|_{\mu_\beta^\varphi}.
\end{aligned}$$

Hereafter, we only need to show the following inequalities,

$$\begin{aligned}
&\limsup_{n\to\infty}n^\alpha\|g\phi^n-h\psi^n\|_{\mu_\beta^\varphi}\preceq\|C_\phi^g-C_\psi^h\|_{e,\mathscr{B}^\alpha\to\mathscr{B}_\beta^\varphi}\\
&\preceq\lim_{r\to1}\sup_{|\phi(z)|>r}\left|\mathscr{T}_\alpha^\beta(g\phi)(z)\right|\rho(z)+\lim_{r\to1}\sup_{|\psi(z)|>r}\left|\mathscr{T}_\alpha^\beta(h\psi)(z)\right|\rho(z)\\
&\quad+\lim_{r\to1}\sup_{\min\{|\phi(z)|,|\psi(z)|\}>r}\left|\mathscr{T}_\alpha^\beta(g\phi)(z)-\mathscr{T}_\alpha^\beta(h\psi)(z)\right|.
\end{aligned}\tag{3.6}$$

To prove the first inequality in (3.6), we choose a bounded sequence $f_n(z)=z^{n+1}/\|z^{n+1}\|_{\mathscr{B}^\alpha}$ in $\mathscr{B}^\alpha$, which converges to zero uniformly on every compact subset of $\mathbb{D}$. For any compact operator $K:\mathscr{B}^\alpha\to\mathscr{B}_\beta^\varphi$, it yields that $\lim_{n\to\infty}\|Kf_n\|_{\mathscr{B}_\beta^\varphi}=0$. Furthermore, we conclude that

$$\begin{aligned}
&\|C_\phi^g-C_\psi^h\|_{e,\mathscr{B}^\alpha\to\mathscr{B}_\beta^\varphi}\\
&\succeq\limsup_{n\to\infty}\inf_K\|(C_\phi^g-C_\psi^h-K)f_n\|_{\mathscr{B}_\beta^\varphi}\\
&\geqslant\limsup_{n\to\infty}\inf_K\left(\|(C_\phi^g-C_\psi^h)f_n\|_{\mathscr{B}_\beta^\varphi}-\|Kf_n\|_{\mathscr{B}_\beta^\varphi}\right)\\
&\geqslant\limsup_{n\to\infty}\|(C_\phi^g-C_\psi^h)f_n\|_{\mathscr{B}_\beta^\varphi}\\
&\succeq\limsup_{n\to\infty}(n+1)^\alpha\|g\phi^n-h\psi^n\|_{\mu_\beta^\varphi}\\
&\approx\limsup_{n\to\infty}n^\alpha\|g\phi^n-h\psi^n\|_{\mu_\beta^\varphi},
\end{aligned}$$

the second line from the bottom is due to $\|z^{n+1}\|_{\mathscr{B}^\alpha}\approx(n+1)^{1-\alpha}$.

Now we turn to the second inequality in (3.6). Let $\{L_n\}$ be the sequence of operators given in Lemma 3.3. Since each operator $L_n^{**}:\mathscr{B}^\alpha\to\mathscr{B}^\alpha$ is compact and $C_\phi^g-C_\psi^h:\mathscr{B}^\alpha\to\mathscr{B}_\beta^\varphi$ is bounded, thus $(C_\phi^g-C_\psi^h)L_n^{**}:\mathscr{B}^\alpha\to\mathscr{B}_\beta^\varphi$ is also compact. As a consequence, we have that

$$\begin{aligned}
&\|C^g_\phi - C^h_\psi\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\varphi_\beta}\\
\leqslant &\limsup_{n\to\infty}\|(C^g_\phi - C^h_\psi)-(C^g_\phi - C^h_\psi)L_n^{**}\|_{\mathscr{B}^\alpha\to\mathscr{B}^\varphi_\beta}\\
= &\limsup_{n\to\infty}\|(C^g_\phi - C^h_\psi)(I-L_n^{**})\|_{\mathscr{B}^\alpha\to\mathscr{B}^\varphi_\beta}\\
= &\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\|(C^g_\phi - C^h_\psi)(I-L_n^{**})f\|_{\mathscr{B}^\varphi_\beta}\\
= &\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{z\in\mathbb{D}}\mu^\varphi_\beta(z)\left|g(z)[(I-L_n^{**})f]'(\phi(z))-h(z)[(I-L_n^{**})f]'(\psi(z))\right|.
\end{aligned}$$

For an arbitrary $r\in(0,1)$, we denote

$$\begin{aligned}
&\mathbb{D}_1=\{z\in\mathbb{D}:\ |\phi(z)|\leqslant r,\ |\psi(z)|\leqslant r\},\ \mathbb{D}_2=\{z\in\mathbb{D}:\ |\phi(z)|\leqslant r,\ |\psi(z)|>r\},\\
&\mathbb{D}_3=\{z\in\mathbb{D}:|\phi(z)|>r,\ |\psi(z)|\leqslant r\},\ \mathbb{D}_4=\{z\in\mathbb{D}:\ |\phi(z)|>r,\ |\psi(z)|>r\};\\
&I_i:=\sup_{z\in\mathbb{D}_i}\mu^\varphi_\beta(z)\left|g(z)[(I-L_n^{**})f]'(\phi(z))-h(z)[(I-L_n^{**})f]'(\psi(z))\right|,
\end{aligned}$$

for $i=1,2,3,4.$ Then Cauchy's integral formula and Lemma 3.3 imply that

$$\begin{aligned}
&\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} I_1\\
\leqslant &\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{|\phi(z)|\leqslant r}(\mu^\varphi_\beta(z)|g(z)|)|[(I-L_n^{**})f]'(\phi(z))|\\
&+\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{|\psi(z)|\leqslant r}(\mu^\varphi_\beta(z)|h(z)|)|[(I-L_n^{**})f]'(\psi(z))|\\
\leqslant &M_g\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{|\phi(z)|\leqslant r}|[(I-L_n^{**})f]'(\phi(z))|\\
&+M_h\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{|\psi(z)|\leqslant r}|[(I-L_n^{**})f]'(\psi(z))|=0.
\end{aligned}\tag{3.7}$$

On the other hand, we formulate that

$$\begin{aligned}
&\mu^\varphi_\beta(z)\left|g(z)[(I-L_n^{**})f]'(\phi(z))-h(z)[(I-L_n^{**})f]'(\psi(z))\right|\\
\preceq &\frac{\mu^\varphi_\beta(z)|g(z)|}{(1-|\phi(z)|^2)^\alpha}|(1-|\phi(z)|^2)^\alpha[(I-L_n^{**})f]'(\phi(z))\\
&-(1-|\psi(z)|^2)^\alpha[(I-L_n^{**})f]'(\psi(z))|\\
&+(1-|\psi(z)|^2)^\alpha\left|[(I-L_n^{**})f]'(\psi(z))\right|\left|\mathscr{T}^\beta_\alpha(g\phi)(z)-\mathscr{T}^\beta_\alpha(h\psi)(z)\right|\\
\preceq &\left|\mathscr{T}^\beta_\alpha(g\phi)(z)\right|\rho(z)+(1-|\psi(z)|^2)^\alpha\left|[(I-L_n^{**})f]'(\psi(z))\right|\\
&\cdot\left|\mathscr{T}^\beta_\alpha(g\phi)(z)-\mathscr{T}^\beta_\alpha(h\psi)(z)\right|.
\end{aligned}\tag{3.8}$$

Analogously, we obtain that

$$\begin{aligned}
&\mu_\beta^\varphi(z)\left|g(z)[(I-L_n^{**})f]'(\phi(z))-h(z)[(I-L_n^{**})f]'(\psi(z))\right|\\
\preceq\, &|\mathscr{T}_\alpha^\beta(h\psi)(z)|\rho(z)+(1-|\phi(z)|^2)^\alpha\left|[(I-L_n^{**})f]'(\phi(z))\right|\\
&\cdot\left|\mathscr{T}_\alpha^\beta(g\phi)(z)-\mathscr{T}_\alpha^\beta(h\psi)(z)\right|. \qquad (3.9)
\end{aligned}$$

Since the operators $C_\phi^g, C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$ are bounded, hence $C_\phi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$ is bounded. Thus Theorem 2.4 ensures $\left|\mathscr{T}_\alpha^\beta(g\phi)(z)-\mathscr{T}_\alpha^\beta(h\psi)(z)\right| < \infty$. Employing Lemma 3.3 we can show that

$$\begin{aligned}
&\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} I_2\\
\leqslant\, &\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{|\psi(z)|>r}|\mathscr{T}_\alpha^\beta(h\psi)(z)|\rho(z)\\
&+\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{|\phi(z)|\leqslant r}(1-|\phi(z)|^2)^\alpha\left|[(I-L_n^{**})f]'(\phi(z))\right|\\
&\cdot\left|\mathscr{T}_\alpha^\beta(g\phi)(z)-\mathscr{T}_\alpha^\beta(h\psi)(z)\right|\\
\preceq\, &\sup_{|\psi(z)|>r}|\mathscr{T}_\alpha^\beta(h\psi)(z)|\rho(z). \qquad (3.10)
\end{aligned}$$

Similarly, employing (3.8) we deduce that

$$\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} I_3 \preceq \sup_{|\phi(z)|>r}|\mathscr{T}_\alpha^\beta(g\phi)(z)|\rho(z). \qquad (3.11)$$

Finally, we deduce from (3.8) that

$$\begin{aligned}
&\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} I_4\\
\preceq\, &\sup_{|\phi(z)|>r}|\mathscr{T}_\alpha^\beta(g\phi)(z)|\rho(z)\\
&+\|(I-L_n^{**})f\|_{\mathscr{B}^\alpha}\sup_{\min\{|\phi(z)|,|\psi(z)|\}>r}\left|\mathscr{T}_\alpha^\beta(g\phi)(z)-\mathscr{T}_\alpha^\beta(h\psi)(z)\right|\\
\preceq\, &\sup_{|\phi(z)|>r}|\mathscr{T}_\alpha^\beta(g\phi)(z)|\rho(z)\\
&+\sup_{\min\{|\phi(z)|,|\psi(z)|\}>r}\left|\mathscr{T}_\alpha^\beta(g\phi)(z)-\mathscr{T}_\alpha^\beta(h\psi)(z)\right|. \qquad (3.12)
\end{aligned}$$

Similarly, (3.9) entails that

$$\begin{aligned}
\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} I_4 \preceq &\sup_{|\psi(z)|>r}|\mathscr{T}_\alpha^\beta(h\psi)(z)|\rho(z)\\
&+\sup_{\min\{|\phi(z)|,|\psi(z)|\}>r}\left|\mathscr{T}_\alpha^\beta(g\phi)(z)-\mathscr{T}_\alpha^\beta(h\psi)(z)\right|. \qquad (3.13)
\end{aligned}$$

Consequently, we combine (3.7), (3.10), (3.11) and (3.12), (3.13) to find that

$$\begin{aligned}
&\|C_\phi^g - C_\psi^h\|_{e,\mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi} \\
&\preceq \lim_{r\to 1} \sup_{|\phi(z)|>r} \left|\mathscr{T}_\alpha^\beta(g\phi)(z)\right| \rho(z) + \lim_{r\to 1} \sup_{|\psi(z)|>r} \left|\mathscr{T}_\alpha^\beta(h\psi)(z)\right| \rho(z) \\
&\quad + \lim_{r\to 1} \sup_{\min\{|\phi(z)|,|\psi(z)|\}>r} \left|\mathscr{T}_\alpha^\beta(g\phi)(z) - \mathscr{T}_\alpha^\beta(h\psi)(z)\right|.
\end{aligned}$$

This ends all the proof for the essential norm estimation. □

At the end of this section, we give three equivalent characterizations for the compactness of $C_\phi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$.

THEOREM 3.5. *Let $0 < \alpha, \beta < \infty$ and $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathscr{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$ such that the operators C_ϕ^g, $C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$ are bounded, then the operator $C_\phi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$ is compact if and only if one of the following statements hold,*

$$\begin{aligned}
(i)\ &\lim_{r\to 1} \sup_{|\phi(z)|>r} \left|\mathscr{T}_\alpha^\beta(g\phi)(z)\right| \rho(z) + \lim_{r\to 1} \sup_{|\psi(z)|>r} \left|\mathscr{T}_\alpha^\beta(h\psi)(z)\right| \rho(z) \\
&\quad + \lim_{r\to 1} \sup_{\min\{|\phi(z)|,|\psi(z)|\}>r} \left|\mathscr{T}_\alpha^\beta(g\phi)(z) - \mathscr{T}_\alpha^\beta(h\psi)(z)\right| = 0; \\
(ii)\ &\limsup_{|w|\to 1} \|(C_\phi^g - C_\psi^h) f_w\|_{\mathscr{B}_\beta^\varphi} + \limsup_{|w|\to 1} \|(C_\phi^g - C_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi} = 0; \\
(iii)\ &\limsup_{n\to\infty} n^\alpha \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} = 0.
\end{aligned}$$

4. Some Corollaries

In this section, we illustrate some characterizations for the boundedness and essential norms of $L^g - L^h$, $V_\phi^g - V_\psi^h$ and $L_g - L_h$ acting from $\mathscr{B}^\alpha$ into $\mathscr{B}_\beta^\varphi$. Indeed, the remaining differences are special cases of $C_\phi^g - C_\psi^h$, and then the following corollaries can be easily verified from Theorem 2.4 and Theorem 3.4. At the same time, some examples are listed to show the efficiency of our results.

(1) Let $\phi = \psi = id$ be the identity maps in $C_\phi^g - C_\psi^h$, then $C_{id}^g - C_{id}^h = L^g - L^h$, and denote these two notations

$$\mathscr{T}_\alpha^\beta g(z) = \frac{\mu_\beta^\varphi(z) g(z)}{(1-|z|^2)^\alpha}, \quad \mathscr{T}_\alpha^\beta h(z) = \frac{\mu_\beta^\varphi(z) h(z)}{(1-|z|^2)^\alpha}.$$

In this special case, we find that $\rho(z) = \rho(\phi(z), \psi(z)) = 0$. And we can only use the test function f_w to describe the boundedness and essential norm of $L^g - L^h : \mathscr{B}^\alpha \to \mathscr{B}_\beta^\varphi$.

COROLLARY 4.1. *Let* $0 < \alpha, \beta < \infty$ *and* $\varphi : [0,\infty) \to [0,\infty)$ *be an* $\mathscr{N}$ *-function. Suppose* $g, h \in H(\mathbb{D})$*, then the followings are equivalent,*

(*i*) $L^g - L^h : \mathscr{B}^\alpha \to \mathscr{B}^\varphi_\beta$ *is bounded;*

(*ii*)

$$\sup_{z\in\mathbb{D}} \left| \mathscr{T}^\beta_\alpha g(z) - \mathscr{T}^\beta_\alpha h(z) \right| < \infty;$$

(*iii*)

$$\sup_{w\in\mathbb{D}} \|(L^g - L^h) f_w\|_{\mathscr{B}^\varphi_\beta} < \infty;$$

(*iv*)

$$\sup_{n\in\mathbb{N}_0} n^\alpha \|gz^n - hz^n\|_{\mu^\varphi_\beta} < \infty.$$

COROLLARY 4.2. *Let* $0 < \alpha, \beta < \infty$ *and* $\varphi : [0,\infty) \to [0,\infty)$ *be an* $\mathscr{N}$ *-function. Suppose* $g, h \in H(\mathbb{D})$ *such that the operators* L^g, $L^h : \mathscr{B}^\alpha \to \mathscr{B}^\varphi_\beta$ *are bounded, then the following equivalences hold,*

$$\begin{aligned} &\|L^g - L^h\|_{e,\mathscr{B}^\alpha \to \mathscr{B}^\varphi_\beta} \\ \approx\; &\lim_{r\to 1} \sup_{|z|>r} \left| \mathscr{T}^\beta_\alpha g(z) - \mathscr{T}^\beta_\alpha h(z) \right| \\ \approx\; &\limsup_{|w|\to 1} \|(L^g - L^h) f_w\|_{\mathscr{B}^\varphi_\beta} \\ \approx\; &\limsup_{n\to\infty} n^\alpha \|gz^n - hz^n\|_{\mu^\varphi_\beta}. \end{aligned}$$

Next we apply the above two corollaries into the following example.

EXAMPLE 4.3. For $\alpha > 0$ and $\beta \geqslant 1$, we consider an $\mathscr{N}$ - function $\varphi_\alpha(t) = t^{1/\alpha}$ and $g = z^2 \in H(\mathbb{D})$ and $h = \frac{1}{2}z^2 \in H(\mathbb{D})$, and then we formulate the characterizations for the boundedness and the essential norm of $L^{z^2} - L^{\frac{1}{2}z^2} : \mathscr{B}^\alpha \to \mathscr{B}^{\varphi_\alpha}_\beta$ in details. It is obvious that $\mu^{\varphi_\alpha}_\beta(z) = (1-|z|^2)^{\alpha\beta}$ and

$$\mathscr{T}^\beta_\alpha g(z) = \frac{(1-|z|^2)^{\alpha\beta}}{(1-|z|^2)^\alpha} z^2, \qquad \mathscr{T}^\beta_\alpha h(z) = \frac{(1-|z|^2)^{\alpha\beta}}{(1-|z|^2)^\alpha} \frac{z^2}{2}.$$

Firstly, employing the statement in Corollary 4.1 (*ii*), we deduce that

$$\sup_{z\in\mathbb{D}} |\mathscr{T}^\beta_\alpha g(z) - \mathscr{T}^\beta_\alpha h(z)| = \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha(\beta-1)} \frac{|z|^2}{2} < \infty. \tag{4.1}$$

Similarly, we use the statement in Corollary 4.1 (iv) to obtain that

$$\begin{aligned}
&\sup_{n\in\mathbb{N}_0} n^{\alpha}\|z^{n+2}-\frac{1}{2}z^{n+2}\|_{\mu_\beta^\varphi}\\
=&\frac{1}{2}\sup_{n\in\mathbb{N}_0} n^{\alpha}\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha\beta}|z|^{n+2}\\
=&\frac{1}{2}\sup_{n\in\mathbb{N}_0} n^{\alpha}\left(1-\frac{1+\frac{n}{2}}{1+\frac{n}{2}+\alpha\beta}\right)^{\alpha\beta}\left(\frac{1+\frac{n}{2}}{1+\frac{n}{2}+\alpha\beta}\right)^{1+\frac{n}{2}}\\
\preceq&\sup_{n\in\mathbb{N}_0} n^{\alpha(1-\beta)}<\infty. \qquad (4.2)
\end{aligned}$$

Each one of (4.1) and (4.2) can verify $L^{z^2}-L^{\frac{1}{2}z^2}:\mathscr{B}^\alpha\to\mathscr{B}_\beta^{\varphi\alpha}$ is bounded.

Secondly, since $\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha(\beta-1)}|z|^2<\infty$, the operators $L^{z^2}:\mathscr{B}^\alpha\to\mathscr{B}_\beta^{\varphi\alpha}$ and $L^{\frac{1}{2}z^2}:\mathscr{B}^\alpha\to\mathscr{B}_\beta^{\varphi\alpha}$ are bounded. Consequently, Corollary 4.2 entails

$$\begin{aligned}
&\|L^{z^2}-L^{\frac{1}{2}z^2}\|_{e,\mathscr{B}^\alpha\to\mathscr{B}_\beta^{\varphi\alpha}}\\
\approx&\lim_{r\to1}\sup_{|z|>r}(1-|z|^2)^{\alpha(\beta-1)}\frac{|z|^2}{2}=0\\
\approx&\limsup_{n\to\infty}\frac{1}{2}\sup_{n\in\mathbb{N}_0} n^{\alpha}\left(1-\frac{1+\frac{n}{2}}{1+\frac{n}{2}+\alpha\beta}\right)^{\alpha\beta}\left(\frac{1+\frac{n}{2}}{1+\frac{n}{2}+\alpha\beta}\right)^{1+\frac{n}{2}}=0.
\end{aligned}$$

Observing the above two displays are zero, so the difference $L^{z^2}-L^{\frac{1}{2}z^2}:\mathscr{B}^\alpha\to\mathscr{B}_\beta^{\varphi\alpha}$ is also compact.

(2) Replacing $g,h\in H(\mathbb{D})$ by $g',h'\in H(\mathbb{D})$ in $C_\phi^g-C_\phi^h$, we have $C_\phi^{g'}-C_\psi^{h'}=V_\phi^g-V_\psi^h$, and denote two notations

$$\mathscr{T}_\alpha^\beta(g'\phi)(z)=\frac{\mu_\beta^\varphi(z)g'(z)}{(1-|\phi(z)|^2)^\alpha},\ \mathscr{T}_\alpha^\beta(h'\psi)(z)=\frac{\mu_\beta^\varphi(z)h'(z)}{(1-|\psi(z)|^2)^\alpha};$$

COROLLARY 4.4. *Let $0<\alpha,\beta<\infty$ and $\varphi:[0,\infty)\to[0,\infty)$ be an $\mathscr{N}$-function. Suppose $\phi,\psi\in S(\mathbb{D})$ and $g,h\in H(\mathbb{D})$, then the followings are equivalent,*

(i) $V_\phi^g-V_\psi^h:\mathscr{B}^\alpha\to\mathscr{B}_\beta^\varphi$ *is bounded;*

(ii)

$$\sup_{z\in\mathbb{D}}|\mathscr{T}_\alpha^\beta(g'\phi)(z)|\rho(z)+\sup_{z\in\mathbb{D}}\left|\mathscr{T}_\alpha^\beta(g'\phi)(z)-\mathscr{T}_\alpha^\beta(h'\psi)(z)\right|<\infty,$$
$$\sup_{z\in\mathbb{D}}|\mathscr{T}_\alpha^\beta(h'\psi)(z)|\rho(z)+\sup_{z\in\mathbb{D}}\left|\mathscr{T}_\alpha^\beta(g'\phi)(z)-\mathscr{T}_\alpha^\beta(h'\psi)(z)\right|<\infty;$$

(iii)

$$\sup_{w\in\mathbb{D}}\|(V_\phi^g - V_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi} + \sup_{w\in\mathbb{D}}\|(V_\phi^g - V_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi} < \infty;$$

(iv)

$$\sup_{n\in\mathbb{N}_0} n^\alpha \|g'\phi^n - h'\psi^n\|_{\mu_\beta^\varphi} < \infty.$$

COROLLARY 4.5. *Let $0<\alpha,\beta<\infty$ and $\varphi:[0,\infty)\to[0,\infty)$ be an $\mathscr{N}$-function. Suppose $\phi,\psi\in S(\mathbb{D})$ and $g,h\in H(\mathbb{D})$ such that the operators V_ϕ^g, $V_\psi^h:\mathscr{B}^\alpha\to\mathscr{B}_\beta^\varphi$ are bounded, then the following equivalences hold,*

$$\begin{aligned}
&\|V_\phi^g - V_\psi^h\|_{e,\mathscr{B}^\alpha\to\mathscr{B}_\beta^\varphi}\\
\approx\ &\lim_{r\to1}\sup_{|\phi(z)|>r}\left|\mathscr{T}_\alpha^\beta(g'\phi)(z)\right|\rho(z)\\
&+\lim_{r\to1}\sup_{|\psi(z)|>r}\left|\mathscr{T}_\alpha^\beta(h'\psi)(z)\right|\rho(z)\\
&+\lim_{r\to1}\sup_{\min\{|\phi(z)|,|\psi(z)|\}>r}\left|\mathscr{T}_\alpha^\beta(g'\phi)(z)-\mathscr{T}_\alpha^\beta(h'\psi)(z)\right|\\
\approx\ &\limsup_{|w|\to1}\|(V_\phi^g - V_\psi^h)f_w\|_{\mathscr{B}_\beta^\varphi} + \limsup_{|w|\to1}\|(V_\phi^g - V_\psi^h)\hat{f}_w\|_{\mathscr{B}_\beta^\varphi}\\
\approx\ &\limsup_{n\to\infty} n^\alpha\|g'\phi^n - h'\psi^n\|_{\mu_\beta^\varphi}.
\end{aligned}$$

(3) Let $\phi=\psi=id$ be the identity maps in $V_\phi^g - V_\psi^h$, then $V_{id}^g - V_{id}^h = L_g - L_h$, and denote two notations

$$\mathscr{T}_\alpha^\beta g'(z) = \frac{\mu_\beta^\varphi(z)g'(z)}{(1-|z|^2)^\alpha},\ \mathscr{T}_\alpha^\beta h'(z) = \frac{\mu_\beta^\varphi(z)h'(z)}{(1-|z|^2)^\alpha};$$

Similarly, in this case $\rho(z)=0$ and the test function f_w is enough to exhibit the boundedness and essential norm of $L_g - L_h:\mathscr{B}^\alpha\to\mathscr{B}_\beta^\varphi$.

COROLLARY 4.6. *Let $0<\alpha,\beta<\infty$ and $\varphi:[0,\infty)\to[0,\infty)$ be an $\mathscr{N}$-function. Suppose $g,h\in H(\mathbb{D})$, then the followings are equivalent,*

(i) *$L_g - L_h:\mathscr{B}^\alpha\to\mathscr{B}_\beta^\varphi$ is bounded;*

(ii)

$$\sup_{z\in\mathbb{D}}\left|\mathscr{T}_\alpha^\beta g'(z) - \mathscr{T}_\alpha^\beta h'(z)\right| < \infty;$$

(iii)

$$\sup_{w\in\mathbb{D}}\|(L_g - L_h)f_w\|_{\mathscr{B}_\beta^\varphi} < \infty;$$

(iv)

$$\sup_{n\in\mathbb{N}_0} n^{\alpha}\|g'z^n - h'z^n\|_{\mu_\beta^\varphi} < \infty.$$

COROLLARY 4.7. *Let $0<\alpha,\beta<\infty$ and $\varphi:[0,\infty)\to[0,\infty)$ be an $\mathscr{N}$-function. Suppose $g,h\in H(\mathbb{D})$ such that the operators L_g, $L_h:\mathscr{B}^\alpha\to\mathscr{B}_\beta^\varphi$ are bounded, then the following equivalences hold,*

$$\begin{aligned}
&\|L_g - L_h\|_{e,\mathscr{B}^\alpha\to\mathscr{B}_\beta^\varphi}\\
&\approx \lim_{r\to 1}\sup_{|z|>r}\left|\mathscr{T}_\alpha^\beta g'(z) - \mathscr{T}_\alpha^\beta h'(z)\right|\\
&\approx \limsup_{|w|\to 1}\|(L_g-L_h)f_w\|_{\mathscr{B}_\beta^\varphi}\\
&\approx \limsup_{n\to\infty} n^{\alpha}\|g'z^n - h'z^n\|_{\mu_\beta^\varphi}.
\end{aligned}$$

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REFERENCES

[1] G. M. ANTÓN, R. E. CASTILLO, J. C. RAMOS-FERNÁNDEZ, *Maximal functions and properties of the weighted composition operator acting on the Korenblum α-Bloch and α-Zygmund spaces*, CUBO A Mathematical Journal, 19 (01) (2017) 39–51.

[2] R. E. CASTILLO, J. C. RAMOS-FERNÁNDEZ, M. SALAZAR, *Bounded superposition operators between Bloch-Orlicz and α-Bloch spaces*, Appl. Math. Comput. 218 (2011) 3441–3450.

[3] C. C. COWEN AND B. D. MACCLUER, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.

[4] J. N. DAI, C. H. OUYANG, *Differences of weighted composition operators on $H_\alpha^\infty(B_N)$*, J. Inequal. Appl. Article ID 127431 (2009), 19 pp.

[5] Z. S. FANG AND Z. H. ZHOU, *Essential norms of composition operators between Bloch type spaces in the polydisk*, Arch. Math. 6 (99) (2012) 547–556.

[6] P. GORKIN AND B. D. MACCLUER, *Essential norms of composition operators*, Integral Equations Operator Theory, 48 (2004) 27–40.

[7] O. HYVÄRINEN AND M. LINDSTRÖM, *Estimates of essential norms of weighted composition operators between Bloch-type spaces*, J. Math. Anal. Appl. 393 (2012) 38–44.

[8] O. HYVÄRINEN, M. KEMPPAINEN, M. LINDSTRÖM, A. RAUTIO, AND E. SAUKKO, *The essential norm of weighted composition operators on weighted Banach spaces of analytic functions*, Integral Equations Operator Theory, 72 (2012) 151–157.

[9] Z. J. JIANG, *On a product-type operator from weighted Bergman-Orlicz space to some weighted type spaces*, Appl. Math. Comput. 256 (2015) 37–51.

[10] S. X. LI, S. STEVIĆ, *Generalized composition operators on Zygmund spaces and Bloch type spaces*, J. Math. Anal. Appl. 338 (2008) 1282–1295.

[11] S. X. LI, *Differences of generalized composition operators on the Bloch space*, J. Math. Anal. Appl. 394 (2012) 706–711.

[12] Y. X. LIANG, *Volterra-type operators from weighted Bergman-Orlicz space to β-Zygmund-Orlicz and γ-Bloch-Orlicz spaces*, Monatsh. Math. 182 (2017) 877–897.

[13] Y. X. LIANG, C. CHEN, *New characterizations for differences of Volterra-type operators from α-weighted-type space to β-Bloch-Orlicz space*, Math. Nachr. 291 (14–15) (2018) 2298–2317.

[14] Y. X. LIANG AND X. T. DONG, *New characterizations for the products of differentiation and composition operators between Bloch-type spaces*, J. Inequal. Appl. (2014) 2014:502, doi:10.1186/1029-242X-2014-502.

[15] Y. X. LIANG AND Z. H. ZHOU, *Essential norm of product of differentiation and composition operators between Bloch-type spaces*, Arch. Math. 100 (4) (2013) 347–360.

[16] Y. X. LIANG AND Z. H. ZHOU, *New estimate of essential norm of composition followed by differentiation between Bloch-type spaces*, Banach J. Math. Anal. 8 (2014) 118–137.

[17] Y. X. LIANG AND Z. H. ZHOU, *Some integral-type operators from $F(p,q,s)$ spaces to mixed-norm spaces on the unit ball*, Math. Nachr. 287 (11–12) (2014) 1298–1311.

[18] J. S. MANHAS AND R. ZHAO, *New estimates of essential norms of weighted composition operators between Bloch type spaces*, J. Math. Anal. Appl. 389 (2012) 32–47.

[19] J. C. RAMOS-FERNÁNDEZ, *Composition operators on Bloch-Orlicz type spaces*, Appl. Math. Comput. 217 (2010) 3392–3402.

[20] J. H. SHAPIRO, *Composition Operators and Classical Function Theory*, Spriger-Verlag, 1993.

[21] H. WULAN, D. ZHENG AND K. ZHU, *Compact composition operators on BMOA and the Bloch space*, Proc. Amer. Math. Soc. 137 (2009) 3861–3868.

[22] C. L. YANG, F. W. CHEN AND P. C. WU, *Generalized composition operators on Zygmund-Orlicz type spaces and Bloch-Orlicz type spaces*, J. Funct. Spaces 2014 (2014) Article ID 549370, 9 pages.

[23] R. ZHAO, *Essential norms of composition operators between Bloch type spaces*, Proc. Amer. Math. Soc. 138 (2010) 2537–2546.

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