

UNIVERSALLY SYMMETRIC NORMING OPERATORS ARE COMPACT

SATISH K. PANDEY

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Abstract. We study a specific family of symmetric norms on the algebra $\mathcal{B}(\mathcal{H})$ of operators on a separable infinite-dimensional Hilbert space. With respect to each symmetric norm in this family the identity operator fails to attain its norm. Using this, we generalize one of the main results from [8]; the hypothesis is relaxed, and consequently, the family of symmetric norms for which the result holds is extended.

We introduce and study the concepts of “universally symmetric norming operators” and “universally absolutely symmetric norming operators” on a separable Hilbert space. These refer to the operators that are, respectively, norming and absolutely norming, with respect to every symmetric norm on $\mathcal{B}(\mathcal{H})$. We establish a characterization theorem for such operators and prove that these classes are identical, and that they coincide with the class of compact operators. In particular, we provide an alternative characterization of compact operators on a separable infinite-dimensional Hilbert space.

1. Introduction

Throughout this paper we shall consider Hilbert spaces over the field \mathbb{C} of complex numbers. A bounded linear transformation, henceforth called an “operator”, $T : \mathcal{H} \rightarrow \mathcal{K}$ between two Hilbert spaces is said to be *norming* or *norm attaining* if there is an element $x \in \mathcal{H}$ with $\|x\| = 1$ such that $\|T\| = \|Tx\|$, where $\|T\| = \sup\{\|Tx\|_{\mathcal{K}} : x \in \mathcal{H}, \|x\|_{\mathcal{H}} \leq 1\}$ is the usual operator norm on the Banach space $\mathcal{B}(\mathcal{H}, \mathcal{K})$ of operators from \mathcal{H} to \mathcal{K} . We say that $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is *absolutely norming* if for every nontrivial closed subspace \mathcal{M} of \mathcal{H} , $T|_{\mathcal{M}}$ is norming. We let $\mathcal{N}(\mathcal{H}, \mathcal{K})$ and $\mathcal{AN}(\mathcal{H}, \mathcal{K})$ respectively denote the sets of norming and absolutely norming operators in $\mathcal{B}(\mathcal{H}, \mathcal{K})$. Throughout this exposition, the term “ideal” will always mean a two-sided ideal. Let us recall the following definition.

DEFINITION 1.1. Let \mathfrak{I} be an ideal of the algebra $\mathcal{B}(\mathcal{H})$ of operators on a separable infinite-dimensional Hilbert space \mathcal{H} . A *symmetric norm* on \mathfrak{I} is defined to be a function $\|\cdot\|_s : \mathfrak{I} \rightarrow [0, \infty)$ which satisfies the following conditions:

1. $\|\cdot\|_s$ is a norm;
2. $\|X\|_s = \|X\|$ for every rank-one operator $X \in \mathfrak{I}$ (crossnorm property); and

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3. $\|AXB\|_s \leq \|A\| \|X\|_s \|B\|$ for every $X \in \mathfrak{J}$ and for every pair A, B of operators in $\mathcal{B}(\mathcal{H})$ (uniformity);

where $\|\cdot\|$ is the usual operator norm.

If we consider the ideal \mathfrak{J} to be the full algebra $\mathcal{B}(\mathcal{H})$ itself, then $\|\cdot\|_s$ is said to be a symmetric norm on $\mathcal{B}(\mathcal{H})$. Moreover, the usual operator norm on any ideal \mathfrak{J} of $\mathcal{B}(\mathcal{H})$ is a symmetric norm, and every symmetric norm on $\mathcal{B}(\mathcal{H})$ is topologically equivalent to the usual operator norm.

1.1. Background and motivation

The class of norming operators on Hilbert spaces have been extensively studied and there is a plethora of information on these operators; see, for instance, [4, 5, 1, 2, 3, 9, 11, 12, 13] and references therein. The class of absolutely norming operators, however, was introduced recently in [6]. In [10], we established the following spectral characterization theorem for these operators.

THEOREM 1.2. [10, Theorem 5.1] *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ and $T = U|T|$ its polar decomposition. Then T is absolutely norming, that is, $T \in \mathcal{AN}(\mathcal{H}, \mathcal{H})$ if and only if $|T|$ is of the form $|T| = \alpha I + K + F$, where $\alpha \geq 0$, K is a positive compact operator, and F is a self-adjoint finite-rank operator.*

The above result served to be the first hint to a more general situation. Suppose $\mathcal{B}(\mathcal{H}, \mathcal{H})$ is equipped with a symmetric norm¹ equivalent to the usual operator norm. Then what does it mean for an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ to be norming and absolutely norming in this setting? What about characterizing these operators?

In [8], we continued the study of absolutely norming operators in this more general setting where $\mathcal{B}(\mathcal{H}, \mathcal{H})$ was equipped with one of the following three (families of) symmetric norms: the Ky Fan k -norm(s), the weighted Ky Fan π, k -norm(s), and the (p, k) -singular norm(s). We defined the notion of absolutely norming operators in each of these cases and thereafter characterized the set of these operators with respect to each of these three (families of) norms; see [8, Theorems 4.22, 5.12 and 5.13]. This detailed study of several particular symmetric norms provided us with the insight of *how* to extend the concept of “norming” and “absolutely norming” from a specific symmetric norm to an arbitrary symmetric norm that is equivalent to the usual operator norm; earlier definitions used the intrinsic nature of each symmetric norm in question. (See Definition 3.1; the attention has been restricted to the algebra $\mathcal{B}(\mathcal{H})$ of operators on a separable infinite-dimensional Hilbert space \mathcal{H} , and the theory of symmetrically-normed ideals and their Banach space duals play a crucial part in arriving at this definition.)

The subsequent discussion in [8] involves positive operators of the form of a non-negative scalar multiple of the identity plus a positive compact plus a self-adjoint finite-rank. It is not clear, *a priori*, if the operators of this form are absolutely norming with

¹By abuse of terminology, we continue to use the term “symmetric” to refer to those norms on $\mathcal{B}(\mathcal{H}, \mathcal{H})$ which are symmetric in the sense of Definition 1.1 whenever \mathcal{H} is separable and $\mathcal{H} = \mathcal{H}$.

respect to every symmetric norm on $\mathcal{B}(\mathcal{H})$. It turns out that there exists a symmetric norm on $\mathcal{B}(\ell^2)$ with respect to which the identity operator does not attain its norm; see Theorem 3.5 [8, Proposition 1.3].

1.2. Overview of this paper

The proof of the result presented in [8] — that there exists a symmetric norm on $\mathcal{B}(\ell^2)$ such that the identity operator does not attain its norm — in fact, presents a constructive method to produce, not merely one, but an infinite family of symmetric norms on $\mathcal{B}(\ell^2)$ with respect to each of which the identity operator does not attain its norm.

In this manuscript, we formally introduce this family of symmetric norms and study them in detail (see Section 3). This leads us to Theorem 3.6 which generalizes the above mentioned result from [8]; the hypothesis is relaxed, and consequently, the family of symmetric norms for which the result holds is extended.

We then introduce and study the notion of “universally symmetric norming operators” or USN operators (see Definition 5.1) and “universally absolutely symmetric norming operators” or UASN operators (see Definition 5.2) on a separable infinite-dimensional Hilbert space. These refer to the operators that are, respectively, norming and absolutely norming, with respect to every symmetric norm. It is known (see Theorem 3.2) that a compact operator in $\mathcal{B}(\mathcal{H})$ is universally absolutely symmetric norming, and hence universally symmetric norming. This renders compacts as prototypical examples of such operators. So, we have

$$\text{compact operators} \subseteq \text{UASN operators} \subseteq \text{USN operators}.$$

It would be desirable to know whether a USN operator is compact. In Section 5 we answer this question affirmatively. The following is the main result of this section which essentially states that an operator in $\mathcal{B}(\mathcal{H})$ is universally symmetric norming if and only if it is compact.

THEOREM 5.6. *Let $T \in \mathcal{B}(\mathcal{H})$ and let Φ_1 denote the maximal s.n.function. Then the following statements are equivalent.*

1. $T \in \mathcal{B}_0(\mathcal{H})$.
2. T is universally absolutely symmetric norming, that is, $T \in \mathcal{AN}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to Φ_1 .
3. T is universally symmetric norming, that is, $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to Φ_1 .

We hence establish a characterization theorem for such operators on $\mathcal{B}(\mathcal{H})$. In particular, this result provides an alternative characterization theorem for compact operators on a separable Hilbert space.

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2. Preliminaries

In this section we recall some notions and results concerning the ideal structure of the algebra of all bounded linear operators acting on a separable Hilbert space. Also, since this work is a continuation of [8], this section essentially builds upon, and hence requires, the preliminary section of [8] where the concept of singular values is extended from compact operators to *any* operator; we refer the reader to [8, Section 2] and [7].

NOTATION 2.1. Consider the algebra $\mathcal{B}(\mathcal{H})$ of operators on a separable Hilbert space \mathcal{H} . We let $\mathcal{B}_{00}(\mathcal{H})$, respectively, $\mathcal{B}_0(\mathcal{H})$ denote the set of all finite-rank operators on \mathcal{H} , respectively, the set of compacts. We use $\mathcal{B}_1(\mathcal{H})$ to denote the trace class operators, with the trace norm $\|\cdot\|_1$. By c_0 we denote the space of all convergent sequences of real numbers with limit 0 and we let $c_{00} \subseteq c_0$ be the linear subspace of c_0 consisting of all sequences with a finite number of nonzero terms. The positive cone of c_{00} is denoted by c_{00}^+ and we use $c_{00}^* \subseteq c_{00}^+$ to denote the cone of all nonincreasing nonnegative sequences from c_{00} .

DEFINITION 2.2. An ideal \mathfrak{S} of the algebra $\mathcal{B}(\mathcal{H})$ is said to be a *symmetrically-normed ideal* (abbreviated an *s.n.ideal*) of $\mathcal{B}(\mathcal{H})$ if on it there is defined a symmetric norm $\|\cdot\|_{\mathfrak{S}}$ which makes \mathfrak{S} a Banach space (that is, \mathfrak{S} is complete in the metric given by this norm). We say that two ideals \mathfrak{S}_I and \mathfrak{S}_{II} *coincide elementwise* if \mathfrak{S}_I and \mathfrak{S}_{II} consist of the same elements.

DEFINITION 2.3. [7, Chapter 3, Page 71] A function $\Phi : c_{00} \rightarrow [0, \infty)$ is said to be *symmetric norming function* (or simply an *s.n.function*) if it satisfies the following properties:

- (i) $\Phi(\xi) \geq 0$ for every $\xi := (\xi_j)_{j \in \mathbb{N}} \in c_{00}$;
- (ii) $\Phi(\xi) = 0 \iff \xi = 0$;
- (iii) $\Phi(\alpha\xi) = |\alpha|\Phi(\xi)$ for every $\xi \in c_{00}$ and for every scalar $\alpha \in \mathbb{R}$; and
- (iv) $\Phi(\xi + \psi) \leq \Phi(\xi) + \Phi(\psi)$ for every pair ξ, ψ of sequences in c_{00} .
- (v) $\Phi((\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots)) = \Phi((|\xi_{j_1}|, |\xi_{j_2}|, \dots, |\xi_{j_n}|, 0, 0, \dots))$ for every $\xi \in c_{00}$ and for every $n \in \mathbb{N}$, where j_1, j_2, \dots, j_n is any permutation of the integers $1, 2, \dots, n$.
- (vi) $\Phi((1, 0, 0, \dots)) = 1$.

DEFINITION 2.4. [7, Chapter 3, Page 76] Let Φ and Ψ be two s.n.functions. Φ and Ψ are said to be *equivalent* if

$$\sup_{\xi \in c_{00}} \frac{\Phi(\xi)}{\Psi(\xi)} < \infty \text{ and } \sup_{\xi \in c_{00}} \frac{\Psi(\xi)}{\Phi(\xi)} < \infty.$$

We say that $\Phi \leq \Psi$ if for every every $\xi \in c_{00}$, we have $\Phi(\xi) \leq \Psi(\xi)$.

REMARK 2.5. A moment’s thought will convince the readers that an s.n.function can be uniquely defined by its values on the cone c_{00}^* . Here the *minimal* s.n.function $\Phi_\infty : c_{00}^* \rightarrow [0, \infty)$ is defined by $\Phi_\infty(\xi) = \xi_1$ for every $\xi = (\xi_j)_j \in c_{00}^*$ and the *maximal* s.n.function $\Phi_1 : c_{00}^* \rightarrow [0, \infty)$ is defined by $\Phi_1(\xi) = \sum_j \xi_j$ for every $\xi = (\xi_j)_j \in c_{00}^*$. For any s.n.function Φ , we have $\Phi_\infty \leq \Phi \leq \Phi_1$ (see [7, Chapter 3, Section 3, Relation 3.12, Page 76]).

DEFINITION 2.6. Let Φ be an s.n.function defined on c_{00}^* . Then the function given by the formula

$$\Phi^*(\eta) = \max \left\{ \sum_j \eta_j \xi_j : \xi \in c_{00}^*, \Phi(\xi) = 1 \right\} \text{ for every } \eta \in c_{00}^*,$$

is defined to be the *adjoint of the function* Φ .

REMARK 2.7. That Φ^* is an s.n.function is a trivial observation. Also, the adjoint of Φ^* is Φ . In particular, the minimal and maximal s.n.functions are the adjoint of each other, that is, $\Phi_1^* = \Phi_\infty$ and $\Phi_\infty^* = \Phi_1$. Therefore, when an s.n.function is equivalent to the maximal (minimal) one, its adjoint is equivalent to the minimal (maximal) one.

REMARK 2.8. It is evident that every s.n.ideal gives rise to an s.n.function. Conversely, to every s.n.function Φ we associate an s.n.ideal \mathfrak{S}_Φ , which is referred to as the s.n.ideal generated by the s.n.function Φ . For a detailed exposition of the construction of the s.n.ideal from an s.n.function we refer the reader to Gohberg and Krein’s text [7, Chapter 3] which elaborately discusses the theory of s.n.ideals. An abridged outline of this construction has also been discussed in [8, Section 6, Notation 6.4]. Since we will be dealing with this theory extensively, we have attempted to duplicate their notation wherever possible.

If Φ, Ψ are s.n.functions and $\mathfrak{S}_\Phi, \mathfrak{S}_\Psi$ are the s.n.ideals generated by these s.n.functions respectively, then \mathfrak{S}_Φ and \mathfrak{S}_Ψ coincide elementwise if and only if Φ and Ψ are equivalent. In particular, if Φ is an s.n.function equivalent to Φ_1 , then \mathfrak{S}_Φ and $\mathcal{B}_1(\mathcal{H})$ coincide elementwise and when Φ is equivalent to Φ_∞ , \mathfrak{S}_Φ and $\mathcal{B}_0(\mathcal{H})$ coincide elementwise.

Two more notations are in order. The Banach space dual of a Banach space X is denoted by X^* in the sequel. To indicate that Banach spaces X and Y are isometrically isomorphic, we write $X \cong Y$ isometrically (or, $Y \cong X$ isometrically).

We conclude this section by an often needed piece of folklore from [7].

PROPOSITION 2.9. [7, Chapter 3, Theorem 12.4] *If Φ is an arbitrary s.n.function equivalent to the maximal s.n.function, then the general form of a continuous linear functional f on the separable space \mathfrak{S}_Φ is given by $f(X) = \text{Tr}(AX)$ for some $A \in \mathcal{B}(\mathcal{H})$ and*

$$\|f\| := \sup\{|\text{Tr}(AX)| : X \in \mathfrak{S}_\Phi, \|X\|_\Phi \leq 1\} = \|A\|_{\Phi^*}.$$

Thus, the Banach space dual \mathfrak{S}_Φ^* is isometrically isomorphic to $(\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$, that is,

$$\mathfrak{S}_\Phi^* \cong (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*}).$$

3. Norms that are not attained by the identity

In this section, we revisit a result from [8] and use it to formally introduce a family of symmetric norms on $\mathcal{B}(\mathcal{H})$ with respect to each of which the identity operator is rendered nonnorming. Thereafter, in Subsection 3.3, we improve upon this above mentioned result by extending it to a larger family of symmetric norms.

3.1. Symmetric norming and absolutely symmetric norming operators

Given an arbitrary s.n.function Φ that is equivalent to the maximal s.n.function, we now recall the definition of operators in $\mathcal{B}(\mathcal{H})$ that attain their Φ^* -norm.

DEFINITION 3.1. [8, Definitions 6.11, 6.14] Let Φ be an s.n.function equivalent to the maximal s.n.function. An operator $T \in (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ is said to be Φ^* -norming or symmetric norming with respect to the symmetric norm $\|\cdot\|_{\Phi^*}$ if there exists an operator $K \in \mathfrak{S}_\Phi = (\mathcal{B}_1(\mathcal{H}), \|\cdot\|_\Phi)$ with $\|K\|_\Phi = 1$ such that $|\text{Tr}(TK)| = \|T\|_{\Phi^*}$. We say that $T \in (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ is absolutely Φ^* -norming or absolutely symmetric norming with respect to the symmetric norm $\|\cdot\|_{\Phi^*}$ if for every nontrivial closed subspace \mathcal{M} of \mathcal{H} , $TP_{\mathcal{M}} \in \mathcal{B}(\mathcal{H})$ is Φ^* -norming (here $P_{\mathcal{M}}$ is the orthogonal projection onto \mathcal{M}).

We let $\mathcal{N}_{\Phi^*}(\mathcal{H})$ and $\mathcal{AN}_{\Phi^*}(\mathcal{H})$ respectively denote the set of Φ^* -norming and absolutely Φ^* -norming operators in $\mathcal{B}(\mathcal{H})$. Needless to mention, every absolutely Φ^* -norming operator is Φ^* -norming, that is, $\mathcal{AN}_{\Phi^*}(\mathcal{H}) \subseteq \mathcal{N}_{\Phi^*}(\mathcal{H})$.

With the above definitions established to guide the way, we prove and collect certain fundamental results concerning symmetric norming and absolutely symmetric norming operators in $\mathcal{B}(\mathcal{H})$. We begin by recalling the following result.

THEOREM 3.2. [8, Theorem 6.17] *Let Φ be an arbitrary s.n.function equivalent to the maximal s.n.function. If T is a compact operator, then $T \in \mathcal{AN}_{\Phi^*}(\mathcal{H})$, that is, $\mathcal{B}_0(\mathcal{H}) \subseteq \mathcal{AN}_{\Phi^*}(\mathcal{H})$.*

Following the well established precedent, we use $s_j(T)$ to denote the j -th singular value (or singular number or s-number) of $T \in \mathcal{B}(\mathcal{H})$. The following proposition allows us to concentrate on the positive operators that are symmetric norming. The

technique of the proof might be elementary but since we do not have a reference for the exact statement, a complete proof is provided.

PROPOSITION 3.3. *Let Φ be an s.n.function equivalent to the maximal s.n.function. Then $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ if and only if $|T| \in \mathcal{N}_{\Phi^*}(\mathcal{H})$.*

Proof. We first assume that $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ and observe that $\|T\|_{\Phi^*} = \||T|\|_{\Phi^*}$ since for each j , $s_j(T) = s_j(|T|)$. Then there exists $K \in \mathcal{B}_1(\mathcal{H})$ with $\|K\|_{\Phi} = 1$ such that $\|T\|_{\Phi^*} = |\text{Tr}(TK)|$. If $T = U|T|$ is the polar decomposition of T , then

$$\||T|\|_{\Phi^*} = \|T\|_{\Phi^*} = |\text{Tr}(TK)| = |\text{Tr}(U|T|K)| = |\text{Tr}(|T|KU)|,$$

where $KU \in \mathcal{B}_1(\mathcal{H})$ with $\|KU\|_{\Phi} = \|IKU\|_{\Phi} \leq \|I\| \|K\|_{\Phi} \|U\| = \|K\|_{\Phi} = 1$. In fact, $\|KU\|_{\Phi} = 1$; for if not, then the operator $S := KU/\|KU\|_{\Phi} \in \mathcal{B}_1(\mathcal{H})$ satisfies $\|S\|_{\Phi} = 1$ and yields

$$|\text{Tr}(|T|S)| = \left| \text{Tr} \left(\frac{|T|KU}{\|KU\|_{\Phi}} \right) \right| = \frac{1}{\|KU\|_{\Phi}} |\text{Tr}(|T|KU)| > |\text{Tr}(|T|KU)| = \||T|\|_{\Phi^*},$$

which contradicts the fact that the supremum of the set

$$\{|\text{Tr}(|T|X)| : X \in \mathcal{B}_1(\mathcal{H}), \|X\|_{\Phi} \leq 1\}$$

is attained at KU . This shows that $|T| \in \mathcal{N}_{\Phi^*}(\mathcal{H})$.

Conversely, if $|T| \in \mathcal{N}_{\Phi^*}(\mathcal{H})$, then by replacing T with $|T|$ in the above argument using $|T| = U^*T$, we can prove the existence of $\hat{K} \in \mathcal{B}_1(\mathcal{H})$ with $\|\hat{K}\|_{\Phi} = 1$ such that $\|T\|_{\Phi^*} = |\text{Tr}(T\hat{K}U^*)|$ where $\hat{K}U^* \in \mathcal{B}_1(\mathcal{H})$ with $\|\hat{K}U^*\|_{\Phi} \leq 1$. It can then be shown that $\|\hat{K}U^*\|_{\Phi} = 1$ and the result follows. \square

We need one more result concerning the computation of the symmetric norm of an operator.

PROPOSITION 3.4. *Let Φ be an s.n.function equivalent to the maximal s.n.function and let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\|T\|_{\Phi^*} = \sup \left\{ \sum_j s_j(T)s_j(K) : K \in \mathcal{B}_1(\mathcal{H}), K = \text{diag}\{s_j(K)\}_j, \|K\|_{\Phi} = 1 \right\}$$

Proof. Since Φ is equivalent to the maximal s.n.function, we know that $\mathfrak{S}_{\Phi}^* \cong (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ isometrically and by Definition 3.1 the $\|\cdot\|_{\Phi^*}$ norm for any operator $T \in \mathcal{B}(\mathcal{H})$ is given by $\|T\|_{\Phi^*} = \sup\{|\text{Tr}(TK)| : K \in \mathfrak{S}_{\Phi}, \|K\|_{\Phi} = 1\}$. But the ideal $\mathcal{B}_1(\mathcal{H})$ and \mathfrak{S}_{Φ} coincide elementwise and hence $\|T\|_{\Phi^*} = \sup\{|\text{Tr}(TK)| : K \in \mathcal{B}_1(\mathcal{H}), \|K\|_{\Phi} = 1\}$.

First we set

$$\alpha := \sup\{|\text{Tr}(TK)| : K \in \mathcal{B}_1(\mathcal{H}), \|K\|_{\Phi} = 1\}$$

and

$$\beta := \sup\left\{\sum_j s_j(T)s_j(K) : K \in \mathcal{B}_1(\mathcal{H}), \|K\|_\Phi = 1\right\},$$

and thereafter we claim that $\alpha = \beta$. That $\alpha \leq \beta$ is a trivial observation since $|\text{Tr}(TK)| \leq \sum_j s_j(TK) \leq \sum_j s_j(T)s_j(K)$. To see $\beta \leq \alpha$, let us choose an operator $K \in \mathcal{B}_1(\mathcal{H})$ with $\|K\|_\Phi = 1$. An easy computation yields

$$\begin{aligned} \sum_j s_j(T)s_j(K) &= \left\langle \begin{bmatrix} s_1(T) \\ \vdots \\ s_j(T) \\ \vdots \end{bmatrix}, \begin{bmatrix} s_1(K) \\ \vdots \\ s_j(K) \\ \vdots \end{bmatrix} \right\rangle \\ &\leq \Phi^* \begin{pmatrix} s_1(T) \\ \vdots \\ s_j(T) \\ \vdots \end{pmatrix} \Phi \begin{pmatrix} s_1(K) \\ \vdots \\ s_j(K) \\ \vdots \end{pmatrix} \\ &= \Phi^* \begin{pmatrix} s_1(T) \\ \vdots \\ s_j(T) \\ \vdots \end{pmatrix} = \|T\|_{\Phi^*} \\ &= \sup\{|\text{Tr}(TK)| : K \in \mathcal{B}_1(\mathcal{H}), \|K\|_\Phi = 1\} = \alpha. \end{aligned}$$

It then follows that $\beta \leq \alpha$ and this proves our first claim.

We next let $\gamma := \sup\{\sum_j s_j(T)s_j(K) : K \in \mathcal{B}_1(\mathcal{H}), K = \text{diag}\{s_j(K)\}, \|K\|_\Phi = 1\}$ and prove that $\gamma = \beta$. That $\gamma \leq \beta$ is obvious. To prove $\beta \leq \gamma$, we choose an operator $K \in \mathcal{B}_1(\mathcal{H})$ with $\|K\|_\Phi = 1$ and define

$$\tilde{K} := \begin{pmatrix} s_1(K) & & & & \\ & s_2(K) & & & \\ & & \mathbf{0} & & \\ & & & \ddots & \\ & \mathbf{0} & & & s_j(K) \\ & & & & & \ddots \end{pmatrix}.$$

Notice that for every j , we have $s_j(\tilde{K}) = s_j(K)$ which implies that $\|\tilde{K}\|_\Phi = \|K\|_\Phi = 1$. Even more, $\tilde{K} \in \mathcal{B}_1(\mathcal{H})$ and hence $\sum_j s_j(T)s_j(K) = \sum_j s_j(T)s_j(\tilde{K})$. But since

$$\sum_j s_j(T)s_j(\tilde{K}) \leq \sup\left\{\sum_j s_j(T)s_j(K) : K \in \mathcal{B}_1(\mathcal{H}), K = \text{diag}\{s_j(K)\}, \|K\|_\Phi = 1\right\},$$

it follows that $\beta \leq \gamma$ which establishes our second claim. From the above two observations we conclude that $\alpha = \gamma$, and consequently the assertion is proved. \square

3.2. Norm(s) that are not attained by the identity

We now recall the following result which states that there exists a symmetric norm on $\mathcal{B}(\mathcal{H})$ with respect to which the identity operator does not attain its norm.

THEOREM 3.5. [8, Proposition 1.3] *There exists a symmetric norm $\|\cdot\|_{\Phi_\pi^*}$ on $\mathcal{B}(\ell^2)$ such that $I \notin \mathcal{N}_{\Phi_\pi^*}(\ell^2)$.*

There is more to this theorem than meets the eye; its proof is constructive and illustrates an elegant technique of producing (a family of) symmetric norms on $\mathcal{B}(\mathcal{H})$ with respect to each of which the identity operator does not attain its norm. More precisely, the proof demonstrates a well-defined family of s.n.functions — henceforth referred to as “s.n.functions affiliated to strictly decreasing weights” — which naturally generate such symmetric norms. In what follows we formally introduce this family of s.n.functions.

Let $\widehat{\Pi}$ denote the set of all strictly decreasing convergent sequences of positive numbers with their first term equal to 1 and positive limit, that is,

$$\widehat{\Pi} = \{ \pi := (\pi_n)_{n \in \mathbb{N}} : \pi_1 = 1, \lim \pi_n > 0, \text{ and } \pi_k > \pi_{k+1} \text{ for every } k \in \mathbb{N} \}.$$

For each $\pi \in \widehat{\Pi}$, let Φ_π denote the symmetric norming function defined by $\Phi_\pi(\xi_1, \xi_2, \dots) = \sum_j \pi_j \xi_j$ and observe that Φ_π is equivalent to the maximal s.n.function Φ_1 . Theorem 3.5 essentially proves that $I \notin \mathcal{N}_{\Phi_\pi^*}(\ell^2)$ whenever Φ_π belongs to the family $\{ \Phi_\pi : \pi \in \widehat{\Pi} \}$ of s.n.functions affiliated to strictly decreasing weights.

3.3. A few more norms that are not attained by the identity

We construct more symmetric norms on $\mathcal{B}(\mathcal{H})$ with respect to which the identity operator does not attain its norm. This subsection aims at extending Theorem 3.5 to a larger family of symmetric norms, and thus generalizing it.

Observe that the proof of Theorem 3.5, in its construction of symmetric norms, assumes that the sequence π belongs to $\widehat{\Pi}$ and is thus strictly decreasing. We claim that the requirement for π to be a strictly decreasing sequence is not necessary for the result to hold, as long as there exists a natural number N such that $\pi_N > \pi_{N+1}$. Let us use Π to denote the set of all nonincreasing sequences of positive numbers with their first term equal to 1 and positive limit (so that $\widehat{\Pi} \subseteq \Pi$)². Thus, if $\pi \in \Pi$ is not the constant sequence $\mathbf{1} = (1, 1, \dots)$, then there exists a natural number N such that $\pi_N > \pi_{N+1}$, and consequently, our claim amounts to showing that the adjoint of the s.n.function defined via π generates a symmetric norm on $\mathcal{B}(\ell^2)$ with respect to which the identity operator does not attain its norm.

We conclude this section by proving the following result which, in effect, extends the preceding theorem to every symmetric norm Φ_π in the family $\{ \Phi_\pi : \pi \in \Pi \}$ of

²For the convenience of the readers, it would perhaps be worth to recall that we have used Π , in [8], to denote the set of all nonincreasing sequences of positive numbers with their first term equal to 1. However, in this manuscript, we use Π to denote the set of all nonincreasing sequences of positive numbers with their first term equal to 1 and positive limit. This abuse of notation deemed necessary to avoid too many symbols.

s.n.functions except when $\pi \in \Pi$ is the constant sequence $\mathbf{1} = (1, 1, \dots)$; earlier, the result was shown to hold for the family $\{\Phi_\pi : \pi \in \widehat{\Pi}\}$ of s.n.functions.

THEOREM 3.6. *Let Π be the set of all nonincreasing convergent sequence of positive numbers with their first term equal to 1 and positive limit, that is,*

$$\Pi = \{\pi := (\pi_n)_{n \in \mathbb{N}} : \pi_1 = 1, \lim_n \pi_n > 0, \text{ and } \pi_k \geq \pi_{k+1} \text{ for each } k \in \mathbb{N}\},$$

and consider the subset $\Pi \setminus \{\mathbf{1}\}$ of Π consisting of all nonincreasing convergent sequence of positive numbers with their first term equal to 1 and positive limit except the constant sequence $\mathbf{1}$. For each $\pi \in \Pi \setminus \{\mathbf{1}\}$, let Φ_π denote the symmetric norming function defined by $\Phi_\pi(\xi_1, \xi_2, \dots) = \sum_j \pi_j \xi_j$. Then

1. Φ_π is equivalent to the maximal s.n.function Φ_1 ; and
2. for every $\pi \in \Pi \setminus \{\mathbf{1}\}$, $I \notin \mathcal{N}_{\Phi_\pi^*}(\ell^2)$.

Alternatively, $I \notin \mathcal{N}_{\Phi_\pi^*}(\ell^2)$ for every Φ_π that belongs to the family $\{\Phi_\pi : \pi \in \Pi \setminus \{\mathbf{1}\}\}$ of s.n.functions.

Proof. That each s.n.function from the family $\{\Phi_\pi : \pi \in \Pi \setminus \{\mathbf{1}\}\}$ of s.n.functions is equivalent to the maximal s.n.function Φ_1 is, now, a trivial observation.

The proof of the second claim is almost along the lines of the proof of Theorem 3.5. Let $\pi = (\pi_n)_{n \in \mathbb{N}} \in \Pi \setminus \{\mathbf{1}\}$ and let Φ_π be the s.n.function generated by π , that is, $\Phi_\pi(\xi_1, \xi_2, \dots) = \sum_j \pi_j \xi_j$. Now, contrapositively assume that $I \in \mathcal{N}_{\Phi_\pi^*}$, then the supremum,

$$\sup \left\{ \sum_j s_j(K) : K \in \mathcal{B}_1(\ell^2), K = \text{diag}\{s_1(K), s_2(K), \dots\}, \|K\|_{\Phi_\pi} = 1 \right\},$$

is attained, that is, there exists $K_0 = \text{diag}\{s_1(K_0), s_2(K_0), \dots\} \in \mathcal{B}_1(\ell^2)$ with $\sum_j \pi_j s_j(K_0) = 1$ such that $\|I\|_{\Phi_\pi^*} = \sum_j s_j(K_0)$. We will prove the existence of an operator $\tilde{K} \in \mathcal{B}_1(\ell^2)$, $\|\tilde{K}\|_{\Phi_\pi} = 1$ of the form $\tilde{K} = \text{diag}\{s_1(\tilde{K}), s_2(\tilde{K}), \dots\}$ such that $\sum_i s_i(\tilde{K}) > \sum_j s_j(K_0)$. To this end, since $\pi \in \Pi \setminus \{\mathbf{1}\}$, there exists a natural number M such that $\pi_M > \pi_{M+1}$. Set

$$\lambda = \frac{\pi_M}{\pi_{M+1}},$$

and choose $\varepsilon > 0$ such that

$$s_M(K_0) - \varepsilon = s_{M+1}(K_0) + \lambda \varepsilon.$$

(Of course, there is only one such ε .) Now define \tilde{K} to be the diagonal operator given

Since the operator K_0 with which we began our discussion is arbitrary, it follows that for any given operator in $\mathcal{B}_1(\ell^2)$ with unit norm, one can find another operator in $\mathcal{B}_1(\ell^2)$ with unit norm with trace of larger magnitude and hence the supremum of the above set can never be attained. This shows that the identity operator does not attain its norm. \square

4. Symmetric norming operators affiliated to strictly decreasing weights and their characterization

In this section we return to the study of the family of symmetric norms on $\mathcal{B}(\mathcal{H})$ generated by the duals (or adjoints) of s.n.functions from the family $\{\Phi_\pi : \pi \in \widehat{\Pi}\}$ of s.n.functions affiliated to strictly decreasing weights, and establish a characterization theorem for operators in $\mathcal{B}(\mathcal{H})$ that are symmetric norming with respect to every such symmetric norm. (Recall that for each $\pi \in \widehat{\Pi}$, Φ_π denotes the s.n.function defined by $\Phi_\pi(\xi_1, \xi_2, \dots) = \sum_j \pi_j \xi_j$, and that Φ_π is equivalent to the maximal s.n.function Φ_1 .) This section also studies the operators in $\mathcal{B}(\mathcal{H})$ that are absolutely symmetric norming with respect to every symmetric norm in the family and presents a characterization theorem for those as well. It turns out that an operator is symmetric norming with respect to every symmetric norm in the family if and only if it is absolutely symmetric norming with respect to every symmetric norm in the family. This ‘‘characterization theorem’’ is the main theorem of this section.

Theorem 3.5 essentially proves that $I \notin \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$ whenever $\pi \in \widehat{\Pi}$. We know that for an arbitrary s.n.function Φ equivalent to Φ_1 , we have $\mathcal{N}_{\Phi^*}(\mathcal{H}) \not\subseteq \mathcal{B}_0(\mathcal{H})$. However, it is of interest to know whether $\mathcal{N}_{\Phi_\pi^*}(\mathcal{H}) \subseteq \mathcal{B}_0(\mathcal{H})$ if $\pi \in \widehat{\Pi}$; for if the answer to this question is affirmative, then Theorem 3.2 would yield $\mathcal{N}_{\Phi_\pi^*}(\mathcal{H}) = \mathcal{B}_0(\mathcal{H})$ for every $\pi \in \widehat{\Pi}$ (and would thus characterize the Φ_π^* -norming operators in $\mathcal{B}(\mathcal{H})$ for every $\pi \in \widehat{\Pi}$). By Proposition 3.3 it suffices to know whether $\mathcal{N}_{\Phi_\pi^*}(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})_+ \subseteq \mathcal{B}_0(\mathcal{H})$ where $\mathcal{B}(\mathcal{H})_+ = \{T \in \mathcal{B}(\mathcal{H}) : T \geq 0\}$. The following lemma and example prove the existence of $\pi \in \widehat{\Pi}$ such that $\mathcal{N}_{\Phi_\pi^*}(\mathcal{H}) \not\subseteq \mathcal{B}_0(\mathcal{H})$.

LEMMA 4.1. ([7]) *If $\Phi_\pi \in \{\Phi_\pi : \pi \in \widehat{\Pi}\}$, then its adjoint Φ_π^* is given by*

$$\Phi_\pi^*(\xi) = \sup_n \left\{ \frac{\sum_{j=1}^n \xi_j}{\sum_{j=1}^n \pi_j} \right\} \text{ for every } \xi = (\xi_i)_{i \in \mathbb{N}} \in c_{00}^*.$$

Moreover, the s.n.function Φ_π^* is equivalent to the minimal s.n.function.

For the proof of the above lemma we refer the reader to [7, Chapter 3, Lemma 15.1, Page 147]; readers can also see Pages 148-149, and the paragraph preceding Theorem 15.2 of the monograph [7].

it follows that

$$\frac{\pi_M}{\pi_{M+1}} > \frac{s_M(P)}{s_{M+1}(P)},$$

and thus we have,

$$s_M(P)s_M(\tilde{K}) + s_{M+1}(P)s_{M+1}(\tilde{K}) > s_M(P)s_M(K) + s_{M+1}(P)s_{M+1}(K),$$

which yields

$$\sum_j s_j(P)s_j(\tilde{K}) > \sum_j s_j(P)s_j(K) = \|P\|_{\Phi_\pi^*},$$

which contradicts the assumption that $\sum_j s_j(P)s_j(K)$ is the supremum of the set

$$\left\{ \sum_j s_j(P)s_j(K) : K \in \mathcal{B}_1(\ell^2), K = \text{diag}\{s_1(K), s_2(K), \dots\}, \|K\|_{\Phi_\pi} = 1 \right\}.$$

This proves our assertion. \square

THEOREM 4.6. *Let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator and $\lim_{j \rightarrow \infty} s_j(P) \neq 0$, that is, P is not compact. Then there exists $\pi \in \widehat{\Pi}$ such that*

$$\frac{\pi_n}{\pi_{n+1}} > \frac{s_n(P)}{s_{n+1}(P)} \text{ for every } n \in \mathbb{N}.$$

Alternatively, if $P \in \mathcal{B}(\mathcal{H})$ is positive noncompact operator then there exists $\pi \in \widehat{\Pi}$ such that $P \notin \mathcal{N}_{\Phi_\pi^}(\mathcal{H})$.*

Proof. Since $P \geq 0$ and $\lim_{j \rightarrow \infty} s_j(P) \neq 0$, there exists $s > 0$ such that $\lim_{j \rightarrow \infty} s_j(P) = s$. If we take $\alpha_n := \frac{1}{e^{1/n^2}}$ for all $n \in \mathbb{N}$ and define a sequence $\pi = (\pi_n)_{n \in \mathbb{N}}$ recursively by

$$\pi_1 = 1 \text{ and } \frac{\pi_{n+1}}{\pi_n} := \alpha_n \frac{s_{n+1}(P)}{s_n(P)} \text{ for all } n \in \mathbb{N},$$

we have $\alpha_n < 1$ for all $n \in \mathbb{N}$. Then the fact that $s_n(P)$ is a nonincreasing sequence implies that $\frac{\pi_{n+1}}{\pi_n} < \frac{s_{n+1}(P)}{s_n(P)}$ for all $n \in \mathbb{N}$. Therefore, $\frac{\pi_n}{\pi_{n+1}} > \frac{s_n(P)}{s_{n+1}(P)}$ for every $n \in \mathbb{N}$.

All that remains is to show that $\pi \in \widehat{\Pi}$. That $\pi_1 = 1$ and $(\pi_n)_{n \in \mathbb{N}}$ is a strictly decreasing sequence of positive real numbers are trivial observations. We complete the proof by showing that $\lim_{n \rightarrow \infty} \pi_n > 0$. An easy calculation shows that

$$\pi_{n+1} = \left(\prod_{m=1}^n \alpha_m \right) \left(\frac{s_{n+1}(P)}{s_1(P)} \right) \text{ for each } n \in \mathbb{N}.$$

Let $x_n = \left(\prod_{m=1}^n \alpha_m \right)$ for every $n \in \mathbb{N}$ and observe that

$$\pi_{n+1} = x_n \left(\frac{s_{n+1}(P)}{s_1(P)} \right),$$

which yields

$$\lim_{n \rightarrow \infty} \pi_{n+1} = \frac{1}{s_1(P)} \lim_{n \rightarrow \infty} x_n \lim_{n \rightarrow \infty} s_{n+1}(P).$$

This observation, together with the facts that $s_1(P) > 0$ and $\lim_{n \rightarrow \infty} s_{n+1}(P) = s > 0$ allows us to infer that $\lim_{n \rightarrow \infty} \pi_{n+1} > 0$ if and only if $\lim_{n \rightarrow \infty} x_n > 0$. But

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \frac{1}{e^{\sum_{m=1}^n 1/m^2}} = \frac{1}{e^{\pi/6}} > 0,$$

and we conclude that $\lim_{n \rightarrow \infty} \pi_n > 0$. This completes the proof. \square

We are now in a position to prove a key result — a characterization theorem for positive operators in $\{\mathcal{N}_{\Phi_\pi^*}(\mathcal{H}) : \pi \in \widehat{\Pi}\}$ — which answers the question we asked in the paragraph preceding the Example 4.3. Moreover, this result is a special case of a more general result that is presented in the next section (see Theorem 5.4).

THEOREM 4.7. *Let P be a positive operator on \mathcal{H} . Then the following statements are equivalent.*

1. $P \in \mathcal{B}_0(\mathcal{H})$.
2. $P \in \mathcal{AN}_{\Phi_\pi^*}(\mathcal{H})$ for every $\pi \in \widehat{\Pi}$.
3. $P \in \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$ for every $\pi \in \widehat{\Pi}$.

Proof. (1) implies (2) follows from Theorem 3.2. (2) implies (3) is obvious. (3) implies (1) is a direct consequence of the Theorem 4.6. \square

We conclude this section by proving the following result that extends the above theorem to bounded operators in $\mathcal{B}(\mathcal{H})$, the above theorem required the operator to be positive. This is the main theorem of this section.

THEOREM 4.8. *If $T \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent.*

1. $T \in \mathcal{B}_0(\mathcal{H})$.
2. $T \in \mathcal{AN}_{\Phi_\pi^*}(\mathcal{H})$ for every $\pi \in \widehat{\Pi}$.
3. $T \in \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$ for every $\pi \in \widehat{\Pi}$.

Proof. (2) implies (3) is obvious, as is (1) implies (2) from the Theorem 3.2. The Proposition 3.3 along with the Theorem 4.6 proves (3) implies (1). \square

The above result, although very important, is transitory. We will see a much more general result than this — the characterization theorem for universally symmetric norming operators (see Theorem 5.6).

5. Universally symmetric norming operators and their characterization

In the preceding section we considered a certain family $\{\Phi_\pi : \pi \in \widehat{\Pi}\}$ of s.n.functions and a family of symmetric norms on $\mathcal{B}(\mathcal{H})$ generated by the dual of these, and we studied the symmetric norming operators and absolutely symmetric norming operators with respect to each of these symmetric norms. The fact that each member of the family $\{\Phi_\pi : \pi \in \widehat{\Pi}\}$ is equivalent to the maximal s.n.function Φ_1 suggests the possibility of extending the Theorem 4.8 to a larger family of s.n.functions. With this in mind, our attention is drawn to the family of all s.n.functions that are equivalent to the maximal s.n.function, that is, the family $\{\Phi : \Phi \text{ is equivalent to } \Phi_1\}$ of s.n.functions. This larger family of s.n.functions provides us with the leading idea on which we develop the notions of “universally symmetric norming operators” and “universally absolutely symmetric norming operators” on a separable Hilbert space. The study of these operators are taken up in this section. Our main result is Theorem 5.6 which, in effect, states that an operator in $\mathcal{B}(\mathcal{H})$ is universally symmetric norming if and only if it is universally absolutely symmetric norming, which holds if and only if it is compact.

We begin by defining the relevant classes of operators.

DEFINITION 5.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *universally symmetric norming* if $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to the maximal s.n.function Φ_1 . Alternatively, an operator $T \in (\mathcal{B}(\mathcal{H}))$ is said to be *universally symmetric norming* if $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ for every Φ from the family $\{\Phi : \Phi \text{ is equivalent to } \Phi_1\}$ of s.n.functions.

DEFINITION 5.2. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *universally absolutely symmetric norming* if $T \in \mathcal{AN}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to the maximal s.n.function Φ_1 .

REMARK 5.3. Since every symmetric norm on $\mathcal{B}(\mathcal{H})$ is topologically equivalent to the usual operator norm, it follows that $T \in \mathcal{B}(\mathcal{H})$ is universally symmetric norming (respectively universally absolutely symmetric norming) if and only if T is symmetric norming (respectively absolutely symmetric norming) with respect to every symmetric norm on $\mathcal{B}(\mathcal{H})$. Another important observation worth mentioning here is that every universally absolutely symmetric norming operator is universally symmetric norming.

The following theorem gives a useful characterization of positive universally symmetric norming operators in $\mathcal{B}(\mathcal{H})$.

THEOREM 5.4. *Let P be a positive operator on \mathcal{H} and let Φ_1 denote the maximal s.n.function. Then the following statements are equivalent.*

1. $P \in \mathcal{B}_0(\mathcal{H})$.
2. P is *universally absolutely symmetric norming*, that is, $P \in \mathcal{AN}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to Φ_1 .

3. P is universally symmetric norming, that is, $P \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to Φ_1 .

Proof. The implication (1) \implies (2) is an immediate consequence of Theorem 3.2 and (2) \implies (3) is straightforward. To prove (3) \implies (1), assume that the positive operator P is universally symmetric norming on \mathcal{H} . Then the statement (3) of Theorem 4.7 holds which implies that P is compact and the proof is complete. \square

We next establish the following result which allows us to extend the above theorem to operators that are not necessarily positive.

PROPOSITION 5.5. *An operator $T \in \mathcal{B}(\mathcal{H})$ is universally symmetric norming if and only if $|T|$ is universally symmetric norming.*

Proof. This follows immediately from the Proposition 3.3. \square

We are now prepared to extend the Theorem 5.4 for an arbitrary operator on a separable Hilbert space.

THEOREM 5.6. *Let $T \in \mathcal{B}(\mathcal{H})$ and let Φ_1 denote the maximal s.n.function. Then the following statements are equivalent.*

1. $T \in \mathcal{B}_0(\mathcal{H})$.
2. T is universally absolutely symmetric norming, that is, $T \in \mathcal{A} \mathcal{N}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to Φ_1 .
3. T is universally symmetric norming, that is, $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to Φ_1 .

Proof. Theorem 5.4 and the preceding proposition yield this result. \square

REMARK 5.7. The preceding result provides an alternative characterization of compact operators on \mathcal{H} .

It is worth noticing that Theorem 4.6 essentially states that given any positive noncompact operator on (an infinite-dimensional separable) Hilbert space \mathcal{H} , there exists a symmetric norm on $\mathcal{B}(\mathcal{H})$ with respect to which the operator does not attain its norm. The following corollary extends Theorem 4.6 to any noncompact operator.

COROLLARY 5.8. *If $T \in \mathcal{B}(\mathcal{H})$ is a noncompact operator then there exists $\pi \in \widehat{\Pi}$ such that $T \notin \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$*

Proof. Contrapositively, if for every $\pi \in \widehat{\Pi}$ the operator $T \in \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$, then by the preceding theorem T must be a compact operator. \square

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Satish K. Pandey
 Department of Mathematics
 Indraprastha Institute of Information Technology Delhi (IIIT-Delhi)
 New Delhi, Delhi 110020, India
 e-mail: satish@iiitd.ac.in
<http://noncommutative.space/>