

## MULTILINEAR HAUSDORFF OPERATOR AND COMMUTATORS ON WEIGHTED MORREY AND HERZ SPACES

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*Abstract.* In this paper, we establish some necessary and sufficient conditions for the boundedness of multilinear Hausdorff operators on weighted central Morrey and Herz type spaces. Moreover, we also discuss some sufficient conditions for the boundedness of commutators of multilinear Hausdorff operators on weighted Morrey–Herz spaces. By these, we generalize some previous known results.

### 1. Introduction

The Hausdorff operator is of fundamental importance in many branches of mathematical analysis, and has been intensively studied since its important applications. The history of the Hausdorff operator can be traced back to the work of Hurwitz and Silverman [22] in 1917, and then was proposed by Hausdorff [21] to study summability of number series (see also [18], [19], [20] for more details). Let us recall that the one-dimensional Hausdorff operator is defined by

$$\mathcal{H}_\varphi(f)(x) = \int_0^\infty \frac{\varphi(t)}{t} f\left(\frac{x}{t}\right) dt, \quad (1.1)$$

where  $\varphi$  is a locally integrable function on the positive half-line. It is worth pointing out that the Hausdorff operator reduces to many other classical operators in analysis such as the Cesàro operator, Hardy–Littlewood–Pólya operator, Riemann–Liouville fractional integral operator and Hardy–Littlewood average operator, by choosing the kernel function  $\varphi$  appropriately, (see, e.g., [17], [26] and references therein).

The Hausdorff operator is extended to the high dimensional space by Brown and Móricz [4] and independently by Lerner and Liflyand [29]. More details, let  $\Phi$  be a locally integrable function on  $\mathbb{R}^n$ . The Hausdorff operator  $\mathcal{H}_{\Phi,A}$  is then defined by

$$\mathcal{H}_{\Phi,A}(f)(x) = \int_{\mathbb{R}^n} \Phi(t) f(A(t)x) dt, \quad x \in \mathbb{R}^n, \quad (1.2)$$

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where  $A(t)$  is an  $n \times n$  matrix satisfying  $\det A(t) \neq 0$  for almost everywhere  $t$  in the support of  $\Phi$  and  $x$  is assumed to be the column  $n$ -vector. It should be pointed out that if we take  $\Phi(t) = \psi(t_1)\chi_{[0,1]^n}(t)$  and  $A(t) = t_1 I_n$  ( $I_n$  is an identity matrix), for  $t = (t_1, t_2, \dots, t_n)$ , where  $\psi: [0, 1] \rightarrow [0, \infty)$  is a measurable function, then  $\mathcal{H}_{\Phi, A}$  reduces to the weighted Hardy–Littlewood operator (see [20] for more details) defined by

$$\mathcal{H}_\psi f(x) = \int_0^1 f(tx)\psi(t)dt, \quad x \in \mathbb{R}^n. \tag{1.3}$$

More information on the weighted Hardy–Littlewood operator as well as its applications can be found in [14], [33], [34] and references therein.

In recent years, the Hausdorff operators, weighted Hardy–Littlewood operators, Hardy–Cesàro operators and their commutators have been significantly developed into different contexts, and studied on many function spaces such as Lebesgue, Morrey, Herz, Morrey–Herz, Hardy and BMO spaces including the weighted settings. For more details, one may find in [1], [2], [3], [4], [5], [7], [8], [10], [13], [16], [24], [25], [27], [28], [29], [31], [32] and references therein. Very recently, Chuong, Duong and Dung [9] have introduced and studied a more general class of multilinear Hausdorff operators defined as follows.

DEFINITION 1. Let  $\Phi: \mathbb{R}^n \rightarrow [0, \infty)$  and  $A_i(y)$  be  $n \times n$  invertible matrices for almost everywhere  $y$  in the support of  $\Phi$ , for all  $i = 1, \dots, m$ . Given  $f_1, f_2, \dots, f_m: \mathbb{R}^n \rightarrow \mathbb{C}$  be measurable functions, the multilinear Hausdorff operator  $H_{\Phi, \vec{A}}$  is defined by

$$H_{\Phi, \vec{A}}(\vec{f})(x) = \int_{\mathbb{R}^d} \Phi(y) \prod_{i=1}^m f_i(A_i(y)x)dy, \quad x \in \mathbb{R}^n, \tag{1.4}$$

for  $\vec{f} = (f_1, \dots, f_m)$  and  $\vec{A} = (A_1, \dots, A_m)$ .

Let us take measurable functions  $s_1(y), \dots, s_m(y) \neq 0$  almost everywhere in  $\mathbb{R}^n$ . Consider a special case where the matrices  $A_i(y) = \text{diag}[s_i(y), \dots, s_i(y)]$ , for all  $i = 1, \dots, m$ . Then, we also investigate the multilinear operator of the form

$$\mathcal{H}_{\Phi, \vec{s}}(\vec{f})(x) = \int_{\mathbb{R}^d} \Phi(y) \left( \prod_{i=1}^m f_i(s_i(y)x) \right) dy, \quad x \in \mathbb{R}^n. \tag{1.5}$$

Note that by letting  $\Phi(y) = \psi(y)\chi_{[0,1]^n}(y)$ , it is evident that  $\mathcal{H}_{\Phi, \vec{s}}$  reduces to the weighted multilinear Hardy–Cesàro operator introduced by Hung and Ky [23] as

$$U_{\psi, \vec{s}}^{m,d}(\vec{f})(x) = \int_{[0,1]^d} \left( \prod_{i=1}^m f_i(s_i(y)x) \right) \psi(y)dy, \quad x \in \mathbb{R}^n. \tag{1.6}$$

Let  $b$  be a measurable function. We denote by  $\mathcal{M}_b$  the multiplication operator defined by  $\mathcal{M}_b f(x) = b(x)f(x)$  for any measurable function  $f$ . If  $\mathcal{H}$  is a linear operator on some measurable function space, the commutator, in the sense of Coifman–Rochberg–Weiss [6], formed by  $\mathcal{M}_b$  and  $\mathcal{H}$  is defined by  $[\mathcal{M}_b, \mathcal{H}]f(x) = (\mathcal{M}_b \mathcal{H} -$

$\mathcal{H}_{\Phi, \vec{b}} f(x)$ . Similarly, the commutators of Coifman–Rochberg–Weiss type of multilinear Hausdorff operator are defined by

$$\mathcal{H}_{\Phi, \vec{A}}^{\vec{b}}(\vec{f})(x) = \int_{\mathbb{R}^d} \Phi(y) \prod_{i=1}^m (b_i(x) - b_i(A_i(y)x)) \prod_{i=1}^m f_i(A_i(y)x) dy, \quad x \in \mathbb{R}^n, \quad (1.7)$$

where  $\vec{b} = (b_1, \dots, b_m)$ , and  $b_i$  are locally integrable functions on  $\mathbb{R}^n$ .

It is well known that the multilinear Hausdorff operators, the weighted multilinear Hardy–Cesàro operator and their commutators have been extended to study on some function spaces. The interested reader is referred to the works [9], [11], [12], [15] and [23] for more details.

Motivated by above results, the main purpose of this paper is to give some necessary and sufficient conditions for the boundedness of multilinear Hausdorff operators on central Morrey and Morrey–Herz spaces with arbitrary weighted functions. Furthermore, some sufficient conditions for the boundedness of commutators of multilinear Hausdorff operators with the Lipschitz functions on weighted Morrey–Herz spaces are also obtained. As a consequence, we generalize some previous known results.

Our paper is organized as follows. In Section 2, we present some notations and preliminaries on weighted Lebesgue spaces, central Morrey spaces, Herz spaces, Morrey–Herz spaces, and Lipschitz spaces. Our main theorems and their proofs are given in Section 3.

### 2. Preliminaries

Let us start with this section by recalling some standard definitions and notations pertaining to our work. Throughout the whole paper,  $n$  denotes the dimensional number of the Euclidean space  $\mathbb{R}^n$ . Let us denote by  $\|T\|_{X \rightarrow Y}$ , the norm of  $T$  between two normed vector spaces  $X$  and  $Y$ . We write  $a \lesssim b$  to mean that there is a positive constant  $C$ , independent of the main parameters, such that  $a \leq Cb$ . The symbol  $f \simeq g$  means that  $f$  is equivalent to  $g$  (i.e.  $C^{-1}f \leq g \leq Cf$ ). The weighted functions are locally integrable non-negative measurable functions on  $\mathbb{R}^n$ . For any measurable set  $\Omega$ , we denote by  $\chi_\Omega$  its characteristic function, and  $(\omega(\Omega))^\alpha = (\int_\Omega \omega(x) dx)^\alpha$  for any weighted function  $\omega$ . Let  $L_\omega^p(\mathbb{R}^n)$  ( $0 < p < \infty$ ) be the space of all measurable functions  $f$  on  $\mathbb{R}^n$  such that

$$\|f\|_{L_\omega^p} = \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty.$$

The space  $L_{\text{loc}}^p(\mathbb{R}^n, \omega)$  is defined as the set of all measurable functions  $f$  on  $\mathbb{R}^n$  satisfying  $\int_\Omega |f(x)|^q \omega(x) dx < \infty$  for any compact subset  $\Omega$  of  $\mathbb{R}^n$ . Also,  $L_{\text{loc}}^p(\mathbb{R}^n \setminus \{0\}, \omega)$  is defined in a similar way to the space  $L_{\text{loc}}^p(\mathbb{R}^n, \omega)$ .

Now, let us recall some definitions of the weighted Morrey, Herz, and Morrey–Herz spaces. For their deep applications in harmonic analysis, one may find in the book [30]. For  $k \in \mathbb{Z}$ , let  $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ ,  $S_k = B_k \setminus B_{k-1}$ , and  $\chi_k$  denotes the characteristic function of the set  $S_k$ . Denote  $B_r = \{x \in \mathbb{R}^n : |x| \leq r\}$  for all  $r > 0$ .

DEFINITION 2. Let  $0 < p < \infty$  and  $-1/p < \lambda < 0$ . Let  $\omega$  be a weighted function. The weighted central Morrey space is defined by

$$\dot{M}_\omega^{\lambda,p}(\mathbb{R}^n) = \{f \in L_{loc}^p(\mathbb{R}^n, \omega) : \|f\|_{\dot{M}_\omega^{\lambda,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{M}_\omega^{\lambda,p}} = \sup_{r>0} \frac{1}{|B_r|^{\lambda+1/p}} \|f\|_{L_\omega^p(B_r)}.$$

It is well known that the Herz space is first introduced by Herz to study of absolutely convergent Fourier transforms. Next, we turn to the definition of weighted Herz spaces, and weighted Morrey–Herz spaces.

DEFINITION 3. Let  $0 < p < \infty$ ,  $0 < q < \infty$  and  $\alpha \in \mathbb{R}$ . Let  $\omega$  be a weighted function. Then, the homogeneous weighted Herz space  $\dot{K}_{p,q}^\alpha(\mathbb{R}^n, \omega)$  is defined to be the set of all functions  $f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}, \omega)$  such that  $\|f\|_{\dot{K}_{p,q}^\alpha(\omega)} < \infty$ , where

$$\|f\|_{\dot{K}_{p,q}^\alpha(\omega)} = \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L_\omega^q}^p \right)^{1/p}.$$

DEFINITION 4. Let  $\alpha \in \mathbb{R}$ ,  $0 < p < \infty$ ,  $0 < q < \infty$ ,  $\lambda \geq 0$  and  $\omega$  be a weighted function. The weighted Morrey–Herz space  $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n, \omega)$  is defined as the space of all functions  $f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}, \omega)$  such that  $\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\omega)} < \infty$ , where

$$\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\omega)} = \sup_{k_0 \in \mathbb{Z}} \left( 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_k\|_{L_\omega^q}^p \right)^{1/p} \right).$$

In particular, for  $\lambda = 0$ , it is easily seen that  $M\dot{K}_{p,q}^{\alpha,0}(\mathbb{R}^n, \omega) = \dot{K}_{p,q}^\alpha(\mathbb{R}^n, \omega)$ . This means that the weighted Herz space is a special case of the weighted Morrey–Herz space. Some applications of weighted Morrey–Herz spaces to the Hardy–Cesàro operators as well as the multilinear Hausdorff operators may be found, for example, in the works [10], [12] and [13].

DEFINITION 5. Let  $0 < \gamma \leq 1$ . The Lipschitz space  $\Lambda_\gamma(\mathbb{R}^n)$  is defined as the set of all functions  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  such that  $\|f\|_{\Lambda_\gamma} < \infty$ , where

$$\|f\|_{\Lambda_\gamma} = \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma}.$$

### 3. The main results

First, for simplicity of notation, we denote

$$\Psi_{\text{sup}}(y) = \text{ess sup}_{x \in \mathbb{R}^n} \prod_{i=1}^m \left( \frac{\omega(x)}{\omega_i(A_i(y)x)} \right)^{1/q_i},$$

and

$$\Psi_{\text{inf}}(y) = \text{ess inf}_{x \in \mathbb{R}^n} \prod_{i=1}^m \left( \frac{\omega(x)}{\omega_i(A_i(y)x)} \right)^{1/q_i}.$$

In what follows, we assume that  $\alpha, \alpha_i \in \mathbb{R}$ ,  $\lambda, \lambda_i$  is non-negative real numbers,  $0 < p, p_i < \infty$ , and  $1 \leq q, q_i < \infty$ , for  $i = 1, \dots, m$ , satisfying

$$\alpha = \alpha_1 + \dots + \alpha_m, \quad \lambda = \lambda_1 + \dots + \lambda_m,$$

and

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}, \quad \frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_m}.$$

For a matrix  $A = (a_{ij})_{n \times n}$ , the norm of  $A$  is defined by

$$\|A\| = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{|Ax|}{|x|}.$$

Clearly, we have  $|Ax| \leq \|A\| \cdot |x|$  for any  $x \in \mathbb{R}^n$ . Especially, if  $A$  is invertible, we then have

$$\|A\|^{-n} \leq |\det(A^{-1})| \leq \|A^{-1}\|^n.$$

Remark that if the matrix  $A(y)$  satisfying

$$\text{ess sup}_{y \in \mathbb{R}^d} \|A(y)\| \cdot \|A^{-1}(y)\| = \rho < \infty, \tag{3.1}$$

then we have  $\|A(y)\| \simeq \|A^{-1}(y)\|^{-1}$  for almost all  $y \in \mathbb{R}^d$ . In addition, it is easy to show that

$$\|A(y)\|^\zeta \lesssim \|A^{-1}(y)\|^{-\zeta}, \quad \text{for all } \zeta \in \mathbb{R},$$

and

$$|A(y)x|^\zeta \gtrsim \|A^{-1}(y)\|^{-\zeta} |x|^\zeta, \quad \text{for all } \zeta \in \mathbb{R}, x \in \mathbb{R}^n.$$

Obviously, the class of real orthogonal matrices satisfies the condition (3.1) above.

Now, we are in a position to state and prove our first main results concerning the boundedness of multilinear Hausdorff operators on the product of central Morrey spaces with arbitrary weights.

THEOREM 1. Let  $-1/q_i < \lambda_i < 0$ , and  $\omega, \omega_i$  be arbitrary weighted functions for all  $i = 1, \dots, m$ .

(i) If

$$\mathcal{A}_{\text{sup}} = \int_{\mathbb{R}^d} |\Phi(y)| \prod_{i=1}^m |\det A_i(y)|^{\lambda_i} \Psi_{\text{sup}}(y) dy < \infty, \tag{3.2}$$

then the operator  $\mathcal{H}_{\Phi, \vec{A}}$  is bounded from  $\prod_{i=1}^m \dot{M}_{\omega_i}^{\lambda_i, q_i}(\mathbb{R}^n)$  to  $\dot{M}_{\omega}^{\lambda, q}(\mathbb{R}^n)$ . Moreover,

$$\|\mathcal{H}_{\Phi, \vec{A}}\|_{\prod_{i=1}^m \dot{M}_{\omega_i}^{\lambda_i, q_i} \rightarrow \dot{M}_{\omega}^{\lambda, q}} \leq \mathcal{A}_{\text{sup}}.$$

(ii) Suppose  $\Phi$  is a non-negative function. If  $\mathcal{H}_{\Phi, \vec{A}}$  is determined as a bounded operator from  $\prod_{i=1}^m \dot{M}_{\omega_i}^{\lambda_i, q_i}(\mathbb{R}^n)$  to  $\dot{M}_{\omega}^{\lambda, q}(\mathbb{R}^n)$ , we have

$$\mathcal{A}_{\text{inf}} = \int_{\mathbb{R}^d} \Phi(y) \prod_{i=1}^m \|A_i(y)\|^{n\lambda_i} \Psi_{\text{inf}}(y) dy < \infty, \tag{3.3}$$

and

$$\|\mathcal{H}_{\Phi, \vec{A}}\|_{\prod_{i=1}^m \dot{M}_{\omega_i}^{\lambda_i, q_i} \rightarrow \dot{M}_{\omega}^{\lambda, q}} \geq \frac{\prod_{i=1}^m n(\lambda_i + 1/q_i)}{n(\lambda + 1/q)} \cdot \mathcal{A}_{\text{inf}}. \tag{3.4}$$

*Proof.* (i) Let  $\vec{f} = (f_1, \dots, f_m) \in \prod_{i=1}^m \dot{M}_{\omega_i}^{\lambda_i, q_i}(\mathbb{R}^n)$  and  $B_r = B(0, r)$  be a ball in  $\mathbb{R}^n$ . By the Minkowski inequality, the Hölder's inequality and change of variables, we have

$$\begin{aligned} \|\mathcal{H}_{\Phi, \vec{A}}(\vec{f})\|_{L_{\omega}^q(B_r)} &= \left( \int_{B_r} \left| \int_{\mathbb{R}^d} \Phi(y) \prod_{i=1}^m f_i(A_i(y)x) dy \right|^q \omega(x) dx \right)^{1/q} \\ &\leq \int_{\mathbb{R}^d} |\Phi(y)| \left( \left\| \prod_{i=1}^m f_i(A_i(y)x) \omega^{1/q}(x) \right\|_{L^q(B_r)} \right) dy \\ &\leq \int_{\mathbb{R}^d} |\Phi(y)| \Psi_{\text{sup}}(y) \left( \prod_{i=1}^m \|f_i(A_i(y)x) \omega_i^{1/q_i}(A_i(y)x)\|_{L^{q_i}(B_r)} \right) dy \\ &= \int_{\mathbb{R}^d} |\Phi(y)| \prod_{i=1}^m |\det A_i(y)|^{-1/q_i} \Psi_{\text{sup}}(y) \left( \prod_{i=1}^m \|f_i\|_{L_{\omega_i}^{q_i}(A_i(y)B_r)} \right) dy \\ &=: \int_{\mathbb{R}^d} \mathcal{H}(y) \left( \prod_{i=1}^m \|f_i\|_{L_{\omega_i}^{q_i}(A_i(y)B_r)} \right) dy, \end{aligned} \tag{3.5}$$

where

$$\mathcal{H}(y) = |\Phi(y)| \prod_{i=1}^m |\det A_i(y)|^{-1/q_i} \Psi_{\text{sup}}(y). \tag{3.6}$$

From the definition of weighted central Morrey space and by (3.5), we have

$$\begin{aligned} \|\mathcal{H}_{\Phi, \vec{A}}(\vec{f})\|_{M_{\omega}^{\lambda, q}} &= \sup_{r>0} \frac{1}{|B_r|^{\lambda+1/q}} \|\mathcal{H}_{\Phi, \vec{A}}(\vec{f})\|_{L_{\omega}^q(B_r)} \\ &\leq \sup_{r>0} \left( \int_{\mathbb{R}^d} \mathcal{K}(y) \frac{\prod_{i=1}^m |A_i(y)B_r|^{\lambda_i+1/q_i}}{|B_r|^{\lambda+1/q}} dy \right) \left( \prod_{i=1}^m \|f_i\|_{M_{\omega_i}^{\lambda_i, q_i}} \right) \\ &\leq \left( \int_{\mathbb{R}^d} \mathcal{K}(y) \prod_{i=1}^m |\det A_i(y)|^{(\lambda_i+1/q_i)} dy \right) \left( \prod_{i=1}^m \|f_i\|_{M_{\omega_i}^{\lambda_i, q_i}} \right) \\ &\leq \mathcal{A}_{\text{sup}} \cdot \left( \prod_{i=1}^m \|f_i\|_{M_{\omega_i}^{\lambda_i, q_i}} \right). \end{aligned}$$

This asserts that the case (i) is proved.

(ii) Let us choose  $\vec{f}_0 = (f_{1,0}, \dots, f_{m,0})$  such that

$$f_{i,0}(x) = |x|^{n\lambda_i} (\omega_i(x))^{-1/q_i}.$$

It is not hard to show that  $f_{i,0} \in \dot{M}_{\omega_i}^{\lambda_i, q_i}(\mathbb{R}^n)$ , and

$$\|f_{i,0}\|_{M_{\omega_i}^{\lambda_i, q_i}} = \frac{|S_{n-1}|^{-\lambda_i}}{n(\lambda_i + 1/q_i)},$$

where  $S_{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ . It is easy to see that  $|A_i(y)x|^{\lambda_i} \geq \|A_i(y)\|^{\lambda_i} |x|^{\lambda_i}$ , for every  $\lambda_i < 0$ . This leads to that

$$\mathcal{H}_{\Phi, \vec{A}}(\vec{f}_0)(x) \geq \left( \int_{\mathbb{R}^d} \Phi(y) \prod_{i=1}^m \|A_i(y)\|^{n\lambda_i} \omega_i^{-1/q_i}(A_i(y)x) dy \right) |x|^{n\lambda}.$$

Therefore,

$$\begin{aligned} \|\mathcal{H}_{\Phi, \vec{A}}(\vec{f}_0)\|_{M_{\omega}^{\lambda, q}} &= \sup_{r>0} \frac{1}{|B_r|^{\lambda+1/q}} \|\mathcal{H}_{\Phi, \vec{A}}(\vec{f}_0)\|_{L_{\omega}^q(B_r)} \\ &\geq \mathcal{A}_{\text{inf}} \cdot \sup_{r>0} \frac{1}{|B_r|^{\lambda+1/q}} \left( \int_{B_r} |x|^{n\lambda q} dx \right)^{1/q} \\ &\geq \frac{\prod_{i=1}^m n(\lambda_i + 1/q_i)}{n(\lambda + 1/q)} \cdot \mathcal{A}_{\text{inf}} \cdot \prod_{i=1}^m \|f_{i,0}\|_{M_{\omega_i}^{\lambda_i, q_i}}. \end{aligned}$$

Because  $\mathcal{H}_{\Phi, \vec{A}}$  is bounded from  $\prod_{i=1}^m \dot{M}_{\omega_i}^{\lambda_i, q_i}(\mathbb{R}^n)$  to  $\dot{M}_{\omega}^{\lambda, q}(\mathbb{R}^n)$ , we have  $\mathcal{A}_{\text{inf}} < \infty$ , and the inequality (3.4) holds. Therefore, Theorem 1 is proved.  $\square$

Let us consider the special case  $A_i(y) = s_i(y) \cdot I_n$ , where  $I_n$  is an identity matrix. Suppose that  $\omega_i(tx) = v_i(t)\omega_i(x)$  for all  $t \in \mathbb{R}$ , where  $v_i$  are non-negative functions, and  $\omega(x) := \prod_{i=1}^m \omega_i(x)^{q/q_i}$ . Then we have

$$\Psi_{\text{sup}}(y) = \Psi_{\text{inf}}(y) = \prod_{i=1}^m v_i(s_i(y))^{-1/q_i}.$$

In particular, let us take  $\omega_i \in \mathcal{W}_{\beta_i}$ , where  $\mathcal{W}_{\beta_i}$  is the set of absolutely homogeneous weighted functions of degree  $\beta_i$  in  $\mathbb{R}^n$ , namely,  $\omega_i(tx) = |t|^{\beta_i} \omega_i(x)$  for all  $t \neq 0, x \in \mathbb{R}^n$  (see [13] for more details). For  $\omega \in \mathcal{W}_{\beta}$  such that  $\beta/q = \sum_{i=1}^m \beta_i/q_i$ , we have

$$\Psi_{\text{sup}}(y) = \Psi_{\text{inf}}(y) = \prod_{i=1}^m |s_i(y)|^{-\beta_i/q_i},$$

for all  $y \in \mathbb{R}^d$ . Thus, by Theorem 1, we have the following result.

**COROLLARY 1.** *Let  $\Phi$  be a non-negative function and  $\omega, \omega_i$  be weighted functions satisfying  $\omega(x) = \prod_{i=1}^m \omega_i(x)^{q_i/q}$ . Suppose that  $\omega_i(tx) = v_i(t)\omega_i(x)$  for all  $t \in \mathbb{R}$ , where  $v_i$  are non-negative functions. Then, we have that  $\mathcal{H}_{\Phi, \vec{s}}$  is bounded from  $\prod_{i=1}^m \dot{M}_{\omega_i}^{\lambda_i, q_i}(\mathbb{R}^n)$  to  $\dot{M}_{\omega}^{\lambda, q}(\mathbb{R}^n)$  if and only if*

$$\mathcal{A} = \int_{\mathbb{R}^d} \Phi(y) \prod_{i=1}^m |s_i(y)|^{n\lambda_i} v_i(s_i(y))^{-1/q_i} dy < \infty.$$

Moreover, in this case, we have

$$\frac{\prod_{i=1}^m n(\lambda_i + 1/q_i)}{n(\lambda + 1/q)} \cdot \mathcal{A} \leq \| \mathcal{H}_{\Phi, \vec{s}} \|_{\prod_{i=1}^m \dot{M}_{\omega_i}^{\lambda_i, q_i} \rightarrow \dot{M}_{\omega}^{\lambda, q}} \leq \mathcal{A}.$$

Next, we also give the boundedness and bound of the multilinear Hausdorff operator on the weighted Morrey–Herz spaces.

**THEOREM 2.** *Let  $\omega, \omega_i$  be arbitrary weighted functions for all  $i = 1, \dots, m$ . Let  $v = v(y)$  be the greatest integer number satisfying*

$$\max_{i=1, \dots, m} \{ \|A_i(y)\| \cdot \|A_i^{-1}(y)\| \} < 2^{-v}, \quad \text{for a.e. } y \in \mathbb{R}^d.$$

(i) *Let  $1 \leq p < \infty$ , or  $0 < p < 1$  and  $\lambda > 0$ . If*

$$\mathcal{B}_{\text{sup}} = \int_{\mathbb{R}^d} |\Phi(y)| \prod_{i=1}^m |\det A_i(y)|^{-1/q_i} \|A_i(y)\|^{\lambda_i - \alpha_i} \left( \sum_{k=v-1}^0 2^{k(\lambda_i - \alpha_i)} \right) \Psi_{\text{sup}}(y) dy < \infty,$$

then  $\mathcal{H}_{\Phi, \vec{A}}$  is bounded from  $\prod_{i=1}^m MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\mathbb{R}^n, \omega_i)$  to  $MK_{p, q}^{\alpha, \lambda}(\mathbb{R}^n, \omega)$ . Moreover,

$$\| \mathcal{H}_{\Phi, \vec{A}} \|_{\prod_{i=1}^m MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\omega_i) \rightarrow MK_{p, q}^{\alpha, \lambda}(\omega)} \lesssim \mathcal{B}_{\text{sup}}.$$

(ii) *Let  $\Phi$  be a non-negative function, and*

$$\text{ess sup}_{y \in \mathbb{R}^d, i=1, \dots, m} \|A_i(y)\| \cdot \|A_i^{-1}(y)\| = \rho < \infty. \tag{3.7}$$

Suppose that  $\mathcal{H}_{\Phi, \vec{A}}$  is a bounded operator from  $\prod_{i=1}^m MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\mathbb{R}^n, \omega_i)$  to  $MK_{p, q}^{\alpha, \lambda}(\mathbb{R}^n, \omega)$  for all  $1 \leq p < \infty$  and  $\lambda_i = 0$ , or  $0 < p < \infty$  and  $\lambda_i > 0$ . We thus have

$$\mathcal{B}_{\text{inf}} = \int_{\mathbb{R}^d} \Phi(y) \prod_{i=1}^m \|A_i(y)\|^{\lambda_i - \alpha_i - n/q_i} \Psi_{\text{inf}}(y) dy < \infty.$$

Moreover,

$$\|\mathcal{H}_{\Phi, \vec{A}}\|_{\prod_{i=1}^m MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\omega_i) \rightarrow MK_{p, q}^{\alpha, \lambda}(\omega)} \gtrsim \mathcal{B}_{\text{inf}}.$$

*Proof.* Using a similar argument as the inequality (3.5), we have

$$\|\mathcal{H}_{\Phi, \vec{A}}(\vec{f})\chi_k\|_{L^q_\omega} \leq \int_{\mathbb{R}^d} \mathcal{H}(y) \left( \prod_{i=1}^m \|f_i\|_{L^{q_i}_{\omega_i}(A_i(y)S_k)} \right) dy, \tag{3.8}$$

where  $\mathcal{H}(y)$  is in (3.6). Because  $\det A_i(y) \neq 0$  almost everywhere  $y$  in  $\mathbb{R}^d$ , there is an integer number  $\ell_i = \ell_i(y)$  such that  $2^{\ell_i - 1} < \|A_i(y)\| \leq 2^{\ell_i}$ . Set  $z = A_i(y)x$  for  $x \in S_k$ . It is evident that

$$|z| \leq \|A_i(y)\| |x| \leq 2^{k + \ell_i},$$

and

$$|z| \geq \frac{\|A_i(y)\| \cdot |x|}{\|A_i^{-1}(y)\|} > 2^{k + \ell_i + \nu - 2}.$$

This implies that

$$\|f_i\|_{L^{q_i}_{\omega_i}(A_i(y)S_k)} \leq \sum_{j=\nu-1}^0 \|f_i\chi_{k+\ell_i+j}\|_{L^{q_i}_{\omega_i}}. \tag{3.9}$$

By the definition of weighted Morrey–Herz spaces, we have, by (3.9), that

$$\begin{aligned} \|\mathcal{H}_{\Phi, \vec{A}}(\vec{f})\|_{MK_{p, q}^{\alpha, \lambda}(\omega)} &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p} \|\mathcal{H}_{\Phi, \vec{A}}(\vec{f})\chi_k\|_{L^q_\omega}^p \right)^{1/p} \\ &\leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p} \left( \int_{\mathbb{R}^d} \mathcal{H}(y) \prod_{i=1}^m \left( \sum_{j=\nu-1}^0 \|f_i\chi_{k+\ell_i+j}\|_{L^{q_i}_{\omega_i}} \right) dy \right)^p \right)^{1/p} \end{aligned} \tag{3.10}$$

Now we consider two cases as follows.

Case 1:  $1 \leq p < \infty$ .

Applying the Minkowski inequality, we have

$$\|\mathcal{H}_{\Phi, \vec{A}}(\vec{f})\|_{MK_{p, q}^{\alpha, \lambda}(\omega)} \leq \int_{\mathbb{R}^d} \mathcal{H}(y) \varphi(y) dy, \tag{3.11}$$

where

$$\varphi(y) := \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} \prod_{i=1}^m 2^{k \alpha_i p} \left( \sum_{j=v-1}^0 \|f_i \chi_{k+\ell_i+j}\|_{L_{\omega_i}^{q_i}} \right)^p \right)^{1/p}.$$

By the Hölder inequality and the Minkowski inequality, we have

$$\begin{aligned} \varphi(y) &\leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \prod_{i=1}^m \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha_i p_i} \left( \sum_{j=v-1}^0 \|f_i \chi_{k+\ell_i+j}\|_{L_{\omega_i}^{q_i}} \right)^{p_i} \right)^{1/p_i} \\ &\leq \prod_{i=1}^m \sum_{j=v-1}^0 2^{(\lambda_i - \alpha_i)(\ell_i + j)} \left( \prod_{i=1}^m \|f_i\|_{MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\omega_i)} \right) \\ &\lesssim \left( \prod_{i=1}^m \|A_i(y)\|^{\lambda_i - \alpha_i} \sum_{j=v-1}^0 2^{j(\lambda_i - \alpha_i)} \right) \left( \prod_{i=1}^m \|f_i\|_{MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\omega_i)} \right). \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we have

$$\|\mathcal{H}_{\Phi, \vec{A}}(\vec{f})\|_{MK_{p, q}^{\alpha, \lambda}(\omega)} \lesssim \mathcal{B}_{\text{sup}} \cdot \left( \prod_{i=1}^m \|f_i\|_{MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\omega_i)} \right),$$

which is the desired result

Case 2:  $0 < p < \infty$  and  $\lambda > 0$ .

It follows from the definition of weighted Morrey–Herz space that

$$\begin{aligned} \|f_i \chi_{k+\ell_i+j}\|_{L_{\omega_i}^{q_i}} &\leq 2^{(\lambda_i - \alpha_i)(k+\ell_i+j)} \|f_i\|_{MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\omega_i)} \\ &\lesssim 2^{k(\lambda_i - \alpha_i)} 2^{j(\lambda_i - \alpha_i)} \|A_i(y)\|^{\lambda_i - \alpha_i} \|f_i\|_{MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\omega_i)}, \quad j = -v, \dots, 0. \end{aligned} \tag{3.13}$$

Set  $\psi(y) = \prod_{i=1}^m \|A_i(y)\|^{\lambda_i - \alpha_i} \sum_{j=v-1}^0 2^{j(\lambda_i - \alpha_i)}$  for simplicity. By (3.10) and (3.13), we have

$$\begin{aligned} &\|\mathcal{H}_{\Phi, \vec{A}}(\vec{f})\|_{MK_{p, q}^{\alpha, \lambda}(\omega)} \\ &\lesssim \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k \lambda p} \right)^{1/p} \left( \int_{\mathbb{R}^d} \mathcal{H}(y) \psi(y) dy \right) \left( \prod_{i=1}^m \|f_i\|_{MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\omega_i)} \right) \\ &\lesssim \left( \int_{\mathbb{R}^d} \mathcal{H}(y) \psi(y) dy \right) \left( \prod_{i=1}^m \|f_i\|_{MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\omega_i)} \right) \\ &\lesssim \mathcal{B}_{\text{sup}} \cdot \left( \prod_{i=1}^m \|f_i\|_{MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\omega_i)} \right). \end{aligned}$$

Thus the proof of part (i) is finished.

(ii) Suppose that  $\mathcal{H}_{\Phi, \vec{A}}$  is a bounded operator from  $\prod_{i=1}^m M\dot{K}_{p_i, q_i}^{\alpha_i, \lambda_i}(\mathbb{R}^n, \omega_i)$  to  $M\dot{K}_{p, q}^{\alpha, \lambda}(\mathbb{R}^n, \omega)$ .

Case 1:  $1 \leq p < \infty$  and  $\lambda_i = 0$ . Then it is clearly that  $M\dot{K}_{p, q}^{\alpha, 0}(\mathbb{R}^n, \omega) = \dot{K}_{p, q}^{\alpha}(\mathbb{R}^n, \omega)$  and  $M\dot{K}_{p_i, q_i}^{\alpha_i, 0}(\mathbb{R}^n, \omega) = \dot{K}_{p_i, q_i}^{\alpha_i}(\mathbb{R}^n, \omega)$ . Let us choose  $\vec{f}_\varepsilon = (f_{1, \varepsilon}, \dots, f_{m, \varepsilon})$  for  $\varepsilon > 0$  such that

$$f_{i, \varepsilon}(x) = \begin{cases} 0, & |x| < \rho^{-1}, \\ |x|^{-\alpha_i - n/q_i - \varepsilon} (\omega_i(x))^{-1/q_i}, & \text{otherwise.} \end{cases}$$

By similar arguments as the proof of Theorem 2 in [11], we have

$$\|f_{i, \varepsilon}\|_{\dot{K}_{p_i, q_i}^{\alpha_i}(\omega_i)} \simeq \left(\frac{2^{(1-\theta)\varepsilon p_i}}{2^{\varepsilon p_i} - 1}\right)^{1/p_i} \left(\frac{2^{q_i(\alpha_i + \varepsilon)} - 1}{q_i(\alpha_i + \varepsilon)}\right)^{1/q_i}, \tag{3.14}$$

where  $\theta$  is the smallest integer number satisfying  $\theta \geq -\ln\rho/\ln 2$ . Set

$$U_x = \{y \in \mathbb{R}^d : |A_i(y)x| \geq \rho^{-1} \text{ for all } i = 1, \dots, m\},$$

and

$$V_\varepsilon = \{y \in \mathbb{R}^d : \|A_i(y)\| \geq \varepsilon, \text{ for all } i = 1, \dots, m\}.$$

It is not hard to check that

$$V_\varepsilon \subset U_x \text{ for all } x \in \mathbb{R}^n \setminus B(0, \varepsilon^{-1}). \tag{3.15}$$

Hence, for any  $x \in \mathbb{R}^n \setminus B(0, \varepsilon^{-1})$ , by (3.15), we deduce that

$$\begin{aligned} \mathcal{H}_{\Phi, \vec{A}}(\vec{f}_\varepsilon)(x) &\geq \int_{U_x} \Phi(y) \prod_{i=1}^m |A_i(y)x|^{-\alpha_i - n/q_i - \varepsilon} \omega_i^{-1/q_i}(A_i(y)x) dy \\ &\geq \int_{V_\varepsilon} \Phi(y) \prod_{i=1}^m |A_i(y)x|^{-\alpha_i - n/q_i - \varepsilon} \omega_i^{-1/q_i}(A_i(y)x) dy \\ &\gtrsim \left( \int_{V_\varepsilon} \Phi(y) \prod_{i=1}^m \|A_i(y)\|^{-\alpha_i - n/q_i - \varepsilon} \omega_i^{-1/q_i}(A_i(y)x) dy \right) g_\varepsilon(x), \end{aligned}$$

where  $g_\varepsilon(x) = |x|^{-\alpha - n/q - m\varepsilon} \chi_{\mathbb{R}^n \setminus B(0, \varepsilon^{-1})}$ . Observe that one may find the integer number  $k_0$  such that  $2^{k_0 - 2} < \varepsilon^{-1} \leq 2^{k_0 - 1}$ . Thus we have for any  $k \geq k_0$ ,

$$\|\mathcal{H}_{\Phi, \vec{A}}(\vec{f}_\varepsilon)\chi_k\|_{L_\omega^q} \geq \mathcal{B}(\varepsilon) \|g_\varepsilon \chi_k\|_{L^q},$$

where

$$\mathcal{B}(\varepsilon) = \int_{V_\varepsilon} \Phi(y) \prod_{i=1}^m \|A_i(y)\|^{-\alpha_i - n/q_i - \varepsilon} \Psi_{\text{inf}}(y) dy.$$

So,

$$\begin{aligned} \|\mathcal{H}_{\Phi, \vec{A}}(\vec{f}_\varepsilon)\|_{\dot{K}_{p,q}^\alpha(\omega)} &\geq \mathcal{B}(\varepsilon) \left( \sum_{k=k_0}^\infty 2^{k\alpha p} \left( \int_{S_k} |x|^{-\alpha q - n - m\varepsilon q} dx \right)^{p/q} \right)^{1/p} \\ &\gtrsim \mathcal{B}(\varepsilon) \left( \frac{2^{-k_0 m \varepsilon p}}{1 - 2^{-m \varepsilon p}} \right)^{1/p} \left( \frac{2^{q(\alpha + m\varepsilon) - 1}}{q(\alpha + m\varepsilon)} \right)^{1/q}. \end{aligned} \tag{3.16}$$

By assuming that  $\mathcal{H}_{\Phi, \vec{A}}$  is bounded from  $\prod_{i=1}^m \dot{K}_{p_i, q_i}^{\alpha_i}(\mathbb{R}^n, \omega_i)$  to  $\dot{K}_{p,q}^\alpha(\mathbb{R}^n, \omega)$ , by (3.14) and (3.16), letting  $\varepsilon \rightarrow 0$ , it immediately follows from the Fatou lemma that

$$\int_{\mathbb{R}^d} \Phi(y) \prod_{i=1}^m \|A_i(y)\|^{-\alpha_i - n/q_i} \Psi_{\inf}(y) dy < \infty.$$

This completes the proof of the case 1.

Case 2:  $0 < p < \infty$  and  $\lambda_i > 0$ .

Similarly, we also take  $\vec{g}_0 = (g_{1,0}, \dots, g_{m,0})$  such that

$$g_{i,0}(x) = |x|^{\lambda_i - \alpha_i - n/q_i} \omega_i(x)^{-1/q_i}, \quad i = 1, \dots, m.$$

A simple calculation gives us that

$$\|g_{i,0}\|_{MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\omega_i)} < \infty. \tag{3.17}$$

It is evident that

$$\begin{aligned} \mathcal{H}_{\Phi, \vec{A}}(\vec{g}_0)(x) &= \int_{\mathbb{R}^d} \Phi(y) \prod_{i=1}^m |A_i(y)x|_{h_i}^{\lambda_i - \alpha_i - n/q_i} \omega_i^{-1/q_i}(A_i(y)x) dy \\ &\gtrsim \mathcal{B}_{\inf} \cdot h(x), \end{aligned}$$

where  $h(x) = |x|^{\lambda - \alpha - n/q} \omega(x)^{-1/q}$ . It is not hard to show that  $\|h\|_{MK_{p,q}^{\alpha, \lambda}(\omega)} < \infty$ . Combining this with (3.17), we obtain

$$\begin{aligned} \|\mathcal{H}_{\Phi, \vec{A}}(\vec{g}_0)\|_{MK_{p,q}^{\alpha, \lambda}(\omega)} &\gtrsim \mathcal{B}_{\inf} \cdot \|h\|_{MK_{p,q}^{\alpha, \lambda}(\omega)} \\ &\gtrsim \mathcal{B}_{\inf} \cdot \prod_{i=1}^m \|g_{i,0}\|_{MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\omega_i)}. \end{aligned}$$

Therefore, Theorem 2 is proved.  $\square$

In particular, we have necessary and sufficient conditions of  $\mathcal{H}_{\Phi, \vec{A}}$  on the product of weighted Lebesgue spaces in the case  $\alpha = \lambda = \lambda_i = \alpha_i = 0$  and  $p_i = q_i$  for all  $i = 1, \dots, m$ . More precisely, we have the following result.

COROLLARY 2. Let  $1 \leq q, q_i < \infty$ , and  $\omega, \omega_i$  be arbitrary weighted functions.

(i) If

$$\mathcal{C}_{\text{sup}} = \int_{\mathbb{R}^d} |\Phi(y)| \prod_{i=1}^m |\det A_i(y)|^{-1/q_i} \Psi_{\text{sup}}(y) dy < \infty,$$

then  $\mathcal{H}_{\Phi, \vec{A}}$  is bounded from  $L_{\omega_1}^{q_1}(\mathbb{R}^n) \times \dots \times L_{\omega_m}^{q_m}(\mathbb{R}^n)$  to  $L_{\omega}^q(\mathbb{R}^n)$ . Moreover,

$$\|\mathcal{H}_{\Phi, \vec{A}}\|_{\prod_{i=1}^m L_{\omega_i}^{q_i}(\mathbb{R}^n) \rightarrow L_{\omega}^q(\mathbb{R}^n)} \leq \mathcal{C}_{\text{sup}}.$$

(ii) Let  $\Phi$  be a non-negative function, and

$$\text{ess sup}_{y \in \mathbb{R}^d, i=1, \dots, m} \|A_i(y)\| \cdot \|A_i^{-1}(y)\| = \rho < \infty.$$

Suppose  $\mathcal{H}_{\Phi, \vec{A}}$  is bounded from  $L_{\omega_1}^{q_1}(\mathbb{R}^n) \times \dots \times L_{\omega_m}^{q_m}(\mathbb{R}^n)$  to  $L_{\omega}^q(\mathbb{R}^n)$ . We thus have

$$\mathcal{C}_{\text{inf}} = \int_{\mathbb{R}^d} \Phi(y) \prod_{i=1}^m \|A_i(y)\|^{-n/q_i} \Psi_{\text{inf}}(y) dy < \infty.$$

Moreover,

$$\|\mathcal{H}_{\Phi, \vec{A}}\|_{\prod_{i=1}^m L_{\omega_i}^{q_i}(\mathbb{R}^n) \rightarrow L_{\omega}^q(\mathbb{R}^n)} \geq \mathcal{C}_{\text{inf}}.$$

It is worth pointing out that Corollary 3 extends and strengthens the results of Theorem 3.1 and Theorem 3.2 in [3] to the setting of multilinear Hausdorff operator.

By virtue of Theorem 2 and Corollary 2, we have some necessary and sufficient conditions for the boundedness of  $\mathcal{H}_{\Phi, \vec{s}}$  on the product of weighted Herz spaces, and sharp bound of  $\mathcal{H}_{\Phi, \vec{s}}$  on the product of Lebesgue spaces with power weights. Namely, the following is true.

COROLLARY 3. Let  $1 \leq p < \infty$ , and  $\Phi$  be a non-negative function. Let  $\omega(x) = |x|^\beta$ ,  $\omega_i(x) = |x|^{\beta_i}$  for all  $i = 1, \dots, m$ , such that

$$\sum_{i=1}^m \frac{\beta_i}{q_i} = \frac{\beta}{q}.$$

Then,  $\mathcal{H}_{\Phi, \vec{s}}$  is a bounded operator from  $\prod_{i=1}^m \dot{K}_{p_i, q_i}^{\alpha_i}(\mathbb{R}^n, \omega_i)$  to  $\dot{K}_{p, q}^{\alpha}(\mathbb{R}^n, \omega)$  if and only if

$$\mathcal{D} = \int_{\mathbb{R}^d} \Phi(y) \prod_{i=1}^m |s_i(y)|^{-\alpha_i - (n + \beta_i)/q_i} dy < \infty.$$

Moreover,

$$\|\mathcal{H}_{\Phi, \vec{s}}\|_{\prod_{i=1}^m \dot{K}_{p_i, q_i}^{\alpha_i}(\omega_i) \rightarrow \dot{K}_{p, q}^{\alpha}(\omega)} \simeq \mathcal{D}.$$

In particular, we have

$$\|\mathcal{H}_{\Phi, \vec{s}}\|_{\prod_{i=1}^m L_{\omega_i}^{q_i}(\mathbb{R}^n) \rightarrow L_{\omega}^q(\mathbb{R}^n)} = \int_{\mathbb{R}^d} \Phi(y) \prod_{i=1}^m |s_i(y)|^{-(n + \beta_i)/q_i} dy.$$

Finally, we shall discuss the boundedness of commutators of  $\mathcal{H}_{\Phi, \vec{A}}$  with the symbols in the Lipschitz spaces on the product of weighted Morrey–Herz spaces.

**THEOREM 3.** *Let  $0 < p < \infty$ ,  $1 \leq q^* \leq q < \infty$ , and  $\lambda_i \geq 0$ ,  $\gamma, \gamma_i > 0$ ,  $r_i > 0$ ,  $\alpha^*, \alpha_i$  be real numbers such that*

$$\frac{1}{q^*} = \frac{1}{q} + \sum_{i=1}^m \frac{1}{r_i}, \quad \gamma = \gamma_1 + \dots + \gamma_m,$$

$$\alpha^* = \alpha - \gamma - \sum_{i=1}^m \frac{n}{r_i}.$$

Let  $b_i \in \Lambda_{\gamma_i}(\mathbb{R}^n)$ , and  $\omega, \omega_i$  be arbitrary weighted functions. Assume that

$$\mathcal{D}_{\text{sup}} = \int_{\mathbb{R}^d} |\Phi(y)| \Psi_{\text{sup}}(y) \prod_{i=1}^m \mathcal{K}_i(y) dy < \infty,$$

where

$$\mathcal{K}_i(y) := \|A_i(y) - I\|^\gamma \|A_i(y)\|^{(\lambda_i - \alpha_i)} |\det A_i(y)|^{-1/q_i} \left( \sum_{k=v-1}^0 2^{k(\lambda_i - \alpha_i)} \right),$$

and  $v$  is given in Theorem 2. Then the commutator  $\mathcal{H}_{\Phi, \vec{A}}^{\vec{b}}$  is determined as a bounded operator from  $\prod_{i=1}^m \dot{MK}_{p_i, q_i}^{\alpha_i, \lambda_i}(\mathbb{R}^n, \omega_i)$  to  $\dot{MK}_{p, q^*}^{\alpha^*, \lambda}(\mathbb{R}^n, \omega^{q^*/q})$  for any  $\lambda > 0$  and  $0 < p < \infty$ , or  $\lambda \geq 0$  and  $1 \leq p < \infty$ .

*Proof.* For simplicity of exposition, we denote

$$\Phi_1(y) = \Psi_{\text{sup}}(y) |\Phi(y)|, \quad \Phi_2(y) = \left( \prod_{i=1}^m \|A_i(y) - I_n\|^\gamma \right) \Phi_1(y),$$

$$\Phi_3(y) = \left( \prod_{i=1}^m |\det A_i(y)|^{-1/q_i} \right) \Phi_2(y), \quad \text{and} \quad \|\vec{b}\|_{\Lambda_\gamma} = \prod_{i=1}^m \|b_i\|_{\Lambda_{\gamma_i}}.$$

It is obvious that for all  $b_i \in \Lambda_{\gamma_i}(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ , we have

$$\prod_{i=1}^m |b_i(x) - b_i(A_i(y)x)| \leq \|\vec{b}\|_{\Lambda_\gamma} \prod_{i=1}^m \|A_i(y) - I_n\|^\gamma |x|^{\gamma_i}.$$

Let us write  $v = \omega^{q^*/q}$  for simplicity. By the Minkowski inequality, the Hölder in-

equality, and change of variables, we have

$$\begin{aligned}
 & \| \mathcal{H}_{\Phi, \vec{A}}^{\vec{b}}(\vec{f}) \chi_k \|_{L_{\omega}^{q^*}} \\
 & \leq \int_{\mathbb{R}^d} \left( \int_{S_k} \prod_{i=1}^m |f_i(A_i(y)x)|^{q^*} \prod_{i=1}^m |b_i(x) - b_i(A_i(y)x)|^{q^*} \nu(x) dx \right)^{1/q^*} |\Phi(y)| dy \\
 & \leq \int_{\mathbb{R}^d} \left( \int_{S_k} \prod_{i=1}^m (|f_i(A_i(y)x)|^{q^*} \omega_i(A_i(y)x)^{q^*/q_i}) \prod_{i=1}^m |b_i(x) - b_i(A_i(y)x)|^{q^*} dx \right)^{1/q^*} \Phi_1(y) dy \\
 & \leq \int_{\mathbb{R}^d} \prod_{i=1}^m \left( \int_{S_k} |f_i(A_i(y)x)|^{q_i} \omega_i(A_i(y)x) dx \right)^{1/q_i} \\
 & \quad \times \prod_{i=1}^m \left( \int_{S_k} |b_i(x) - b_i(A_i(y)x)|^{r_i} dx \right)^{1/r_i} \Phi_1(y) dy \\
 & \leq \|\vec{b}\|_{\Lambda_\gamma} \int_{\mathbb{R}^d} \prod_{i=1}^m \left( \int_{S_k} |f_i(A_i(y)x)|^{q_i} \omega_i(A_i(y)x) dx \right)^{1/q_i} \prod_{i=1}^m \left( \int_{S_k} |x|^{r_i} dx \right)^{1/r_i} \Phi_2(y) dy \\
 & \lesssim \|\vec{b}\|_{\Lambda_\gamma} 2^{k(\alpha - \alpha^*)} \int_{\mathbb{R}^d} \prod_{i=1}^m \left( \sum_{j=v-1}^0 \|f_i \chi_{k+\ell_i+j}\|_{L_{\omega_i}^{q_i}} \right) \Phi_3(y) dy, \tag{3.18}
 \end{aligned}$$

where recall that  $2^{\ell_i-1} < \|A_i(y)\| \leq 2^{\ell_i}$ . From the definition of weighted Morrey–Herz spaces, we have

$$\begin{aligned}
 & \| \mathcal{H}_{\Phi, \vec{A}}^{\vec{b}}(\vec{f}) \|_{MK_{p,q^*}^{\alpha^*, \lambda}(\nu)} \\
 & = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha^* p} \| \mathcal{H}_{\Phi, \vec{A}}^{\vec{b}}(\vec{f}) \chi_k \|_{L_{\omega}^{q^*}}^p \right)^{1/p} \\
 & \lesssim \|\vec{b}\|_{\Lambda_\gamma} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p} \left( \int_{\mathbb{R}^d} \prod_{i=1}^m \left( \sum_{j=v-1}^0 \|f_i \chi_{k+\ell_i+j}\|_{L_{\omega_i}^{q_i}} \right) \Phi_3(y) dy \right)^p \right)^{1/p}
 \end{aligned}$$

Analogous to the proof of Theorem 2 for the case (i), we can prove that

$$\| \mathcal{H}_{\Phi, \vec{A}}^{\vec{b}}(\vec{f}) \|_{MK_{p,q^*}^{\alpha^*, \lambda}(\nu)} \lesssim \mathcal{D}_{\text{sup}} \cdot \|\vec{b}\|_{\Lambda_\gamma} \cdot \prod_{i=1}^m \|f_i\|_{MK_{p_i, q_i}^{\alpha_i, \lambda_i}(\omega_i)},$$

for all  $0 < p < 1$  and  $\lambda > 0$ , or  $1 \leq p < \infty$  and  $\lambda \geq 0$ . Thus, Theorem 3 is proved completely.  $\square$

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