

A PSEUDOSPECTRAL MAPPING THEOREM FOR OPERATOR PENCILS

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Abstract. Let X be a complex Banach space and $BL(X)$ is the Banach algebra of all bounded linear operators on X . For $A, B \in BL(X)$, $n \in \mathbb{Z}_+$, and $\varepsilon > 0$, we define the (n, ε) -pseudospectrum of linear operator pencil (A, B) as

$$\Lambda_{n,\varepsilon}(A, B) = \sigma(A, B) \cup \left\{ \lambda \in \mathbb{C} : \left\| (\lambda B - A)^{-2^n} \right\|^{\frac{1}{2^n}} \geq \varepsilon^{-1} \right\}.$$

Here $\sigma(A, B)$ denotes the spectrum of the linear operator pencil (A, B) . This article establishes certain properties of (n, ε) -pseudospectrum of operator pencils. We prove the Spectral Mapping Theorem for operator pencils. We also find an analogue of the Spectral Mapping Theorem for pseudospectrum and (n, ε) -pseudospectrum of operator pencils. Some examples are provided to illustrate the findings.

1. Introduction

Throughout this article \mathbb{Z}_+ denotes the set of all positive integers, X denotes a complex Banach space, and $BL(X)$ is the Banach algebra of all bounded linear operators on X . For $\lambda \in \mathbb{C}$ and $r > 0$, define

$$D(\lambda, r) = \{z \in \mathbb{C} : |z - \lambda| \leq r\}.$$

DEFINITION 1.1. Let $A \in BL(X)$, $n \in \mathbb{Z}_+$, and $\varepsilon > 0$. The (n, ε) -pseudospectrum of A is defined by

$$\Lambda_{n,\varepsilon}(A) = \sigma(A) \cup \left\{ \lambda \in \mathbb{C} : \left\| (\lambda I - A)^{-2^n} \right\|^{\frac{1}{2^n}} \geq \varepsilon^{-1} \right\}.$$

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The (n, ε) -pseudospectrum generalizes the pseudospectrum. For $A \in BL(X)$, we have $\Lambda_{0,\varepsilon}(A) = \Lambda_\varepsilon(A)$, the ε -pseudospectrum of A . The (n, ε) -pseudospectrum evolved from the basic question on how to approximate the spectrum of a linear operator on separable Hilbert spaces. In [8], Hansen observed that numerical computation of the spectrum of an operator can result in the spectrum of a slightly perturbed operator. This is due to the discontinuous behaviour of the spectrum and it is a concern for numerical analysts. As a better approximation to the spectrum Hansen introduced the (n, ε) -pseudospectrum of an operator on a separable Hilbert space (see [7]). The approximating properties of (n, ε) -pseudospectrum may found in [8, 9]. For more information on pseudospectrum and (n, ε) -pseudospectrum, one may refer to [4, 10, 16, 17].

Let $A, B \in BL(X)$. The generalized eigenvalue problem is defined by

$$Ax = \lambda Bx,$$

where $x \neq 0$ and $\lambda \in \mathbb{C}$. The generalized eigenvalue problems lie at the heart of dynamical problems concerning various engineering structures. Operator pencils arise in quantum mechanics, control theory, epidemic models in biology, numerical solutions to differential equations and hence play an important role in numerical analysis and perturbation theory.

DEFINITION 1.2. Let $A, B \in BL(X)$. The spectrum of the linear operator pencil (A, B) is defined by

$$\sigma(A, B) = \{\lambda \in \mathbb{C} : \lambda B - A \text{ is not invertible}\}.$$

Throughout this paper, we are considering the linear operator pencil, and call it the operator pencil subsequently. The generalized resolvent of the operator pencil (A, B) is defined by

$$\rho(A, B) = \mathbb{C} \setminus \sigma(A, B).$$

EXAMPLE 1.3. Consider $A, B: \ell^2 \rightarrow \ell^2$ defined by $A(x_1, x_2, \dots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots, \frac{x_n}{n}, 0, \dots)$ and $B(x_1, x_2, \dots) = (x_1, x_2, \dots, x_n, 0, \dots)$. Then $\sigma(A, B) = \mathbb{C}$.

EXAMPLE 1.4. Consider $A, B: \ell^1 \rightarrow \ell^1$ defined by $A(x_1, x_2, \dots) = (x_1, x_2, \dots)$ and $B(x_1, x_2, \dots) = (x_2, x_3, \dots, x_n, 0, \dots)$. Then $\sigma(A, B) = \emptyset$.

REMARK 1.5. If B is invertible, then $\sigma(A, B) = \sigma(AB^{-1})$ and hence $\sigma(A, B)$ is non-empty and compact.

DEFINITION 1.6. Let $A, B \in BL(X)$ and $\varepsilon > 0$, the ε -pseudospectrum of the operator pencil (A, B) is defined by

$$\Lambda_\varepsilon(A, B) = \sigma(A, B) \cup \{\lambda \in \mathbb{C} : \|(\lambda B - A)^{-1}\| \geq \varepsilon^{-1}\}.$$

This article is devoted to studying (n, ε) -pseudospectrum of operator pencils. The following problems are addressed in the article.

1. To obtain the Spectral Mapping Theorem for operator pencils.
2. To find an analogue of the Spectral Mapping Theorem for pseudospectrum of operator pencils.
3. To find an analogue of the Spectral Mapping Theorem for (n, ε) -pseudospectrum of operator pencils.

The spectral mapping theorem for unbounded normal operators is proved in [2]. The analogue of the spectral mapping theorem for pseudospectrum, determinant spectrum and condition spectrum are also available, refer [1, 11, 12, 13, 14, 18]. The following is the outline of the article.

In Section 2, we give the definition of (n, ε) -pseudospectrum of an operator pencil and develop certain properties of the same. We also find various equivalent definitions of (n, ε) -pseudospectrum of an operator pencil (Theorem 2.7). In Section 3, we prove the Spectral Mapping Theorem for operator pencils (Theorem 3.1). We also find an analogue of the Spectral Mapping Theorem for pseudospectrum of operator pencils (Theorem 3.4, Theorem 3.5). In section 4, we find an analogue of the Spectral Mapping Theorem for (n, ε) -pseudospectrum of operator pencils (Theorem 4.2, Theorem 4.3). Examples are provided in each section to demonstrate the results developed.

2. Properties of (n, ε) -pseudospectrum of operator pencils

This section develops certain properties of (n, ε) -pseudospectrum of operator pencils. These results are used subsequently in the article.

DEFINITION 2.1. Let $A, B \in BL(X)$, $n \in \mathbb{Z}_+$, and $\varepsilon > 0$. The (n, ε) -pseudospectrum of the operator pencil (A, B) is denoted by $\Lambda_{n, \varepsilon}(A, B)$ and is defined by

$$\Lambda_{n, \varepsilon}(A, B) = \sigma(A, B) \cup \left\{ \lambda \in \mathbb{C} : \left\| (\lambda B - A)^{-2n} \right\|^{\frac{1}{2n}} \geq \varepsilon^{-1} \right\}.$$

The generalized (n, ε) -pseudoresolvent of the operator pencil (A, B) is defined by

$$\rho_{n, \varepsilon}(A, B) = \rho(A, B) \cap \left\{ \lambda \in \mathbb{C} : \left\| (\lambda B - A)^{-2n} \right\|^{\frac{1}{2n}} < \varepsilon^{-1} \right\}.$$

REMARK 2.2.

1. $\sigma(A, B) \subseteq \Lambda_{n, \varepsilon}(A, B)$ for every $n \in \mathbb{Z}_+$ and $\varepsilon > 0$.
2. If $B = I$, then $\Lambda_{n, \varepsilon}(A, I) = \Lambda_{n, \varepsilon}(A)$.
3. If $n = 0$, then $\Lambda_{0, \varepsilon}(A, B) = \Lambda_{\varepsilon}(A, B)$.

DEFINITION 2.3. Let $A, B \in BL(X)$, $n \in \mathbb{Z}_+ \cup \{0\}$, and $\lambda \in \mathbb{C}$. Define $\gamma_{A,B}^n : \mathbb{C} \rightarrow [0, \infty)$ as

$$\gamma_{A,B}^n(\lambda) = \begin{cases} \|(\lambda B - A)^{-2^n}\|^{\frac{1}{2^n}}, & \text{if } \lambda \notin \sigma(A, B) \\ 0, & \text{if } \lambda \in \sigma(A, B). \end{cases}$$

THEOREM 2.4. Let $A, B \in BL(X)$, $n \in \mathbb{Z}_+ \cup \{0\}$, and $\varepsilon > 0$. Then the following holds.

(i) If B is invertible, then $\gamma_{A,B}^n$ is continuous.

(ii) $\Lambda_{n,\varepsilon}(A, B) = \{ \lambda \in \mathbb{C} : \gamma_{A,B}^n(\lambda) \leq \varepsilon \}$.

(iii) $\Lambda_{n+1,\varepsilon}(A, B) \subseteq \Lambda_{n,\varepsilon}(A, B)$.

(iv) $\Lambda_{n,\varepsilon_1}(A, B) \subseteq \Lambda_{n,\varepsilon_2}(A, B)$ for every $0 < \varepsilon_1 \leq \varepsilon_2$.

(v) $\sigma(A, B) = \bigcap_{\varepsilon > 0} \Lambda_{n,\varepsilon}(A, B)$.

(vi) $\Lambda_{n,\varepsilon}(\alpha A, \alpha B) = \Lambda_{n,\frac{\varepsilon}{|\alpha|}}(A, B)$ for $\alpha \neq 0$.

(vii) $\Lambda_{n,\varepsilon}(\beta A + \alpha B, B) = \alpha + \beta \Lambda_{n,\frac{\varepsilon}{|\beta|}}(A, B)$ for $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$.

(viii) If X is a Hilbert space, then $\lambda \in \Lambda_{n,\varepsilon}(A, B) \iff \bar{\lambda} \in \Lambda_{n,\varepsilon}(A^*, B^*)$.

Proof.

(i) If B is invertible, then

$$\begin{aligned} \sigma(A, B) &= \{ \lambda \in \mathbb{C} : \lambda B - A \text{ is not invertible} \} \\ &= \{ \lambda \in \mathbb{C} : \lambda I - AB^{-1} \text{ is not invertible} \} \\ &= \sigma(AB^{-1}). \end{aligned}$$

Suppose $\lambda_m \in \mathbb{C} \setminus \sigma(A, B)$ and $\lambda_m \rightarrow \lambda$ for some $\lambda \notin \sigma(A, B)$. Then

$$\gamma_{A,B}^n(\lambda_m) = \|(\lambda_m B - A)^{-2^n}\|^{\frac{1}{2^n}} \rightarrow \|(\lambda B - A)^{-2^n}\|^{\frac{1}{2^n}} = \gamma_{A,B}^n(\lambda).$$

If $\lambda_m \in \mathbb{C} \setminus \sigma(A, B)$ and $\lambda_m \rightarrow \lambda$ for some $\lambda \in \sigma(A, B)$. From Lemma 10.17 of [15], $\|(\lambda_m I - AB^{-1})^{-2^n}\| \rightarrow \infty$ and $\gamma_{A,B}^n(\lambda_m) \rightarrow 0 = \gamma_{A,B}^n(\lambda)$. Hence $\gamma_{A,B}^n$ is continuous.

(ii) This follows from the Definition 2.1 and 2.3.

(iii) For $\lambda \in \rho(A, B)$, denote $(\lambda B - A)^{-1}$ by $R(\lambda, A, B)$. Then

$$\begin{aligned} \Lambda_{n,\varepsilon}(A, B) &= \sigma(A, B) \cup \left\{ \lambda \in \mathbb{C} : \left\| R(\lambda, A, B)^{2^n} \right\|^{\frac{1}{2^n}} \geq \varepsilon^{-1} \right\} \\ &= \sigma(A, B) \cup \left\{ \lambda \in \mathbb{C} : \frac{1}{\left\| R(\lambda, A, B)^{2^n} \right\|^{\frac{1}{2^n}}} \leq \varepsilon \right\}. \end{aligned}$$

The result is true from the following fact,

$$\begin{aligned} \frac{1}{\left\| R(\lambda, A, B)^{2^{n+1}} \right\|^{\frac{1}{2^{n+1}}}} &\geq \frac{1}{\left\| R(\lambda, A, B)^{2^n} \right\|^{\frac{1}{2^{n+1}}} \left\| R(\lambda, A, B)^{2^n} \right\|^{\frac{1}{2^{n+1}}}} \\ &= \frac{1}{\left\| R(\lambda, A, B)^{2^n} \right\|^{\frac{1}{2^n}}}. \end{aligned}$$

(iv) Let $\lambda \in \Lambda_{n,\varepsilon_1}(A, B)$ and $0 < \varepsilon_1 \leq \varepsilon_2$. Then

$$\left\| (\lambda B - A)^{-2^n} \right\|^{\frac{1}{2^n}} \geq \frac{1}{\varepsilon_1} \geq \frac{1}{\varepsilon_2}.$$

Hence $\lambda \in \Lambda_{n,\varepsilon_2}(A, B)$.

(v) Observe that

$$\begin{aligned} \lambda \in \bigcap_{\varepsilon > 0} \Lambda_{n,\varepsilon}(A, B) &\iff \gamma_{A,B}^n(\lambda) \leq \varepsilon \text{ for every } \varepsilon > 0 \\ &\iff \gamma_{A,B}^n(\lambda) = 0 \iff \lambda \in \sigma(A, B). \end{aligned}$$

(vi) Let $\alpha \neq 0$, then $\sigma(\alpha A, \alpha B) = \sigma(A, B)$. Also

$$\begin{aligned} \lambda \in \Lambda_{n,\varepsilon}(\alpha A, \alpha B) \setminus \sigma(\alpha A, \alpha B) &\iff \left\| (\lambda \alpha B - \alpha A)^{-2^n} \right\|^{\frac{1}{2^n}} \geq \varepsilon^{-1} \\ &\iff \left\| (\lambda B - A)^{-2^n} \right\|^{\frac{1}{2^n}} \geq \frac{|\alpha|}{\varepsilon} \\ &\iff \lambda \in \Lambda_{n, \frac{\varepsilon}{|\alpha|}}(A, B) \setminus \sigma(A, B). \end{aligned}$$

(vii) Let $\alpha, \beta \in \mathbb{C}$ and $\beta \neq 0$, then

$$\begin{aligned} \sigma(\beta A + \alpha B, B) &= \{ \lambda \in \mathbb{C} : \lambda B - \beta A - \alpha B \text{ is not invertible} \} \\ &= \left\{ \lambda \in \mathbb{C} : \frac{\lambda - \alpha}{\beta} B - A \text{ is not invertible} \right\}. \end{aligned}$$

i.e., $\lambda \in \sigma(\beta A + \alpha B, B) \iff \frac{\lambda - \alpha}{\beta} \in \sigma(A, B) \iff \lambda \in \alpha + \beta \sigma(A, B)$. Also

$$\begin{aligned} \lambda \in \Lambda_{n,\varepsilon}(\beta A + \alpha B, B) \setminus \sigma(\beta A + \alpha B, B) &\iff \left\| (\lambda B - \beta A - \alpha B)^{-2n} \right\|^{\frac{1}{2^n}} \geq \frac{1}{\varepsilon} \\ &\iff \frac{1}{|\beta|} \left\| \left(\frac{\lambda - \alpha}{\beta} B - A \right)^{-2n} \right\|^{\frac{1}{2^n}} \geq \frac{1}{\varepsilon} \\ &\iff \frac{\lambda - \alpha}{\beta} \in \Lambda_{n, \frac{\varepsilon}{|\beta|}}(A, B) \setminus \sigma(A, B) \\ &\iff \lambda \in \alpha + \beta \Lambda_{n, \frac{\varepsilon}{|\beta|}}(A, B) \setminus \sigma(A, B). \end{aligned}$$

(viii) If $\lambda \in \sigma(A, B) \iff \bar{\lambda} \in \sigma(A^*, B^*)$. If $\lambda \in \Lambda_{n,\varepsilon}(A, B) \setminus \sigma(A, B)$, then

$$\left\| (\lambda B - A)^{-2n} \right\|^{\frac{1}{2^n}} = \left\| \left(\bar{\lambda} B^* - A^* \right)^{-2n} \right\|^{\frac{1}{2^n}} \geq \frac{1}{\varepsilon}.$$

Hence $\Lambda_{n,\varepsilon}(A, B) \subseteq \Lambda_{n,\varepsilon}(A^*, B^*)$. The other inclusion also follows similarly. \square

THEOREM 2.5. Let $A, B \in BL(X)$, $n \in \mathbb{Z}_+ \cup \{0\}$, and $\varepsilon > 0$. If B is invertible, then

(i) $\Lambda_{n,\varepsilon}(A, B) \subseteq D(0, \|AB^{-1}\| + \varepsilon \|B^{-1}\|)$.

(ii) $\Lambda_{n,\varepsilon}(A, B)$ is compact.

(iii) If $AB = BA$, then

$$\Lambda_{n, \frac{\varepsilon}{\|B^{2n}\|}}(AB^{-1}) \subseteq \Lambda_{n,\varepsilon}(A, B) \subseteq \Lambda_{n,\varepsilon \|B^{-2n}\|} \frac{1}{\|B^{2n}\|}(AB^{-1}).$$

(iv) If A is invertible, $AB = BA$, and $k_1 = |\lambda^{-1}| \|A^{2n}\| \frac{1}{\|B^{-2n}\|}$ for some $\lambda \neq 0$. Then

$$\lambda \in \Lambda_{n,\varepsilon}(A^{-1}, B) \implies \frac{1}{\lambda} \in \Lambda_{n,\varepsilon k_1}(A, B^{-1}).$$

Further if $k_2 = |\lambda^{-1}| \|A^{-2n}\| \frac{1}{\|B^{2n}\|}$ for some $\lambda \neq 0$, then

$$\lambda \in \Lambda_{n,\varepsilon}(A, B^{-1}) \implies \frac{1}{\lambda} \in \Lambda_{n,\varepsilon k_2}(A^{-1}, B).$$

Proof.

(i) If B is invertible, then $\sigma(A, B) = \sigma(AB^{-1})$. Hence $\lambda \in \sigma(A, B)$ implies $\lambda \in D(0, \|AB^{-1}\|)$. If $\lambda \in \Lambda_{n,\varepsilon}(A, B) \setminus \sigma(A, B)$, then

$$\left\| (\lambda B - A)^{-2n} \right\|^{\frac{1}{2^n}} = \left\| [B^{-1}(\lambda I - AB^{-1})^{-1}]^{2n} \right\|^{\frac{1}{2^n}} \leq \|B^{-1}\| \|(\lambda I - AB^{-1})^{-1}\|.$$

Suppose $|\lambda| > \|AB^{-1}\| + \varepsilon \|B^{-1}\| > \|AB^{-1}\|$, then $(\lambda I - AB^{-1})$ is invertible.

$$\left\| (\lambda B - A)^{-2n} \right\|^{\frac{1}{2^n}} \leq \|B^{-1}\| \left\| (\lambda I - AB^{-1})^{-1} \right\| \leq \|B^{-1}\| \frac{1}{|\lambda| - \|AB^{-1}\|} < \frac{1}{\varepsilon}.$$

Hence $\lambda \notin \Lambda_{n,\varepsilon}(A, B)$ and $\Lambda_{n,\varepsilon}(A, B) \subseteq D(0, \|AB^{-1}\| + \varepsilon \|B^{-1}\|)$.

(ii) This follows from (i), (ii) of Theorem 2.4 and (i) of Theorem 2.5.

(iii) We have $\sigma(A, B) = \sigma(AB^{-1})$. Suppose $AB = BA$ and $\lambda \in \Lambda_{n, \frac{\varepsilon}{\|B^{2n}\|^{\frac{1}{2^n}}}}(AB^{-1}) \setminus \sigma(AB^{-1})$, then

$$\begin{aligned} \frac{\|B^{2n}\|^{\frac{1}{2^n}}}{\varepsilon} &\leq \left\| (\lambda I - AB^{-1})^{-2n} \right\|^{\frac{1}{2^n}} \\ &= \left\| [B(\lambda B - A)^{-1}]^{2n} \right\|^{\frac{1}{2^n}} \\ &\leq \left\| B^{2n} \right\|^{\frac{1}{2^n}} \left\| (\lambda B - A)^{-2n} \right\|^{\frac{1}{2^n}}. \end{aligned}$$

Hence $\lambda \in \Lambda_{n,\varepsilon}(A, B) \setminus \sigma(A, B)$. Further assume that $AB = BA$ and $\lambda \in \Lambda_{n,\varepsilon}(A, B) \setminus \sigma(A, B)$, then

$$\begin{aligned} \frac{1}{\varepsilon} &\leq \left\| (\lambda B - A)^{-2n} \right\|^{\frac{1}{2^n}} \\ &= \left\| [(\lambda I - AB^{-1})B]^{-2n} \right\|^{\frac{1}{2^n}} \\ &\leq \left\| B^{-2n} \right\|^{\frac{1}{2^n}} \left\| (\lambda I - AB^{-1})^{-2n} \right\|^{\frac{1}{2^n}}. \end{aligned}$$

Hence $\lambda \in \Lambda_{n,\varepsilon\|B^{-2n}\|^{\frac{1}{2^n}}}(AB^{-1}) \setminus \sigma(AB^{-1})$.

(iv) Suppose A, B are invertible and $AB = BA$. Let $\lambda \neq 0$, then

$$\begin{aligned} \lambda \in \sigma(A^{-1}, B) &\iff \lambda B - A^{-1} \text{ is not invertible} \\ &\iff -\lambda B \left(\frac{B^{-1}}{\lambda} - A \right) A^{-1} \text{ is not invertible} \\ &\iff \frac{1}{\lambda} \in \sigma(A, B^{-1}). \end{aligned}$$

Further if $\lambda \in \Lambda_{n,\varepsilon}(A^{-1}, B) \setminus \sigma(A^{-1}, B)$, then

$$\begin{aligned} \frac{1}{\varepsilon} &\leq \left\| (\lambda B - A^{-1})^{-2^n} \right\|^{\frac{1}{2^n}} \\ &= \left\| \left[-(\lambda)^{-1} A \left(\frac{B^{-1}}{\lambda} - A \right)^{-1} B^{-1} \right]^{2^n} \right\|^{\frac{1}{2^n}} \\ &\leq |\lambda^{-1}| \left\| A^{2^n} \right\|^{\frac{1}{2^n}} \left\| \left[\left(\frac{B^{-1}}{\lambda} - A \right)^{-1} \right]^{2^n} \right\|^{\frac{1}{2^n}} \|B^{-2^n}\|^{\frac{1}{2^n}}. \end{aligned}$$

Define $k_1 = |\lambda^{-1}| \left\| A^{2^n} \right\|^{\frac{1}{2^n}} \|B^{-2^n}\|^{\frac{1}{2^n}}$, then $\frac{1}{\varepsilon} \in \Lambda_{n,\varepsilon k_1}(A, B^{-1}) \setminus \sigma(A, B^{-1})$.

Next assume that A, B are invertible, $AB = BA$, and $\lambda \in \Lambda_{n,\varepsilon}(A, B^{-1}) \setminus \sigma(A, B^{-1})$. Then,

$$\begin{aligned} \frac{1}{\varepsilon} &\leq \left\| (\lambda B^{-1} - A)^{-2^n} \right\|^{\frac{1}{2^n}} \\ &= \left\| \left[-(\lambda)^{-1} A^{-1} \left(\frac{B}{\lambda} - A^{-1} \right)^{-1} B \right]^{2^n} \right\|^{\frac{1}{2^n}} \\ &\leq |\lambda^{-1}| \left\| A^{-2^n} \right\|^{\frac{1}{2^n}} \left\| B^{2^n} \right\|^{\frac{1}{2^n}} \left\| \left[\left(\frac{B}{\lambda} - A^{-1} \right)^{-1} \right]^{2^n} \right\|^{\frac{1}{2^n}}. \end{aligned}$$

Define $k_2 = |\lambda^{-1}| \left\| A^{-2^n} \right\|^{\frac{1}{2^n}} \|B^{2^n}\|^{\frac{1}{2^n}}$, then $\frac{1}{\varepsilon} \in \Lambda_{n,\varepsilon k_2}(A^{-1}, B) \setminus \sigma(A^{-1}, B)$. \square

PROPOSITION 2.6. *Let $A, B, V \in BL(X)$, V is invertible, $n \in \mathbb{Z}_+ \cup \{0\}$, and $\varepsilon > 0$. Define $T = VAV^{-1}$ and $S = VBV^{-1}$. Then*

$$\Lambda_{n,\frac{\varepsilon}{k}}(T, S) \subseteq \Lambda_{n,\varepsilon}(A, B) \subseteq \Lambda_{n,\varepsilon k}(T, S),$$

where $k = \|V\|^{\frac{1}{2^n}} \|V^{-1}\|^{\frac{1}{2^n}}$.

Proof. Since $T = VAV^{-1}$ and $S = VBV^{-1}$,

$$\sigma(A, B) = \{\lambda \in \mathbb{C} : \lambda V^{-1} S V - V^{-1} T V \text{ is not invertible}\} = \sigma(T, S).$$

Define $k = \|V\|^{\frac{1}{2n}} \|V^{-1}\|^{\frac{1}{2n}}$, then

$$\begin{aligned} \left\| [(\lambda S - T)^{-1}]^{2n} \right\|^{\frac{1}{2n}} &= \left\| [(\lambda V B V^{-1} - V A V^{-1})^{-1}]^{2n} \right\|^{\frac{1}{2n}} \\ &= \left\| [V^{-1}(\lambda B - A)^{-1}V]^{2n} \right\|^{\frac{1}{2n}} \\ &\leq \|V^{-1}\|^{\frac{1}{2n}} \left\| (\lambda B - A)^{-2n} \right\|^{\frac{1}{2n}} \|V\|^{\frac{1}{2n}} \\ &\leq k \left\| (\lambda B - A)^{-2n} \right\|^{\frac{1}{2n}}. \end{aligned}$$

Hence $\Lambda_{n, \frac{\varepsilon}{k}}(T, S) \subseteq \Lambda_{n, \varepsilon}(A, B)$. Similarly

$$\begin{aligned} \left\| [(\lambda B - A)^{-1}]^{2n} \right\|^{\frac{1}{2n}} &= \left\| [V(\lambda S - T)^{-1}V^{-1}]^{2n} \right\|^{\frac{1}{2n}} \\ &\leq \|V\|^{\frac{1}{2n}} \left\| (\lambda S - T)^{-2n} \right\|^{\frac{1}{2n}} \|V^{-1}\|^{\frac{1}{2n}} \\ &\leq k \left\| (\lambda S - T)^{-2n} \right\|^{\frac{1}{2n}}. \end{aligned}$$

Hence $\Lambda_{n, \varepsilon}(A, B) \subseteq \Lambda_{n, \varepsilon k}(T, S)$. \square

The following theorem presents some equivalent definitions for (n, ε) -pseudospectrum of operator pencils.

THEOREM 2.7. *Let $A, B \in BL(X)$, $n \in \mathbb{Z}_+ \cup \{0\}$, and $\varepsilon > 0$. Then the following are equivalent.*

- (i) $\lambda \in \Lambda_{n, \varepsilon}(A, B)$.
- (ii) $\|(\lambda B - A)^{2n} v\| \leq \varepsilon^{2n}$ for some $v \in X$ with $\|v\| = 1$.
- (iii) $[(\lambda B - A)^{2n} - E] v = 0$ for some $E \in BL(X)$ with $\|E\| \leq \varepsilon^{2n}$ and $v \in X$ with $\|v\| = 1$.

Proof. If $\lambda \in \sigma(A, B)$, then take v as the normalized eigenvector corresponding to the generalized eigenvalue λ and $E = 0$. Hence (i), (ii) and (iii) are equivalent.

(i) \implies (ii). If $\lambda \in \Lambda_{n, \varepsilon}(A, B) \setminus \sigma(A, B)$, then $\|(\lambda B - A)^{-2n}\|^{\frac{1}{2n}} \geq \frac{1}{\varepsilon}$. Hence there exists $u \in X$ with $u \neq 0$ such that $\|(\lambda B - A)^{-2n} u\|^{\frac{1}{2n}} \geq \frac{\|u\|^{\frac{1}{2n}}}{\varepsilon}$. Define $\tilde{v} = (\lambda B - A)^{-2n} u$, then $u = (\lambda B - A)^{2n} \tilde{v}$ and

$$\varepsilon \geq \frac{\|u\|^{\frac{1}{2n}}}{\|\tilde{v}\|^{\frac{1}{2n}}} = \frac{\|(\lambda B - A)^{2n} \tilde{v}\|^{\frac{1}{2n}}}{\|\tilde{v}\|^{\frac{1}{2n}}} = \left\| (\lambda B - A)^{2n} \frac{\tilde{v}}{\|\tilde{v}\|} \right\|^{\frac{1}{2n}}.$$

Define $v = \frac{\tilde{v}}{\|\tilde{v}\|}$, then $\|v\| = 1$ and $\|(\lambda B - A)^{2n} v\| \leq \varepsilon^{2n}$.

(ii) \implies (iii). Suppose there exists $v \in X$ with $\|v\| = 1$ and $\|(\lambda B - A)^{2^n} v\| \leq \varepsilon^{2^n}$. Then there exists $\phi \in X'$ with $\|\phi\| = 1$ and $\phi(v) = 1$ (Corollary 3.3 of [15]). Define a rank one operator $E : X \rightarrow X$

$$E(w) = \phi(w)(\lambda B - A)^{2^n} v.$$

Then $\|E\| \leq \varepsilon^{2^n}$ and $[(\lambda B - A)^{2^n} - E]v = 0$.

(iii) \implies (i). Suppose there exists $E \in BL(X)$ with $\|E\| \leq \varepsilon^{2^n}$ and a unit vector $v \in X$ such that $[(\lambda B - A)^{2^n} - E]v = 0$. For $\lambda \notin \sigma(A, B)$, define $v = (\lambda B - A)^{-2^n} E v$. Then

$$1 = \|v\| = \left\| (\lambda B - A)^{-2^n} E v \right\| \leq \left\| (\lambda B - A)^{-2^n} \right\| \varepsilon^{2^n}.$$

Hence $\left\| (\lambda B - A)^{-2^n} \right\| \geq \frac{1}{\varepsilon^{2^n}}$ and $\lambda \in \Lambda_{n,\varepsilon}(A, B) \setminus \sigma(A, B)$. \square

COROLLARY 2.8. *Let $A, B \in BL(X)$, $n \in \mathbb{Z}_+ \cup \{0\}$, $0 \in \sigma_\varepsilon(B)$ and $AB = BA$. Define*

$$\varepsilon^* = \min \left\{ \left\| A^{2^n} v \right\|^{\frac{1}{2^n}} : Bv = 0, v \in X, \|v\| = 1 \right\}$$

then, $\Lambda_{n,\varepsilon}(A, B) = \mathbb{C}$ for $\varepsilon \geq \varepsilon^$.*

Proof. This follows from (ii) of Theorem 2.7. \square

Operator pencils (A, B) with singular B arise in various applications including fluid mechanics and wave propagations. For more information one may refer to [5, 6]. The following result shows that $\Lambda_{n,\varepsilon}(A, B)$ contains a neighbourhood of $\sigma(A, B)$.

THEOREM 2.9. *Let $A, B \in BL(X)$, B is invertible, $n \in \mathbb{Z}_+ \cup \{0\}$, and $\varepsilon > 0$. Then for each $\lambda \in \sigma(A, B)$ there exists $r > 0$ such that $D(\lambda, r) \subseteq \Lambda_{n,\varepsilon}(A, B)$.*

Proof. If there exists $\lambda \in \sigma(A, B)$ such that $D(\lambda, r) \cap \Lambda_{n,\varepsilon}(A, B)^c \neq \emptyset$ for every $r > 0$. Then there exists a sequence $\lambda_m \in \Lambda_{n,\varepsilon}(A, B)^c$ such that $\lambda_m \rightarrow \lambda$. From (i),(ii) of Theorem 2.4,

$$\Lambda_{n,\varepsilon}(A, B) = \{ \lambda \in \mathbb{C} : \gamma_{A,B}^n(\lambda) \leq \varepsilon \},$$

and $\gamma_{A,B}^n(\lambda_m) \rightarrow \gamma_{A,B}^n(\lambda)$. Thus $\gamma_{A,B}^n(\lambda_m) > \varepsilon$ for every m and $\gamma_{A,B}^n(\lambda) = 0$. This is a contradiction. \square

THEOREM 2.10. *Let $A, B \in BL(X)$ and $n \in \mathbb{Z}_+ \cup \{0\}$. Further $\Phi : BL(X) \rightarrow BL(X)$ such that $\Phi(\alpha A) = \alpha \Phi(A)$ for every $\alpha > 0$. If $\Lambda_{n,\varepsilon_0}(\Phi(\alpha A), B) = \Lambda_{n,\varepsilon_0}(\alpha A, B)$ for every $\alpha > 0$ and some $\varepsilon_0 > 0$, then*

$$\Lambda_{n,\varepsilon}(\Phi(A), B) = \Lambda_{n,\varepsilon}(A, B) \text{ for every } \varepsilon > 0.$$

In particular $\sigma(\Phi(A), B) = \sigma(A, B)$.

Proof. Let $\alpha > 0$, then $\Lambda_{n,\varepsilon_0}(\alpha\Phi(A), B) = \Lambda_{n,\varepsilon_0}(\alpha A, B)$. From (vii) of Theorem 2.4,

$$\alpha \Lambda_{n,\frac{\varepsilon_0}{\alpha}}(\Phi(A), B) = \alpha \Lambda_{n,\frac{\varepsilon_0}{\alpha}}(A, B).$$

Since $\alpha > 0$ is arbitrary,

$$\Lambda_{n,\varepsilon}(\Phi(A), B) = \Lambda_{n,\varepsilon}(A, B) \quad \text{for every } \varepsilon > 0.$$

Also

$$\sigma(\Phi(A), B) = \bigcap_{\varepsilon > 0} \Lambda_{n,\varepsilon}(\Phi(A), B) = \bigcap_{\varepsilon > 0} \Lambda_{n,\varepsilon}(A, B) = \sigma(A, B). \quad \square$$

The following results may be proved similarly.

COROLLARY 2.11. *Let $A, A', B, B' \in BL(X)$ and $n \in \mathbb{Z}_+ \cup \{0\}$. If $\Lambda_{n,\varepsilon_0}(\alpha A, B) = \Lambda_{n,\varepsilon_0}(\alpha A', B')$ for every $\alpha > 0$ and some $\varepsilon_0 > 0$, then*

$$\Lambda_{n,\varepsilon}(\alpha A, B) = \Lambda_{n,\varepsilon}(\alpha A', B') \quad \text{for every } \varepsilon > 0.$$

In particular $\sigma(A, B) = \sigma(A', B')$.

COROLLARY 2.12. *Let $U, V \in BL(X)$ and $n \in \mathbb{Z}_+ \cup \{0\}$. If $\Lambda_{n,\varepsilon_0}(UA, B) = \Lambda_{n,\varepsilon_0}(VA, B)$ for every $A, B \in BL(X)$ and some $\varepsilon_0 > 0$. Then for every $A, B \in BL(X)$,*

$$\Lambda_{n,\varepsilon}(UA, B) = \Lambda_{n,\varepsilon}(VA, B) \quad \text{for every } \varepsilon > 0.$$

In particular $\sigma(UA, B) = \sigma(VA, B)$.

3. Spectral mapping theorem and pseudospectral mapping theorem for operator pencils

For $A \in BL(X)$ and an analytic function f on an open set containing $\sigma(A)$, the Spectral Mapping Theorem gives $f(\sigma(A)) = \sigma(f(A))$. In this section, we prove the Spectral Mapping Theorem and Pseudospectral Mapping Theorem for operator pencils. The classical Spectral Mapping Theorem is shown as a special case of this result. Throughout the section, the operator B considered is invertible.

THEOREM 3.1. *Let $A, B \in BL(X)$, f analytic on Ω , an open set containing $\sigma(A, B)$, and Γ be the any closed contour enclosing Ω . Define*

$$f(A, B) = \frac{1}{2\pi i} \int_{\Gamma} f(z) (zB - A)^{-1} dz.$$

Then

$$f(\sigma(A, B)) = \sigma(Bf(A, B)).$$

Proof. Since B is invertible,

$$\begin{aligned} f(A, B) &= \frac{1}{2\pi i} \int_{\Gamma} f(z) B^{-1} (zI - AB^{-1})^{-1} dz \\ &= \frac{B^{-1}}{2\pi i} \int_{\Gamma} f(z) (zI - AB^{-1})^{-1} dz \\ &= B^{-1} f(AB^{-1}). \end{aligned}$$

The last step is true from the the operator analogue of the Cauchy integral formula or the Dunford Taylor integral. Thus

$$\begin{aligned} f(\sigma(A, B)) &= \{f(\lambda) : \lambda B - A \text{ is not invertible}\} \\ &= \{f(\lambda) : (\lambda I - AB^{-1})B \text{ is not invertible}\} \\ &= f(\sigma(AB^{-1})). \end{aligned}$$

From the usual Spectral Mapping Theorem for operators,

$$f(\sigma(A, B)) = f(\sigma(AB^{-1})) = \sigma(f(AB^{-1})) = \sigma(Bf(A, B)). \quad \square$$

REMARK 3.2. In particular, if $B = I$, then

$$f(\sigma(A)) = f(\sigma(A, I)) = \sigma(f(A, I)) = \sigma(f(A)).$$

The following example shows that the Spectral Mapping Theorem is not true for pseudospectrum of operator pencils. We find $A, B \in BL(\ell^1)$ and an analytic function f on an open set containing $\sigma(A, B)$ such that $f(\Lambda_{\varepsilon}(A, B)) \neq \Lambda_{\varepsilon}(Bf(A, B))$ for every $\varepsilon > 0$.

EXAMPLE 3.3. Consider $A, B \in (BL(\ell^1), \|\cdot\|_1)$ defined by $A(x_1, x_2, x_3, \dots) = (x_2, 0, 0, \dots)$ and $B(x_1, x_2, x_3, \dots) = (2x_1, x_2, x_3, \dots)$. Then $\sigma(A, B) = \{0\}$ and for $\lambda \neq 0$,

$$(\lambda B - A)^{-1} = \left(\frac{x_1}{2\lambda} + \frac{x_2}{2\lambda^2}, \frac{x_2}{\lambda}, \frac{x_3}{\lambda}, \dots\right) \quad \text{and} \quad \|(\lambda B - A)^{-1}\|_1 = \frac{1}{|\lambda|} + \frac{1}{2|\lambda|^2}.$$

For $\varepsilon > 0$,

$$\begin{aligned} \Lambda_{\varepsilon}(A, B) &= \left\{ \lambda \in \mathbb{C} : \|(\lambda B - A)^{-1}\|_1 \geq \frac{1}{\varepsilon} \right\} \\ &= \left\{ \lambda \in \mathbb{C} : \frac{1}{|\lambda|} + \frac{1}{2|\lambda|^2} \geq \frac{1}{\varepsilon} \right\} \\ &= \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{\varepsilon}{2} + \sqrt{\frac{\varepsilon^2 + 2\varepsilon}{4}} \right\}. \\ (\Lambda_{\varepsilon}(A, B))^2 &= \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \left(\frac{\varepsilon}{2} + \sqrt{\frac{\varepsilon^2 + 2\varepsilon}{4}} \right)^2 \right\}. \end{aligned}$$

For $f(z) = z^2$, we have $B(A, B)^2 = (AB^{-1})^2$. For $x = (x_1, x_2, x_3, \dots) \in \ell^1$,

$$(AB^{-1})x = A \left(\frac{x_1}{2}, x_2, x_3, \dots \right) = (x_2, 0, 0, \dots),$$

and $(AB^{-1})^2x = (0, 0, \dots)$. Then

$$\Lambda_\varepsilon(B(A, B)^2) = \Lambda_\varepsilon((AB^{-1})^2) = \{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon\}.$$

Hence for every $\varepsilon > 0$,

$$(\Lambda_\varepsilon(A, B))^2 \neq \Lambda_\varepsilon(B(A, B)^2).$$

THEOREM 3.4. *Let $A, B \in BL(X)$ and let f be an analytic function defined on Ω , an open set containing $\sigma(A, B)$. For $\varepsilon \geq 0$, define*

$$\phi(\varepsilon) = \sup_{\lambda \in \Lambda_\varepsilon(A, B)} \frac{1}{\left\| (f(\lambda)I - f(AB^{-1}))^{-1} \right\|}.$$

Then $\phi(\varepsilon)$ is well defined, $\phi(0) = 0$ and $f(\Lambda_\varepsilon(A, B)) \subseteq \Lambda_{\phi(\varepsilon)}(Bf(A, B))$.

Proof. Define $g : \mathbb{C} \rightarrow \mathbb{R}^+$ by

$$g(\lambda) = \frac{1}{\left\| (f(\lambda)I - f(AB^{-1}))^{-1} \right\|}.$$

We claim that g is continuous. Let $\lambda \notin \sigma(A, B)$, then $f(\lambda) \notin f(\sigma(AB^{-1}))$. Suppose $\lambda_m \in \mathbb{C} \setminus \sigma(A, B)$ such that $\lambda_m \rightarrow \lambda$. Since f is analytic,

$$g(\lambda_m) = \left\| (f(\lambda_m)I - f(AB^{-1}))^{-1} \right\|^{-1} \rightarrow \left\| (f(\lambda)I - f(AB^{-1}))^{-1} \right\|^{-1} = g(\lambda).$$

Hence g is continuous on $\mathbb{C} \setminus \sigma(A, B)$. Further, let $\lambda \in \sigma(A, B)$ and $\lambda_m \in \mathbb{C} \setminus \sigma(A, B)$ such that $\lambda_m \rightarrow \lambda$. Then

$$f(\lambda) \in f(\sigma(A, B)) = \sigma(Bf(A, B)) = \sigma(f(AB^{-1})).$$

Since $f(\lambda_m) \rightarrow f(\lambda)$, from Lemma 10.17 of [15],

$$\left\| (f(\lambda_m)I - f(AB^{-1}))^{-1} \right\| \rightarrow \infty.$$

Thus $g(\lambda_m) \rightarrow 0 = g(\lambda)$. This proves the claim. Also,

$$\phi(\varepsilon) = \sup \{g(\lambda) : \lambda \in \Lambda_\varepsilon(A, B)\}.$$

Since $\Lambda_\varepsilon(A, B)$ is compact, $\phi(\varepsilon)$ is well defined. Next we claim that $\phi(0) = 0$.

$$\phi(0) = \sup_{\lambda \in \sigma(A, B)} \frac{1}{\left\| (f(\lambda)I - f(AB^{-1}))^{-1} \right\|}$$

If $\lambda \in \sigma(A, B)$, then $f(\lambda) \in f(\sigma(A, B)) = \sigma(Bf(A, B)) = \sigma(f(AB^{-1}))$ and

$$\left\| (f(\lambda)I - f(AB^{-1}))^{-1} \right\|^{-1} = 0.$$

Thus $\phi(0) = 0$. Now if $\lambda \in \Lambda_\varepsilon(A, B)$, then

$$\frac{1}{\left\| (f(\lambda)I - f(AB^{-1}))^{-1} \right\|} = g(\lambda) \leq \phi(\varepsilon),$$

and

$$\left\| (f(\lambda)I - f(AB^{-1}))^{-1} \right\| = \frac{1}{g(\lambda)} \geq \frac{1}{\phi(\varepsilon)}.$$

i.e.,

$$f(\lambda) \in \Lambda_{\phi(\varepsilon)}(f(AB^{-1})) = \Lambda_{\phi(\varepsilon)}(Bf(A, B)).$$

Thus

$$f(\Lambda_\varepsilon(A, B)) \subseteq \Lambda_{\phi(\varepsilon)}(Bf(A, B)). \quad \square$$

THEOREM 3.5. *Let $A, B \in BL(X)$ and let f be an analytic injective function defined on Ω , an open set containing $\sigma(A, B)$. Suppose there exists $\varepsilon' > 0$ such that $\Lambda_{\varepsilon'}(Bf(A, B)) \subseteq f(\Omega)$. For $0 \leq \varepsilon < \varepsilon'$, define*

$$\psi(\varepsilon) = \sup_{\omega \in f^{-1}(\Lambda_\varepsilon(Bf(A, B))) \cap \Omega} \frac{\|B\|}{\|(\omega I - AB^{-1})^{-1}\|}.$$

Then $\psi(\varepsilon)$ is well defined, $\psi(0) = 0$ and $\Lambda_\varepsilon(Bf(A, B)) \subseteq f(\Lambda_{\psi(\varepsilon)}(A, B))$.

Proof. Define $h : \mathbb{C} \rightarrow \mathbb{R}^+$ by

$$h(\omega) = \frac{\|B\|}{\|(\omega I - AB^{-1})^{-1}\|}.$$

We claim that h is continuous. Suppose $\omega \notin \sigma(A, B)$, then $\omega \notin \sigma(AB^{-1})$. If $\omega_m \in \mathbb{C} \setminus \sigma(A, B)$ such that $\omega_m \rightarrow \omega$, then

$$\|(\omega_m I - AB^{-1})^{-1}\| \rightarrow \|(\omega I - AB^{-1})^{-1}\|$$

and consequently $h(\omega_m) \rightarrow h(\omega)$. So h is continuous on $\mathbb{C} \setminus \sigma(A, B)$. Next if $\omega \in \sigma(A, B)$ and $\omega_m \in \mathbb{C} \setminus \sigma(A, B)$ such that $\omega_m \rightarrow \omega$. Then from Lemma 10.17 of [15],

$$\|(\omega_m I - AB^{-1})^{-1}\| \rightarrow \infty$$

and $h(\omega_m) \rightarrow 0 = h(\omega)$. This proves the claim. Also

$$\psi(\varepsilon) = \sup\{h(\omega) : \omega \in f^{-1}(\Lambda_\varepsilon(Bf(A, B)))\}.$$

Since $f^{-1}(\Lambda_\varepsilon(Bf(A, B)))$ is compact $\psi(\varepsilon)$ is well defined. Next we claim that $\psi(0) = 0$.

$$\psi(0) = \sup_{\omega \in f^{-1}(\sigma(Bf(A, B)))} \frac{\|B\|}{\|(\omega I - AB^{-1})^{-1}\|}$$

If $\omega \in f^{-1}(\sigma(Bf(A, B)))$, then $f(\omega) \in \sigma(Bf(A, B)) = f(\sigma(A, B)) = f(\sigma(AB^{-1}))$.

Since f is injective, $\omega \in \sigma(A, B)$ and $h(\omega) = \|(\omega I - AB^{-1})^{-1}\| = \infty$. Thus $\psi(0) = 0$.

Let $0 \leq \varepsilon < \varepsilon'$, then

$$\Lambda_\varepsilon(Bf(A, B)) \subseteq \Lambda_{\varepsilon'}(Bf(A, B)) \subseteq f(\Omega).$$

If $z \in \Lambda_\varepsilon(Bf(A, B))$, then $z \in \Lambda_\varepsilon(f(AB^{-1})) \cap f(\Omega)$. Consider $\omega \in \Omega$ such that $z = f(\omega)$, then $\omega \in f^{-1}(\Lambda_\varepsilon(Bf(A, B))) \cap \Omega$. Also

$$\frac{\|B\|}{\|(\omega I - AB^{-1})^{-1}\|} = h(\omega) \leq \psi(\varepsilon).$$

Hence

$$\|(\omega I - AB^{-1})^{-1}\| = \frac{\|B\|}{h(\omega)} \geq \frac{\|B\|}{\psi(\varepsilon)}.$$

From (iii) of Theorem 2.5, $\omega \in \Lambda_{\frac{\psi(\varepsilon)}{\|B\|}}(AB^{-1}) \subseteq \Lambda_{\psi(\varepsilon)}(A, B)$. Thus

$$z = f(\omega) \in f\left(\Lambda_{\frac{\psi(\varepsilon)}{\|B\|}}(AB^{-1})\right) \subseteq f(\Lambda_{\psi(\varepsilon)}(A, B)).$$

Hence for $0 \leq \varepsilon < \varepsilon'$,

$$\Lambda_\varepsilon(Bf(A, B)) \subseteq f(\Lambda_{\psi(\varepsilon)}(A, B)). \quad \square$$

REMARK 3.6.

1. If $B = I$, then $\phi(\varepsilon)$ and $\psi(\varepsilon)$ coincide with the functions defined for the Pseudospectral Mapping Theorem in [13].
2. Combining the above two inclusions,

$$f(\Lambda_\varepsilon(A, B)) \subseteq \Lambda_{\phi(\varepsilon)}(Bf(A, B)) \subseteq f(\Lambda_{\psi(\phi(\varepsilon))}(A, B)),$$

and

$$\Lambda_\varepsilon(Bf(A, B)) \subseteq f(\Lambda_{\psi(\varepsilon)}(A, B)) \subseteq \Lambda_{\phi(\psi(\varepsilon))}(Bf(A, B)).$$

3. Since $\phi(0) = 0 = \psi(0)$ and $\sigma(A, B) = \Lambda_0(A, B)$, the usual Spectral Mapping Theorem for operator pencils can be deduced from the above theorems. However, the proof itself uses the Spectral Mapping Theorem for operator pencils.

4. From the definitions of ϕ and ψ , it is clear that the set inclusions are sharp because other functions cannot replace them. The functions ϕ and ψ may look unwieldy at first glance, but they can be explicitly calculated for certain cases, and we have illustrated it through Example 3.8.
5. If $f(z) = \alpha + \beta z$, where $\alpha, \beta \in \mathbb{C}$ and $\beta \neq 0$. Then

$$\phi(\varepsilon) = \sup_{\lambda \in \Lambda_\varepsilon(A, B)} \frac{1}{|\beta^{-1}| \|(\lambda I - AB^{-1})^{-1}\|} \leq |\beta| \|B^{-1}\| \varepsilon.$$

Also

$$\begin{aligned} \psi(\varepsilon) &\leq \sup_{\omega \in \frac{\Lambda_\varepsilon(\alpha I + \beta AB^{-1}) - \alpha}{\beta}} \frac{\|B\|}{\|(\omega I - AB^{-1})^{-1}\|} \\ &= \sup_{\omega \in \Lambda_{\frac{\varepsilon}{|\beta|}}(AB^{-1})} \frac{\|B\|}{\|(\omega I - AB^{-1})^{-1}\|} \\ &= |\beta^{-1}| \|B\| \varepsilon. \end{aligned}$$

Then $\psi(\phi(\varepsilon)) = \phi(\psi(\varepsilon)) \leq \|B\| \|B^{-1}\| \varepsilon$. Hence $\psi(\phi(\varepsilon)) = \phi(\psi(\varepsilon)) = \varepsilon$, whenever $\|B\| \|B^{-1}\| = 1$. Thus $\Lambda_\varepsilon(Bf(A, B)) = f\left(\Lambda_{|\beta^{-1}| \|B\| \varepsilon}(A, B)\right)$ when $f(z) = \alpha + \beta z$ and $\beta \neq 0$.

The following theorem shows that the class of functions for which the inclusion relation in the Pseudospectral Mapping Theorem becomes equality as in the Spectral Mapping Theorem are only affine functions.

THEOREM 3.7. *Suppose f is a nonconstant analytic function defined on a non empty open set Ω in the complex plane. Then there exists a non negative real-valued function $\eta(A, B, \varepsilon)$ such that for every $A, B \in BL(X)$ with $\sigma(A, B) \subset \Omega$ and $\|B\| \|B^{-1}\| = 1$ we have*

$$f(\Lambda_\varepsilon(A, B)) = \Lambda_{\eta(A, B, \varepsilon)}(Bf(A, B))$$

for all ε sufficiently small if and only if $f(z) = \alpha + \beta z$ for some $\alpha, \beta \in \mathbb{C}$.

Proof. If $f(z) = \alpha + \beta z$. From (5) of Remark 3.6, $f(\Lambda_\varepsilon(A, B)) = \Lambda_{\eta(A, B, \varepsilon)}(f(A, B))$ with $\eta(A, B, \varepsilon) = |\beta| \|B^{-1}\| \varepsilon$. The other part follows from Theorem 2.2 of [13]. (Take $A = aI$, $B = I$ with $a \in \Omega$ and $f'(a) \neq 0$). \square

EXAMPLE 3.8. Consider $A, B \in BL(\mathbb{C}^3, \|\cdot\|_1)$ defined by $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B =$

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \text{ Then } \sigma(A, B) = \{0\} \text{ and for } \lambda \neq 0 \text{ and } \varepsilon > 0, (\lambda B - A)^{-1} = \begin{pmatrix} \frac{1}{3\lambda} & \frac{1}{3\lambda^2} & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{pmatrix}.$$

$$\begin{aligned} \Lambda_\varepsilon(A, B) &= \left\{ \lambda \in \mathbb{C} : \|(\lambda B - A)^{-1}\|_1 \geq \frac{1}{\varepsilon} \right\} \\ &= \left\{ \lambda \in \mathbb{C} : \frac{1}{|\lambda|} + \frac{1}{3|\lambda|^2} \geq \frac{1}{\varepsilon} \right\} \\ &= \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{\varepsilon}{2} + \frac{\sqrt{9\varepsilon^2 + 12\varepsilon}}{6} \right\}. \end{aligned}$$

If $f(z) = z^3$, then

$$\phi(\varepsilon) = \sup_{\lambda \in \Lambda_\varepsilon(A, B)} \frac{1}{\|(\lambda^3 I - (AB^{-1})^3)^{-1}\|_1} = \sup_{\lambda \in \Lambda_\varepsilon(A, B)} |\lambda^3| = \left(\frac{\varepsilon}{2} + \frac{\sqrt{9\varepsilon^2 + 12\varepsilon}}{6} \right)^3.$$

$$\Lambda_\varepsilon(Bf(A, B)) = \Lambda_\varepsilon((AB^{-1})^3) = \{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon\}.$$

$$\begin{aligned} \psi(\varepsilon) &= \sup_{\omega^3 \in \Lambda_\varepsilon((AB^{-1})^3)} \frac{\|B\|_1}{\|(\omega I - AB^{-1})^{-1}\|_1} \\ &= \sup_{\omega^3 \in \Lambda_\varepsilon((AB^{-1})^3)} \frac{3|\omega|^2}{(|\omega| + 1)} \\ &\leq 3\varepsilon^{\frac{2}{3}}. \end{aligned}$$

Consequently,

$$\begin{aligned} (\Lambda_\varepsilon(A, B))^3 &\subseteq \Lambda_{\left(\frac{\varepsilon}{2} + \frac{\sqrt{9\varepsilon^2 + 12\varepsilon}}{6}\right)^3}((AB^{-1})^3), \\ \Lambda_\varepsilon(AB^{-1})^3 &\subseteq \left(\Lambda_{\frac{\varepsilon}{3}}(A, B)\right)^3. \end{aligned}$$

4. (n, ε) -pseudospectral mapping theorem for operator pencils

This section develops an analogue of the Spectral Mapping Theorem for (n, ε) -pseudospectrum of operator pencils. The following example shows the Spectral Mapping Theorem of operator pencils, as it is not valid for (n, ε) -pseudospectrum. Throughout this section, we assume that B is invertible.

EXAMPLE 4.1. Consider $A, B \in (BL(\ell^1), \|\cdot\|_1)$ defined by $A(x_1, x_2, \dots) = (3x_2, 0, \dots)$ and $B(x_1, x_2, \dots) = (3x_1, x_2, \dots)$. Then $\sigma(A, B) = \{0\}$ and for $\lambda \neq 0$,

$$(\lambda B - A)^{-2}(x_1, x_2, \dots) = \left(\frac{x_1}{9\lambda^2} + \frac{4x_2}{3\lambda^3}, \frac{x_2}{\lambda^2}, \frac{x_3}{\lambda^2}, \dots \right).$$

Also

$$\|(\lambda B - A)^{-2}\|_1 = \frac{1}{|\lambda|^2} + \frac{4}{3|\lambda|^3} = \frac{3|\lambda| + 4}{3|\lambda|^3}.$$

For $f(z) = z^2$ and $\varepsilon > 0$,

$$\Lambda_{1,\varepsilon}(A, B) = \left\{ \lambda \in \mathbb{C} : \frac{3|\lambda| + 4}{3|\lambda|^3} \geq \frac{1}{\varepsilon^2} \right\}$$

Then

$$(\Lambda_{1,\varepsilon}(A, B))^2 = \left\{ \lambda^2 \in \mathbb{C} : \frac{3|\lambda|^3}{3|\lambda| + 4} \leq \varepsilon^2 \right\}.$$

Also

$$(AB^{-1})^2(x_1, x_2, \dots) = AB^{-1}(3x_2, 0, \dots) = (0, 0, \dots).$$

Since $B(A, B)^2 = (AB^{-1})^2$,

$$\Lambda_{1,\varepsilon}(B(A, B)^2) = \left\{ \lambda \in \mathbb{C} : \frac{1}{|\lambda|^2} \geq \frac{1}{\varepsilon^2} \right\} = \{ \lambda \in \mathbb{C} : |\lambda| \leq \varepsilon \}.$$

Thus $(\Lambda_{1,\varepsilon}(A, B))^2 \neq \Lambda_{1,\varepsilon}(B(A, B)^2)$.

THEOREM 4.2. *Let $A, B \in BL(X)$, $n \in \mathbb{Z}_+$, and let f be an analytic function defined on Ω , an open set containing $\sigma(A, B)$. For $\varepsilon \geq 0$, define*

$$\phi(\varepsilon) = \sup_{\lambda \in \Lambda_{n,\varepsilon}(A, B)} \frac{1}{\| (f(\lambda)I - f(AB^{-1}))^{-2n} \|^{1/2^n}}.$$

Then $\phi(\varepsilon)$ is well defined, $\phi(0) = 0$ and $f(\Lambda_{n,\varepsilon}(A, B)) \subseteq \Lambda_{n,\phi(\varepsilon)}(Bf(A, B))$.

Proof. Define $g : \mathbb{C} \rightarrow \mathbb{R}^+$ by

$$g(\lambda) = \frac{1}{\| (f(\lambda)I - f(AB^{-1}))^{-2n} \|^{1/2^n}}.$$

We claim that g is continuous. If $\lambda \notin \sigma(A, B)$, then $f(\lambda) \notin f(\sigma(A, B)) = \sigma(f(AB^{-1}))$. Suppose $\lambda_m \in \mathbb{C} \setminus \sigma(A, B)$ such that $\lambda_m \rightarrow \lambda$. Since f is analytic

$$\| (f(\lambda_m)I - f(AB^{-1}))^{-2n} \|^{1/2^n} \rightarrow \| (f(\lambda)I - f(AB^{-1}))^{-2n} \|^{1/2^n}.$$

Hence $g(\lambda_m) \rightarrow g(\lambda)$ and g is continuous on $\mathbb{C} \setminus \sigma(A, B)$. Next let $\lambda \in \sigma(A, B)$, by Spectral Mapping Theorem for operator pencils

$$f(\lambda) \in f(\sigma(A, B)) = \sigma(Bf(A, B)) = \sigma(f(AB^{-1})).$$

If $\lambda_m \in \mathbb{C} \setminus \sigma(A, B)$ be such that $\lambda_m \rightarrow \lambda$. From Lemma 10.17 of [15],

$$\left\| (f(\lambda_m)I - f(AB^{-1}))^{-2^n} \right\| \rightarrow \infty.$$

Hence $g(\lambda_m) \rightarrow 0 = g(\lambda)$. This proves the claim. Also

$$\phi(\varepsilon) = \sup \{g(\lambda) : \lambda \in \Lambda_{n,\varepsilon}(A, B)\}.$$

Since $\Lambda_{n,\varepsilon}(A, B)$ is compact $\phi(\varepsilon)$ is well defined. Next we claim that $\phi(0) = 0$.

$$\phi(0) = \sup_{\lambda \in \sigma(A, B)} \frac{1}{\left\| (f(\lambda)I - f(AB^{-1}))^{-2^n} \right\|^{\frac{1}{2^n}}} = 0.$$

If $\lambda \in \Lambda_{n,\varepsilon}(A, B)$, then

$$\frac{1}{\left\| (f(\lambda)I - f(AB^{-1}))^{-2^n} \right\|^{\frac{1}{2^n}}} = g(\lambda) \leq \phi(\varepsilon),$$

and

$$\left\| (f(\lambda)I - f(AB^{-1}))^{-2^n} \right\|^{\frac{1}{2^n}} = \frac{1}{g(\lambda)} \geq \frac{1}{\phi(\varepsilon)}.$$

Thus $f(\lambda) \in \Lambda_{n,\phi(\varepsilon)}(f(AB^{-1}))$. Hence

$$f(\Lambda_{n,\varepsilon}(A, B)) \subseteq \Lambda_{n,\phi(\varepsilon)}(Bf(A, B)). \quad \square$$

THEOREM 4.3. *Let $A, B \in BL(X)$, $AB = BA$, and $n \in \mathbb{Z}_+$. Further, let f be an analytic injective function defined on Ω , an open set containing $\sigma(A, B)$ also there exists $\varepsilon' > 0$ such that $\Lambda_{n,\varepsilon'}(Bf(A, B)) \subseteq f(\Omega)$. For $0 \leq \varepsilon < \varepsilon'$ define*

$$\psi(\varepsilon) = \sup_{\omega \in f^{-1}(\Lambda_{n,\varepsilon}(Bf(A, B))) \cap \Omega} \frac{\|B^{2^n}\|^{\frac{1}{2^n}}}{\|(\omega I - AB^{-1})^{-2^n}\|^{\frac{1}{2^n}}}.$$

Then $\psi(\varepsilon)$ is well defined, $\psi(0) = 0$ and $\Lambda_{n,\varepsilon}(Bf(A, B)) \subseteq f(\Lambda_{n,\psi(\varepsilon)}(A, B))$.

Proof. Define $h : \mathbb{C} \rightarrow \mathbb{R}^+$ by

$$h(\omega) = \frac{\|B^{2^n}\|^{\frac{1}{2^n}}}{\|(\omega I - AB^{-1})^{-2^n}\|^{\frac{1}{2^n}}}.$$

We claim that h is continuous. Suppose $\omega \notin \sigma(A, B)$ and $\omega_m \in \mathbb{C} \setminus \sigma(A, B)$ such that $\omega_m \rightarrow \omega$. Then

$$\left\| (\omega_m I - AB^{-1})^{-2^n} \right\| \rightarrow \left\| (\omega I - AB^{-1})^{-2^n} \right\|,$$

and $h(\omega_m) \rightarrow h(\omega)$. Hence h is continuous on $\mathbb{C} \setminus \sigma(A, B)$. Next let $\omega \in \sigma(A, B)$, then $\omega \in \sigma(AB^{-1})$ and $h(\omega) = 0$. If $\omega_m \in \mathbb{C} \setminus \sigma(A, B)$ such that $\omega_m \rightarrow \omega$, then from Lemma 10.17 of [15], $\|(\omega_m I - AB^{-1})^{-2^n}\| \rightarrow \infty$. Thus $h(\omega_m) \rightarrow 0 = h(\omega)$. Hence h is continuous on $\sigma(A, B)$. This proves the claim. Also

$$\psi(\varepsilon) = \sup\{h(\omega) : \omega \in f^{-1}(\Lambda_{n,\varepsilon}(Bf(A, B)))\}.$$

Since $f^{-1}(\Lambda_{n,\varepsilon}(Bf(A, B)))$ is compact $\psi(\varepsilon)$ is well defined. Next we claim that $\psi(0) = 0$. If $\omega \in f^{-1}(\sigma(Bf(A, B)))$, then $f(\omega) \in \sigma(Bf(A, B)) = f(\sigma(A, B)) = f(\sigma(AB^{-1}))$. Since f is injective, $\omega \in \sigma(A, B)$ and $\|(\omega I - AB^{-1})^{-2^n}\|^{\frac{1}{2^n}} = \infty$. Thus $h(\omega) = 0$ and

$$\psi(0) = \sup_{\omega \in f^{-1}(\sigma(Bf(A, B)))} \frac{\|B^{2^n}\|^{\frac{1}{2^n}}}{\|(\omega I - AB^{-1})^{-2^n}\|^{\frac{1}{2^n}}} = 0.$$

If $0 \leq \varepsilon < \varepsilon'$, then $\Lambda_{n,\varepsilon}(Bf(A, B)) \subseteq \Lambda_{n,\varepsilon'}(Bf(A, B)) \subseteq f(\Omega)$. For $z \in \Lambda_{n,\varepsilon}(Bf(A, B)) \cap f(\Omega)$, consider $\omega \in \Omega$ such that $z = f(\omega)$, then $\omega \in f^{-1}(\Lambda_{n,\varepsilon}(Bf(A, B)) \cap f(\Omega))$. Thus

$$\frac{\|B^{2^n}\|^{\frac{1}{2^n}}}{\|(\omega I - AB^{-1})^{-2^n}\|^{\frac{1}{2^n}}} = h(\omega) \leq \psi(\varepsilon),$$

and

$$\|(\omega I - AB^{-1})^{-2^n}\|^{\frac{1}{2^n}} = \frac{\|B^{2^n}\|^{\frac{1}{2^n}}}{h(\omega)} \geq \frac{\|B^{2^n}\|^{\frac{1}{2^n}}}{\psi(\varepsilon)}.$$

From (iii) of Theorem 2.5,

$$\omega \in \Lambda_{n, \frac{\psi(\varepsilon)}{\|B^{2^n}\|^{\frac{1}{2^n}}}}(AB^{-1}) \subseteq \Lambda_{n,\psi(\varepsilon)}(A, B),$$

and

$$z = f(\omega) \in f\left(\Lambda_{n, \frac{\psi(\varepsilon)}{\|B^{2^n}\|^{\frac{1}{2^n}}}}(AB^{-1})\right) \subseteq f(\Lambda_{n,\psi(\varepsilon)}(A, B)).$$

Hence for $0 \leq \varepsilon < \varepsilon'$,

$$\Lambda_{n,\varepsilon}(Bf(A, B)) \subseteq f(\Lambda_{n,\psi(\varepsilon)}(A, B)). \quad \square$$

REMARK 4.4.

1. Combining these two inclusions,

$$f(\Lambda_{n,\varepsilon}(A, B)) \subseteq \Lambda_{n,\phi(\varepsilon)}(Bf(A, B)) \subseteq f(\Lambda_{n,\psi(\phi(\varepsilon))}(A, B)),$$

and

$$\Lambda_{n,\varepsilon}(Bf(A, B)) \subseteq f(\Lambda_{n,\psi(\varepsilon)}(A, B)) \subseteq \Lambda_{n,\phi(\psi(\varepsilon))}(Bf(A, B)).$$

- 2. If $B = I$, the above theorems becomes the analogue of the Spectral Mapping Theorem for (n, ϵ) -pseudospectrum of operators (Theorem 4.2, Theorem 4.3).
- 3. Since $\phi(0) = 0 = \psi(0)$, we have $\sigma(A, B) = \Lambda_{n,0}(A, B)$. Thus the usual Spectral Mapping Theorem for operator pencils can be deduced from the above theorems.
- 4. The set inclusions are sharp because other functions cannot replace the functions ϕ and ψ . Through Example 4.7, we illustrate that ϕ and ψ can be calculated explicitly.

REMARK 4.5. If $AB = BA$, $f(z) = \alpha + \beta z$ where $\alpha, \beta \in \mathbb{C}$ and $\beta \neq 0$. Then

$$\phi(\epsilon) = \sup_{\lambda \in \Lambda_{n,\epsilon}(A,B)} \frac{1}{|\beta^{-1}| \|(\lambda I - AB^{-1})^{-2^n}\|^{\frac{1}{2^n}}} \leq |\beta| \|B^{-2^n}\|^{\frac{1}{2^n}} \epsilon.$$

In the similar way,

$$\psi(\epsilon) \leq |\beta^{-1}| \|B^{2^n}\|^{\frac{1}{2^n}} \epsilon.$$

Whenever $\|B\| \|B^{-1}\| = 1$ we have $\psi(\phi(\epsilon)) = \phi(\psi(\epsilon)) = \epsilon$. Thus for $AB = BA$ and $f(z) = \alpha + \beta z$ with $\beta \neq 0$,

$$\Lambda_{n,\epsilon}(Bf(A,B)) = f\left(\Lambda_{n,|\beta^{-1}| \|B^{2^n}\|^{\frac{1}{2^n}} \epsilon}(A,B)\right).$$

The following theorem shows that the class of functions for which the inequality in the (n, ϵ) -pseudospectral mapping theorem becomes equality are only affine functions.

THEOREM 4.6. Suppose f is a nonconstant analytic function defined on a non empty open set Ω in the complex plane. Then there exists a non negative real-valued function $\eta(A, B, \epsilon, n)$ such that for every $A, B \in BL(X)$ with $\sigma(A, B) \subset \Omega$, $AB = BA$ and $\|B\| \|B^{-1}\| = 1$ we have

$$f(\Lambda_{n,\epsilon}(A,B)) = \Lambda_{n,\eta(A,B,\epsilon,n)}(B(f(A,B)))$$

for all ϵ sufficiently small if and only if $f(z) = \alpha + \beta z$ for some $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$.

Proof. The if part follows from Remark 4.5. The only if part follows from Theorem 2.2 of [13] (take $A = aI$ with $a \in \Omega$ and $f'(a) \neq 0$ and $B = I$). □

EXAMPLE 4.7. Consider $A, B \in BL(\mathbb{C}^2, \|\cdot\|_2)$ defined by $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$,

$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Then $\sigma(A, B) = \left\{ -\frac{1}{2}, \frac{1}{2} \right\}$. For $\lambda \notin \sigma(A, B)$, $n = 1$, and $\varepsilon > 0$,

$$\begin{aligned} \Lambda_{1,\varepsilon}(A, B) &= \{ \lambda \in \mathbb{C} : \|(\lambda B - A)^{-2}\|_2^{\frac{1}{2}} \geq \varepsilon^{-1} \} \\ &= \left\{ \lambda \in \mathbb{C} : \frac{(2\lambda)^6}{(2\lambda)^2 + 2 + 2\sqrt{(2\lambda)^2 + 1}} \leq \varepsilon^2 \right\} \\ &= \{ \lambda \in \mathbb{C} : (2\lambda)^4 - 4\lambda\varepsilon - \varepsilon^2 \leq 0 \} \\ &= \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{\sqrt{2k} + \sqrt{-2z + \frac{4\varepsilon}{\sqrt{2k}}}}{2} \right\} \end{aligned}$$

where $k = \left(\frac{\varepsilon^2}{4} + \varepsilon^2 \sqrt{\frac{\varepsilon^2}{27} + \frac{1}{16}} \right)^{\frac{1}{3}} - \frac{\varepsilon^2}{3} \left(\frac{\varepsilon^2}{4} + \varepsilon^2 \sqrt{\frac{\varepsilon^2}{27} + \frac{1}{16}} \right)^{-\frac{1}{3}}$.

For $f(z) = z^2$, we have

$$\begin{aligned} \phi(\varepsilon) &= \sup_{\lambda \in \Lambda_{1,\varepsilon}(A, B)} \frac{1}{\| (f(\lambda)I - f(AB^{-1}))^{-2} \|_2^{\frac{1}{2}}} \\ &= \sup_{\lambda \in \Lambda_{1,\varepsilon}(A, B)} |\lambda^2| \\ &= \frac{4\varepsilon}{\sqrt{2k}} + 2\sqrt{\sqrt{2k}\varepsilon - z^2} \end{aligned}$$

where k is defined above.

$$\Lambda_{1,\varepsilon}(Bf(A, B)) = \Lambda_{1,\varepsilon}((AB^{-1})^2) = \{ \lambda \in \mathbb{C} : |\lambda| \leq \varepsilon \}.$$

We also have

$$\begin{aligned} \psi(\varepsilon) &= \sup_{\omega^2 \in \Lambda_{1,\varepsilon}((AB^{-1})^2)} \frac{\|B^2\|_2^{\frac{1}{2}}}{\|(\omega I - AB^{-1})^{-2}\|_2^{\frac{1}{2}}} \\ &= \sup_{\omega^2 \in \Lambda_{1,\varepsilon}((AB^{-1})^2)} \frac{2\sqrt{2}\omega^3}{\sqrt{2\omega^2 + 1 + \sqrt{4\omega^2 + 1}}} \\ &\leq 2\varepsilon^{\frac{3}{2}}. \end{aligned}$$

Consequently,

$$(\Lambda_{1,\varepsilon}(A, B))^2 \subseteq \Lambda_{1,\phi(\varepsilon)}((AB^{-1})^2),$$

and

$$\Lambda_{1,\varepsilon}((AB^{-1})^2) \subseteq \left(\Lambda_{1,2\varepsilon^{\frac{3}{2}}}(A, B) \right)^2.$$

Concluding remarks

The results are generally valid for elements of a complex unital Banach algebra. We propose the following problems as future work.

1. The weak versions of the pseudospectral and (n, ε) -pseudospectral mapping theorems for operator pencils.
2. The analogue of spectral mapping theorem for operator pencil (A, B) for B singular, and the analogues results for polynomial eigenvalue problem.
3. The (n, ε) -pseudospectrum of tensors (multilinear operators) and multivalued linear operators.

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