Differential
Equations
& Applications

Volume 1, Number 1 (2009), 1–25

# A NEW VERSION OF THE DYNAMICAL SYSTEMS METHOD (DSM) FOR SOLVING NONLINEAR EQUATIONS WITH MONOTONE OPERATORS

N. S. HOANG AND A. G. RAMM

(*communicated by R. Spigler*)

*Abstract.* A version of the Dynamical Systems Method for solving ill-posed nonlinear monotone operator equations is studied in this paper. A discrepancy principle is proposed and justified. A numerical experiment was carried out with the new stopping rule. Numerical experiments show that the proposed stopping rule is efficient.

## 1. Introduction

In this paper we study a version of the Dynamical Systems Method (DSM) for solving the equation

$$F(u) = f, \tag{1}$$

where $F$ is a nonlinear, Fréchet differentiable, monotone operator in a real Hilbert space $H$, and equation (1) is assumed solvable, possibly nonuniquely. Monotonicity means that

$$\langle F(u) - F(v), u - v \rangle \geqslant 0, \quad \forall u, v \in H. \tag{2}$$

It is known (see, e.g., [7]), that the set $\mathscr{N} := \{u : F(u) = f\}$ is closed and convex if $F$ is monotone and continuous. A closed and convex set in a Hilbert space has a unique minimal-norm element. This element in $\mathscr{N}$ we denote by $y$, $F(y) = f$, and call it the minimal-norm solution to equation (1). We assume that

$$\sup_{\|u-u_0\| \leqslant R} \|F'(u)\| \leqslant M_1(R), \tag{3}$$

where $u_0 \in H$ is an element of $H$, $R > 0$ is arbitrary, and $f = F(y)$ is not known but $f_\delta$, the noisy data, are known, and $\|f_\delta - f\| \leqslant \delta$. If $F'(u)$ is not boundedly invertible then solving equation (1) for $u$ given noisy data $f_\delta$ is often (but not always) an ill-posed problem. When $F$ is a linear bounded operator many methods for stable solution of (1) were proposed (see [5]–[7] and references therein). However, when $F$ is nonlinear then the theory is less complete.

*Mathematics subject classification* (2000): 47J05, 47J06, 47J35, 65R30.

*Keywords and phrases*: Dynamical systems method (DSM), nonlinear operator equations, monotone operators, discrepancy principle.


Paper DEA-01-01

DSM consists of finding a nonlinear map $\Phi(t,u)$ such that the Cauchy problem

$$\dot{u} = \Phi(t,u), \qquad u(0) = u_0,$$

has a unique solution for all $t \geqslant 0$, there exists $\lim_{t\to\infty} u(t) := u(\infty)$, and $F(u(\infty)) = f$,

$$\exists!\ u(t) \quad \forall t \geqslant 0; \qquad \exists u(\infty); \qquad F(u(\infty)) = f. \tag{4}$$

Various choices of $\Phi$ were proposed in [7] for (4) to hold. Each such choice yields a version of the DSM.

The DSM for solving equation (1) was extensively studied in [7]–[15]. In [7], the following version of the DSM was investigated for monotone operators $F$:

$$\dot{u}_\delta = -\big(F'(u_\delta) + a(t)I\big)^{-1}\big(F(u_\delta) + a(t)u_\delta - f_\delta\big), \quad u_\delta(0) = u_0. \tag{5}$$

The convergence of this method was justified with some *a apriori* choice of stopping rule. A DSM gradient method was formulated and justified in [4].

In this paper we consider a version of the DSM for solving equation (1):

$$\dot{u}_\delta = -\big(F(u_\delta) + a(t)u_\delta - f_\delta\big), \quad u_\delta(0) = u_0, \tag{6}$$

where $F$ is a monotone operator.

The advantage of this version compared with (5) is the absence of the inverse operator in the algorithm, which makes the algorithm (6) less expensive than (5). On the other hand, algorithm (5) converges faster than (6) in many cases. The algorithm (6) is cheaper than the DSM gradient algorithm proposed in [4].

The convergence of the method (6) for any initial value $u_0$ is proved for a stopping rule based on a discrepancy principle. This *a posteriori* choice of stopping time $t_\delta$ is justified provided that $a(t)$ is suitably chosen.

The advantage of method (6), a modified version of the simple iteration method, over the Gauss-Newton method and the version (5) of the DSM is the following: neither inversion of matrices nor evaluation of $F'$ is needed in a discretized version of (6). Although the convergence rate of the DSM (6) maybe slower than that of the DSM (5), the DSM (6) might be faster than the DSM (5) for large-scale systems due to its lower computation cost.

In this paper we investigate a stopping rule based on a discrepancy principle (DP) for the DSM (6). The main results of this paper are Theorem 17 and Theorem 19 in which a DP is formulated, the existence of a stopping time $t_\delta$ is proved, and the convergence of the DSM with the proposed DP is justified under some natural assumptions.

## 2. Auxiliary results

The inner product in $H$ is denoted $\langle u, v\rangle$. Let us consider the following equation

$$F(V_\delta) + aV_\delta - f_\delta = 0, \qquad a > 0, \tag{7}$$

where $a = const$. It is known (see, e.g., [7], [16]) that equation (7) with monotone continuous operator $F$ has a unique solution for any $f_\delta \in H$.

Let us recall the following result from [7]:

LEMMA 1. *Assume that equation* (1) *is solvable,* $y$ *is its minimal-norm solution, assumption* (2) *holds, and* $F$ *is continuous. Then*

$$\lim_{a\to 0}\|V_a - y\| = 0,$$

*where* $V_a$ *solves* (7) *with* $\delta = 0$.

Clearly, under our assumption (3), $F$ is continuous.

LEMMA 2. *If* (2) *holds and* $F$ *is continuous, then* $\|V_\delta\| = O(\frac{1}{a})$ *as* $a \to \infty$*, and*

$$\lim_{a\to\infty}\|F(V_\delta) - f_\delta\| = \|F(0) - f_\delta\|. \tag{8}$$

*Proof.* Rewrite (7) as

$$F(V_\delta) - F(0) + aV_\delta + F(0) - f_\delta = 0.$$

Multiply this equation by $V_\delta$, use inequality $\langle F(V_\delta) - F(0), V_\delta - 0\rangle \geqslant 0$ and get:

$$a\|V_\delta\|^2 \leqslant \|f_\delta - F(0)\|\|V_\delta\|.$$

Therefore,

$$\|V_\delta\| = O(\frac{1}{a}).$$

This and the continuity of $F$ imply (8). □

Let $a = a(t)$ be a strictly monotonically decaying continuous positive function on $[0,\infty)$, $0 < a(t) \searrow 0$, and assume $a \in C^1[0,\infty)$. These assumptions hold throughout the paper and often are not repeated. Then the solution $V_\delta$ of (7) is a function of $t$, $V_\delta = V_\delta(t)$. From the triangle inequality one gets:

$$\|F(V_\delta(0)) - f_\delta\| \geqslant \|F(0) - f_\delta\| - \|F(V_\delta(0)) - F(0)\|.$$

From Lemma 2 it follows that for large $a(0)$ one has:

$$\|F(V_\delta(0)) - F(0)\| \leqslant M_1\|V_\delta(0)\| = O\left(\frac{1}{a(0)}\right).$$

Therefore, if $\|F(0) - f_\delta\| > C\delta$, then $\|F(V_\delta(0)) - f_\delta\| \geqslant (C - \epsilon)\delta$, where $\epsilon > 0$ is sufficiently small and $a(0) > 0$ is sufficiently large.

Below the words decreasing and increasing mean strictly decreasing and strictly increasing.

LEMMA 3. *Assume* $\|F(0) - f_\delta\| > 0$. *Let* $0 < a(t) \searrow 0$*, and* $F$ *be monotone. Denote*

$$\psi(t) := \|V_\delta(t)\|, \qquad \phi(t) := a(t)\psi(t) = \|F(V_\delta(t)) - f_\delta\|,$$

*where* $V_\delta(t)$ *solves* (7) *with* $a = a(t)$. *Then* $\phi(t)$ *is decreasing, and* $\psi(t)$ *is increasing.*

*Proof.* Since $\|F(0)-f_\delta\| > 0$, one has $\psi(t) \neq 0, \forall t \geqslant 0$. Indeed, if $\psi(t)\big|_{t=\tau} = 0$, then $V_\delta(\tau) = 0$, and equation (7) implies $\|F(0)-f_\delta\| = 0$, which is a contradiction. Note that $\phi(t) = a(t)\|V_\delta(t)\|$. One has

$$\begin{aligned} 0 &\leqslant \langle F(V_\delta(t_1)) - F(V_\delta(t_2)), V_\delta(t_1) - V_\delta(t_2)\rangle \\ &= \langle -a(t_1)V_\delta(t_1) + a(t_2)V_\delta(t_2), V_\delta(t_1) - V_\delta(t_2)\rangle \\ &= (a(t_1)+a(t_2))\langle V_\delta(t_1), V_\delta(t_2)\rangle - a(t_1)\|V_\delta(t_1)\|^2 - a(t_2)\|V_\delta(t_2)\|^2. \end{aligned} \tag{9}$$

Thus,

$$\begin{aligned} 0 &\leqslant (a(t_1)+a(t_2))\langle V_\delta(t_1), V_\delta(t_2)\rangle - a(t_1)\|V_\delta(t_1)\|^2 - a(t_2)\|V_\delta(t_2)\|^2 \\ &\leqslant (a(t_1)+a(t_2))\|V_\delta(t_1)\|\|V_\delta(t_2)\| - a(t_1)\|V_\delta(t_1)\|^2 - a(t_2)\|V_\delta(t_2)\|^2 \\ &= (a(t_1)\|V_\delta(t_1)\| - a(t_2)\|V_\delta(t_2)\|)(\|V_\delta(t_2)\| - \|V_\delta(t_1)\|) \\ &= (\phi(t_1) - \phi(t_2))(\psi(t_2) - \psi(t_1)). \end{aligned} \tag{10}$$

If $\psi(t_2) > \psi(t_1)$ then (10) implies $\phi(t_1) \geqslant \phi(t_2)$, so

$$a(t_1)\psi(t_1) \geqslant a(t_2)\psi(t_2) > a(t_2)\psi(t_1).$$

Thus, if $\psi(t_2) > \psi(t_1)$ then $a(t_2) < a(t_1)$ and, therefore, $t_2 > t_1$, because $a(t)$ is strictly decreasing.

Similarly, if $\psi(t_2) < \psi(t_1)$ then $\phi(t_1) \leqslant \phi(t_2)$. This implies $a(t_2) > a(t_1)$, so $t_2 < t_1$.

Suppose $\psi(t_1) = \psi(t_2)$, i.e., $\|V_\delta(t_1)\| = \|V_\delta(t_2)\|$. From (9), one has

$$\|V_\delta(t_1)\|^2 \leqslant \langle V_\delta(t_1), V_\delta(t_2)\rangle \leqslant \|V_\delta(t_1)\|\|V_\delta(t_2)\| = \|V_\delta(t_1)\|^2.$$

This implies $V_\delta(t_1) = V_\delta(t_2)$, and then equation (7) implies $a(t_1) = a(t_2)$. Hence, $t_1 = t_2$, because $a(t)$ is strictly decreasing.

Therefore $\phi(t)$ is decreasing and $\psi(t)$ is increasing. □

LEMMA 4. *Suppose that $\|F(0)-f_\delta\| > C\delta$, $C > 1$, and $a(0)$ is sufficiently large. Then, there exists a unique $t_1 > 0$ such that $\|F(V_\delta(t_1)) - f_\delta\| = C\delta$.*

*Proof.* The uniqueness of $t_1$ follows from Lemma 3 because $\|F(V_\delta(t)) - f_\delta\| = \phi(t)$, and $\phi$ is decreasing. We have $F(y) = f$, and

$$\begin{aligned} 0 &= \langle F(V_\delta) + aV_\delta - f_\delta, F(V_\delta) - f_\delta\rangle \\ &= \|F(V_\delta) - f_\delta\|^2 + a\langle V_\delta - y, F(V_\delta) - f_\delta\rangle + a\langle y, F(V_\delta) - f_\delta\rangle \\ &= \|F(V_\delta) - f_\delta\|^2 + a\langle V_\delta - y, F(V_\delta) - F(y)\rangle + a\langle V_\delta - y, f - f_\delta\rangle + a\langle y, F(V_\delta) - f_\delta\rangle \\ &\geqslant \|F(V_\delta) - f_\delta\|^2 + a\langle V_\delta - y, f - f_\delta\rangle + a\langle y, F(V_\delta) - f_\delta\rangle. \end{aligned}$$

Here the inequality $\langle V_\delta - y, F(V_\delta) - F(y)\rangle \geqslant 0$ was used. Therefore

$$\begin{aligned}\|F(V_\delta) - f_\delta\|^2 &\leqslant -a\langle V_\delta - y, f - f_\delta\rangle - a\langle y, F(V_\delta) - f_\delta\rangle \\ &\leqslant a\|V_\delta - y\|\|f - f_\delta\| + a\|y\|\|F(V_\delta) - f_\delta\| \\ &\leqslant a\delta\|V_\delta - y\| + a\|y\|\|F(V_\delta) - f_\delta\|.\end{aligned} \tag{11}$$

On the other hand, we have

$$\begin{aligned}0 &= \langle F(V_\delta) - F(y) + aV_\delta + f - f_\delta, V_\delta - y\rangle \\ &= \langle F(V_\delta) - F(y), V_\delta - y\rangle + a\|V_\delta - y\|^2 + a\langle y, V_\delta - y\rangle + \langle f - f_\delta, V_\delta - y\rangle \\ &\geqslant a\|V_\delta - y\|^2 + a\langle y, V_\delta - y\rangle + \langle f - f_\delta, V_\delta - y\rangle,\end{aligned}$$

where the inequality $\langle V_\delta - y, F(V_\delta) - F(y)\rangle \geqslant 0$ was used. Therefore,

$$a\|V_\delta - y\|^2 \leqslant a\|y\|\|V_\delta - y\| + \delta\|V_\delta - y\|.$$

This implies

$$a\|V_\delta - y\| \leqslant a\|y\| + \delta. \tag{12}$$

From (11) and (12), and an elementary inequality $ab \leqslant \epsilon a^2 + \frac{b^2}{4\epsilon}$, $\forall\epsilon > 0$, one gets:

$$\begin{aligned}\|F(V_\delta) - f_\delta\|^2 &\leqslant \delta^2 + a\|y\|\delta + a\|y\|\|F(V_\delta) - f_\delta\| \\ &\leqslant \delta^2 + a\|y\|\delta + \epsilon\|F(V_\delta) - f_\delta\|^2 + \frac{1}{4\epsilon}a^2\|y\|^2,\end{aligned} \tag{13}$$

where $\epsilon > 0$ is fixed, independent of $t$, and can be chosen arbitrary small. Let $t \to \infty$ and $a = a(t) \searrow 0$. Then (13) implies

$$\overline{\lim}_{t\to\infty}(1-\epsilon)\|F(V_\delta) - f_\delta\|^2 \leqslant \delta^2.$$

This, the continuity of $F$, the continuity of $V_\delta(t)$ on $[0,\infty)$, and the assumption $\|F(0) - f_\delta\| > C\delta$ imply that equation $\|F(V_\delta(t)) - f_\delta\| = C\delta$ must have a solution $t_1 > 0$. The uniqueness of this solution was already established. □

REMARK 5. From the proof of Lemma 4 one obtains the following result: *If $t_n \nearrow \infty$ then there exists a unique $n_1 > 0$ such that*

$$\|F(V_{n_1+1}) - f_\delta\| \leqslant C\delta < \|F(V_{n_1}) - f_\delta\|, \qquad V_n := V_\delta(t_n).$$

REMARK 6. From Lemma 2 and Lemma 3 one concludes that

$$a_n\|V_n\| = \|F(V_n) - f_\delta\| \leqslant \|F(0) - f_\delta\|, \qquad a_n := a(t_n), \quad \forall n \geqslant 0.$$

REMARK 7. Let $V := V_\delta(t)|_{\delta=0}$, so

$$F(V) + a(t)V - f = 0.$$

Let $y$ be the minimal-norm solution to equation (1). We claim that

$$\|V_\delta - V\| \leqslant \frac{\delta}{a}. \tag{14}$$

Indeed, from (7) one gets

$$F(V_\delta) - F(V) + a(V_\delta - V) = f - f_\delta.$$

Multiply this equality with $(V_\delta - V)$ and use the monotonicity of $F$ to get

$$a\|V_\delta - V\|^2 \leqslant \delta\|V_\delta - V\|.$$

This implies (14).

Similarly, multiplying the equation

$$F(V) + aV - F(y) = 0,$$

by $V - y$ one derives the inequality:

$$\|V\| \leqslant \|y\|. \tag{15}$$

Similar arguments one can find in [7].

From (14) and (15), one gets the following estimate:

$$\|V_\delta\| \leqslant \|V\| + \frac{\delta}{a} \leqslant \|y\| + \frac{\delta}{a}. \tag{16}$$

LEMMA 8. *Suppose* $a(t) = \frac{d}{(c+t)^b}$, $\varphi(t) = \int_0^t \frac{a(s)}{2} ds$ *where* $b \in (0, \frac{1}{2}]$, $d$ *and* $c$ *are positive constants. Then*

$$\frac{d}{2}\left(1 - \frac{2b}{c^\theta d}\right)\int_0^t \frac{e^{\varphi(s)}}{(s+c)^{2b}} ds < \frac{e^{\varphi(t)}}{(c+t)^b}, \qquad \forall t > 0, \quad \theta = 1 - b > 0. \tag{17}$$

*Proof.* We have

$$\varphi(t) = \int_0^t \frac{d}{2(c+s)^b} ds = \frac{d}{2(1-b)}\left((c+t)^{1-b} - c^{1-b}\right) = p(c+t)^\theta - C_3, \tag{18}$$

where $\theta := 1 - b$, $p := \frac{d}{2\theta}$, $C_3 := pc^\theta$. One has

$$\begin{aligned}
\frac{d}{dt}\frac{e^{p(c+t)^\theta}}{(c+t)^b} &= \frac{p\theta e^{p(c+t)^\theta}}{(c+t)^{b+1-\theta}} - \frac{be^{p(c+t)^\theta}}{(c+t)^{b+1}} \\
&= \frac{e^{p(c+t)^\theta}}{(c+t)^b}\left(\frac{d}{2(c+t)^b} - \frac{b}{c+t}\right) \\
&\geqslant \frac{e^{p(c+t)^\theta}}{(c+t)^b}\frac{d}{2(c+t)^b}\left(1 - \frac{2b}{c^\theta d}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d}{2}\left(1 - \frac{2b}{c^\theta d}\right)\int_0^t \frac{e^{p(c+s)^\theta}}{(s+c)^{2b}} ds &\leqslant \int_0^t \frac{d}{ds}\frac{e^{p(c+s)^\theta}}{(c+s)^b} ds \\
&\leqslant \frac{e^{p(c+t)^\theta}}{(c+t)^b} - \frac{e^{pc^\theta}}{c^b} \leqslant \frac{e^{p(c+t)^\theta}}{(c+t)^b}.
\end{aligned}$$

Multiplying this inequality by $e^{-C_3}$ and using (18), one obtains (17). Lemma 8 is proved. □

LEMMA 9. *Let* $a(t)=\frac{d}{(c+t)^b}$ *and* $\varphi(t):=\int_0^t \frac{a(s)}{2}ds$ *where* $d,c>0$, $b\in(0,\frac{1}{2}]$ *and* $c^{1-b}d\geqslant 6b$. *One has*

$$e^{-\varphi(t)}\int_0^t e^{\varphi(s)}|\dot a(s)|\|V_\delta(s)\|ds\leqslant \frac{1}{2}a(t)\|V_\delta(t)\|,\qquad t\geqslant 0. \tag{19}$$

*Proof.* From Lemma 8, one has

$$\frac{1}{2}\left(1-\frac{2b}{c^\theta d}\right)\int_0^t e^{\varphi(s)}\frac{d^2}{(s+c)^{2b}}ds<e^{\varphi(t)}\frac{d}{(c+t)^b},\qquad \forall c,b\geqslant 0,\quad \theta=1-b>0. \tag{20}$$

Since $c^{1-b}d\geqslant 6b$ or $\frac{6b}{c^\theta d}\leqslant 1$, one has

$$1-\frac{2b}{c^\theta d}\geqslant\frac{4b}{c^\theta d}\geqslant\frac{4b}{(c+s)^{1-b}d},\qquad s\geqslant 0.$$

This implies

$$\frac{a^2(s)}{2}\left(1-\frac{2b}{c^\theta d}\right)=\frac{d^2}{2(c+s)^{2b}}\left(1-\frac{2b}{c^\theta d^2}\right)\geqslant\frac{4db}{2(c+s)^{b+1}}=2|\dot a(s)|,\qquad s\geqslant 0. \tag{21}$$

Multiplying (20) by $\|V_\delta(t)\|$, using inequality (21) and the fact that $\|V_\delta(t)\|$ is increasing, one gets, for all $t\geqslant 0$, the following inequalities:

$$e^{\varphi(t)}a(t)\|V_\delta(t)\|>\int_0^t e^{\varphi(s)}\|V_\delta(t)\|\frac{a^2(s)}{2}\left(1-\frac{2b}{c^\theta d}\right)ds\geqslant 2\int_0^t e^{\varphi(s)}|\dot a(s)|\|V_\delta(s)\|ds.$$

This implies inequality (19). Lemma 9 is proved. □

Let us recall the following lemma, which is basic in our proofs.

LEMMA 10. ([7], p. 97) *Let* $\alpha(t)$, $\beta(t)$, $\gamma(t)$ *be continuous nonnegative functions on* $[t_0,\infty)$, $t_0\geqslant 0$ *is a fixed number. If there exists a function*

$$\mu\in C^1[t_0,\infty),\quad \mu>0,\quad \lim_{t\to\infty}\mu(t)=\infty,$$

*such that*

$$0\leqslant\alpha(t)\leqslant\frac{\mu}{2}\left[\gamma-\frac{\dot\mu(t)}{\mu(t)}\right],\qquad \dot\mu:=\frac{d\mu}{dt}, \tag{22}$$

$$\beta(t)\leqslant\frac{1}{2\mu}\left[\gamma-\frac{\dot\mu(t)}{\mu(t)}\right], \tag{23}$$

$$\mu(0)g(0)<1, \tag{24}$$

*and* $g(t)\geqslant 0$ *satisfies the inequality*

$$\dot g(t)\leqslant-\gamma(t)g(t)+\alpha(t)g^2(t)+\beta(t),\quad t\geqslant t_0, \tag{25}$$

*then* $g(t)$ *exists on* $[t_0,\infty)$ *and*

$$0\leqslant g(t)<\frac{1}{\mu(t)}\to 0,\quad \text{as}\quad t\to\infty. \tag{26}$$

*If inequalities* (22)–(24) *hold on an interval* $[t_0,T)$, *then* $g(t)$ *exists on this interval and inequality* (26) *holds on* $[t_0,T)$.

LEMMA 11. *Suppose $M_1$ and $c_1$ are positive constants and $0 \neq y \in H$. Then there exist a number $\lambda > 0$ and a function $a(t) \in C^1[0,\infty)$, $0 < a(t) \searrow 0$, such that*

$$|\dot{a}(t)| \leqslant \frac{a^2(t)}{2},$$

*and the following conditions hold*

$$\frac{M_1}{\|y\|} \leqslant \lambda, \tag{27}$$

$$0 \leqslant \frac{\lambda}{2a(t)}\left[a(t) - \frac{|\dot{a}(t)|}{a(t)}\right], \tag{28}$$

$$c_1\frac{|\dot{a}(t)|}{a(t)} \leqslant \frac{a(t)}{2\lambda}\left[a(t) - \frac{|\dot{a}(t)|}{a(t)}\right], \tag{29}$$

$$\frac{\lambda}{a(0)}g(0) < 1. \tag{30}$$

*Proof.* Take

$$a(t) = \frac{d}{(c+t)^b}, \quad 0 < b \leqslant \frac{1}{2}, \quad 2b \leqslant c^{1-b}d, \quad c \geqslant 1. \tag{31}$$

Note that $|\dot{a}| = -\dot{a}$. We have

$$\frac{|\dot{a}|}{a^2} = \frac{b}{d(c+t)^{1-b}} \leqslant \frac{b}{dc^{1-b}} \leqslant \frac{1}{2}.$$

Hence,

$$\frac{a(t)}{2} \leqslant a(t) - \frac{|\dot{a}(t)|}{a(t)}. \tag{32}$$

Thus, inequality (28) is satisfied. Take

$$\lambda \geqslant \frac{M_1}{\|y\|}, \tag{33}$$

then (27) is satisfied. For any given $g(0)$, choose $a(0)$ sufficiently large so that

$$\frac{\lambda}{a(0)}g(0) < 1.$$

Therefore, inequality (30) is satisfied.

Choose $\kappa \geqslant 1$ such that

$$\kappa > \max\left(\frac{4\lambda c_1 b}{d^2}, 1\right). \tag{34}$$

Define

$$\nu(t) := \kappa a(t), \qquad \lambda_\kappa := \kappa\lambda. \tag{35}$$

Note that (28) holds for $a(t) = \nu(t)$, $\lambda = \lambda_\kappa$ since (32) holds as well under this transformation, i.e.,

$$\frac{\nu(t)}{2} \leqslant \nu(t) - \frac{|\dot{\nu}(t)|}{\nu(t)}. \tag{36}$$

Using the inequalities (34) and $c \geqslant 1$ and the definition (35), one obtains

$$4\lambda_\kappa c_1 \frac{|\dot{\nu}(t)|}{\nu^3(t)} = 4\lambda c_1 \frac{b}{\kappa d^2 (c+t)^{1-2b}} \leqslant 4\lambda c_1 \frac{b}{\kappa d^2} \leqslant 1.$$

This implies

$$c_1 \frac{|\dot{\nu}|}{\nu(t)} \leqslant \frac{\nu^2(t)}{4\lambda_\kappa} \leqslant \frac{\nu(t)}{2\lambda_\kappa}\left[\nu - \frac{2|\dot{\nu}|}{\nu}\right].$$

Thus, one can replace the function $a(t)$ by $\nu(t) = \kappa a(t)$ and $\lambda$ by $\lambda_\kappa = \kappa\lambda$ in the inequalities (27)–(30). □

LEMMA 12. *Suppose $M_1$, $c_1$ and $\tilde{\alpha}$ are positive constants and $0 \neq y \in H$. Then there exist a number $\lambda > 0$ and a sequence $0 < (a_n)_{n=0}^\infty \searrow 0$ such that the following conditions hold*

$$\frac{a_n}{a_{n+1}} \leqslant 2, \tag{37}$$

$$\|f_\delta - F(0)\| \leqslant \frac{a_0^2}{\lambda}, \tag{38}$$

$$\frac{M_1}{\lambda} \leqslant \|y\|, \tag{39}$$

$$\frac{a_n}{\lambda} - \frac{\tilde{\alpha} a_n^2}{\lambda} + \frac{a_n - a_{n+1}}{a_{n+1}} c_1 \leqslant \frac{a_{n+1}}{\lambda}. \tag{40}$$

*Proof.* Let us show that if $a_0 > 0$ is sufficiently large, then the following sequence

$$a_n = \frac{a_0}{(1+n)^b}, \qquad b = \frac{1}{2}, \tag{41}$$

satisfies conditions (38)–(40) if

$$\lambda \geqslant \frac{M_1}{\|y\|}. \tag{42}$$

Condition (37) is satisfied by the sequence (41). Inequality (39) is satisfied since (42) holds. Choose $a(0)$ so that

$$a_0 \geqslant \sqrt{\|f_\delta - F(0)\|\lambda}, \tag{43}$$

then (38) is satisfied.

Assume that $(a_n)_{n=0}^\infty$ and $\lambda$ satisfy (37), (38) and (39). Choose $\kappa \geqslant 1$ such that

$$\kappa \geqslant \max\left(\frac{1}{\tilde{\alpha} a_0 \sqrt{2}}, \frac{\lambda c_1}{\tilde{\alpha} a_0^2}\right). \tag{44}$$

It follows from (44) that

$$\frac{1}{\kappa a_0 \sqrt{2}} \leqslant \tilde{\alpha}, \qquad \frac{\lambda c_1}{\kappa a_0^2} \leqslant \tilde{\alpha}. \tag{45}$$

Define

$$(b_n)_{n=0}^{\infty} := (\kappa a_n)_{n=0}^{\infty}, \qquad \lambda_\kappa := \kappa\lambda. \tag{46}$$

For all $n \geqslant 0$ one has

$$\frac{a_n - a_{n+1}}{a_n^2} = \frac{a_n^2 - a_{n+1}^2}{a_n^2(a_n + a_{n+1})} \leqslant \frac{a_n^2 - a_{n+1}^2}{2a_{n+1}a_n^2} = \frac{\frac{a_0^2}{n+1} - \frac{a_0^2}{n+2}}{2\frac{a_0}{\sqrt{n+2}}\frac{a_0^2}{n+1}} = \frac{1}{a_0 2\sqrt{n+2}} \leqslant \frac{1}{a_0 2\sqrt{2}}. \tag{47}$$

Since $a_n$ is decreasing, one has

$$\begin{aligned} \frac{a_n - a_{n+1}}{a_n^2 a_{n+1}} &= \frac{a_n^2 - a_{n+1}^2}{a_n^2 a_{n+1}(a_n + a_{n+1})} \\ &\leqslant \frac{a_n^2 - a_{n+1}^2}{2a_n^2 a_{n+1}^2} = \frac{\frac{a_0^2}{n+1} - \frac{a_0^2}{n+2}}{2\frac{a_0^2}{n+2}\frac{a_0^2}{n+1}} \leqslant \frac{1}{2a_0^2}, \qquad \forall n \geqslant 0. \end{aligned} \tag{48}$$

Using inequalities (47) and (45), one gets

$$\frac{2(a_n - a_{n+1})}{\kappa a_n^2} \leqslant \frac{1}{\kappa a_0\sqrt{2}} \leqslant \tilde{\alpha}. \tag{49}$$

Similarly, using inequalities (48) and (45), one gets

$$\frac{2\lambda(a_n - a_{n+1})c_1}{\kappa a_n^2 a_{n+1}} \leqslant \frac{\lambda c_1}{\kappa a_0^2} \leqslant \tilde{\alpha}. \tag{50}$$

Inequalities (49) and (50) imply

$$\begin{aligned} \frac{b_n - b_{n+1}}{\lambda_\kappa} + \frac{b_n - b_{n+1}}{b_{n+1}}c_1 &= \frac{a_n - a_{n+1}}{\lambda} + \frac{a_n - a_{n+1}}{a_{n+1}}c_1 \\ &= \frac{\kappa a_n^2}{2\lambda}\frac{2(a_n - a_{n+1})}{\kappa a_n^2} + \frac{\kappa a_n^2}{2\lambda}\frac{2\lambda(a_n - a_{n+1})c_1}{\kappa a_n^2 a_{n+1}} \\ &\leqslant \frac{\kappa a_n^2}{2\lambda}\tilde{\alpha} + \frac{\kappa a_n^2}{2\lambda}\tilde{\alpha} = \frac{\kappa a_n^2\tilde{\alpha}}{\lambda} = \frac{\tilde{\alpha} b_n^2}{\lambda_\kappa}. \end{aligned}$$

Thus, inequality (40) holds for $a_n$ replaced by $b_n = \kappa a_n$ and $\lambda$ replaced by $\lambda_\kappa = \kappa\lambda$, where $\kappa$ satisfies (44). Inequalities (37)–(39) hold as well under this transformation. Thus, the choices $a_n = b_n$ and $\lambda := \kappa\frac{M_1}{\|y\|}$, where $\kappa$ satisfies (44), satisfy all the conditions of Lemma 12. □

REMARK 13. The constant $c_1$, used in Lemmas 11 and 12, will be used in Theorems 17 and 19. This constant is defined in equation (62). The constant $\tilde{\alpha}$, used in Lemma 12, is the one from Theorem 19. This constant is defined in (89).

REMARK 14. Using similar arguments one can show that the sequence $a_n = \frac{d}{(c+n)^b}$, where $c \geqslant 1$, $0 < b \leqslant \frac{1}{2}$, satisfy all conditions of Lemma 4 provided that $d$ is sufficiently large and $\lambda$ is chosen so that inequality (42) holds.

REMARK 15. In the proof of Lemmas 11 and 12 the numbers $a_0$ and $\lambda$ can be chosen so that $\frac{a_0}{\lambda}$ is uniformly bounded as $\delta \to 0$ regardless of the rate of growth of the constant $M_1 = M_1(R)$ from formula (3) when $R \to \infty$, i.e., regardless of the strength of the nonlinearity $F(u)$.

To satisfy (42) one can choose $\lambda = M_1 \frac{1}{\|y\|}$. To satisfy (43) one can choose

$$a_0 = \sqrt{\lambda(\|f - F(0)\| + \|f\|)} \geqslant \sqrt{\lambda\|f_\delta - F(0)\|},$$

where we have assumed without loss of generality that $0 < \|f_\delta - f\| < \|f\|$. With this choice of $a_0$ and $\lambda$, the ratio $\frac{a_0}{\lambda}$ is bounded uniformly with respect to $\delta \in (0,1)$ and does not depend on $R$. The dependence of $a_0$ on $\delta$ is seen from (43) since $f_\delta$ depends on $\delta$. In practice one has $\|f_\delta - f\| < \|f\|$. Consequently,

$$\sqrt{\|f_\delta - F(0)\|\lambda} \leqslant \sqrt{(\|f - F(0)\| + \|f\|)\lambda}.$$

Thus, we can practically choose $a(0)$ independent of $\delta$ from the following inequality

$$a_0 \geqslant \sqrt{\lambda(\|f - F(0)\| + \|f\|)}.$$

Indeed, with the above choice one has $\frac{a_0}{\lambda} \leqslant c(1 + \sqrt{\lambda^{-1}}) \leqslant c$, where $c > 0$ is a constant independent of $\delta$, and one can assume that $\lambda \geqslant 1$ without loss of generality.

This Remark is used in the proof of the main result in Section 3. Specifically, it is used to prove that an iterative process (88) generates a sequence which stays in the ball $B(u_0, R)$ for all $n \leqslant n_0 + 1$, where the number $n_0$ is defined by formula (99) (see below), and $R > 0$ is sufficiently large. An upper bound on $R$ is given in the proof of Theorem 19, below formula (112).

REMARK 16. One can choose $u_0 \in H$ such that

$$g_0 := \|u_0 - V_0\| \leqslant \frac{\|F(0) - f_\delta\|}{a_0}. \tag{51}$$

Indeed, if, for example, $u_0 = 0$, then by Remark 6 one gets

$$g_0 = \|V_0\| = \frac{a_0\|V_0\|}{a_0} \leqslant \frac{\|F(0) - f_\delta\|}{a_0}.$$

If (38) and (51) hold then $g_0 \leqslant \frac{a_0}{\lambda}$.

## 3. Main results

### 3.1. Dynamical systems method

Assume:

$$0 < a(t) \searrow 0, \quad \lim_{t\to\infty} \frac{\dot{a}(t)}{a(t)} = 0, \quad \frac{|\dot{a}(t)|}{a^2(t)} \leqslant \frac{1}{2}. \tag{52}$$

Let $u_\delta(t)$ solve the following Cauchy problem:

$$\dot{u}_\delta = -[F(u_\delta) + a(t)u_\delta - f_\delta], \quad u_\delta(0) = u_0. \tag{53}$$

THEOREM 17. *Assume that* $F : H \to H$ *is a monotone operator, condition (3) holds, and* $u_0$ *is an element of* $H$*, satisfying inequality* (83) *(see below). Let* $a(t)$ *satisfy conditions of Lemma 11. For example, one can choose* $a(t) = \frac{d}{(c+t)^b}$*, where* $b \in (0, \frac{1}{2}]$*,* $c \geqslant 1$ *and* $d > 0$ *are constants, and* $d$ *is sufficiently large. Assume that equation* $F(u) = f$ *has a solution* $y \in B(u_0, R)$*, possibly nonunique, and* $y$ *is the minimal-norm solution to this equation. Let* $f$ *be unknown but* $f_\delta$ *be given,* $\|f_\delta - f\| \leqslant \delta$*. Then the solution* $u_\delta(t)$ *to problem* (53) *exists on an interval* $[0, T_\delta]$*,* $\lim_{\delta\to 0} T_\delta = \infty$*, and there exists* $t_\delta$*,* $t_\delta \in (0, T_\delta)$*, not necessarily unique, such that*

$$\|F(u_\delta(t_\delta)) - f_\delta\| = C_1\delta^\zeta, \quad \lim_{\delta\to 0} t_\delta = \infty, \tag{54}$$

*where* $C_1 > 1$ *and* $0 < \zeta \leqslant 1$ *are constants. If* $\zeta \in (0,1)$ *and* $t_\delta$ *satisfies* (54)*, then*

$$\lim_{\delta\to 0} \|u_\delta(t_\delta) - y\| = 0. \tag{55}$$

REMARK 18. One can easily choose $u_0$ satisfying inequality (83). Note that inequality (83) is a sufficient condition for (86) to hold. In our proof inequality (86) is used at $t = t_\delta$. The stopping time $t_\delta$ is often sufficiently large for the quantity $e^{-\varphi(t_\delta)}h_0$ to be small. In this case inequality (86) with $t = t_\delta$ is satisfied for a wide range of $u_0$.

*Proof.* [Proof of Theorem 17] Denote

$$C := \frac{C_1 + 1}{2}. \tag{56}$$

Let

$$w := u_\delta - V_\delta, \quad g(t) := \|w\|.$$

One has

$$\dot{w} = -\dot{V}_\delta - \big[F(u_\delta) - F(V_\delta) + a(t)w\big]. \tag{57}$$

Multiplying (57) by $w$ and using (2) one gets

$$g\dot{g} \leqslant -ag^2 + \|\dot{V}_\delta\|g. \tag{58}$$

Let $t_0 > 0$ be such that

$$\frac{\delta}{a(t_0)} = \frac{1}{C-1}\|y\|, \qquad C > 1. \tag{59}$$

This $t_0$ exists and is unique since $a(t) > 0$ monotonically decays to 0 as $t \to \infty$. By Lemma 4, there exists $t_1$ such that

$$\|F(V_\delta(t_1)) - f_\delta\| = C\delta, \quad F(V_\delta(t_1)) + a(t_1)V_\delta(t_1) - f_\delta = 0. \tag{60}$$

We claim that $t_1 \in [0, t_0]$.

Indeed, from (7) and (16) one gets

$$C\delta = a(t_1)\|V_\delta(t_1)\| \leqslant a(t_1)\left(\|y\| + \frac{\delta}{a(t_1)}\right) = a(t_1)\|y\| + \delta, \quad C > 1,$$

so

$$\delta \leqslant \frac{a(t_1)\|y\|}{C-1}.$$

Thus,

$$\frac{\delta}{a(t_1)} \leqslant \frac{\|y\|}{C-1} = \frac{\delta}{a(t_0)}.$$

Since $a(t) \searrow 0$, the above inequality implies $t_1 \leqslant t_0$. Differentiating both sides of (7) with respect to $t$, one obtains

$$A_{a(t)}\dot{V}_\delta = -\dot{a}V_\delta, \quad A := F'(V_\delta), \quad A_a := A + aI.$$

This implies

$$\|\dot{V}_\delta\| \leqslant |\dot{a}|\|A^{-1}_{a(t)}V_\delta\| \leqslant \frac{|\dot{a}|}{a}\|V_\delta\| \leqslant \frac{|\dot{a}|}{a}\left(\|y\| + \frac{\delta}{a}\right) \leqslant \frac{|\dot{a}|}{a}\|y\|\left(1+\frac{1}{C-1}\right), \quad \forall t \leqslant t_0. \tag{61}$$

Since $g \geqslant 0$, inequalities (58) and (61) imply

$$\dot{g} \leqslant -a(t)g(t) + \frac{|\dot{a}(t)|}{a(t)}c_1, \quad c_1 = \|y\|\left(1 + \frac{1}{C-1}\right). \tag{62}$$

Inequality (62) is of the type (25) with

$$\gamma(t) = a(t), \quad \alpha(t) = 0, \quad \beta(t) = c_1\frac{|\dot{a}(t)|}{a(t)}.$$

Let us check assumptions (22)–(24). Take

$$\mu(t) = \frac{\lambda}{a(t)}, \quad \lambda = \text{const}.$$

By Lemma 11 there exist $\lambda$ and $a(t)$ such that conditions (22)–(24) hold. Thus, Lemma 10 yields

$$g(t) < \frac{a(t)}{\lambda}, \quad \forall t \leqslant t_0. \tag{63}$$

Therefore,

$$\begin{aligned}\|F(u_\delta(t)) - f_\delta\| \leqslant& \|F(u_\delta(t)) - F(V_\delta(t))\| + \|F(V_\delta(t)) - f_\delta\| \\ \leqslant& M_1 g(t) + \|F(V_\delta(t)) - f_\delta\| \\ \leqslant& \frac{M_1 a(t)}{\lambda} + \|F(V_\delta(t)) - f_\delta\|, \qquad \forall t \leqslant t_0.\end{aligned} \tag{64}$$

It follows from Lemma 3 that $\|F(V_\delta(t)) - f_\delta\|$ is decreasing. Since $t_1 \leqslant t_0$, one gets

$$\|F(V_\delta(t_0)) - f_\delta\| \leqslant \|F(V_\delta(t_1)) - f_\delta\| = C\delta. \tag{65}$$

This, inequality (64), the inequality $\frac{M_1}{\lambda} \leqslant \|y\|$ (see (33)), the relation (59), and the definition $C_1 = 2C - 1$ (see (56)) imply

$$\begin{aligned}\|F(u_\delta(t_0)) - f_\delta\| \leqslant& \frac{M_1 a(t_0)}{\lambda} + C\delta \\ \leqslant& \frac{M_1\delta(C-1)}{\lambda\|y\|} + C\delta \leqslant (2C-1)\delta = C_1\delta.\end{aligned} \tag{66}$$

Thus, if

$$\|F(u_\delta(0)) - f_\delta\| \geqslant C_1\delta^\zeta, \quad 0 < \zeta \leqslant 1,$$

then there exists $t_\delta \in (0, t_0)$ such that

$$\|F(u_\delta(t_\delta)) - f_\delta\| = C_1\delta^\zeta \tag{67}$$

for any given $\zeta \in (0, 1]$, and any fixed $C_1 > 1$.

*Let us prove* (55)*. If this is done, then Theorem 17 is proved.*

First, we prove that $\lim_{\delta\to 0} \frac{\delta}{a(t_\delta)} = 0$.

From (64) with $t = t_\delta$, and from (16), one gets

$$\begin{aligned} C_1\delta^\zeta &\leqslant M_1\frac{a(t_\delta)}{\lambda} + a(t_\delta)\|V_\delta(t_\delta)\| \\ &\leqslant M_1\frac{a(t_\delta)}{\lambda} + \|y\|a(t_\delta) + \delta. \end{aligned}$$

Thus, for sufficiently small $\delta$, one gets

$$\tilde{C}\delta^\zeta \leqslant a(t_\delta)\left(\frac{M_1}{\lambda} + \|y\|\right), \quad \tilde{C} > 0,$$

where $\tilde{C} < C_1$ is a constant. Therefore,

$$\lim_{\delta\to 0} \frac{\delta}{a(t_\delta)} \leqslant \lim_{\delta\to 0} \frac{\delta^{1-\zeta}}{\tilde{C}}\left(\frac{M_1}{\lambda} + \|y\|\right) = 0, \quad 0 < \zeta < 1. \tag{68}$$

*Secondly, we prove that*

$$\lim_{\delta\to 0} t_\delta = \infty. \tag{69}$$

Using (53), one obtains:

$$\frac{d}{dt}\big(F(u_\delta) + au_\delta - f_\delta\big) = A_a\dot{u}_\delta + \dot{a}u_\delta = -A_a\big(F(u_\delta) + au_\delta - f_\delta\big) + \dot{a}u_\delta,$$

where $A_a := F'(u_\delta) + a$. This and (7) imply:

$$\frac{d}{dt}\big[F(u_\delta) - F(V_\delta) + a(u_\delta - V_\delta)\big] = -A_a\big[F(u_\delta) - F(V_\delta) + a(u_\delta - V_\delta)\big] + \dot{a}u_\delta. \tag{70}$$

Denote

$$v := F(u_\delta) - F(V_\delta) + a(u_\delta - V_\delta), \quad h = \|v\|.$$

Multiplying (70) by $v$ and using monotonicity of $F$, one obtains

$$\begin{aligned} h\dot{h} &= -\langle A_a v, v\rangle + \langle v, \dot{a}(u_\delta - V_\delta)\rangle + \dot{a}\langle v, V_\delta\rangle \\ &\leqslant -h^2 a + h|\dot{a}|\|u_\delta - V_\delta\| + |\dot{a}|h\|V_\delta\|, \qquad h \geqslant 0. \end{aligned} \tag{71}$$

Again, we have used the inequality $\langle F'(u_\delta)v, v\rangle \geqslant 0$ which follows from the monotonicity of $F$. Thus,

$$\dot{h} \leqslant -ha + |\dot{a}|\|u_\delta - V_\delta\| + |\dot{a}|\|V_\delta\|. \tag{72}$$

Since $\langle F(u_\delta)-F(V_\delta), u_\delta-V_\delta\rangle \geqslant 0$, one obtains two inequalities

$$a\|u_\delta-V_\delta\|^2 \leqslant \langle v, u_\delta-V_\delta\rangle \leqslant \|u_\delta-V_\delta\|h, \tag{73}$$

and

$$\|F(u_\delta)-F(V_\delta)\|^2 \leqslant \langle v, F(u_\delta)-F(V_\delta)\rangle \leqslant h\|F(u_\delta)-F(V_\delta)\|. \tag{74}$$

Inequalities (73) and (74) imply:

$$a\|u_\delta-V_\delta\| \leqslant h, \quad \|F(u_\delta)-F(V_\delta)\| \leqslant h. \tag{75}$$

Inequalities (72) and (75) imply

$$\dot h \leqslant -h\Big(a-\frac{|\dot a|}{a}\Big)+|\dot a|\|V_\delta\|. \tag{76}$$

Since $a-\frac{|\dot a|}{a} \geqslant \frac{a}{2}$ by the last inequality in (52), it follows from inequality (76) that

$$\dot h \leqslant -\frac{a}{2}h+|\dot a|\|V_\delta\|. \tag{77}$$

Inequality (77) implies:

$$h(t) \leqslant h(0)e^{-\int_0^t \frac{a(s)}{2}ds}+e^{-\int_0^t \frac{a(s)}{2}ds}\int_0^t e^{\int_0^s \frac{a(\xi)}{2}d\xi}|\dot a(s)|\|V_\delta(s)\|ds. \tag{78}$$

Denote

$$\varphi(t) := \int_0^t \frac{a(s)}{2}ds.$$

From (78) and (75), one gets

$$\|F(u_\delta(t))-F(V_\delta(t))\| \leqslant h(0)e^{-\varphi(t)}+e^{-\varphi(t)}\int_0^t e^{\varphi(s)}|\dot a(s)|\|V_\delta(s)\|ds. \tag{79}$$

Therefore,

$$\begin{aligned}\|F(u_\delta(t))-f_\delta\| &\geqslant \|F(V_\delta(t))-f_\delta\|-\|F(V_\delta(t))-F(u_\delta(t))\|\\ &\geqslant a(t)\|V_\delta(t)\|-h(0)e^{-\varphi(t)}-e^{-\varphi(t)}\int_0^t e^{\varphi(s)}|\dot a|\|V_\delta\|ds.\end{aligned} \tag{80}$$

From Lemma 9 it follows that there exists an $a(t)$ such that

$$\frac{1}{2}a(t)\|V_\delta(t)\| \geqslant e^{-\varphi(t)}\int_0^t e^{\varphi(s)}|\dot a|\|V_\delta(s)\|ds. \tag{81}$$

For example, one can choose

$$a(t)=\frac{d}{(c+t)^b}, \quad b\in(0,\frac{1}{2}], \quad dc^{1-b}\geqslant 6b, \tag{82}$$

where $d,c>0$. Moreover, one can always choose $u_0$ such that

$$h(0)=\|F(u_0)+a(0)u_0-f_\delta\| \leqslant \frac{1}{4}a(0)\|V_\delta(0)\|, \tag{83}$$

because the equation

$$F(u_0) + a(0)u_0 - f_\delta = 0$$

is solvable.

If (83) holds, then

$$h(0)e^{-\varphi(t)} \leqslant \frac{1}{4}a(0)\|V_\delta(0)\|e^{-\varphi(t)}, \qquad t \geqslant 0. \tag{84}$$

If (82) holds, $c \geqslant 1$ and $2b \leqslant d$, then it follows that

$$e^{-\varphi(t)}a(0) \leqslant a(t). \tag{85}$$

Indeed, inequality $a(0) \leqslant a(t)e^{\varphi(t)}$ is obviously true for $t = 0$, and $\left(a(t)e^{\varphi(t)}\right)'_t \geqslant 0$, provided that $c \geqslant 1$ and $2b \leqslant d$.

Inequalities (84) and (46) imply

$$e^{-\varphi(t)}h(0) \leqslant \frac{1}{4}a(t)\|V_\delta(0)\| \leqslant \frac{1}{4}a(t)\|V_\delta(t)\|, \quad t \geqslant 0. \tag{86}$$

where we have used the inequality $\|V_\delta(t)\| \leqslant \|V_\delta(t')\|$ for $t \leqslant t'$, established in Lemma 3. From (67) and (80)–(86), one gets

$$C\delta^\zeta = \|F(u_\delta(t_\delta)) - f_\delta\| \geqslant \frac{1}{4}a(t_\delta)\|V_\delta(t_\delta)\|.$$

Thus,

$$\lim_{\delta\to 0} a(t_\delta)\|V_\delta(t_\delta)\| \leqslant \lim_{\delta\to 0} 4C\delta^\zeta = 0.$$

Since $\|V_\delta(t)\|$ is increasing, this implies $\lim_{\delta\to 0} a(t_\delta) = 0$. Since $0 < a(t) \searrow 0$, it follows that (69) holds.

From the triangle inequality, inequalities (63) and (14), one obtains

$$\begin{aligned}\|u_\delta(t_\delta) - y\| &\leqslant \|u_\delta(t_\delta) - V_\delta\| + \|V(t_\delta) - V_\delta(t_\delta)\| + \|V(t_\delta) - y\| \\ &\leqslant \frac{a(t_\delta)}{\lambda} + \frac{\delta}{a(t_\delta)} + \|V(t_\delta) - y\|.\end{aligned} \tag{87}$$

From (68), (69), inequality (87) and Lemma 1, one obtains (55). Theorem 17 is proved. □

### 3.2. An iterative scheme

Let $V_{n,\delta}$ solve the equation:

$$F(V_{n,\delta}) + a_n V_{n,\delta} - f_\delta = 0.$$

Denote $V_n := V_{n,\delta}$.

Consider the following iterative scheme:

$$u_{n+1} = u_n - \alpha_n[F(u_n) + a_n u_n - f_\delta], \quad u_0 = u_0, \tag{88}$$

where $u_0$ is chosen so that inequality (51) holds, and $\{\alpha_n\}_{n=1}^{\infty}$ is a positive sequence such that

$$0 < \tilde{\alpha} \leqslant \alpha_n \leqslant \frac{2}{a_n + (M_1 + a_n)}, \qquad M_1 = \sup_{u \in B(u_0, R)} \|F'(u)\|. \tag{89}$$

It follows from this condition that

$$\|1 - \alpha_n(J_n + a_n)\| = \sup_{a_n \leqslant \lambda \leqslant M_1 + a_n} |1 - \alpha_n \lambda| \leqslant 1 - \alpha_n a_n. \tag{90}$$

Here, $J_n$ is an operator in $H$ such that $J_n \geqslant 0$ and $\|J_n\| \leqslant M_1, \forall u \in B(u_0, R)$. A specific choice of $J_n$ is made in formula (96) below.

Let $a_n$ and $\lambda$ satisfy conditions (37)–(40). Assume that equation $F(u) = f$ has a solution $y \in B(u_0, R)$, possibly nonunique, and $y$ is the minimal-norm solution to this equation. Let $f$ be unknown but $f_\delta$ be given, and $\|f_\delta - f\| \leqslant \delta$. We prove the following result:

THEOREM 19. *Assume* $a_n = \frac{d}{(c+n)^b}$ *where* $c \geqslant 1$, $0 < b \leqslant \frac{1}{2}$, *and* $d$ *is sufficiently large so that conditions* (37)–(40) *hold. Let* $u_n$ *be defined by* (88). *Assume that* $u_0$ *is chosen so that* (51) *holds. Then there exists a unique* $n_\delta$ *such that*

$$\|F(u_{n_\delta}) - f_\delta\| \leqslant C_1 \delta^\zeta, \quad C_1 \delta^\zeta < \|F(u_n) - f_\delta\|, \quad \forall n < n_\delta, \tag{91}$$

*where* $C_1 > 1$, $0 < \zeta \leqslant 1$.

*Let* $0 < (\delta_m)_{m=1}^{\infty}$ *be a sequence such that* $\delta_m \to 0$. *If the sequence* $\{n_m := n_{\delta_m}\}_{m=1}^{\infty}$ *is bounded, and* $\{n_{m_j}\}_{j=1}^{\infty}$ *is a convergent subsequence, then*

$$\lim_{j \to \infty} u_{n_{m_j}} = \tilde{u}, \tag{92}$$

*where* $\tilde{u}$ *is a solution to the equation* $F(u) = f$. *If*

$$\lim_{m \to \infty} n_m = \infty, \tag{93}$$

*where* $\zeta \in (0, 1)$, *then*

$$\lim_{m \to \infty} \|u_{n_m} - y\| = 0. \tag{94}$$

*Proof.* Denote

$$C := \frac{C_1 + 1}{2}. \tag{95}$$

Let

$$z_n := u_n - V_n, \quad g_n := \|z_n\|.$$

One has

$$F(u_n) - F(V_n) = J_n z_n, \qquad J_n = \int_0^1 F'(u_0 + \xi z_n) d\xi. \tag{96}$$

Since $F'(u) \geqslant 0, \forall u \in H$ and $\|F'(u)\| \leqslant M_1, \forall u \in B(u_0, R)$, it follows that $J_n \geqslant 0$ and $\|J_n\| \leqslant M_1$. From (88) and (96) one obtains

$$\begin{aligned} z_{n+1} &= z_n - \alpha_n [F(u_n) - F(V_n) + a_n z_n] - (V_{n+1} - V_n) \\ &= (1 - \alpha_n(J_n + a_n)) z_n - (V_{n+1} - V_n). \end{aligned} \tag{97}$$

From (97) and (90), one gets

$$\begin{aligned} g_{n+1} &\leqslant g_n\|1-\alpha_n(J_n+a_n)\| + \|V_{n+1}-V_n\| \\ &\leqslant g_n(1-\alpha_n a_n) + \|V_{n+1}-V_n\|. \end{aligned} \tag{98}$$

Since $0 < a_n \searrow 0$, for any fixed $\delta > 0$ there exists $n_0$ such that

$$\frac{\delta}{a_{n_0+1}} > \frac{1}{C-1}\|y\| \geqslant \frac{\delta}{a_{n_0}}, \qquad C > 1. \tag{99}$$

By (37), one has $\frac{a_n}{a_{n+1}} \leqslant 2, \forall n \geqslant 0$. This and (99) imply

$$\frac{2}{C-1}\|y\| \geqslant \frac{2\delta}{a_{n_0}} > \frac{\delta}{a_{n_0+1}} > \frac{1}{C-1}\|y\| \geqslant \frac{\delta}{a_{n_0}}, \qquad C > 1. \tag{100}$$

Thus,

$$\frac{2}{C-1}\|y\| > \frac{\delta}{a_n}, \quad \forall n \leqslant n_0+1. \tag{101}$$

The number $n_0$, satisfying (101), exists and is unique since $a_n > 0$ monotonically decays to 0 as $n \to \infty$. By Remark 5, there exists a number $n_1$ such that

$$\|F(V_{n_1+1}) - f_\delta\| \leqslant C\delta < \|F(V_{n_1}) - f_\delta\|, \tag{102}$$

where $V_n$ solves the equation $F(V_n) + a_n V_n - f_\delta = 0$.

*We claim that* $n_1 \in [0, n_0]$.

Indeed, one has $\|F(V_{n_1}) - f_\delta\| = a_{n_1}\|V_{n_1}\|$, and $\|V_{n_1}\| \leqslant \|y\| + \frac{\delta}{a_{n_1}}$ (cf. (16)), so

$$C\delta < a_{n_1}\|V_{n_1}\| \leqslant a_{n_1}\left(\|y\| + \frac{\delta}{a_{n_1}}\right) = a_{n_1}\|y\| + \delta, \quad C > 1. \tag{103}$$

Therefore,

$$\delta < \frac{a_{n_1}\|y\|}{C-1}. \tag{104}$$

Thus, by (100),

$$\frac{\delta}{a_{n_1}} < \frac{\|y\|}{C-1} < \frac{\delta}{a_{n_0+1}}. \tag{105}$$

Here the last inequality is a consequence of (100). Since $a_n$ decreases monotonically, inequality (105) implies $n_1 \leqslant n_0$. One has

$$\begin{aligned} a_{n+1}\|V_n - V_{n+1}\|^2 &= \langle (a_{n+1}-a_n)V_n - F(V_n) + F(V_{n+1}), V_n - V_{n+1}\rangle \\ &\leqslant \langle (a_{n+1}-a_n)V_n, V_n - V_{n+1}\rangle \\ &\leqslant (a_n - a_{n+1})\|V_n\|\|V_n - V_{n+1}\|. \end{aligned} \tag{106}$$

By (16), $\|V_n\| \leqslant \|y\| + \frac{\delta}{a_n}$, and, by (101), $\frac{\delta}{a_n} \leqslant \frac{2\|y\|}{C-1}$ for all $n \leqslant n_0+1$. Therefore,

$$\|V_n\| \leqslant \|y\|\left(1 + \frac{2}{C-1}\right), \qquad \forall n \leqslant n_0+1, \tag{107}$$

and, by (106),

$$\|V_n - V_{n+1}\| \leqslant \frac{a_n - a_{n+1}}{a_{n+1}} \|V_n\| \leqslant \frac{a_n - a_{n+1}}{a_{n+1}} \|y\| \left(1 + \frac{2}{C-1}\right), \quad \forall n \leqslant n_0 + 1. \quad (108)$$

Inequalities (98) and (108) imply

$$g_{n+1} \leqslant (1 - \alpha_n a_n) g_n + \frac{a_n - a_{n+1}}{a_{n+1}} c_1, \qquad \forall n \leqslant n_0 + 1, \tag{109}$$

where the constant $c_1$ is defined in (62).

By Lemma 4 and Remark 14, the sequence $(a_n)_{n=1}^{\infty}$, satisfies conditions (37)–(40), provided that $a_0$ is sufficiently large and $\lambda > 0$ is chosen so that (42) holds. Let us show by induction that

$$g_n < \frac{a_n}{\lambda}, \qquad 0 \leqslant n \leqslant n_0 + 1. \tag{110}$$

Inequality (110) holds for $n = 0$ by Remark 16. Suppose (110) holds for some $n \geqslant 0$. From (109), (110) and (40), one gets

$$\begin{aligned} g_{n+1} &\leqslant (1 - \alpha_n a_n) \frac{a_n}{\lambda} + \frac{a_n - a_{n+1}}{a_{n+1}} c_1 \\ &= -\frac{\alpha_n a_n^2}{\lambda} + \frac{a_n}{\lambda} + \frac{a_n - a_{n+1}}{a_{n+1}} c_1 \\ &\leqslant \frac{a_{n+1}}{\lambda}. \end{aligned} \tag{111}$$

Thus, by induction, inequality (110) holds for all $n$ in the region $0 \leqslant n \leqslant n_0 + 1$.

From (16) one has $\|V_n\| \leqslant \|y\| + \frac{\delta}{a_n}$. This and the triangle inequality imply

$$\|u_0 - u_n\| \leqslant \|u_0\| + \|z_n\| + \|V_n\| \leqslant \|u_0\| + \|z_n\| + \|y\| + \frac{\delta}{a_n}. \tag{112}$$

Inequalities (107), (110), and (112) guarantee that the sequence $u_n$, generated by the iterative process (88), remains in the ball $B(u_0, R)$ for all $n \leqslant n_0 + 1$, where $R \leqslant \frac{a_0}{\lambda} + \|u_0\| + \|y\| + \frac{\delta}{a_n}$. This inequality and the estimate (101) imply that the sequence $u_n$, $n \leqslant n_0 + 1$, stays in the ball $B(u_0, R)$, where

$$R \leqslant \frac{a_0}{\lambda} + \|u_0\| + \|y\| + \|y\| \frac{C+1}{C-1}. \tag{113}$$

By Remark 15, one can choose $a_0$ and $\lambda$ so that $\frac{a_0}{\lambda}$ is uniformly bounded as $\delta \to 0$ even if $M_1(R) \to \infty$ as $R \to \infty$ at an arbitrary fast rate. Thus, the sequence $u_n$ stays in the ball $B(u_0, R)$ for $n \leqslant n_0 + 1$ when $\delta \to 0$. An upper bound on $R$ is given above. It does not depend on $\delta$ as $\delta \to 0$.

One has:

$$\begin{aligned} \|F(u_n) - f_\delta\| &\leqslant \|F(u_n) - F(V_n)\| + \|F(V_n) - f_\delta\| \\ &\leqslant M_1 g_n + \|F(V_n) - f_\delta\| \\ &\leqslant \frac{M_1 a_n}{\lambda} + \|F(V_n) - f_\delta\|, \qquad \forall n \leqslant n_0 + 1, \end{aligned} \tag{114}$$

where (110) was used and $M_1$ is the constant from (3). Since $\|F(V_n) - f_\delta\|$ is decreasing, by Lemma 3, and $n_1 \leqslant n_0$, one gets

$$\|F(V_{n_0+1}) - f_\delta\| \leqslant \|F(V_{n_1+1}) - f_\delta\| \leqslant C\delta. \tag{115}$$

From (39), (114), (115), the relation (99), and the definition $C_1 = 2C - 1$ (see (95)), one concludes that

$$\begin{aligned}\|F(u_{n_0+1}) - f_\delta\| \leqslant & \frac{M_1 a_{n_0+1}}{\lambda} + C\delta \\ \leqslant & \frac{M_1\delta(C-1)}{\lambda\|y\|} + C\delta \leqslant (2C-1)\delta = C_1\delta.\end{aligned} \tag{116}$$

*Thus, if*

$$\|F(u_0) - f_\delta\| > C_1\delta^\zeta, \quad 0 < \zeta \leqslant 1,$$

*then one concludes from* (116) *that there exists* $n_\delta$, $0 < n_\delta \leqslant n_0 + 1$, *such that*

$$\|F(u_{n_\delta}) - f_\delta\| \leqslant C_1\delta^\zeta < \|F(u_n) - f_\delta\|, \quad 0 \leqslant n < n_\delta, \tag{117}$$

*for any given* $\zeta \in (0,1]$, *and any fixed* $C_1 > 1$.

*Let us prove* (92).

If $n > 0$ is fixed, then $u_{\delta,n}$ is a continuous function of $f_\delta$. Denote

$$\tilde{u} := \tilde{u}_N = \lim_{\delta\to 0} u_{\delta,n_{m_j}}, \tag{118}$$

where

$$\lim_{j\to\infty} n_{m_j} = N.$$

From (118) and the continuity of $F$, one obtains:

$$\|F(\tilde{u}) - f_\delta\| = \lim_{j\to\infty} \|F(u_{n_{m_j}}) - f_\delta\| \leqslant \lim_{\delta\to 0} C_1\delta^\zeta = 0.$$

Thus, $\tilde{u}$ is a solution to the equation $F(u) = f$, and (92) is proved.

*Let us prove* (94) *assuming that* (93) *holds.*

From (91) and (114) with $n = n_\delta - 1$, and from (117), one gets

$$C_1\delta^\zeta \leqslant M_1\frac{a_{n_\delta-1}}{\lambda} + a_{n_\delta-1}\|V_{n_\delta-1}\| \leqslant M_1\frac{a_{n_\delta-1}}{\lambda} + \|y\|a_{n_\delta-1} + \delta.$$

If $\delta > 0$ is sufficiently small, then the above equation implies

$$\tilde{C}\delta^\zeta \leqslant a_{n_\delta-1}\left(\frac{M_1}{\lambda} + \|y\|\right), \quad \tilde{C} > 0,$$

where $\tilde{C} < C_1$ is a constant. Therefore, by (37),

$$\lim_{\delta\to 0}\frac{\delta}{2a_{n_\delta}} \leqslant \lim_{\delta\to 0}\frac{\delta}{a_{n_\delta-1}} \leqslant \lim_{\delta\to 0}\frac{\delta^{1-\zeta}}{\tilde{C}}\left(\frac{M_1}{\lambda} + \|y\|\right) = 0, \quad 0 < \zeta < 1. \tag{119}$$

In particular, for $\delta = \delta_m$, one gets

$$\lim_{\delta_m \to 0} \frac{\delta_m}{a_{n_m}} = 0. \tag{120}$$

From the triangle inequality and inequalities (14) and (110) one obtains

$$\begin{aligned} \|u_{n_m} - y\| &\leqslant \|u_{n_m} - V_{n_m}\| + \|V_n - V_{n_m,0}\| + \|V_{n_m,0} - y\| \\ &\leqslant \frac{a_{n_m}}{\lambda} + \frac{\delta_m}{a_{n_m}} + \|V_{n_m,0} - y\|. \end{aligned} \tag{121}$$

From (93), (120), inequality (121) and Lemma 1, one obtains (94). Theorem 19 is proved. □

## 4. Numerical experiments

Let us do a numerical experiment solving nonlinear equation (1) with

$$F(u) := B(u) + \frac{u^3}{6} := \int_0^1 e^{-|x-y|} u(y)dy + \frac{u^3}{6}, \quad f(x) := \frac{13}{6} - e^{-x} - \frac{e^x}{e}. \tag{122}$$

One can check that $u(x) \equiv 1$ solves the equation $F(u) = f$. The operator $B$ is compact in $H = L^2[0,1]$. The operator $u \longmapsto u^3$ is defined on a dense subset $D$ of of $L^2[0,1]$, for example, on $D := C[0,1]$. If $u, v \in D$, then

$$\langle u^3 - v^3, u - v \rangle = \int_0^1 (u^3 - v^3)(u - v)dx \geqslant 0.$$

Moreover,

$$e^{-|x|} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1 + \lambda^2} d\lambda.$$

Therefore, $\langle B(u-v), u-v \rangle \geqslant 0$, so

$$\langle F(u-v), u-v \rangle \geqslant 0, \qquad \forall u, v \in D.$$

Note that $D$ does not contain subsets, open in $H = L^2[0,1]$, i.e., it does not contain interior points of $H$. This is a reflection of the fact that the operator $G(u) = \frac{u^3}{6}$ is unbounded on any open subset of $H$. For example, in any ball $\|u\| \leqslant C$, $C = const > 0$, where $\|u\| := \|u\|_{L^2[0,1]}$, there is an element $u$ such that $\|u^3\| = \infty$. As such an element one can take, for example, $u(x) = c_1 x^{-b}$, $\frac{1}{3} < b < \frac{1}{2}$. here $c_1 > 0$ is a constant chosen so that $\|u\| \leqslant C$. The operator $u \longmapsto F(u) = G(u) + B(u)$ is maximal monotone on $D_F := \{u : u \in H,\ F(u) \in H\}$ (see [2, p.102]), so that equation (7) is uniquely solvable for any $f_\delta \in H$.

The Fréchet derivative of $F$ is:

$$F'(u)h = \frac{u^2 h}{2} + \int_0^1 e^{-|x-y|} h(y)dy. \tag{123}$$

If $u(x)$ vanishes on a set of positive Lebesgue's measure, then $F'(u)$ is obviously not boundedly invertible. If $u \in C[0,1]$ vanishes even at one point $x_0$, then $F'(u)$ is not boundedly invertible in $H$.

Let us use the iterative process (88):

$$\begin{aligned} u_{n+1} &= u_n - \alpha_n(F(u_n) + a_n u_n - f_\delta), \\ u_0 &= 0. \end{aligned} \tag{124}$$

We stop iterations at $n := n_\delta$ such that the following inequality holds

$$\|F(u_{n_\delta}) - f_\delta\| < C\delta^\zeta, \quad \|F(u_n) - f_\delta\| \geqslant C\delta^\zeta, \quad n < n_\delta, \quad C > 1, \quad \zeta \in (0,1). \tag{125}$$

Integrals of the form $\int_0^1 e^{-|x-y|}h(y)dy$ in (122) and (123) are computed by using the trapezoidal rule. The noisy function used in the test is

$$f_\delta(x) = f(x) + \kappa f_{noise}(x), \quad \kappa > 0.$$

The noise level $\delta$ and the relative noise level are determined by

$$\delta = \kappa\|f_{noise}\|, \quad \delta_{rel} := \frac{\delta}{\|f\|}.$$

In the test, $\kappa$ is computed in such a way that the relative noise level $\delta_{rel}$ equals to some desired value, i.e.,

$$\kappa = \frac{\delta}{\|f_{noise}\|} = \frac{\delta_{rel}\|f\|}{\|f_{noise}\|}.$$

We have used the relative noise level as an input parameter in the test.

The version of DSM, developed in this paper and denoted by DSMS, is compared with the version of DSM in [3], denoted by DSMN. Indeed, the DSMN is the following iterative scheme

$$u_{n+1} = u_n - A_n^{-1}(F'(u_n) + a_n u_n - f_\delta), \quad u_0 = u_0, \qquad n \geqslant 0, \tag{126}$$

where $a_n = \frac{a_0}{1+n}$. This iterative scheme is used with a stopping time $n_\delta$ defined by (91). The existence of this stopping time and the convergence of the method is proved in [3].

As we have proved, the DSMS converges when $a_n = \frac{a_0}{(1+n)^b}$, $b \in (0, \frac{1}{2}]$, and $a_0$ is sufficiently large. However, in practice, if we choose $a_0$ too large then the method will use too many iterations before reaching the stopping time $n_\delta$ in (125). This means that the computation time is large. Since

$$\|F(V_\delta) - f_\delta\| = a(t)\|V_\delta\|,$$

and $\|V_\delta(t_\delta) - u_\delta(t_\delta)\| = O(a(t_\delta))$, we have

$$C\delta^\zeta = \|F(u_\delta(t_\delta)) - f_\delta\| \sim a(t_\delta).$$

Thus, we choose

$$a_0 = C_0\delta^\zeta, \qquad C_0 > 0.$$

The parameter $a_0$ used in the DSMN is also chosen by this formula.

In all figures, the $x$-axis represents the variable $x$. In all figures, by *DSMS* we denote the numerical solutions obtained by the DSMS, by *DSMN* we denote solutions by the DSMN and by *exact* we denote the exact solution.

In experiments, we found that the DSMS works well with $a_0 = C_0\delta^{\zeta}$, $C_0 \in [0.5,2]$. Indeed, in the test the DSMS is implemented with $a_n := C_0\frac{\delta^{0.99}}{(n+1)^{0.5}}$, $C_0 = 1$ while the DSMN is implemented with $a_n := C_0\frac{\delta^{0.99}}{(n+1)}$, $C_0 = 1$. For $C_0 > 3$ the convergence rate of DSMS is much slower while the DSMN still works well if $C_0 \in [1,4]$. In all experiments, the noise function $f_{noise}$ is a vector with random entries normally distributed of mean 0 and variant 1.

Figure 1 plots the solutions using relative noise levels $\delta = 0.01$ and $\delta = 0.001$. The exact solution used in these experiments is $u = 1$. In the test the DSMS is implemented with $\alpha_n = 1$, $C = 1.01$, $\zeta = 0.99$ and $\alpha_n = 1, \forall n \geqslant 0$. The number of iterations of the DSMS for $\delta = 0.01$ and $\delta = 0.001$ were 98 and 99 while the number of iteration for the DSMN are 10 and 10, respectively. The CPU time for the DSMS are 0.0139 and 0.0147 second while the CPU time for the DSMN are 0.0153 and 0.0169 corresponding to $\delta_{rel} = 0.01$ and $\delta_{rel} = 0.001$. The number of node points used in computing integrals in (122) and (123) was $N = 100$. Figure 1 shows that the solutions by the DSMN and DSMS are nearly the same in this figure.

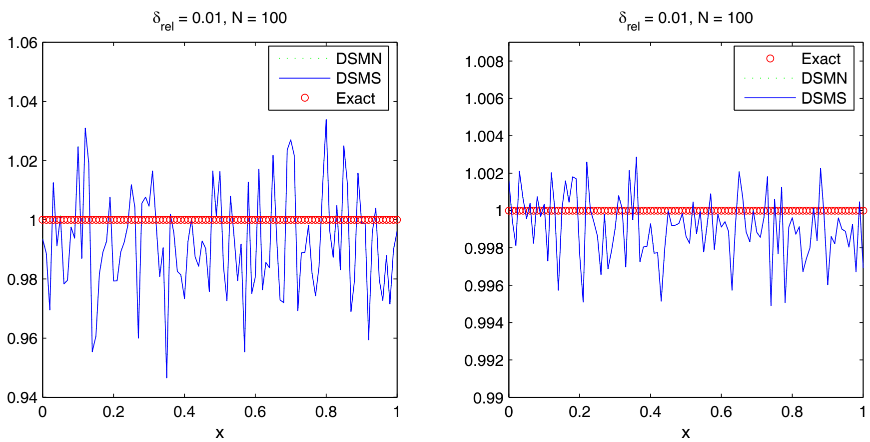


Figure 1. *Plots of solutions obtained by the DSMN and DSMS when* $N = 100$, $u = 1$, $x \in [0,1]$, $\delta_{rel} = 0.01$ *(left) and* $N = 100$, $u = 1$, $x \in [0,1]$, $\delta_{rel} = 0.001$ *(right).*

Figure 2 presents the numerical results when $N = 100$ with $\delta = 0.01$ $u(x) = \sin(2\pi x)$, $x \in [0,1]$ (left) and with $\delta = 0.001$, $u(x) = \sin(\pi x)$, $x \in [0,1]$ (right). In these cases, the DSMN took 10 and 12 iterations to give the numerical solutions while the DSMS took 56 and 67 iterations for $\delta = 0.01$ and $\delta = 0.001$, respectively. The computation time for the DSMS are 0.0102 and 0.0132 second while those for the DSMN are 0.0169 and 0.0186 second for $\delta = 0.01$ and $\delta = 0.001$, respectively. For larger number of node points experiments show that the DSMS is much faster than the

DSMN. Figure 2 show that the numerical results of the DSMS are better than those of the DSMN.

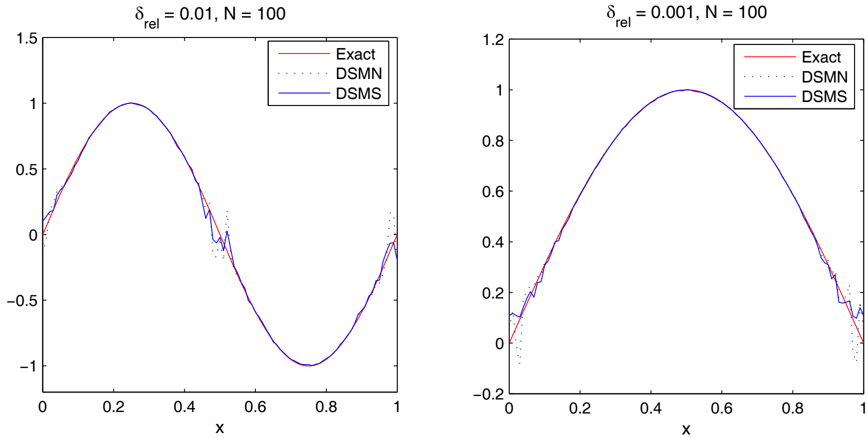


Figure 2. *Plots of solutions obtained by the DSMN and DSMS when* $N = 100$, $u(x) = \sin(2\pi x)$, $x \in [0,1]$, $\delta_{rel} = 0.01$ *(left) and* $N = 100$, $u(x) = \sin(\pi x)$, $x \in [0,1]$, $\delta_{rel} = 0.001$ *(right).*

In our experiments, the DSMS requires about the same or less time of computation than the DSMN. For larger number of node points, we found out that the DSMS runs faster than the DSMN. Moreover, the DSMS yields numerical results with the same accuracy as the DSMN does.

All the computations were carried out using MATLAB in double-precision arithmetic on a PC computer with an Intel Centrino Duo CPU of 1.62 GHz and 3 GB RAM.

## 5. Concluding remarks

Numerical experiments agree with the theory that the convergence rate of the DSMS is slower than that of the DSMN. It is because the rate of decay of the sequence $\{\frac{1}{(1+n)^{\frac{1}{2}}}\}_{n=1}^{\infty}$ is much slower than that of the sequence $\{\frac{1}{1+n}\}_{n=1}^{\infty}$. However, since the cost for one iteration of the DSMS is $O(N^2)$, which is much smaller than that of DSMN (the cost of one iteration of the DSMN is $O(N^3)$), the DSMS required less time to yield a numerical result than the DSMN. Here $N$ is the number of the nodal points. Thus, for large scale problems, the DSMS is an alternative to the DSMN. Also, as is shown in Figure 2, the DSMS may yield more accurate solutions.

Experiments show that the DSMN still works with $a_n = \frac{a_0}{(1+n)^b}$ for $\frac{1}{2} \leqslant b \leqslant 1$. So, in practice one may use faster decaying sequence $a_n$ to reduce the time of computation.

From the numerical results we conclude that the proposed DSM with the discrepancy type stopping rule could be a good alternative for the DSMN for large scale problems.

REMARK. After the completion of this work, we saw the paper [1] in which an iterative process for solving equation (1) with monotone operator is proposed. In [1] some unnatural assumptions are made. For example, assumption (2.4) in [1] implies that the growth of the nonlinearity is not faster than linear, assumption (2.5) is not verifiable practically, in Theorem 2.1 the existence of $N(\delta)$ is not proved, so the result is actually not proved. A "generalized discrepancy principle" (2.8) in [1] is therefore not justified.

## REFERENCES


[1] A. BAKUSHINSKY AND A. SMIRNOVA, *Iterative regularization and generalized discrepancy principle for monotone operators*, Numer. Funct. Anal. and Optimization, **28**, (2007), 13–25.
[2] K. DEIMLING, *Nonlinear functional analysis*, Springer Verlag, Berlin, 1985.
[3] N. S. HOANG AND A. G. RAMM, *An iterative scheme for solving nonlinear equations with monotone operators*. (submitted)
[4] N. S. HOANG AND A. G. RAMM, *Dynamical Systems Gradient method for solving nonlinear equations with monotone operators*, Acta Appl. Math., (to appear)
[5] V. IVANOV, V. TANANA AND V. VASIN, *Theory of ill-posed problems*, VSP, Utrecht, 2002.
[6] V. A. MOROZOV, *Methods of solving incorrectly posed problems*, Springer Verlag, New York, 1984.
[7] A. G. RAMM, *Dynamical systems method for solving operator equations*, Elsevier, Amsterdam, 2007.
[8] A. G. RAMM, *Global convergence for ill-posed equations with monotone operators: the dynamical systems method*, J. Phys A, **36**, (2003), L249–L254.
[9] A. G. RAMM, *Dynamical systems method for solving nonlinear operator equations*, International Jour. of Applied Math. Sci., **1**, N1, (2004), 97–110.
[10] A. G. RAMM, *Dynamical systems method for solving operator equations*, Communic. in Nonlinear Sci. and Numer. Simulation, **9**, N2, (2004), 383–402.
[11] A. G. RAMM, *DSM for ill-posed equations with monotone operators*, Comm. in Nonlinear Sci. and Numer. Simulation, **10**, N8, (2005), 935–940.
[12] A. G. RAMM, Discrepancy principle for the dynamical systems method, Communic. in Nonlinear Sci. and Numer. Simulation, **10**, N1, (2005), 95–101
[13] A. G. RAMM, *Dynamical systems method (DSM) and nonlinear problems, in the book: Spectral Theory and Nonlinear Analysis*, World Scientific Publishers, Singapore, 2005, 201–228. (ed J. Lopez-Gomez).
[14] A. G. RAMM, *Dynamical systems method (DSM) for unbounded operators*, Proc. Amer. Math. Soc., **134**, N4, (2006), 1059–1063.
[15] A. G. RAMM, *Dynamical Systems Method (DSM) for general nonlinear equations*, Nonlinear Analysis: Theory, Methods and Applications, **69**, (2008), 1934–1940.
[16] E. ZEIDLER, *Nonlinear functional analysis*, Springer, New York, 1985.


(Received June 17, 2008)

*N. S. Hoang*
*Mathematics Department*
*Kansas State University*
*Manhattan, KS 66506-2602*
*USA*
*e-mail:* nguyenhs@math.ksu.edu

*A. G. Ramm*
*Mathematics Department*
*Kansas State University*
*Manhattan, KS 66506-2602*
*USA*
*e-mail:* ramm@math.ksu.edu