

ORDER BOUNDEDNESS AND ESSENTIAL NORM OF GENERALIZED WEIGHTED COMPOSITION OPERATORS ON BERGMAN SPACES WITH DOUBLING WEIGHTS

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Abstract. In this paper, the order boundedness and essential norm of generalized weighted composition operators on Bergman spaces with doubling weights are characterized. Specially, we estimate the essential norm of these operators on weighted Bergman spaces by using the reduce order method.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in the complex plane \mathbb{C} . Let $\omega : \mathbb{D} \rightarrow [0, \infty)$ be an integrable function and radial, that is, $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. Denote $\hat{\omega}(z) = \int_{|z|}^1 \omega(s) ds$ for all $z \in \mathbb{D}$. A weight ω belongs to the class $\hat{\mathcal{D}}$ if $\hat{\omega}(r) \leq C\hat{\omega}(\frac{1+r}{2})$, where $C = C(\omega) \geq 1$ and $0 \leq r < 1$. Furthermore, we write $\omega \in \check{\mathcal{D}}$ if there exist constants $\vartheta = \vartheta(\omega) > 1$ and $C = C(\omega) > 1$ such that $\hat{\omega}(r) \geq C\hat{\omega}(1 - \frac{1-r}{\vartheta})$ for all $0 \leq r < 1$. We denote $\mathcal{D} = \check{\mathcal{D}} \cap \hat{\mathcal{D}}$ and $\omega(E) = \int_E \omega dA$ for each measurable set $E \subset \mathbb{D}$.

Let $H(\mathbb{D})$ be the space of analytic functions on \mathbb{D} . For $0 < p < \infty$ and the radial weight ω , the Bergman space A_{ω}^p associated to ω is defined by

$$A_{\omega}^p = \left\{ f \in H(\mathbb{D}) : \|f\|_{A_{\omega}^p}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty \right\},$$

where $dA(z) = \frac{1}{\pi} dx dy$ is the normalized Lebesgue area measure on \mathbb{D} . As usual, let A_{α}^p stand for the classical weighted Bergman space induced by radial weight $\omega(z) = (1 - |z|^2)^{\alpha}$, where $-1 < \alpha < \infty$. A_{ω}^p is a Banach space for $1 \leq p < \infty$ under the norm $\|\cdot\|_{A_{\omega}^p}$. See [8, 23] for the theory of weighted Bergman spaces. Let $q > 0$ and μ be a finite positive Borel measure on \mathbb{D} . We say that $f \in L_{\mu}^q$ if the measurable function f satisfies

$$\|f\|_{L_{\mu}^q}^q = \int_{\mathbb{D}} |f(w)|^q d\mu(w) < \infty.$$

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Suppose φ is an analytic map of \mathbb{D} into itself. Every analytic self-map φ induces a composition operator C_φ on $H(\mathbb{D})$ by

$$C_\varphi(f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}),$$

for all $z \in \mathbb{D}$. See [1] and [27] for the theory of composition operators.

For $n \in \mathbb{N}$, $D^n f = f^{(n)}$ is the differential operator on $H(\mathbb{D})$. Let $n \in \mathbb{N} \cup \{0\}$ and $u \in H(\mathbb{D})$. The generalized weighted composition operator, denoted by $D_{\varphi,u}^n$, is defined by

$$D_{\varphi,u}^n(f)(z) = u(z)f^{(n)}(\varphi(z)), \quad f \in H(\mathbb{D}).$$

The generalized weighted composition operator was coined by Zhu in [36]. Clearly, if $n = 0$ and $u \equiv 1$, the operator $D_{\varphi,u}^n$ becomes the composition operator C_φ . When $n = 0$, the operator $D_{\varphi,u}^n$ is called the weighted composition operator, usually denoted by uC_φ . As $n = 1$ and $u(z) = \varphi'(z)$, then $D_{\varphi,u}^n = DC_\varphi$. If $n = 1$ and $u(z) \equiv 1$, then $D_{\varphi,u}^n = C_\varphi D$. The operators DC_φ and $C_\varphi D$ have been studied in [10, 19, 20, 21]. For some recent work on generalized weighted composition operators, we refer the interested readers to [34, 37, 38] and [18].

Let \mathbf{X} be a quasi-Banach space and μ be a positive measure on \mathbb{D} . Assume $0 < q < \infty$ and let $T : \mathbf{X} \rightarrow L_\mu^q$ be an operator. We say that T is order bounded if T maps the closed unit ball $B_{\mathbf{X}}$ of \mathbf{X} into an order interval of L_μ^q . In other words, there exists a non-negative element $h \in L_\mu^q$ such that $|Tf| \leq h$ almost everywhere with respect to μ for all $f \in B_{\mathbf{X}}$. This concept has been studied in several references [31, 29, 28].

The order bounded composition operators on Hardy spaces was introduced by Hunziker and Jarchow in [11]. Motivated by [11], Hibscheweiler [9] characterized order bounded composition operators acting on standard weighted Bergman spaces. Later, Ueki [31] considered the order boundedness of weighted composition operators on standard weighted Bergman spaces. Subsequently, Wolf [33] studied order bounded weighted composition operators acting on Bergman spaces with general weights. Recently, the order boundedness of weighted composition operators acting between Banach spaces like Hardy spaces, weighted Bergman spaces, weighted Dirichlet spaces and derivative Hardy spaces were discussed (see [29, 28, 7, 12]). Motivated by [31, 33], we investigate the order boundedness of $D_{\varphi,u}^n$ on Bergman spaces with doubling weights.

Let X and Y be Banach spaces. The essential norm of linear operator $T : X \rightarrow Y$ is defined as

$$\|T\|_{e, X \rightarrow Y} = \inf_K \|T - K\|_{X \rightarrow Y},$$

where K is any compact operator and $\|\cdot\|_{X \rightarrow Y}$ is the operator norm. It is obvious that $\|T\|_{e, X \rightarrow Y} = 0$ if and only if T is a compact operator.

The study of the essential norm of composition operators on Hardy spaces and Bergman spaces was dated back to Shapiro [26]. Čučković and Zhao extended Shapiro's [26] results to standard weighted Bergman spaces and Hardy spaces in [2, 3]. After their works, Demazeux [4] considered the essential norm of weighted composition operators on Hardy spaces in terms of pullback measure for $1 \leq p, q \leq \infty$. In light of their work,

the authors [5] investigated the boundedness and essential norm of weighted composition operators on Bergman spaces induced by doubling weights. Based on their work and inspired by the idea from [37], Liu [13] studied the boundedness and compactness of generalized weighted composition operators $D_{\varphi,u}^n$ between different Bergman spaces with doubling weights. See [5, 15, 14, 13] for more results of composition operators on Bergman spaces A_{ω}^p . The essential norm of composition operators on Bergman spaces with admissible Békollé weights was studied by [30]. Recently, Esmaeili and Kellay [6] considered the essential norm of weighted composition operators on weighted Bergman spaces. Many authors considered the essential norm of composition operators on different weighted Bergman spaces, see [32, 17] and references therein.

Motivated by the idea from [5, 6, 30], we estimate the essential norm of generalized weighted composition operators on Bergman spaces with doubling weights.

In this paper, we denote constants by C which are positive and may differ from one occurrence to the other. The notation $a \lesssim b$ means that there is a positive constant C such that $a \leq Cb$. The symbol $a \asymp b$ means that both $a \lesssim b$ and $b \lesssim a$ hold.

2. Preliminary results

The pseudo-hyperbolic metric ρ on \mathbb{D} is defined as

$$\rho(z, w) = |\varphi_w(z)| = \left| \frac{w - z}{1 - \bar{w}z} \right|,$$

for $z, w \in \mathbb{D}$. For $r \in (0, 1)$, the pseudo-hyperbolic disk is defined by

$$\Delta(w, r) = \{z \in \mathbb{D}, \rho(z, w) < r\}.$$

For $z \in \mathbb{D} \setminus \{0\}$,

$$S(z) = \left\{ \xi \in \mathbb{D} : |z| \leq |\xi| < 1, |\arg \xi - \arg z| < \frac{1 - |z|}{2} \right\}$$

is called a Carleson square. We set $S(0) = \mathbb{D}$.

LEMMA 2.1. [13, Lemma 2.1] *Let $\omega \in \mathcal{D}$, $0 < p < \infty$ and $n \in \mathbb{N} \cup \{0\}$. If $f \in A_{\omega}^p$, then there exists a constant $C = C(\omega) > 0$ such that*

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{A_{\omega}^p}}{(\omega(S(z)))^{1/p} (1 - |z|)^n}$$

for all $z \in \mathbb{D}$.

LEMMA 2.2. [13, Proposition 3.1] *Let $0 < p \leq q < \infty$, $\omega \in \mathcal{D}$ and $n \in \mathbb{N} \cup \{0\}$. Let μ be a positive Borel measure on \mathbb{D} . Then there exists $r = r(\omega) \in (0, 1)$ such that the following statements hold.*

(i) $D^n : A_{\omega}^p \rightarrow L_{\mu}^q$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{\mu(\Delta(z, r))}{(\omega(S(z)))^{q/p}(1 - |z|)^{nq}} < \infty. \tag{2.1}$$

Moreover,

$$\|D^n\|_{A_{\omega}^p \rightarrow L_{\mu}^q}^q \asymp \sup_{z \in \mathbb{D}} \frac{\mu(\Delta(z, r))}{(1 - |z|)^{nq}(\omega(S(z)))^{q/p}}.$$

(ii) $D^n : A_{\omega}^p \rightarrow L_{\mu}^q$ is compact if and only if

$$\lim_{|z| \rightarrow 1^-} \frac{\mu(\Delta(z, r))}{(\omega(S(z)))^{q/p}(1 - |z|)^{nq}} = 0. \tag{2.2}$$

In light of Lemma 2.2, we get the following lemma.

LEMMA 2.3. Let $0 < p \leq q < \infty$, $\omega \in \mathcal{D}$ and $n \in \mathbb{N} \cup \{0\}$. Assume that μ is a positive Borel measure on \mathbb{D} , $r = r(\omega) \in (0, 1)$. Then there exist a large enough $\delta = \delta(\omega, p) > 0$ such that

$$\|D^n\|_{A_{\omega}^p \rightarrow L_{\mu}^q}^q \asymp \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|)^{\delta q}}{|1 - \bar{a}w|^{(\delta+n)q}(\omega(S(a)))^{q/p}} d\mu(w).$$

Proof. For $a \in \mathbb{D}$ and $r \in (0, 1)$, we have

$$\begin{aligned} \frac{\mu(\Delta(a, r))}{(1 - |a|)^{nq}(\omega(S(a)))^{q/p}} &= \int_{\Delta(a, r)} \frac{1}{(1 - |a|)^{nq}(\omega(S(a)))^{q/p}} d\mu(w) \\ &\asymp \int_{\Delta(a, r)} \frac{(1 - |a|)^{\delta q}}{|1 - \bar{a}w|^{(\delta+n)q}(\omega(S(a)))^{q/p}} d\mu(w) \\ &\lesssim \int_{\mathbb{D}} \frac{(1 - |a|)^{\delta q}}{|1 - \bar{a}w|^{(\delta+n)q}(\omega(S(a)))^{q/p}} d\mu(w). \end{aligned}$$

By Lemma 2.2, we find that

$$\begin{aligned} \|D^n\|_{A_{\omega}^p \rightarrow L_{\mu}^q}^q &\asymp \sup_{a \in \mathbb{D}} \frac{\mu(\Delta(a, r))}{(1 - |a|)^{nq}(\omega(S(a)))^{q/p}} \\ &\lesssim \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|)^{\delta q}}{|1 - \bar{a}w|^{(\delta+n)q}(\omega(S(a)))^{q/p}} d\mu(w). \end{aligned}$$

By [22, Lemma 3.1], we can choose some large enough $\delta = \delta(\omega, p) > 0$ and

$$f_a(z) = \left(\frac{1 - |a|}{1 - \bar{a}z}\right)^{\delta} \omega(S(a))^{-1/p}, \quad a, z \in \mathbb{D}, \tag{2.3}$$

then $\|f_a\|_{A_{\omega}^p} \lesssim 1$. Thus, we get

$$\int_{\mathbb{D}} |f_a^{(n)}(z)|^q d\mu(z) = \int_{\mathbb{D}} \frac{|a|^n(1 - |a|)^{\delta q}}{|1 - \bar{a}z|^{(\delta+n)q}(\omega(S(a)))^{q/p}} d\mu(z) \lesssim \|D^n\|_{A_{\omega}^p \rightarrow L_{\mu}^q}^q.$$

The proof is complete. \square

We use the pullback measure as an important tool to study the generalize weighted composition operators between different Bergman spaces with doubling weights. Let φ be an analytic self-map of \mathbb{D} and $0 < q < \infty$. Assume that $u \in H(\mathbb{D})$, we define a finite positive Borel measure $\mu_{\varphi, u}^v$ on \mathbb{D} as follows:

$$\mu_{\varphi, u}^v(E) = \int_{\varphi^{-1}(E)} |u(z)|^q v(z) dA(z),$$

where E is a Borel subset of unit disk \mathbb{D} . For $D_{\varphi, u}^n : A_{\omega}^p \rightarrow A_v^q$, it can be clearly seen that

$$\|D_{\varphi, u}^n f\|_{A_v^q} = \int_{\mathbb{D}} |f^{(n)}(z)|^q d\mu_{\varphi, u}^v(z), \quad f \in A_{\omega}^p. \tag{2.4}$$

LEMMA 2.4. *Let $\omega \in \mathcal{D}$, $n \in \mathbb{N} \cup \{0\}$, $0 < p < \infty$ and $0 < r = r(\omega) < 1$. If $f \in A_{\omega}^p$, there exists a constant $C = C(\omega) > 0$ such that*

$$|f^{(n)}(z)|^p \leq \frac{C}{\omega(S(z))} \int_{\Delta(z, r)} \frac{|f(w)|^p}{(1 - |w|)^{np}} \tilde{\omega}(w) dA(w)$$

for $z \in \mathbb{D}$. Here $\tilde{\omega}(z) = \frac{\hat{\omega}(z)}{1 - |z|}$.

Proof. It is clear that $1 - |z| \asymp 1 - |w|$ for $w \in \Delta(z, r)$. Since $\omega \in \mathcal{D}$, by [22, Lemma 2.1] and [24, (2.27)], there exist constants $0 < \alpha = \alpha(\omega) < \beta = \beta(\omega) < \infty$ and $C = C(\omega) \geq 1$ such that

$$\frac{1}{C} \left(\frac{1 - r}{1 - t} \right)^{\alpha} \leq \frac{\hat{\omega}(r)}{\hat{\omega}(t)} \leq C \left(\frac{1 - r}{1 - t} \right)^{\beta}, \tag{2.5}$$

where $0 \leq r \leq t < 1$. By (2.5), we know that $\hat{\omega}(z) \asymp \hat{\omega}(w)$ for $w \in \Delta(z, r)$. By a direct calculation, we know that $\hat{\omega}(z)(1 - |z|) \asymp \omega(S(z))$ for $\omega \in \mathcal{D}$. By [16, Lemma 2.1], we claim that

$$\begin{aligned} |f^{(n)}(z)|^p &\leq \frac{C}{(1 - |z|)^{2+np}} \int_{\Delta(z, r)} |f(w)|^p dA(w) \\ &\asymp \frac{C}{\hat{\omega}(z)(1 - |z|)(1 - |z|)^{np}} \int_{\Delta(z, r)} |f(w)|^p \frac{\hat{\omega}(w)}{(1 - |w|)} dA(w) \\ &\asymp \frac{C}{\omega(S(z))} \int_{\Delta(z, r)} \frac{|f(w)|^p}{(1 - |w|)^{np}} \tilde{\omega}(w) dA(w). \quad \square \end{aligned}$$

LEMMA 2.5. [13, Theorem 1.3] *Let $0 < p \leq q < \infty$ and $\omega, v \in \mathcal{D}$. Assume that φ is an analytic self-map of \mathbb{D} , $u \in A_v^q$ and $n \in \mathbb{N} \cup \{0\}$. Then $D_{\varphi, u}^n : A_{\omega}^p \rightarrow A_v^q$ is bounded if and only if there exists a large enough $\delta = \delta(\omega, p) > 0$ such that*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|)^{\delta q} |u(\xi)|^q v(\xi)}{|1 - \bar{a}\varphi(\xi)|^{(\delta+n)q} (\omega(S(a)))^{q/p}} dA(\xi) < \infty. \tag{2.6}$$

3. Order boundness of $D_{\varphi,u}^n : A_{\omega}^p \rightarrow A_{\nu}^q$ for $0 < p, q < \infty$

Next, we will study the order boundedness of $D_{\varphi,u}^n : A_{\omega}^p \rightarrow A_{\nu}^q$ for $0 < p, q < \infty$.

THEOREM 3.1. *Let $0 < p, q < \infty$ and $\omega, \nu \in \mathcal{D}$. Suppose $n \in \mathbb{N} \cup \{0\}$. Let φ be an analytic self-map of \mathbb{D} and $u \in H(\mathbb{D})$. Then $D_{\varphi,u}^n : A_{\omega}^p \rightarrow A_{\nu}^q$ is order bounded if and only if*

$$\int_{\mathbb{D}} \frac{|u(z)|^q \nu(z)}{(1 - |\varphi(z)|^2)^{nq} (\omega(S(\varphi(z))))^{q/p}} dA(z) < \infty. \tag{3.1}$$

Proof. Assume that $D_{\varphi,u}^n : A_{\omega}^p \rightarrow A_{\nu}^q$ is order bounded. There exists a non-negative function $h \in L^q_{\nu}$ such that $|D_{\varphi,u}^n f(z)| \leq h(z)$ for all $z \in \mathbb{D}$ and $f \in A_{\omega}^p$ with $\|f\|_{A_{\omega}^p} \lesssim 1$. To get (3.1), we set

$$I_1 = \int_{\{z \in \mathbb{D}, |\varphi(z)| > \frac{1}{2}\}} \frac{|u(z)|^q \nu(z)}{(1 - |\varphi(z)|^2)^{nq} (\omega(S(\varphi(z))))^{q/p}} dA(z) \tag{3.2}$$

and

$$I_2 = \int_{\{z \in \mathbb{D}, |\varphi(z)| \leq \frac{1}{2}\}} \frac{|u(z)|^q \nu(z)}{(1 - |\varphi(z)|^2)^{nq} (\omega(S(\varphi(z))))^{q/p}} dA(z). \tag{3.3}$$

By (2.3) and for $z \in \mathbb{D}$, take

$$f_{\varphi(z)}(w) = \frac{(1 - |\varphi(z)|^2)^{\delta}}{(1 - \overline{\varphi(z)}w)^{\delta} (\omega(S(\varphi(z))))^{1/p}}, \quad w \in \mathbb{D}.$$

For some large enough $\delta = \delta(\omega, p) > 0$, we know that $f_{\varphi(z)} \in A_{\omega}^p$ and $\|f_{\varphi(z)}\|_{A_{\omega}^p} \lesssim 1$. Then

$$f_{\varphi(z)}^{(n)}(w) = C_{\delta,n} \frac{(1 - |\varphi(z)|^2)^{\delta} (\overline{\varphi(z)})^n}{(1 - \overline{\varphi(z)}w)^{\delta+n} (\omega(S(\varphi(z))))^{1/p}}, \tag{3.4}$$

where $C_{\delta,n} = \delta(\delta + 1)(\delta + 2) \dots (\delta + n - 1)$. By a direct computation, for $z \in \mathbb{D}$, we have

$$|D_{\varphi,u}^n f_{\varphi(z)}(w)| = \frac{C_{\delta,n} (1 - |\varphi(z)|^2)^{\delta} |u(w)| |\varphi(z)|^n}{|1 - \overline{\varphi(z)}\varphi(w)|^{n+\delta} (\omega(S(\varphi(z))))^{1/p}} \leq h(w).$$

So, by taking $w = z$, we can get

$$\frac{C_{\delta,n} |u(z)| |\varphi(z)|^n}{(1 - |\varphi(z)|^2)^n (\omega(S(\varphi(z))))^{1/p}} = |D_{\varphi,u}^n f_{\varphi(z)}(z)| \leq h(z).$$

For $z \in \mathbb{D}$ such that $|\varphi(z)| > \frac{1}{2}$, we get $|\varphi(z)|^n > \frac{1}{2^n}$. Therefore,

$$\begin{aligned}
 I_1 &= \int_{\{z \in \mathbb{D}, |\varphi(z)| > \frac{1}{2}\}} \frac{|u(z)|^q}{(1 - |\varphi(z)|^2)^{nq} (\omega(S(\varphi(z))))^{q/p}} \nu(z) dA(z) \\
 &\leq \frac{2^{nq}}{C_{\delta,n}} \int_{\{z \in \mathbb{D}, |\varphi(z)| > \frac{1}{2}\}} \left| \frac{C_{\delta,n} |u(z)| |\varphi(z)|^n}{(1 - |\varphi(z)|^2)^n (\omega(S(\varphi(z))))^{1/p}} \right|^q \nu(z) dA(z) \\
 &\lesssim \int_{\mathbb{D}} \left| \frac{C_{\delta,n} |u(z)| |\varphi(z)|^n}{(1 - |\varphi(z)|^2)^n (\omega(S(\varphi(z))))^{1/p}} \right|^q \nu(z) dA(z) \\
 &\leq \int_{\mathbb{D}} |h(z)|^q \nu(z) dA(z) < \infty.
 \end{aligned} \tag{3.5}$$

For $z \in \mathbb{D}$ such that $|\varphi(z)| \leq \frac{1}{2}$, we can find a constant $C > 0$ such that

$$\frac{1}{(1 - |\varphi(z)|^2)^n (\omega(S(\varphi(z))))^{1/p}} \leq C. \tag{3.6}$$

On the other hand, since $P_n(z) = \frac{z^n}{\|z^n\|_{A_\omega^p}}$ is in A_ω^p and $\|P_n\|_{A_\omega^p} \leq 1$, by the order boundedness of the operator $D_{\varphi,u}^n$, for $z \in \mathbb{D}$, we obtain

$$\frac{n!}{\|z^n\|_{A_\omega^p}} |u(z)| = |D_{\varphi,u}^n P_n(z)| \leq h(z). \tag{3.7}$$

Since n is fixed, from (3.6) and (3.7), for $z \in \mathbb{D}$, we get

$$\begin{aligned}
 I_2 &= \int_{\{z \in \mathbb{D}, |\varphi(z)| \leq \frac{1}{2}\}} \frac{|u(z)|^q}{(1 - |\varphi(z)|^2)^{nq} (\omega(S(\varphi(z))))^{q/p}} \nu(z) dA(z) \\
 &\leq C \int_{\{z \in \mathbb{D}, |\varphi(z)| \leq \frac{1}{2}\}} |u(z)|^q \nu(z) dA(z) \lesssim \int_{\mathbb{D}} |u(z)|^q \nu(z) dA(z) \\
 &\leq \int_{\mathbb{D}} |h(z)|^q \nu(z) dA(z) < \infty.
 \end{aligned} \tag{3.8}$$

By (3.5) and (3.8), we see that

$$\int_{\mathbb{D}} \frac{|u(z)|^q}{(1 - |\varphi(z)|^2)^{nq} (\omega(S(\varphi(z))))^{q/p}} \nu(z) dA(z) = I_1 + I_2 < \infty.$$

Thus, the condition (3.1) holds.

Conversely, assume that condition (3.1) holds. Define

$$h(z) = \frac{|u(z)|}{(1 - |\varphi(z)|^2)^n (\omega(S(\varphi(z))))^{1/p}}.$$

Then h is a nonnegative function in L^q_ν . For any function $f \in A_\omega^p$ with $\|f\|_{A_\omega^p} \leq 1$, by Lemma 2.1, there is a constant $C = C(\omega) > 0$ such that

$$|D_{\varphi,u}^n f(z)| = |u(z) f^{(n)}(\varphi(z))| \leq C \frac{|u(z)|}{(1 - |\varphi(z)|^2)^n (\omega(S(\varphi(z))))^{1/p}} = Ch(z)$$

for any $z \in \mathbb{D}$. Thus, $D_{\varphi,u}^n : A_\omega^p \rightarrow A_\nu^q$ is order bounded. The proof is complete. \square

4. Essential norm of $D_{\phi,u}^p : A_\omega^p \rightarrow A_\nu^q$ for $1 \leq p \leq q < \infty$

We begin this section with an approximation of the essential norm of the bounded operator $D_{\phi,u}^p : A_\omega^p \rightarrow A_\nu^q$ for $1 \leq p \leq q < \infty$. If $f \in H(\mathbb{D})$, then $f(z) = \sum_{k=0}^\infty a_k z^k$. For any $m \geq 1$, let $R_m f(z) = \sum_{k=m}^\infty a_k z^k$ and $T_m = I - R_m$, where $I f = f$ is the identity operator. In order to prove one of the main results, we need the following lemmas.

LEMMA 4.1. [35, Proposition 1] *Suppose X is a Banach space of holomorphic functions in \mathbb{D} with the property that the polynomials are dense in X . Then $\|T_m f - f\|_X \rightarrow 0$ as $m \rightarrow \infty$ for each $f \in X$ if and only if $\sup\{\|T_m\| : m \geq 1\} < \infty$.*

LEMMA 4.2. [35, Corollary 3] *The Taylor series of every function in H^p converges in norm if and only if $1 < p < \infty$.*

LEMMA 4.3. *For $1 < p < \infty$ and ω is a radial weight, then $\|T_m f - f\|_{A_\omega^p} \rightarrow 0$ as $m \rightarrow \infty$ for each $f \in A_\omega^p$. Moreover, $\sup\{\|R_m\|_{A_\omega^p \rightarrow A_\omega^p} : m \geq 1\} < \infty$ and $\sup\{\|T_m\|_{A_\omega^p \rightarrow A_\omega^p} : m \geq 1\} < \infty$, where $R_m = I - T_m$.*

Proof. It follows from Lemmas 4.1 and 4.2 that T_m is bounded uniformly on H^p for $1 < p < \infty$. Thus, there exists a constant $C > 0$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} |T_m f(re^{i\theta})|^p d\theta \leq C \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta,$$

for $p > 1$ and any $m \geq 1$. Applying polar coordinates, we see that

$$\|T_m f\|_{A_\omega^p}^p \leq C \int_0^1 \omega(r) r dr \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq C \|f\|_{A_\omega^p}^p.$$

Therefore $\|T_m\|_{A_\omega^p \rightarrow A_\omega^p} \leq C$ for any $m \geq 1$. By Lemma 4.1, we obtain that $\|T_m f - f\|_{A_\omega^p} \rightarrow 0$ as $m \rightarrow \infty$. Since $R_m = I - T_m$, we have

$$\|R_m\|_{A_\omega^p \rightarrow A_\omega^p} = \|I - T_m\|_{A_\omega^p \rightarrow A_\omega^p} \leq 1 + \|T_m\|_{A_\omega^p \rightarrow A_\omega^p} \leq 1 + C. \quad \square$$

LEMMA 4.4. *Suppose that $\omega \in \hat{\mathcal{G}}$ and $1 < p < \infty$. Let $\varepsilon > 0$ and $r \in (0, 1)$. Then there exists a $m_0 \in \mathbb{N}$, for any $m \geq m_0$,*

$$|R_m f(z)| \lesssim \varepsilon \|f\|_{A_\omega^p}, \tag{4.1}$$

for every $z \in D_r = \{z \in \mathbb{D}, |z| \leq r\}$ and each $f \in A_\omega^p$.

Proof. Let $\omega_n = \int_0^1 r^n \omega(r) dr$. By [25, p. 665], we see that

$$B_z^\omega(\xi) = \sum_{n=0}^\infty \frac{(\xi \bar{z})^n}{2\omega_{2n+1}}$$

is the reproducing kernel of A_ω^p for $p \geq 1$. Then, we have

$$|R_m f(z)| = |\langle R_m f, B_z^\omega \rangle| = |\langle f, R_m B_z^\omega \rangle| \leq \int_{\mathbb{D}} |f(w) \overline{R_m B_z^\omega(w)}| \omega(w) dA(w) \tag{4.2}$$

$$\lesssim \|f\|_{A_\omega^p} \|R_m B_z^\omega\|_{A_\omega^q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. For $z \in D_r$, we show that

$$\|R_m B_z^\omega\|_{A_\omega^q} = \left(\int_{\mathbb{D}} |R_m B_z^\omega(w)|^q \omega(w) dA(w) \right)^{\frac{1}{q}} \lesssim \sum_{k=m}^\infty \frac{r^k}{2\omega_{2k+1}}.$$

By [25, Lemma 6], we deduce that

$$\lim_{m \rightarrow \infty} \sum_{k=m}^\infty \frac{r^k}{2\omega_{2k+1}} = 0. \tag{4.3}$$

Therefore, for any $\varepsilon > 0$, there exists a $m_0 \in \mathbb{N}$ and $m \geq m_0$, such that

$$\|R_m B_z^\omega\|_{A_\omega^p} \leq \varepsilon.$$

By (4.2), we get $|R_m f(z)| \lesssim \varepsilon \|f\|_{A_\omega^p}$ for any $f \in A_\omega^p$. \square

For $p = 1$, let $\mathcal{T}_m f(z) = \sum_{k=0}^{m-1} (1 - \frac{k}{m}) a_k z^k$ and $\mathcal{R}_m = I - \mathcal{T}_m$. We get the following lemma.

LEMMA 4.5. *Let $\omega \in \hat{\mathcal{G}}$ and $f \in A_\omega^1$, then $\|\mathcal{T}_m f - f\|_{A_\omega^1} \rightarrow 0$ as $m \rightarrow \infty$ for each $f \in A_\omega^1$. Moreover, $\sup\{\|\mathcal{T}_m\|_{A_\omega^1 \rightarrow A_\omega^1} : m \geq 1\} < \infty$ and $\sup\{\|\mathcal{R}_m\|_{A_\omega^1 \rightarrow A_\omega^1} : m \geq 1\} < \infty$, where $\mathcal{R}_m = I - \mathcal{T}_m$.*

Proof. By [4, p. 196], $\|\mathcal{T}_m\|_{H^1 \rightarrow H^1} \leq 1$. Using the same way of Lemma 4.3, we know that $\|\mathcal{T}_m\|_{A_\omega^1 \rightarrow A_\omega^1} \leq C$ for any $m \geq 1$. We claim that

$$\|\mathcal{R}_m\|_{A_\omega^1 \rightarrow A_\omega^1} = \|I - \mathcal{T}_m\|_{A_\omega^1 \rightarrow A_\omega^1} \leq 1 + \|\mathcal{T}_m\|_{A_\omega^1 \rightarrow A_\omega^1} < 1 + C \tag{4.4}$$

for any $m \geq 1$ and C is a positive constant. \square

LEMMA 4.6. *Assume that $\omega \in \hat{\mathcal{G}}$. Let $\varepsilon > 0$ and $r \in (0, 1)$. Then there exists a $m_0 \in \mathbb{N}$, for any $m \geq m_0$,*

$$|\mathcal{R}_m f(w)| \lesssim \varepsilon \|f\|_{A_\omega^1}, \tag{4.5}$$

for every $w \in D_r = \{w \in \mathbb{D}, |w| \leq r\}$ and each $f \in A_\omega^1$.

Proof. By the proof of Lemma 4.4, we deduce that

$$|\mathcal{R}_m f(w)| = |\langle \mathcal{R}_m f, B_w^\omega \rangle| = |\langle f, \mathcal{R}_m B_w^\omega \rangle| \lesssim \|f\|_{A_\omega^1} \|\mathcal{R}_m B_w^\omega\|_{H^\infty}.$$

Take $|w| \leq r$, we can prove that

$$\|\mathcal{R}_m B_w^\omega\|_{H^\infty} = \sup_{\xi \in \mathbb{D}} |\mathcal{R}_m B_w^\omega(\xi)| = \sup_{\xi \in \mathbb{D}} |(I - \mathcal{T}_m) B_w^\omega(\xi)| \leq \frac{1}{m} \sum_{k=1}^\infty \frac{kr^{k-1}}{2\omega_{2k+1}} + \sum_{k=m}^\infty \frac{r^k}{2\omega_{2k+1}}.$$

By [25, Lemma 6], we see that $\sum_{k=1}^\infty \frac{kr^{k-1}}{2\omega_{2k+1}}$ is convergent and (4.3) holds. Therefore, for any $\varepsilon > 0$, there exists a $m_0 \in \mathbb{N}$ and $m \geq m_0$, such that

$$\|\mathcal{R}_m B_w^\omega\|_{H^\infty} \leq \varepsilon.$$

Thus $|\mathcal{R}_m f(w)| \lesssim \varepsilon \|f\|_{A_{\omega}^1}$ for any $f \in A_{\omega}^1$. \square

The following lemma is very useful to prove the compactness of composition operators and its generalizations on some function spaces.

LEMMA 4.7. [13, Lemma 2.2] *Suppose $0 < p, q < \infty$, $\omega, \nu \in \mathcal{D}$. Suppose $u \in H(\mathbb{D})$ and $n \in \mathbb{N} \cup \{0\}$. Let φ be an analytic self-map of \mathbb{D} such that $D_{\varphi,u}^n : A_{\omega}^p \rightarrow A_{\nu}^q$ is bounded. Then $D_{\varphi,u}^n : A_{\omega}^p \rightarrow A_{\nu}^q$ is compact if and only if whenever $\{f_k\}$ is bounded in A_{ω}^p and $f_k \rightarrow 0$ uniformly on compact subsets of \mathbb{D} as $k \rightarrow \infty$, $\lim_{k \rightarrow \infty} \|D_{\varphi,u}^n(f_k)\|_{A_{\nu}^q} = 0$.*

THEOREM 4.1. *Let $1 \leq p \leq q < \infty$ and $\omega, \nu \in \mathcal{D}$. Suppose $n \in \mathbb{N} \cup \{0\}$. Let φ be an analytic self-map of \mathbb{D} and $u \in A_{\nu}^q$. If $D_{\varphi,u}^n : A_{\omega}^p \rightarrow A_{\nu}^q$ is bounded, then there exists a large enough $\delta = \delta(\omega, p) > 0$ such that*

$$\|D_{\varphi,u}^n\|_{e, A_{\omega}^p \rightarrow A_{\nu}^q}^q \asymp \limsup_{|a| \rightarrow 1} \int_{\mathbb{D}} \frac{(1 - |a|)^{\delta q} |u(\xi)|^q \nu(\xi)}{|1 - \bar{a}\varphi(\xi)|^{(\delta+n)q} (\omega(S(a)))^{q/p}} dA(\xi). \tag{4.6}$$

Proof. Lower estimate. Let $f_a(z) = \left(\frac{1-|a|}{1-\bar{a}z}\right)^{\delta} \omega(S(a))^{-1/p}$ for some large enough $\delta = \delta(\omega, p) > 0$. Then $\{f_a\}$ is a bounded sequence in A_{ω}^p converging to zero uniformly on compact subsets of \mathbb{D} as $|a| \rightarrow 1$. Fix a compact operator $K : A_{\omega}^p \rightarrow A_{\nu}^q$, by Lemma 4.7, we know that $\|Kf_a\|_{A_{\nu}^q} \rightarrow 0$ as $|a| \rightarrow 1$. Therefore

$$\begin{aligned} \|D_{\varphi,u}^n - K\|_{A_{\omega}^p \rightarrow A_{\nu}^q} &\gtrsim \limsup_{|a| \rightarrow 1} \|(D_{\varphi,u}^n - K)f_a\|_{A_{\nu}^q} \\ &\gtrsim \limsup_{|a| \rightarrow 1} (\|D_{\varphi,u}^n f_a\|_{A_{\nu}^q} - \|Kf_a\|_{A_{\nu}^q}) \\ &= \limsup_{|a| \rightarrow 1} \|D_{\varphi,u}^n f_a\|_{A_{\nu}^q}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \|D_{\varphi,u}^n\|_{e, A_{\omega}^p \rightarrow A_{\nu}^q} &= \inf_K \|D_{\varphi,u}^n - K\|_{A_{\omega}^p \rightarrow A_{\nu}^q} \gtrsim \limsup_{|a| \rightarrow 1} \|D_{\varphi,u}^n f_a\|_{A_{\nu}^q} \\ &= \limsup_{|a| \rightarrow 1} \left(\int_{\mathbb{D}} \frac{|a|^n (1 - |a|)^{\delta q} |u(\xi)|^q \nu(\xi)}{|1 - \bar{a}\varphi(\xi)|^{(\delta+n)q} (\omega(S(a)))^{q/p}} dA(\xi) \right)^{1/q}. \end{aligned}$$

We get

$$\|D_{\varphi,u}^n\|_{e, A_\omega^p \rightarrow A_\nu^q}^q \gtrsim \limsup_{|a| \rightarrow 1} \int_{\mathbb{D}} \frac{(1 - |a|)^{\delta q} |u(\xi)|^q \nu(\xi)}{|1 - \bar{a}\varphi(\xi)|^{(\delta+n)q} (\omega(S(a)))^{q/p}} dA(\xi). \tag{4.7}$$

Upper estimate. The case $1 < p \leq q < \infty$. Considering the compact operator $T_m : A_\omega^p \rightarrow A_\nu^q$ by $T_m f = \sum_{k=0}^{m-1} b_k z^k$ and letting $R_m = I - T_m$, where I is identity operator. We can see that

$$\|D_{\varphi,u}^n\|_{e, A_\omega^p \rightarrow A_\nu^q} \leq \|D_{\varphi,u}^n \circ R_m\|_{e, A_\omega^p \rightarrow A_\nu^q} + \|D_{\varphi,u}^n \circ T_m\|_{e, A_\omega^p \rightarrow A_\nu^q} = \|D_{\varphi,u}^n \circ R_m\|_{e, A_\omega^p \rightarrow A_\nu^q}.$$

Thus

$$\|D_{\varphi,u}^n\|_{e, A_\omega^p \rightarrow A_\nu^q}^q \leq \liminf_{m \rightarrow \infty} \|D_{\varphi,u}^n \circ R_m\|_{e, A_\omega^p \rightarrow A_\nu^q}^q \leq \liminf_{m \rightarrow \infty} \|D_{\varphi,u}^n \circ R_m\|_{A_\omega^p \rightarrow A_\nu^q}^q. \tag{4.8}$$

Fix $f \in A_\omega^p$ with $\|f\|_{A_\omega^p} \leq 1$ and $r \in (0, 1)$. Suppose $D_r = \{z \in \mathbb{D}, |z| \leq r\}$. Then

$$\begin{aligned} \|(D_{\varphi,u}^n \circ R_m)f\|_{A_\omega^p \rightarrow A_\nu^q}^q &\leq \int_{\mathbb{D}} |(R_m f)^{(n)}(\varphi(\xi))|^q |u(\xi)|^q \nu(\xi) dA(\xi) \\ &= \int_{\mathbb{D}} |(R_m f)^{(n)}(z)|^q d\mu_{\varphi,u}^\nu(z), \end{aligned}$$

where $\mu_{\varphi,u}^\nu = \int_{\varphi^{-1}(E)} |u(z)|^q \nu(z) dA(z)$ for all E is Borel subsets of \mathbb{D} .

From Lemma 2.1, we have

$$|f(z)|^{q-p} \lesssim \frac{\|f\|_{A_\omega^p}^{q-p}}{(\omega(S(z)))^{(q-p)/p}}. \tag{4.9}$$

By Lemma 2.4 and (4.9), we obtain

$$\begin{aligned} &\int_{\mathbb{D}} |(R_m f)^{(n)}(z)|^q d\mu_{\varphi,u}^\nu(z) \\ &\leq \int_{\mathbb{D}} d\mu_{\varphi,u}^\nu(z) \frac{C}{\omega(S(z))} \int_{\Delta(z,r)} \frac{|R_m f(w)|^q}{(1 - |w|)^{nq}} \tilde{\omega}(w) dA(w) \\ &\asymp \int_{\mathbb{D}} d\mu_{\varphi,u}^\nu(z) \int_{\Delta(z,r)} \frac{|R_m f(w)|^{q-p+p}}{(1 - |w|)^{nq} \omega(S(w))} \tilde{\omega}(w) dA(w) \\ &\lesssim \|R_m f\|_{A_\omega^p}^{q-p} \int_{\mathbb{D}} d\mu_{\varphi,u}^\nu(z) \int_{\Delta(z,r)} \frac{|R_m f(w)|^p}{(1 - |w|)^{nq} (\omega(S(w)))^{(q-p)/p} \omega(S(w))} \tilde{\omega}(w) dA(w) \\ &= \|R_m f\|_{A_\omega^p}^{q-p} \int_{\mathbb{D}} d\mu_{\varphi,u}^\nu(z) \int_{\Delta(z,r)} \frac{|R_m f(w)|^p}{(1 - |w|)^{nq} (\omega(S(w)))^{q/p}} \tilde{\omega}(w) dA(w) \\ &= \|R_m f\|_{A_\omega^p}^{q-p} \int_{\mathbb{D}} d\mu_{\varphi,u}^\nu(z) \int_{\mathbb{D}} \frac{\chi_{\Delta(z,r)}(w) |R_m f(w)|^p}{(1 - |w|)^{nq} (\omega(S(w)))^{q/p}} \tilde{\omega}(w) dA(w), \end{aligned} \tag{4.10}$$

where $\chi_{\Delta(z,r)}$ is the characteristic function of the set $\Delta(z,r)$. Obviously, $\chi_{\Delta(z,r)}(w) = \chi_{\Delta(w,r)}(z)$. By Fubini's Theorem, we obtain

$$\begin{aligned} & \int_{\mathbb{D}} |(R_m f)^{(n)}(z)|^q d\mu_{\varphi, u}^v(z) \\ & \lesssim \|R_m f\|_{A_{\omega}^p}^{q-p} \int_{\mathbb{D}} \frac{\mu_{\varphi, u}^v(\Delta(w,r))}{(1-|w|)^{nq}(\omega(S(w)))^{q/p}} |R_m f(w)|^p \tilde{\omega}(w) dA(w). \end{aligned} \tag{4.11}$$

Set

$$J_{1, m} = \int_{D_r} \frac{\mu_{\varphi, u}^v(\Delta(w,r))}{(1-|w|)^{nq}(\omega(S(w)))^{q/p}} |R_m f(w)|^p \tilde{\omega}(w) dA(\xi)$$

and

$$J_{2, m} = \int_{\mathbb{D} \setminus D_r} \frac{\mu_{\varphi, u}^v(\Delta(w,r))}{(1-|w|)^{nq}(\omega(S(w)))^{q/p}} |R_m f(w)|^p \tilde{\omega}(w) dA(w).$$

Then we get

$$\|(D_{\varphi, u}^n \circ R_m) f\|_{A_{\omega}^p \rightarrow A_{\omega}^q}^q \lesssim \|R_m f\|_{A_{\omega}^p}^{q-p} (J_{1, m} + J_{2, m}), \tag{4.12}$$

with $m \geq 1$. Since $D_{\varphi, u}^n : A_{\omega}^p \rightarrow A_{\omega}^q$ is bounded, Lemma 2.5 implies that there exists a large enough $\delta = \delta(\omega, p) > 0$ such that

$$\begin{aligned} M &= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|)^{\delta q} |u(w)|^q v(w)}{|1-\bar{a}\varphi(w)|^{(\delta+n)q}(\omega(S(a)))^{q/p}} dA(w) \\ &= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|)^{\delta q}}{|1-\bar{a}\xi|^{(\delta+n)q}(\omega(S(a)))^{q/p}} d\mu_{\varphi, u}^v(\xi) \\ &< \infty. \end{aligned} \tag{4.13}$$

For $\xi \in \Delta(w,r)$, we have

$$\begin{aligned} \frac{\mu_{\varphi, u}^v(\Delta(w,r))}{(1-|w|)^{nq}(\omega(S(w)))^{q/p}} &= \int_{\Delta(w,r)} \frac{1}{(1-|w|)^{nq}(\omega(S(w)))^{q/p}} d\mu_{\varphi, u}^v(\xi) \\ &\asymp \int_{\Delta(w,r)} \frac{(1-|w|)^{\delta q}}{|1-\bar{w}\xi|^{(\delta+n)q}(\omega(S(w)))^{q/p}} d\mu_{\varphi, u}^v(\xi) \\ &\lesssim \int_{\mathbb{D}} \frac{(1-|w|)^{\delta q}}{|1-\bar{w}\xi|^{(\delta+n)q}(\omega(S(w)))^{q/p}} d\mu_{\varphi, u}^v(\xi). \end{aligned} \tag{4.14}$$

Fix $\varepsilon > 0$. By (4.14), (4.13) and Lemma 4.4, hence

$$\begin{aligned} J_{1, m} &= \int_{D_r} \frac{\mu_{\varphi, u}^v(\Delta(w,r))}{(1-|w|)^{nq}(\omega(S(w)))^{q/p}} |R_m f(w)|^p \tilde{\omega}(w) dA(w) \\ &\leq \sup_{w \in \mathbb{D}} \frac{\mu_{\varphi, u}^v(\Delta(w,r))}{(1-|w|)^{nq}(\omega(S(w)))^{q/p}} \int_{D_r} |R_m f(w)|^p \tilde{\omega}(w) dA(w) \\ &\leq CM \int_{D_r} |R_m f(w)|^p \tilde{\omega}(w) dA(w) \\ &\leq CM \varepsilon^p \|f\|_{A_{\omega}^p}^p, \end{aligned}$$

for any $m \geq m_0$. Thus,

$$\lim_{m \rightarrow \infty} \sup_{\|f\|_{A_\omega^p} \leq 1} \|R_m f\|_{A_\omega^{q-p}}^{q-p} J_{1,m} = 0. \tag{4.15}$$

For $\omega \in \mathcal{D}$ and $f \in H(\mathbb{D})$, from [25, Proposition 5], we know that

$$\|f\|_{A_\omega^p} \asymp \|f\|_{A_\omega^p}. \tag{4.16}$$

By (4.14), (4.16) and Lemma 4.3, we claim that

$$\begin{aligned} J_{2,m} &= \int_{\mathbb{D} \setminus D_r} \frac{\mu_{\varphi, u}^v(\Delta(w, r))}{(1 - |w|)^{nq}(\omega(S(w)))^{q/p}} |R_m f(w)|^p \tilde{\omega}(w) dA(w) \\ &\leq \sup_{|a| > r} \frac{\mu_{\varphi, u}^v(\Delta(a, r))}{(1 - |a|)^{nq}(\omega(S(a)))^{q/p}} \int_{\mathbb{D} \setminus D_r} |R_m f|^p \tilde{\omega}(w) dA(w) \\ &\lesssim \sup_{m \geq 1} \|R_m\|_{A_\omega^p \rightarrow A_\omega^p}^p \|f\|_{A_\omega^p}^p \sup_{|a| > r} \int_{\mathbb{D}} \frac{(1 - |a|)^{\delta q}}{|1 - \bar{a}\xi|^{(\delta+n)q}(\omega(S(a)))^{q/p}} d\mu_{\varphi, u}^v(\xi). \end{aligned}$$

Hence,

$$\lim_{m \rightarrow \infty} \sup_{\|f\|_{A_\omega^p} \leq 1} \|R_m f\|_{A_\omega^{q-p}}^{q-p} J_{2,m} \lesssim \sup_{|a| > r} \int_{\mathbb{D}} \frac{(1 - |a|)^{\delta q}}{|1 - \bar{a}\xi|^{(\delta+n)q}(\omega(S(a)))^{q/p}} d\mu_{\varphi, u}^v(\xi). \tag{4.17}$$

Combining (4.8), (4.12), (4.15), (4.17) and (4.13), we deduce that

$$\begin{aligned} \|D_{\varphi, u}^n\|_{e, A_\omega^p \rightarrow A_\omega^q}^q &\leq \liminf_{m \rightarrow \infty} \|D_{\varphi, u}^n \circ R_m\|_{A_\omega^p \rightarrow A_\omega^q}^q \\ &\lesssim \sup_{|a| > r} \int_{\mathbb{D}} \frac{(1 - |a|)^{\delta q} |u(w)|^q v(w)}{|1 - \bar{a}\varphi(w)|^{(\delta+n)q} \omega(S(a))^{q/p}} dA(w). \end{aligned}$$

Letting $r \rightarrow 1$, we have

$$\|D_{\varphi, u}^n\|_{e, A_\omega^p \rightarrow A_\omega^q}^q \lesssim \limsup_{|a| \rightarrow 1} \int_{\mathbb{D}} \frac{(1 - |a|)^{\delta q} |u(w)|^q v(w)}{|1 - \bar{a}\varphi(w)|^{(\delta+n)q} \omega(S(a))^{q/p}} dA(w). \tag{4.18}$$

When $1 = p \leq q < \infty$, by Lemmas 4.5 and 4.6, we can use the same way to get that (4.18) holds. We omit the details. The proof of the Theorem 4.1 is complete. \square

Data Availability. All data generated or analyzed during this study are included in this article and in its bibliography.

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