

## SHARP GENERALIZED UNCERTAINTY PRINCIPLES VIA FACTORIZATIONS

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**Abstract.** Using the factorizations of suitable operators, we establish several identities that give simple and direct understandings as well as provide the remainders and optimizers of the sharp generalized uncertainty principles.

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