Operators
and
Matrices
Volume 16, Number 1 (2022), 35–68 doi:10.7153/oam-2022-16-05

CONVERGENCE OF THE CONJUGATE GRADIENT METHOD WITH UNBOUNDED OPERATORS

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Abstract. In the framework of inverse linear problems on infinite-dimensional Hilbert space, we prove the convergence of the conjugate gradient iterates to an exact solution to the inverse problem in the most general case where the self-adjoint, non-negative operator is unbounded and with minimal, technically unavoidable assumptions on the initial guess of the iterative algorithm. The convergence is proved to always hold in the Hilbert space norm (error convergence), as well as at other levels of regularity (energy norm, residual, etc.) depending on the regularity of the iterates. We also discuss, both analytically and through a selection of numerical tests, the main features and differences of our convergence result as compared to the case, already available in the literature, where the operator is bounded.

Mathematics subject classification (2020): 33C47, 41A65, 46N40, 47B25, 47N40.
Keywords and phrases: Inverse linear problems, infinite-dimensional Hilbert space, ill-posed problems, Krylov subspaces methods, conjugate gradient, self-adjoint operators, spectral measure, orthogonal polynomials.

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Paper OaM-16-05

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