

GENERALIZED CORE INVERSE IN BANACH *-ALGEBRAS

HUANYIN CHEN AND MARJAN SHEIBANI*

(Communicated by F. Kittaneh)

Abstract. We introduce a new generalized inverse, called generalized core inverse in a Banach *-algebra. This new inverse is an extension of weak core inverse defined for complex square matrix and bounded linear operators over Hilbert spaces. We present various characterizations of this new inverse. The relationship between the generalized core inverse and other generalized inverses is investigated. Finally, we consider the necessary and sufficient conditions under which generalized core inverse and generalized core-EP inverse coincide with each other in a Banach *-algebra.

1. Introduction

Let $\mathbb{C}^{n \times n}$ be the Banach algebra of all $n \times n$ complex matrices with conjugate transpose $*$. Let $A \in \mathbb{C}^{n \times n}$. A matrix $X \in \mathbb{C}^{n \times n}$ is called the Drazin inverse of A if there exists some $k \in \mathbb{N}$ such that

$$AX^2 = X, \quad XA = AX, \quad A^k = A^{k+1}X.$$

Such X is unique, and we denote X by A^D . A matrix $X \in \mathbb{C}^{n \times n}$ is called the core-EP inverse of A if there exists some $k \in \mathbb{N}$ such that

$$AX^2 = X, \quad AX = (AX)^*, \quad A^k = XA^{k+1}$$

(see [8, 15, 18, 23]). Such X is unique if it exists, and we denote X by $A^{\textcircled{D}}$. A matrix $X \in \mathbb{C}^{n \times n}$ is called the weak group inverse of A if X satisfies

$$AX^2 = X, \quad AX = A^{\textcircled{D}}A$$

(see [19, 20, 21]). Such X is unique if it exists, and we denote X by $A^{\textcircled{W}}$. A matrix $X \in \mathbb{C}^{n \times n}$ is called the Moore-Penrose inverse of A if X satisfies

$$A = AXA, \quad X = XAX, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

Such X is unique, and we denote X by A^\dagger .

Mathematics subject classification (2020): 15A09, 16U99, 46H05.

Keywords and phrases: Core-EP inverse, generalized group inverse, weak group inverse, weak core inverse, Moore-Penrose inverse, Banach algebra.

* Corresponding author.

A matrix $X \in \mathbb{C}^{n \times n}$ is called the weak core inverse of $A \in \mathbb{C}^{n \times n}$ if X satisfies

$$XAX = X, \quad AX = AA^{\textcircled{w}}AA^\dagger, \quad XA = A^DAA^{\textcircled{w}}A.$$

Such X is unique if it exists. We denote it by $A^{\textcircled{w}, \dagger}$. Recently, many authors studied weak core inverse of complex matrices from different views, e.g., [9, 11, 12, 15, 16, 22]. In [9], Ferreyra et al. considered the weak core inverse for complex square matrices. In [22], Zhou and Chen studied the weak core inverse in a ring with proper involution. In [16], Mosić and Marovt investigated the weak core inverse for the bounded linear operator over a Hilbert space.

An involution of a Banach algebra \mathcal{A} is an anti-automorphism whose square is the identity map 1. A Banach algebra \mathcal{A} with involution $*$ is called a Banach $*$ -algebra. Let \mathcal{A} be a Banach $*$ -algebra. The involution $*$ is proper if $x^*x = 0 \implies x = 0$ for any $x \in \mathcal{A}$. Throughout the paper, all Banach algebras are complex with a proper involution $*$. Evidently, $\mathbb{C}^{n \times n}$ is the Banach $*$ -algebra with conjugate transpose as the proper involution.

In [3, 4], the authors introduced and studied generalized core-EP inverse and generalized group inverse for an element in a Banach $*$ -algebra. An element $a \in \mathcal{A}$ has generalized core-EP inverse if there exists a $x \in \mathcal{A}$ such that

$$x = ax^2, \quad (ax)^* = ax, \quad \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Such x is unique if it exists, and denote it by $a^{\textcircled{d}}$ (see [3]). An element $a \in \mathcal{A}$ has generalized group inverse if there exists a $x \in \mathcal{A}$ such that

$$x = ax^2, \quad (a^*a^2x)^* = a^*a^2x, \quad \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Such x is unique if it exists, and denote it by $a^{\textcircled{g}}$ (see [4]). An element $a \in \mathcal{A}$ has Moore-Penrose inverse provided that there exists some $x \in \mathcal{A}$ such that

$$a = axa, \quad x = xax, \quad (ax)^* = ax, \quad (xa)^* = xa.$$

Such x is unique if it exists, and we denote it by a^\dagger .

Let $\mathcal{A}^{\textcircled{d}}, \mathcal{A}^{\textcircled{g}}$ and \mathcal{A}^\dagger denote the sets of all generalized core-EP invertible, generalized group invertible and Moore-Penrose invertible elements in \mathcal{A} , respectively. We now introduce a new generalized inverse which is an extension of the weak core inverse mentioned above.

DEFINITION 1.1. An element $a \in \mathcal{A}$ has generalized core inverse provided that $a \in \mathcal{A}^{\textcircled{d}} \cap \mathcal{A}^\dagger$.

The purpose of this paper is to investigate various properties of generalized core inverse in a Banach $*$ -algebra. The content of this paper is organized as follows. In Section 2, equivalent characterizations and fundamental properties of the generalized core inverse are proved. In Section 3, we establish the relationship between the generalized core inverse and other generalized inverses. Finally, in Section 4, we present necessary and sufficient conditions under which generalized core inverse and generalized core-EP inverse coincide with each other. Many properties of weak core inverse are thereby extended to the general setting.

2. Characterizations of generalized core inverse

In this section we introduce generalized core inverse and investigate its equivalent characterizations. Our starting point is the following:

THEOREM 2.1. *Let $a \in \mathcal{A}^{\odot} \cap \mathcal{A}^{\dagger}$. Then there exists a unique $x \in \mathcal{A}$ such that*

$$xax = x, \quad ax = aa^{\odot}aa^{\dagger}, \quad xa = a^{\odot}a.$$

Proof. Take $x = a^{\odot}aa^{\dagger}$. Then

$$\begin{aligned} xax &= a^{\odot}aa^{\dagger}aa^{\odot}aa^{\dagger} = a^{\odot}aa^{\dagger} = x, \\ ax &= aa^{\odot}aa^{\dagger}, \\ xa &= a^{\odot}aa^{\dagger}a = a^{\odot}a \end{aligned}$$

Suppose that x' satisfies the preceding equations. Then one checks that

$$x' = x'ax' = a^{\odot}ax' = a^{\odot}aa^{\odot}aa^{\dagger} = a^{\odot}aa^{\dagger} = x,$$

as required. \square

DEFINITION 2.2. Let $a \in \mathcal{A}^{\odot} \cap \mathcal{A}^{\dagger}$. Then a is called generalized core invertible. The unique x satisfying the three equations in Theorem 2.1 is called the generalized core inverse of a and denoted by a^{\odot} . The symbol \mathcal{A}^{\odot} denotes the set of all generalized core invertible elements in \mathcal{A} .

Recall that $a \in \mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) if there exists $x \in \mathcal{A}$ such that $ax^2 = x$, $ax = xa$, $a - a^2x \in \mathcal{A}^{qnil}$. Such x is unique, if exists, and denote x by a^d . Here, $\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + \lambda a \in \mathcal{A}^{-1}\}$. Evidently, $a \in \mathcal{A}^{qnil} \Leftrightarrow \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} = 0$. The g-Drazin inverse plays an important role in ring and matrix theory (see [1]). We note that the weak core inverse in [22] is a special case of generalized core inverse as the following shows.

PROPOSITION 2.3. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

- (1) $a \in \mathcal{A}^{\odot}$.
- (2) *There exists a unique $x \in \mathcal{A}$ such that*

$$xax = x, \quad ax = aa^{\odot}aa^{\dagger}, \quad xa = a^d aa^{\odot}a.$$

Proof. In view of [4, Theorem 3.4], $a^d aa^{\odot} = a^{\odot}$; hence, $xa = a^d aa^{\odot}a$ if and only if $xa = a^{\odot}a$, the result follows. \square

Let $a \in \mathcal{A}^{\odot}$. In view of Theorem 2.1, $a^{\odot} = a^{\odot}aa^{\dagger}$. Set $c = aa^{\odot}a$. We now find necessary and sufficient conditions on a and x so that the generalized core inverse of a is x .

THEOREM 2.4. *The following are equivalent:*

- (1) $a^{\odot} = x$.
- (2) $ax = ca^{\dagger}$ and $x\mathcal{A} \subseteq a^d\mathcal{A}$.
- (3) $ax = ca^{\dagger}$ and $ax^2 = x$.

Proof. (1) \Rightarrow (2) We directly check that

$$\begin{aligned} ax &= aa^{\odot} \\ &= a(a^{\otimes}aa^{\dagger}) \\ &= (aa^{\otimes}a)a^{\dagger} \\ &= ca^{\dagger}. \end{aligned}$$

In view of Theorem 2.1 and [4, Theorem 3.4], $x\mathcal{A} = (a^{\otimes}aa^{\dagger})\mathcal{A} \subseteq a^{\otimes}\mathcal{A} \subseteq a^d\mathcal{A}$.

(2) \Rightarrow (1) Since $ax = ca^{\dagger}$, we have $ax = aa^{\otimes}aa^{\dagger}$. It follows by $x\mathcal{A} \subseteq a^d\mathcal{A}$ that $(1 - aa^d)x = 0$, and then $aa^dx = x$. This implies that $x = a^d(ax) = a^dca^{\dagger} = a^daa^{\otimes}aa^{\dagger} = a^{\otimes}aa^{\dagger}$. Then

$$\begin{aligned} xax &= (a^{\otimes}aa^{\dagger})a(a^{\otimes}aa^{\dagger}) \\ &= (a^{\otimes}(aa^{\dagger})a)(a^{\otimes}aa^{\dagger}) \\ &= (a^{\otimes}aa^{\otimes})aa^{\dagger} \\ &= a^{\otimes}aa^{\dagger} \\ &= x. \end{aligned}$$

Moreover, we have $xa = a^{\otimes}(aa^{\dagger}a) = a^{\otimes}a$. Therefore $a^{\odot} = x$, as desired.

(1) \Rightarrow (3) By the preceding discussion, we have $ax = ca^{\dagger}$. In view of Theorem 2.1, $x = a^{\otimes}aa^{\dagger}$. Then we check that

$$ax^2 = aa^{\otimes}aa^{\dagger}a^{\otimes}aa^{\dagger} = a(a^{\otimes})^2aa^{\dagger} = a^{\otimes}aa^{\dagger} = x,$$

as required.

(3) \Rightarrow (2) Since $ax = ca^{\dagger}$, we see that $ax = aa^{\otimes}aa^{\dagger}$. As $ax^2 = x$, by induction, we have $a^n x^{n+1} = x$. We observe that

$$x = (a^n - a^d a^{n+1})x^{n+1} + a^d a^{n+1}x^{n+1},$$

hence,

$$\begin{aligned} \|x - a^d a(ax^2)\|^{1/n} &= \|x - a^d a(a^n x^{n+1})\|^{1/n} \\ &\leq \| (a^n - a^d a^{n+1})x^{n+1} \|^{1/n} \\ &= \| (a - a^d a^2)^n \|^{1/n} \|a\|^{1 - 1/n} \|x\|^{1 + 1/n}. \end{aligned}$$

Since $a - a^d a^2 \in \mathcal{A}^{qnil}$, we have $\lim_{n \rightarrow \infty} \| (a - a^d a^2)^n \|^{1/n} = 0$. Hence, we get

$$\lim_{n \rightarrow \infty} \|x - a^d a(ax^2)\|^{1/n} = 0,$$

and so $x = a^d ax$. Therefore $x\mathcal{A} \subseteq a^d\mathcal{A}$, as desired. \square

Let $X \in \mathbb{C}^{n \times n}$. The symbol $R(X)$ denote the range space of X . We now derive

COROLLARY 2.5. *Let $A \in \mathbb{C}^{n \times n}$. The following are equivalent:*

- (1) $A^{\textcircled{w}, \dagger} = X$.
- (2) $AX = AA^{\textcircled{w}}AA^\dagger$ and $R(X) \subseteq R(A^D)$.
- (3) $AX = AA^{\textcircled{w}}AA^\dagger$ and $AX^2 = X$.

Proof. Since $A \in \mathbb{C}^{n \times n}$, it follows from Proposition 2.3 that $A^{\textcircled{c}} = A^{\textcircled{w}, \dagger}$. Therefore we complete the proof by Theorem 2.4. \square

We are now ready to prove the following.

THEOREM 2.6. *The following are equivalent:*

- (1) $a^{\textcircled{c}} = x$.
- (2) $xcx = x$, $cx = ca^\dagger$ and $xc = a^d c$.

Proof. (1) \Rightarrow (2) In view of Theorem 2.1, $x = a^{\textcircled{g}}aa^\dagger$. We check that

$$\begin{aligned} xcx &= a^{\textcircled{g}}aa^\dagger aa^{\textcircled{g}}aa^{\textcircled{g}}aa^\dagger \\ &= a^{\textcircled{g}}aa^{\textcircled{g}}aa^{\textcircled{g}}aa^\dagger \\ &= a^{\textcircled{g}}aa^\dagger \\ &= x, \\ cx &= aa^{\textcircled{g}}aa^{\textcircled{g}}aa^\dagger \\ &= aa^{\textcircled{g}}aa^\dagger \\ &= ca^\dagger, \\ xc &= a^{\textcircled{g}}aa^\dagger aa^{\textcircled{g}}a \\ &= a^{\textcircled{g}}aa^{\textcircled{g}}a \\ &= a^{\textcircled{g}}a \\ &= a^d aa^{\textcircled{g}}a \\ &= a^d c, \end{aligned}$$

as desired.

(2) \Rightarrow (1) One directly verifies that $x = xcx = x(cx) = x(ca^\dagger) = (xc)a^\dagger = (a^d c)a^\dagger = a^d aa^{\textcircled{g}}aa^\dagger = a^{\textcircled{g}}aa^\dagger$. This completes the proof by Theorem 2.1. \square

Recall that $a \in \mathcal{A}$ has group inverse x if x satisfies the equations: $ax^2 = x$, $xa^2 = a$ and $ax = xa$, and denote x by $a^\#$. We now derive

COROLLARY 2.7. *The following are equivalent:*

- (1) $a^{\textcircled{c}} = x$.
- (2) $xax = x$, $xa = a^{\textcircled{g}}a$ and $cx = ca^\dagger$.
- (3) $xax = x$, $xa = a^{\textcircled{g}}a$ and $c^d x = c^d a^\dagger$.

Proof. (1) \Rightarrow (2) In view of Theorem 2.1, $xax = x$ and $xa = a^{\textcircled{g}}a$. By virtue of Theorem 2.4, $cx = ca^\dagger$, as desired.

(2) \Rightarrow (3) In view of [4, Corollary 6.2], $a^2a^{\textcircled{g}} = (a^{\textcircled{g}})^g$. Hence, $a^2a^{\textcircled{g}} \in \mathcal{A}^d$. By using Cline’s formula (see [13, Theorem 2.1]), $c = aa^{\textcircled{g}}a \in \mathcal{A}^d$. Since $cx = ca^\dagger$, we see that $c^d x = (c^d)^2(cx) = (c^d)^2(ca^\dagger) = c^d a^\dagger$.

(3) \Rightarrow (1) By the argument above, $a^2a^{\textcircled{g}} = (a^{\textcircled{g}})^\#$, and so $(a^2a^{\textcircled{g}})^\# = a^{\textcircled{g}}$. By Cline’s formula again, we have

$$c^d = (aa^{\textcircled{g}}a)^d = aa^{\textcircled{g}}[(a^2a^{\textcircled{g}})^\#]^2a = aa^{\textcircled{g}}(a^{\textcircled{g}})^2a = (a^{\textcircled{g}})^2a.$$

Since $c^d x = c^d a^\dagger$, we have

$$(a^{\textcircled{g}})^2ax = (a^{\textcircled{g}})^2aa^\dagger.$$

Accordingly,

$$a^2(a^{\textcircled{g}})^2ax = a^2(a^{\textcircled{g}})^2aa^\dagger;$$

hence,

$$cx = aa^{\textcircled{g}}ax = aa^{\textcircled{g}}aa^\dagger = ca^\dagger.$$

The corollary is therefore established by Theorem 2.6. \square

3. Relations with other generalized inverses

In this section we establish relations among generalized core-EP inverse, Moore-Penrose inverse and $\{2\}$ -inverse, etc. We come now to the demonstration for which this section has been developed.

THEOREM 3.1. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{c}}$.

(2) $a \in \mathcal{A}^{\textcircled{d}} \cap \mathcal{A}^\dagger$.

In this case, $a^{\textcircled{d}} = aa^{\textcircled{g}}a^\dagger$.

Proof. (1) \Rightarrow (2) Let $x = a^{\textcircled{g}}$ and $y = a^\dagger$. Then

$$\begin{aligned} \||axy - (a^d a^2 x)y\| &= \|(a^n - a^d a^{n+1})(a^{\textcircled{c}})^n y\| \\ &\leq \|a^n - a^d a^{n+1}\| \|(a^{\textcircled{c}})^n y\|. \end{aligned}$$

Since $a \in \mathcal{A}^d$, we have

$$\lim_{n \rightarrow \infty} \|a^n - a^d a^{n+1}\|^{\frac{1}{n}} = 0.$$

Accordingly,

$$\lim_{n \rightarrow \infty} \||axy - (a^d a^2 x)y\|^{\frac{1}{n}} = 0.$$

Hence, $axy = (a^d a^2 x)y$. Therefore we have

$$\begin{aligned} axy &= (a^d a^2 x)y \\ &= a^d (ay)a^2 xy \\ &= a^d (y^* a^*) a^2 xy \\ &= a^d y^* (a^* a^2 x)y \\ &= a^d y^* (a^* a^2 x)^* y \\ &= a^d y^* x^* (a^2)^* ay. \end{aligned}$$

Taking $z = axy$. Then we verify that

$$\begin{aligned} az^2 &= a(a^d y^* x^* (a^2)^* ay)axy = aa^d y^* x^* (a^2)^* axy \\ &= aa^d y^* [x^* (a^2)^* a]xy = aa^d y^* [x^* (a^2)^* a]^* xy \\ &= aa^d aya^2 x^2 y = aa^d a^2 x^2 y = a^3 x^3 y \\ &= axy = z, \\ az &= a^2 xy = (aya)(axy) = (ay)a^2 xy = (y^* a^*) a^2 xy \\ &= y^* (a^* a^2 x)y = y^* (a^* a^2 x)^* y = (a^* a^2 xy)^* y \\ &= (y^* a^* a^2 xy)^* = (aya^2 xy)^* = (a^2 xy)^* = (az)^*. \end{aligned}$$

Moreover, we check that

$$\begin{aligned} za^{n+1} &= a^d y^* x^* (a^2)^* (aya)a^{n+1} a^d \\ &= a^d y^* [x^* (a^2)^* a]a^{n+1} a^d \\ &= a^d y^* [a^* (a^2)x]^* a^{n+1} a^d \\ &= a^d y^* [a^* (a^2)x]a^{n+1} a^d \\ &= a^d (ay)^* a^2 (xa^{n+1})a^d \end{aligned}$$

Then we derive

$$\begin{aligned} & \|a^n - za^{n+1}\| \\ &= \|a^n - a^d a^{n+1} + a^d a^2 a^n a^d - a^d y^* x^* (a^2)^* aya^{n+1}\| \\ &= \|a^n - a^d a^{n+1}\| + \|a^d y^* x^* (a^2)^* aya^{n+1} a a^d - a^d y^* x^* (a^2)^* aya^{n+1}\| \\ &\quad + \|a^d (ay)a^2 a^n a^d - a^d y^* x^* (a^2)^* aya^{n+1} a a^d\| \\ &\leq \|a^n - a^d a^{n+1}\| + \|a^d y^* x^* (a^2)^* aya\| \|a^n - a^d a^{n+1}\| \\ &\quad + \|a^d (ay)a^2 a^n a^d - a^d y^* [x^* (a^2)^* a]a^{n+1} a^d\| \\ &\leq \|a^n - a^d a^{n+1}\| + \|a^d y^* x^* (a^2)^* aya\| \|a^n - a^d a^{n+1}\| \\ &\quad + \|a^d (ay)^* a^2 a^n a^d - a^d y^* [a^* (a^2)x]^* a^{n+1} a^d\| \\ &\leq [1 + \|a^d y^* x^* (a^2)^* aya\|] \|a^n - a^d a^{n+1}\| \\ &\quad + \|a^d (ay)^* a^2\| \|a^n - xa^{n+1}\| \|a^d\|, \end{aligned}$$

and so

$$\lim_{n \rightarrow \infty} \|a^n - za^{n+1}\|^{\frac{1}{n}} = 0.$$

Therefore $a \in \mathcal{A}^{(d)}$ and $a^{(d)} = z$, as required.

(2) \Rightarrow (1) Since $a \in \mathcal{A}^{(d)}$, it follows by [4, Theorem 6.1] that $a \in \mathcal{A}^{(\otimes)}$. This completes the proof by Theorem 2.1. \square

By the preceding discussion, we have $a^{(\odot)} = a^{(\otimes)} a a^\dagger$, $a^{(d)} = a a^{(\otimes)} a^\dagger$. Recall that x is called the core inverse of $a \in \mathcal{A}$ if $ax^2 = x$, $x a^2 = a$ and $(ax)^* = ax$, and we denote x by $x = a^{(\ominus)}$ (see [14]). Moreover, we can derive

COROLLARY 3.2. Let $a \in \mathcal{A}^{\odot}$. Then a^{\odot} is a reflexive inverse of $aa^{\otimes}a$ and

$$\begin{aligned} a^{\odot} &= (a^{\odot})^2 a^2 a^{\dagger} \\ &= (aa^{\odot}a)^{\#} aa^{\dagger} \\ &= a^d (a^2 a^d)^{\oplus} a^2 a^{\dagger}. \end{aligned}$$

Proof. One easily checks that

$$\begin{aligned} a^{\odot} (aa^{\otimes}a) a^{\odot} &= a^{\odot} aa^{\odot} = a^{\otimes} aa^{\dagger} aa^{\odot} = a^{\otimes} aa^{\odot} = a^{\odot}, \\ (aa^{\otimes}a) a^{\odot} (aa^{\otimes}a) &= aa^{\otimes} aa^{\dagger} aa^{\otimes} a = aa^{\otimes} aa^{\otimes} a = aa^{\otimes} a. \end{aligned}$$

Then a^{\odot} is a reflexive inverse of $aa^{\otimes}a$.

By virtue of Theorem 3.1, $a \in \mathcal{A}^{\odot}$. It follows by [4, Theorem 6.1] that $a^{\otimes} = (a^{\odot})^2 a$. Hence

$$a^{\odot} = a^{\otimes} aa^{\dagger} = (a^{\odot})^2 a^2 a^{\dagger}.$$

We directly verify that

$$\begin{aligned} (aa^{\odot}a)(a^{\odot})^2 a &= a(a^{\odot})^2 a = a^{\odot} a = (a^{\odot})^2 a(aa^{\odot}a), \\ (aa^{\odot}a)[(a^{\odot})^2 a]^2 &= a^{\odot} a(a^{\odot})^2 a = (a^{\odot})^2 a, \\ (aa^{\odot}a)^2 (a^{\odot})^2 a &= (aa^{\odot}a)a^{\odot} a = aa^{\odot} a. \end{aligned}$$

Thus, $(aa^{\odot}a)^{\#} = (a^{\odot})^2 a$, and so $a^{\odot} = (aa^{\odot}a)^{\#} aa^{\dagger}$. By virtue of [4, Theorem 3.4], we obtain $(a^2 a^d)(a^{\odot})^2 = a^{\odot}$, $((a^2 a^d)(a^{\odot}))^* = (aa^{\odot})^* = aa^{\odot} = (a^2 a^d)(a^{\odot})$. Moreover, we have

$$\begin{aligned} \|a^{\odot} a^3 a^d - a^2 a^d\| &= \|a^{\odot} a^{n+1} a(a^d)^n - a^2 a^d\| \\ &= \|(a^n - a^{\odot} a^{n+1}) a(a^d)^n\| \\ &= \|a^n - a^{\odot} a^{n+1}\| \|a(a^d)^n\|. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|a^n - a^{\odot} a^{n+1}\|^{\frac{1}{n}} = 0$, we derive

$$\lim_{n \rightarrow \infty} \|a^{\odot} a^3 a^d - a^2 a^d\|^{\frac{1}{n}} = 0.$$

This implies that $a^{\odot} a^3 a^d = a^2 a^d$. Hence, $a^{\odot} (a^2 a^d)^2 = a^{\odot} a^3 a^d = a^2 a^d$, and then $(a^2 a^d)^{\oplus} = a^{\odot}$. In view of [4, Theorem 3.4], we have

$$\begin{aligned} \|(a^{\odot})^2 - a^d (a^2 a^d)^{\oplus}\| &= \|a^{\odot} (aa^{\odot}a^{\odot}) - a^d a^{\odot}\| \\ &= \|a^{\odot} a^{n+1} (a^d)^{n+1} a^{\odot} - a^n (a^d)^{n+1} a^{\odot}\| \\ &= \|(a^n - a^{\odot} a^{n+1}) (a^d)^{n+1} a^{\odot}\| \\ &\leq \|a^n - a^{\odot} a^{n+1}\| \|(a^d)^{n+1} a^{\odot}\|. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|a^n - a^{\odot} a^{n+1}\|^{\frac{1}{n}} = 0$, we prove that

$$\lim_{n \rightarrow \infty} \|(a^{\odot})^2 - a^d (a^2 a^d)^{\oplus}\|^{\frac{1}{n}} = 0,$$

whence, $(a^{\odot})^2 = a^d (a^2 a^d)^{\oplus}$. Accordingly, $a^{\odot} = (a^{\odot})^2 a^2 a^{\dagger} = a^d (a^2 a^d)^{\oplus} a^2 a^{\dagger}$, as asserted. \square

LEMMA 3.3. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

- (1) $a \in \mathcal{A}^{\odot}$.
- (2) *The system of equations $a^{\pi}x = 0$ and $ax = a^{(\odot)}a^2a^{\dagger}$ is consistent.*

Here, $a^{\pi} = 1 - aa^d$. In this case, the solution x is unique and $a^{\odot} = x$.

Proof. (1) \Rightarrow (2) Let $x = a^{\odot}$. Then $x = a^{(\otimes)}aa^{\dagger}$. In view of [4, Theorem 6.1], $a^{(\otimes)} = (a^{(\odot)})^2a = a^d q$ for an idempotent $q \in \mathcal{A}$. One easily checks that

$$\begin{aligned} a^{\pi}x &= a^{\pi}a^d q = 0, \\ ax &= a(a^{(\odot)})^2aaa^{\dagger} = a^{(\odot)}a^2a^{\dagger}. \end{aligned}$$

(2) \Rightarrow (1) By hypothesis, there exists $x \in \mathcal{A}$ such that $a^{\pi}x = 0$ and $ax = a^{(\odot)}a^2a^{\dagger}$. Obviously, $a^{(\odot)} = a(a^{(\odot)})^2$. By virtue of [4, Theorem 6.1], $a^{(\otimes)} = (a^{(\odot)})^2a$. In light of [4, Theorem 3.4], $aa^d a^{(\otimes)} = a^{(\otimes)}$. Then

$$\begin{aligned} x &= a^d(ax) = a^d(a^{(\odot)}a^2a^{\dagger}) = aa^d(a^{(\odot)})^2aa^{\dagger} \\ &= aa^d a^{(\otimes)}aa^{\dagger} = a^{(\otimes)}aa^{\dagger} = a^{\odot}. \end{aligned}$$

Therefore $a \in \mathcal{A}^{\odot}$.

Suppose that there exist $x, y \in \mathcal{A}$ such that

$$\begin{aligned} a^{\pi}x &= 0, \quad ax = a^{(\odot)}a^2a^{\dagger}; \\ a^{\pi}y &= 0, \quad ay = a^{(\odot)}a^2a^{\dagger}. \end{aligned}$$

Then $ax = ay$, and so $a^d x = (a^d)^2(ax) = (a^d)^2(ay) = a^d y$. Accordingly, $x = a(a^d x) = a(a^d y) = aa^d y = y$, as asserted. \square

THEOREM 3.4. *Let $a \in \mathcal{A}^{\odot}$. Then the following are equivalent:*

- (1) $a^{\odot} = x$.
- (2) $xax = x$, $ax = a^{(\odot)}a^2a^{\dagger}$ and $xa = a^{(\otimes)}a$.
- (3) $xaa^{\dagger} = x$ and $xa = a^{(\otimes)}a$.
- (4) $xa^{(\odot)}a^2x = x$, $a^{(\odot)}a^2x = a^{(\odot)}a^2a^{\dagger}$ and $xa^{(\odot)}a^2 = a^{(\otimes)}a$.

Proof. (1) \Rightarrow (2) By virtue of Theorem 2.1, $xax = x$ and $xa = a^{(\otimes)}a$. In view of Lemma 3.3, $ax = a^{(\odot)}a^2a^{\dagger}$.

(2) \Rightarrow (1) Obviously, $a^{(\odot)} = a(a^{(\odot)})^2$. By virtue of [4, Theorem 6.1], $a^{(\otimes)} = (a^{(\odot)})^2a$. Therefore $ax = a^{(\odot)}a^2a^{\dagger} = a[(a^{(\odot)})^2a](aa^{\dagger}) = aa^{(\otimes)}aa^{\dagger}$. According to Theorem 2.1, $a^{\odot} = x$.

(1) \Rightarrow (3) By virtue of Theorem 2.1, $x = a^{(\otimes)}aa^{\dagger}$. Then $xa = a^{(\otimes)}a$. Moreover, $xaa^{\dagger} = a^{(\otimes)}aa^{\dagger}aa^{\dagger} = a^{(\otimes)}aa^{\dagger} = x$, as desired.

(3) \Rightarrow (1) By hypothesis, we have $x = (xa)a^{\dagger} = (a^{(\otimes)}a)a^{\dagger} = a^{(\otimes)}aa^{\dagger}$. According to Theorem 2.1, $a^{\odot} = x$.

(1) \Rightarrow (4) Set $x = a^{\otimes}aa^{\dagger}$. In view of Lemma 3.3, $ax = a^{\textcircled{d}}a^2a^{\dagger}$, and then $a^{\textcircled{d}}a^2x = a^{\textcircled{d}}aa^{\textcircled{d}}a^2a^{\dagger} = a^{\textcircled{d}}a^2a^{\dagger}$. By virtue of [4, Theorem 6.1], we have

$$\begin{aligned} x(a^{\textcircled{d}}a^2) &= a^{\otimes}aa^{\dagger}(a^{\textcircled{d}}a^2) \\ &= a^{\otimes}(aa^{\dagger}a)(a^{\textcircled{d}})^2a^2 \\ &= a^{\otimes}a[(a^{\textcircled{d}})^2a]a \\ &= (a^{\otimes}aa^{\otimes})a \\ &= a^{\otimes}a. \end{aligned}$$

Moreover, we derive

$$\begin{aligned} xa^{\textcircled{d}}a^2x &= x(a^{\textcircled{d}}a^2x) \\ &= x(a^{\textcircled{d}}a^2a^{\dagger}) \\ &= (xa^{\textcircled{d}}a^2)a^{\dagger} \\ &= a^{\otimes}aa^{\dagger}. \end{aligned}$$

(4) \Rightarrow (1) By hypothesis, we have $x = x(a^{\textcircled{d}}a^2x) = (xa^{\textcircled{d}}a^2)a^{\dagger} = a^{\otimes}aa^{\dagger}$, thus yielding the result. \square

Let $a \in \mathcal{A}$. We say that a has $\{2\}$ -inverse x provided that $x = xax$. We denote $a_{T,S}^{(2)} = \{x \in \mathcal{A} \mid xax = x, im(a) = T, ker(a) = S\}$. Here, $im(a) = \{ar \mid r \in \mathcal{A}\}, ker(a) = \{r \in \mathcal{A} \mid ar = 0\}$. We now derive

THEOREM 3.5. *Let $a \in \mathcal{A}^{\textcircled{c}}$. Then*

$$a^{\textcircled{c}} = a_{im(a^d), ker((a^d)^*a^2a^{\dagger})}^{(2)}$$

Proof. Let $x = a^{\textcircled{c}}$. In view of Theorem 2.1, we have $x = xax$.

Step 1. $im(x) = im(a^d)$. Since $x = a^{\otimes}aa^{\dagger} = aa^d a^{\otimes}aa^{\dagger}$, we see that $x\mathcal{A} \subseteq im(a^d)$. We observe that

$$\begin{aligned} \|a^d - a^{\textcircled{c}}aa^d\|_n^{\frac{1}{n}} &= \|a^n(a^d)^{n+1} - a^{\textcircled{c}}a^{n+1}(a^d)^{n+1}\|_n^{\frac{1}{n}} \\ &\leq \|a^n - a^{\textcircled{c}}a^{n+1}\|_n^{\frac{1}{n}} \|a^d\|^{1+\frac{1}{n}}. \end{aligned}$$

As in the proof of Theorem 3.1, $\lim_{n \rightarrow \infty} \|a^n - a^{\textcircled{c}}a^{n+1}\|_n^{\frac{1}{n}} = 0$. Then we have $\lim_{n \rightarrow \infty} \|a^d - a^{\textcircled{c}}aa^d\|_n^{\frac{1}{n}} = 0$, and so $a^d = a^{\textcircled{c}}aa^d$. Therefore $a^d\mathcal{A} \subseteq x\mathcal{A}$. This completes that $im(a^{\textcircled{c}}) = im(a^d)$.

Step 2. $ker(x) = ker((a^d)^*a^2a^{\dagger})$. We observe that

$$\begin{aligned} ker(x) &= ker(ax) \\ &= ker(aa^{\textcircled{c}}aa^{\dagger}) \\ &= ker(aa^{\textcircled{d}}a^2a^{\dagger}), \end{aligned}$$

Then $r \in ker(x)$ if and only if $a^2a^{\dagger}r \in ker(a^{\textcircled{d}}) = ker(a^d)^*$. Accordingly, $ker(x) = ker((a^d)^*a^2a^{\dagger})$. \square

We now provide a new property of the weak core inverse for a complex matrix.

COROLLARY 3.6. Let $A \in \mathbb{C}^{n \times n}$. Then

$$A^{\textcircled{w}, \dagger} = A_{R(A^D), N(A^D)^*A^2A^\dagger}^{(2)}$$

Proof. This is an immediate consequence of Theorem 3.5. \square

Let $a, b, c \in \mathcal{A}$. The element a has (b, c) -inverse provide that there exists $x \in \mathcal{A}$ such that

$$xab = b, \quad cax = c \quad \text{and} \quad x \in bAx \cap xAc.$$

If such x exists, it is unique and denote it by $a^{(b,c)}$ (see [6]). We now derive the following.

THEOREM 3.7. Let $a \in \mathcal{A}^{\textcircled{c}}$. Then $a \in \mathcal{A}^d \cap \mathcal{A}^\dagger$ and a has $(a^d, (a^d)^*a^2a^\dagger)$ -inverse. In this case, $a^{\textcircled{c}} = a^{(a^d, (a^d)^*a^2a^\dagger)}$.

Proof. Since $a \in \mathcal{A}^{\textcircled{c}}$, $a \in \mathcal{A}^d \cap \mathcal{A}^\dagger$. Let $x = a^{\textcircled{c}}$. Then we verify that

$$\begin{aligned} x &= x(ax) = (a^{\textcircled{g}}aa^\dagger)ax = aa^da^{\textcircled{g}}aa^\dagger ax \in a^dAx, \\ x &= (xa)x = (xa)(a^{\textcircled{g}}aa^\dagger) = xa(a^{\textcircled{d}})^2a^2a^\dagger = xa^{\textcircled{d}}a^2a^\dagger \\ &= xa^d(aa^{\textcircled{d}})a^2a^\dagger = xa^d(aa^{\textcircled{d}})^*a^2a^\dagger = xa^d(a^da^2a^{\textcircled{d}})^*a^2a^\dagger \\ &= xa^d(a^2a^{\textcircled{d}})^*(a^d)^*a^2a^\dagger \in x\mathcal{A}(a^d)^*a^2a^\dagger, \\ xaa^d &= (a^{\textcircled{g}}aa^\dagger)aa^d = a^{\textcircled{g}}aa^d = a^d, \\ (a^d)^*a^2a^\dagger ax &= (a^d)^*a^2a^\dagger a(a^{\textcircled{g}}aa^\dagger) = (a^d)^*a^2(a^{\textcircled{g}}aa^\dagger) \\ &= (a^d)^*a^2(a^{\textcircled{d}})^2a^2a^\dagger = (a^d)^*aa^{\textcircled{d}}a^2a^\dagger \\ &= (a^d)^*aa^{\textcircled{d}}a^2a^\dagger = (a^d)^*(aa^{\textcircled{d}})^*a^2a^\dagger \\ &= [a(a^d)^2]^*a^2a^\dagger = (a^d)^*a^2a^\dagger. \end{aligned}$$

Therefore a has $(a^d, (a^d)^*a^2a^\dagger)$ -inverse, as desired. \square

COROLLARY 3.8. Let $A \in \mathbb{C}^{n \times n}$. Then

$$A^{\textcircled{w}, \dagger} = A^{(A^D, (A^D)^*A^2A^\dagger)}.$$

Proof. This is obvious by Theorem 3.7. \square

COROLLARY 3.9. Let $a, x \in \mathcal{A}$, $ax = xa$ and $a^*x = xa^*$. If $a \in \mathcal{A}^{\textcircled{c}}$, then $a^{\textcircled{c}}x = xa^{\textcircled{c}}$.

Proof. Since $a^*x = xa^*$, it follows by [7, Theorem 2.3] that $a^{(a^*, a^*)}x = xa^{(a^*, a^*)}$. Similarly, we have $a^\dagger x = xa^\dagger$. As in the proof of [2, Corollary 4.6], $a^d x = xa^d$. This implies that $(a^d)^*a^2a^\dagger x = x(a^d)^*a^2a^\dagger$. In light of Theorem 3.7 and [7, Theorem 2.3], we have

$$\begin{aligned} a^{\textcircled{c}}x &= a^{(a^d, (a^d)^*a^2a^\dagger)}x \\ &= xa^{(a^d, (a^d)^*a^2a^\dagger)} \\ &= xa^{\textcircled{c}}. \end{aligned}$$

This completes the proof. \square

We now establish the reverse order law for generalized core inverse in a Banach $*$ -algebra.

COROLLARY 3.10. *Let $a, b \in \mathcal{A}^{\odot}$. If $ab = ba$, $a^*b = ba^*$, then $ab \in \mathcal{A}^{\odot}$. In this case,*

$$(ab)^{\odot} = a^{\odot}b^{\odot} = b^{\odot}a^{\odot}.$$

Proof. By hypothesis, we have $a^*b = ba^*$, $a^*b^* = (ba)^* = (ab)^* = b^*a^*$. In light of Corollary 3.9, $a^*b^{\odot} = b^{\odot}a^*$. Likewise, we prove that $ab^{\odot} = b^{\odot}a$. By using Corollary 3.9 again, $a^{\odot}b^{\odot} = b^{\odot}a^{\odot}$.

Since $ab = ba$ and $a^*b = ba^*$, it follows by [5, Theorem 2.3] that $ab \in \mathcal{A}^{\otimes}$ and $(ab)^{\otimes} = a^{\otimes}b^{\otimes}$. Since $a^*b = ba^*$, by using [7, Theorem 2.3], we prove that $a^{\dagger}b = a^{(a^*, a^*)}b = ba^{(a^*, a^*)} = ba^{\dagger}$. One directly verifies that $ab \in \mathcal{A}^{\dagger}$ and $(ab)^{\dagger} = a^{\dagger}b^{\dagger}$. By virtue of Theorem 2.1, $ab \in \mathcal{A}^{\odot}$.

Since $ab = ba$ and $ab^* = b^*a$, it follows by [5, Lemma 2.2] that $ab^{\otimes} = b^{\otimes}a$. Moreover, we have $a^{\dagger}b = ba^{\dagger}$. Likewise, $a^{\dagger}b^* = b^*a^{\dagger}$. By using [5, Lemma 2.2] again, $a^{\dagger}b^{\otimes} = b^{\otimes}a^{\dagger}$. Then

$$\begin{aligned} a^{\odot}b^{\odot} &= a^{\otimes}aa^{\dagger}b^{\otimes}bb^{\dagger} \\ &= a^{\otimes}b^{\otimes}bb^{\dagger}aa^{\dagger} \\ &= a^{\otimes}b^{\otimes}aba^{\dagger}b^{\dagger} \\ &= (ab)^{\otimes}ab(ab)^{\dagger} \\ &= (ab)^{\odot}. \end{aligned}$$

This completes the proof. \square

4. Generalized core elements

An element $a \in \mathcal{A}$ is called a generalized core element provided that $a^{\odot} = a^{\#}$. Evidently, generalized core and weak core matrices coincide with each other for a complex matrix (see [11]). We are focusing our attention on the investigation of generalized core elements in a Banach $*$ -algebra. An element $a \in \mathcal{A}$ is an EP element provided that there exists $x \in \mathcal{A}$ such that $xa^2 = a$, $ax^2 = x$, $(xa)^* = xa$. As is well known, $a \in \mathcal{A}$ is EP if and only if there exists $x \in \mathcal{A}$ such that $a^2x = a$, $ax = xa$, $(ax)^* = ax$ if and only if $a \in \mathcal{A}^{\#}$ and $(aa^{\#})^* = aa^{\#}$.

THEOREM 4.1. *Let $a \in \mathcal{A}^{\odot}$. Then the following are equivalent:*

- (1) $a \in \mathcal{A}$ is a generalized core element.
- (2) $a^{\otimes} = a^{\#}$.
- (3) a^{\odot} is EP.

Proof. (1) \Rightarrow (2) In view of [4, Theorem 6.1], $a^{\textcircled{g}}a = (a^{\textcircled{g}}aa^\dagger)a = a^{\textcircled{c}}a = a^{\textcircled{d}}a = a(a^{\textcircled{d}})^2a = aa^{\textcircled{g}}$. Therefore $a^{\textcircled{g}} = a^d$, as required.

(2) \Rightarrow (3) In view of Theorem 3.1, we have

$$\begin{aligned} a^{\textcircled{d}} &= aa^{\textcircled{g}}a^\dagger = aa^da^\dagger = a^daa^\dagger \\ &= a^{\textcircled{g}}aa^\dagger = a^{\textcircled{c}}. \end{aligned}$$

In view of [3, Theorem 1.2], $a^{\textcircled{d}} \in \mathcal{A}^\#$ and $(a^{\textcircled{d}})^\# = a^2a^{\textcircled{d}}$. Furthermore, we check that

$$\begin{aligned} &a^{\textcircled{d}}a^2a^{\textcircled{d}} - (a^2a^{\textcircled{d}})a^{\textcircled{d}} \\ &= a^{\textcircled{d}}a^{n+1}(a^{\textcircled{d}})^n - aa^{\textcircled{d}} \\ &= a^{\textcircled{d}}a^{n+1}(a^{\textcircled{d}})^n - a^n(a^{\textcircled{d}})^n \\ &= -(a^n - a^{\textcircled{d}}a^{n+1})(a^{\textcircled{d}})^n. \end{aligned}$$

Thus

$$\|a^{\textcircled{d}}a^2a^{\textcircled{d}} - (a^2a^{\textcircled{d}})a^{\textcircled{d}}\|^{1/n} \leq \|a^n - a^{\textcircled{d}}a^{n+1}\|^{1/n} \|a^{\textcircled{d}}\|,$$

and then

$$\lim_{n \rightarrow \infty} \|a^{\textcircled{d}}a^2a^{\textcircled{d}} - (a^2a^{\textcircled{d}})a^{\textcircled{d}}\|^{1/n} = 0.$$

Therefore $a^{\textcircled{d}}(a^2a^{\textcircled{d}}) = (a^2a^{\textcircled{d}})a^{\textcircled{d}}$, Thus, $a^{\textcircled{c}}$ is EP.

(3) \Rightarrow (1) One directly verifies that

$$\begin{aligned} a^2a^{\textcircled{c}}(a^{\textcircled{c}})^2 &= a^2(a^{\textcircled{g}})^3aa^\dagger = a^{\textcircled{g}}aa^\dagger = a^{\textcircled{c}}, \\ a^{\textcircled{c}}(a^2a^{\textcircled{c}})^2 &= aa^{\textcircled{g}}a^2a^{\textcircled{g}}aa^\dagger = a^2a^{\textcircled{g}}aa^\dagger = a^2a^{\textcircled{c}}, \\ a^{\textcircled{c}}(a^2a^{\textcircled{c}}) &= aa^{\textcircled{g}}aa^\dagger = a^2(a^{\textcircled{g}})^2aa^\dagger = (a^2a^{\textcircled{c}})a^{\textcircled{c}}. \end{aligned}$$

Then $a^{\textcircled{c}} \in \mathcal{A}^\#$ and $(a^{\textcircled{c}})^\# = a^2a^{\textcircled{c}}$. Since $a^{\textcircled{c}}$ is EP, we have

$$a^{\textcircled{c}}(a^2a^{\textcircled{c}}) = [a^{\textcircled{c}}(a^2a^{\textcircled{c}})]^*.$$

It is easy to verify that

$$\begin{aligned} a^{\textcircled{c}}(a^2a^{\textcircled{c}}) &= a^{\textcircled{g}}aa^\dagger(a^2a^{\textcircled{g}}aa^\dagger) \\ &= (a^{\textcircled{g}}a^2)a^{\textcircled{g}}aa^\dagger \\ &= a(a^{\textcircled{g}}aa^\dagger) \\ &= aa^{\textcircled{c}}, \end{aligned}$$

and so $aa^{\textcircled{c}} = (aa^{\textcircled{c}})^*$. Obviously, $a(a^{\textcircled{c}})^2 = aa^{\textcircled{g}}aa^\dagger a(a^{\textcircled{g}})^2aa^\dagger = a(a^{\textcircled{g}})^2aa^\dagger = a^{\textcircled{g}}aa^\dagger = a^{\textcircled{c}}$. Also we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|a^n - a^{\textcircled{c}}a^{n+1}\|^{1/n} &= \lim_{n \rightarrow \infty} \|a^n - a^{\textcircled{g}}aa^\dagger a^{n+1}\|^{1/n} \\ &= \lim_{n \rightarrow \infty} \|a^n - a^{\textcircled{g}}a^{n+1}\|^{1/n} \\ &= 0. \end{aligned}$$

Therefore $a^{\textcircled{c}} = a^{\textcircled{d}}$, as asserted. \square

We need the following lemma concerning some properties of the generalized core-EP decomposition of an element in $\mathcal{A}^{\textcircled{c}}$.

LEMMA 4.2. Let $a \in \mathcal{A}^{\odot}$ and $a = a_1 + a_2$ be the generalized core-EP decomposition of a , and let $p = a_1 a_1^{\oplus}$. Then

$$a_1 = \begin{pmatrix} t & s \\ 0 & 0 \end{pmatrix}_p, \quad a_2 = \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}_p,$$

where $t \in (pAp)^{-1}$, $n \in (p^\pi Ap^\pi)^{qnil}$.

Proof. Obviously, $p^2 = p = p^*$. It is easy to verify that $p^\pi a_1 = (1 - a_1 a_1^{\oplus}) a_1 = 0$, and so $a_1 = \begin{pmatrix} t & s \\ 0 & 0 \end{pmatrix}_p$. Here, $t = pap$ and $s = pa(1 - p)$. Moreover, we have $pa_2 = a_1 a_1^{\oplus} a_2 = (a_1^{\oplus})^* (a_1)^* a_2 = 0$ and $p^\pi a_2 p = (1 - a_1 a_1^{\oplus}) a_2 a_1 a_1^{\oplus} = 0$. Hence, $a_2 = \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}_p$, where $n = p^\pi a_2 p^\pi \in \mathcal{A}^{qnil}$. \square

THEOREM 4.3. Let $a \in \mathcal{A}^{\odot}$. Then the following are equivalent:

- (1) $a \in \mathcal{A}$ is a generalized core element.
- (2) $aa^{\odot} = aa^{\oplus}$.
- (3) $a^d a^{\odot} = a^d a^{\oplus}$.

Proof. (1) \Rightarrow (2) By hypothesis, $a^{\odot} = a^{\oplus}$, and so $aa^{\odot} = aa^{\oplus}$.

(2) \Rightarrow (3) Since $a^d = (a^d)^2 a$, it follows by $aa^{\odot} = aa^{\oplus}$ that $a^d a^{\odot} = a^d a^{\oplus}$, as required.

(3) \Rightarrow (1) In view of Lemma 4.2, we have

$$a_1 = \begin{pmatrix} t & s \\ 0 & 0 \end{pmatrix}_p, \quad a_2 = \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}_p,$$

where p, t, s, n constructed as in Lemma 4.2. Then $a = \begin{pmatrix} t & s \\ 0 & n \end{pmatrix}_p$ and $a^{\oplus} = \begin{pmatrix} t^{-1} & 0 \\ 0 & 0 \end{pmatrix}_p$.

By virtue of [4, Theorem 6.1], we have

$$a^{\otimes} = (a^{\oplus})^2 a = \begin{pmatrix} t^{-1} & t^{-2}s \\ 0 & 0 \end{pmatrix}_p.$$

Set $x = a^{\odot}$. In view of Theorem 3.4, $xax = x$ and $xa = a^{\otimes} a$. Write $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}_p$.

Then

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}_p \begin{pmatrix} t & s \\ 0 & n \end{pmatrix}_p = xa = a^{\otimes} a = \begin{pmatrix} t^{-1} & t^{-2}s \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} t & s \\ 0 & n \end{pmatrix}_p,$$

and so $x_{11} = t^{-1}$, $x_{21} = 0$ and $x_{22}n = 0$.

Since $xax = x$, we have

$$\begin{pmatrix} t^{-1} & x_{12} \\ 0 & x_{22} \end{pmatrix}_p \begin{pmatrix} t & s \\ 0 & n \end{pmatrix}_p \begin{pmatrix} t^{-1} & x_{12} \\ 0 & x_{22} \end{pmatrix}_p = \begin{pmatrix} t^{-1} & x_{12} \\ 0 & x_{22} \end{pmatrix}_p.$$

This implies that $x_{22} = (x_{22}n)x_{22} = 0$. Therefore $x = \begin{pmatrix} t^{-1} & y \\ 0 & 0 \end{pmatrix}_p$ for some $z \in \mathcal{A}$. Moreover, we have

$$a^d = \begin{pmatrix} t^{-1} & z \\ 0 & 0 \end{pmatrix}_p \text{ for some } z \in \mathcal{A}, \quad a^{\textcircled{d}} = \begin{pmatrix} t^{-1} & 0 \\ 0 & 0 \end{pmatrix}_p.$$

Accordingly, we have

$$\begin{aligned} a^d a^{\textcircled{c}} &= \begin{pmatrix} t^{-1} & z \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} t^{-1} & y \\ 0 & 0 \end{pmatrix}_p \\ &= \begin{pmatrix} t^{-2} & t^{-1}y \\ 0 & 0 \end{pmatrix}_p, \\ a^d a^{\textcircled{d}} &= \begin{pmatrix} t^{-1} & z \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} t^{-1} & 0 \\ 0 & 0 \end{pmatrix}_p \\ &= \begin{pmatrix} t^{-2} & 0 \\ 0 & 0 \end{pmatrix}_p. \end{aligned}$$

Since $a^d a^{\textcircled{c}} = a^d a^{\textcircled{d}}$, we have $t^{-1}y = 0 = 0$, and so $y = 0$. This implies that $a^{\textcircled{c}} = a^{\textcircled{d}}$, as asserted. \square

COROLLARY 4.4. *Let $a \in \mathcal{A}^{\textcircled{c}}$. Then the following are equivalent:*

- (1) $a \in \mathcal{A}$ is a generalized core element.
- (2) $aa^{\textcircled{d}} = a^{\textcircled{d}}a$.
- (3) $a^d a^{\textcircled{d}} = a^{\textcircled{d}} a^d$.

Proof. (1) \Rightarrow (2) Since $a^{\textcircled{c}} = a^{\textcircled{d}}$, it follows by Theorem 4.3 that $aa^{\textcircled{d}} = aa^{\textcircled{c}} = a^{\textcircled{d}}a$, as desired.

(2) \Rightarrow (3) Since $aa^{\textcircled{d}} = a^{\textcircled{d}}a$, as in the proof of [2, Corollary 4.6] that $a^d a^{\textcircled{d}} = a^{\textcircled{d}} a^d$.

(3) \Rightarrow (1) By hypothesis, $a^d a^{\textcircled{d}} = a^{\textcircled{d}} a^d$. In light of [4, Theorem 6.1], we have

$$\begin{aligned} a^d a^{\textcircled{c}} &= a^d a^{\textcircled{d}} a a^\dagger = a^d (a^{\textcircled{d}})^2 a^2 a^\dagger = (a^d)^2 [a (a^{\textcircled{d}})^2] a^2 a^\dagger \\ &= (a^d)^2 a^{\textcircled{d}} a^2 a^\dagger = a^{\textcircled{d}} (a^d)^2 a^2 a^\dagger = a^{\textcircled{d}} a^d a a^\dagger \\ &= (a^d)^2 (a a^{\textcircled{d}}) (a a^\dagger) = (a^d)^2 (a a^{\textcircled{d}})^* (a a^\dagger)^* \\ &= (a^d)^2 ((a a^\dagger) (a a^{\textcircled{d}}))^* = (a^d)^2 (a a^{\textcircled{d}})^* = (a^d)^2 a a^{\textcircled{d}} = a^d a^{\textcircled{d}}. \end{aligned}$$

This completes the proof by Theorem 4.3. \square

A matrix $A \in \mathbb{C}^{n \times n}$ is weak core matrix if $A^{\textcircled{w}, \dagger} = A^{\textcircled{d}}$ (see [11]). We provide a new characterization of such complex matrix.

COROLLARY 4.5. *Let $A \in \mathbb{C}^{n \times n}$. Then the following are equivalent:*

- (1) *A is a weak core matrix.*
- (2) $AA^{\textcircled{D}} = A^{\textcircled{D}}A$.
- (3) $A^DA^{\textcircled{D}} = A^{\textcircled{D}}A^D$.

Proof. This is immediate from Corollary 4.4. \square

Acknowledgement. The authors are highly grateful to the referee for his/her careful reading and helpful suggestions.

REFERENCES

- [1] H. CHEN AND M. SHEIBANI ABDOLYOUSEFI, *Theory of Clean Rings and Matrices*, Singapore: World Scientific, 2023.
- [2] H. CHEN AND M. SHEIBANI, *Generalized weighted core inverse in Banach algebras*, preprint, 2023, <https://doi.org/10.21203/rs.3.rs-3318243/v1>.
- [3] H. CHEN AND M. SHEIBANI, *On generalized core-EP invertibility in a Banach algebra*, arXiv:2309.09862 [math.FA], <https://doi.org/10.48550/arXiv.2309.09862>.
- [4] H. CHEN AND M. SHEIBANI, *Generalized group inverse in a Banach *-algebra*, preprint, 2023, <https://doi.org/10.21203/rs.3.rs-3338906/v1>.
- [5] H. CHEN AND M. SHEIBANI, *Some new results in generalized weighted core inverse in Banach *-algebras*, preprint, 2023, <https://doi.org/10.21203/rs.3.rs-3329741/v1>.
- [6] M. P. DRAZIN, *A class of outer generalized inverses*, Linear Algebra Appl., **436** (2012), 1909–1923.
- [7] M. P. DRAZIN, *Commuting properties of generalized inverses*, Linear Multilinear Algebra, **61** (2013), 1675–1681.
- [8] Y. GAO AND J. CHEN, *Pseudo core inverses in rings with involution*, Comm. Algebra, **46** (2018), 38–50.
- [9] D. E. FERREYRA, F. E. LEVIS, A. N. PRIORI AND N. THOME, *The weak core inverse*, Aequat. Math., **95** (2021), 351–373.
- [10] D. E. FERREYRA, V. ORQUERA AND N. THOME, *Representations of weighted WG inverse and a rank equation's solution*, Linear and Multilinear Algebra, **71** (2023), 226–241.
- [11] Z. FU, K. ZUO AND Y. CHEN, *Further characterizations of the weak core inverse of matrices and the weak core matrix*, AIMS Math., **7** (2021), 3630–3647.
- [12] R. KUANG, C. DENG, *Common properties among various generalized inverses and constrained binary relations*, Linear Multilinear Algebra, **71** (2023), 1295–1322.
- [13] Y. LIAO, J. CHEN AND J. CUI, *Cline's formula for the generalized Drazin inverse*, Bull. Malays. Math. Sci. Soc., **37** (2014), 37–42.
- [14] N. MIHAJLOVIC, *Group inverse and core inverse in Banach and C^* -algebras*, Comm. Algebra, **48** (2020), 1803–1818.
- [15] D. MOSIĆ, *Core-EP inverses in Banach algebras*, Linear Multilinear Algebra, **69** (2021), 2976–2989.
- [16] D. MOSIĆ AND J. MAROVT, *Weighted weak core inverse of operators*, Linear Multilinear Algebra, **70** (2022), 4991–5013.
- [17] D. MOSIĆ, P. S. STANIMIROVIC, *Expressions and properties of weak core inverse*, Appl. Math. Comput., **415** (2022), Article ID 126704, 23 p.
- [18] H. WANG, *Core-EP decomposition and its applications*, Linear Algebra Appl., **508** (2016), 289–300.
- [19] H. WANG, J. CHEN, *Weak group inverse*, Open Math., **16** (2018), 1218–1232.
- [20] M. ZHOU, J. CHEN AND Y. ZHOU, *Weak group inverses in proper *-rings*, J. Algebra Appl., **19** (2020), doi:10.1142/S0219498820502382.

- [21] M. ZHOU, J. CHEN, Y. ZHOU AND N. THOME, *Weak group inverses and partial isometries in proper *-rings*, *Linear Multilinear Algebra*, **70** (2021), 1–16.
- [22] Y. ZHOU AND J. CHEN, *Weak core inverses and pseudo core inverses in a ring with involution*, *Linear Multilinear Algebra*, **22** (2022), 6876–6890.
- [23] H. ZHU AND P. PATRICIO, *Characterizations for pseudo core inverses in a ring with involution*, *Linear Multilinear Algebra*, **67** (2019), 1109–1120.

(Received September 5, 2023)

Huanyin Chen
School of Big Data
Fuzhou University of International Studies and Trade
Fuzhou 350202, China
e-mail: huanyinchenfz@163.com

Marjan Sheibani
Farzanegan Campus
Semnan University
Semnan, Iran
e-mail: m.sheibani@semnan.ac.ir