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ON FREDHOLM TYPE INTEGRAL EQUATION IN TWO VARIABLES

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Abstract. The aim of this paper is to study some basic properties of solutions of a certain Fredholm type integral equation in two variables. The tools employed in the analysis are based on the applications of the Banach fixed point theorem and the new integral inequality with explicit estimate.

1. Introduction

In [1], Bica, Căuş and Mureşan iniciated the study of Fredholm integral equation of the form

$$x(t) = f(t) + \int_0^a g(t,s,x(s),x'(s))ds, \tag{1.1}$$

in Banach space setting. Motivated by the results in [1], recently in [7,8], Pachpatte studied the qualitative behavior of solutions of equation (1.1) and its further generalization. Inspired by the results in [1,7,8], in this paper we consider the Fredholm type integral equation of the form

$$u(x,y) = f(x,y) + \int_0^a \int_0^b g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t))dtds, \tag{1.2}$$

where f,g are given functions and u is the unknown function to be found.

Let $R_+ = [0,\infty)$, $I_a = [0,a]$, $I_b = [0,b]$ $(a > 0, b > 0)$ be the given subsets of R, the set of real numbers, $\Delta = I_a \times I_b$ and $C(A,B)$ denote the class of continuous functions from the set A to the set B. The partial derivatives of a function $z(x,y)$ $(x,y \in R)$ with respect to x and y are denoted by $D_1z(x,y) = \frac{\partial}{\partial x}z(x,y), D_2z(x,y) = \frac{\partial}{\partial y}z(x,y)$. Throughout, we assume that $f \in C(\Delta,R)$, $g \in C(\Delta^2 \times R^3,R)$ and $D_if \in C(\Delta,R)$, $D_ig \in C(\Delta^2 \times R^3,R)$ for $i = 1,2$. In fact, the study of qualitative properties of solutions of equation (1.2) is challenging and requires new ideas in handling the equations of the form (1.2). The main objective of the present paper is to study some fundamental qualitative properties of solutions of equation (1.2) under some suitable conditions on the

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functions involved therein. The tools employed in the analysis are based on the applications of the Banach fixed point theorem (see [4, p. 37]) coupled with Bielecki type norm (see [2]) and a suitable version of the integral inequality established by Pachpatte (see [5, p.111]). Here, our approach is elementary and provide some useful basic results which may be considered as a foundation for future advanced studies in the field.

2. Existence and uniqueness

By a solution of equation (1.2) we mean a continuous function $u : \Delta \to R$ which is continuously differentiable with respect to x and y for $(x,y) \in \Delta$ and satisfies the equation (1.2). Differentiating both sides of (1.2) partially with respect to x and y, it is easy to observe that the solution $u(x,y)$ of equation (1.2) for $i = 1,2$ satisfies the following integral equations

$$D_i u(x,y) = D_i f(x,y) + \int_0^a \int_0^b D_i g(x,y,s,t,u(s,t),D_1 u(s,t),D_2 u(s,t))dtds, \tag{2.1}$$

for $(x,y) \in \Delta$ (see [3, p. 318]). For $z,\ D_1 z,\ D_2 z \in C(\Delta,R)$ we denote by $|z(x,y)|_1 = |z(x,y)| + |D_1 z(x,y)| + |D_2 z(x,y)|$. Let E be the space of functions $z,\ D_1 z,\ D_2 z \in C(\Delta,R)$ which fulfill the condition

$$|z(x,y)|_1 = O(\exp(\lambda(x+y))), \tag{2.2}$$

for $(x,y) \in \Delta$, where λ is a positive constant. In the space E we define the norm (see [2,6])

$$|z|_E = \sup_{(x,y)\in\Delta} \{|z(x,y)|_1 \exp(-\lambda(x+y))\}. \tag{2.3}$$

It is easy to see that E with the norm defined in (2.3) is a Banach space. We note that the condition (2.2) implies that there is a constant $M \geqslant 0$ such that $|z(x,y)|_1 \leqslant M\exp(\lambda(x+y))$. Using this fact in (2.3) we observe that

$$|z|_E \leqslant M. \tag{2.4}$$

We need the following special version of the integral inequality established by Pachpatte (see [5, p. 111]). We shall state it in the following lemma for completeness.

LEMMA. *Let $z,\ p,\ q,\ r \in C(\Delta,R_+)$. Suppose that*

$$z(x,y) \leqslant p(x,y) + q(x,y)\int_0^a \int_0^b r(s,t)z(s,t)dtds, \tag{2.5}$$

for $(x,y) \in \Delta$. If

$$d = \int_0^a \int_0^b r(s,t)q(s,t)dtds < 1, \tag{2.6}$$

then

$$z(x,y) \leqslant p(x,y) + q(x,y)\left\{\frac{1}{1-d}\int_0^a\int_0^b r(s,t)p(s,t)dtds\right\}, \tag{2.7}$$

for $(x,y) \in \Delta$.

Our main result in this section is given in the following theorem.

THEOREM 1. *Suppose that*

(i) *the function* g *in equation (1.2) and its derivatives* $D_i g$ *for* $i = 1,2$ *satisfy the conditions*

$$|g(x,y,s,t,u,v,w) - g(x,y,s,t,\bar{u},\bar{v},\bar{w})| \leqslant r(x,y,s,t)[|u-\bar{u}| + |v-\bar{v}| + |w-\bar{w}|], \tag{2.8}$$

and

$$|D_i g(x,y,s,t,u,v,w) - D_i g(x,y,s,t,\bar{u},\bar{v},\bar{w})| \leqslant r_i(x,y,s,t)[|u-\bar{u}| + |v-\bar{v}| + |w-\bar{w}|], \tag{2.9}$$

(ii) *for* λ *as in (2.2),*

(a_1) *there exists a nonnegative constant* α *such that* $\alpha < 1$ *and*

$$\int_0^a\int_0^b [r(x,y,s,t) + r_1(x,y,s,t) + r_2(x,y,s,t)]\exp(\lambda(s+t))dtds \leqslant \alpha\exp(\lambda(x+y)), \tag{2.10}$$

for $(x,y) \in \Delta$,

(a_2) *there exists a nonnegative constant* β *such that*

$$|f(x,y)|_1 + \int_0^a\int_0^b |g(x,y,s,t,0,0,0)|_1 dtds \leqslant \beta\exp(\lambda(x+y)), \tag{2.11}$$

for $(x,y) \in \Delta$.

Then the equation (1.2) has a unique solution $u(x,y)$ *on* Δ *in* E.

Proof. Let $u(x,y) \in E$ and define the operator T by

$$(Tu)(x,y) = f(x,y) + \int_0^a\int_0^b g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t))dtds. \tag{2.12}$$

Differentiating both sides of (2.12) partially with respect to x and y we have

$$D_i(Tu)(x,y) = D_i f(x,y) + \int_0^a\int_0^b D_i g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t))dtds, \tag{2.13}$$

for $i=1,2$. Now, we show that Tu maps E into itself. Evidently, for $i=1,2$ $Tu, D_i(Tu)$ are continuous on Δ and $Tu, D_i(Tu)\in R$. We verify that (2.2) is fulfilled. From (2.12), (2.13), (2.4) and using the hypotheses, we have

$$\begin{aligned}
|(Tu)(x,y)|_1 \leqslant |f(x,y)|_1 &+ \int_0^a\int_0^b |g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)) \\
&\qquad - g(x,y,s,t,0,0,0)|dtds \\
&+ \int_0^a\int_0^b |g(x,y,s,t,0,0,0)|dtds \\
&+ \int_0^a\int_0^b |D_1g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)) \\
&\qquad - D_1g(x,y,s,t,0,0,0)|dtds \\
&+ \int_0^a\int_0^b |D_1g(x,y,s,t,0,0,0)|dtds \\
&+ \int_0^a\int_0^b |D_2g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)) \\
&\qquad - D_2g(x,y,s,t,0,0,0)|dtds \\
&+ \int_0^a\int_0^b |D_2g(x,y,s,t,0,0,0)|dtds \\
\leqslant |f(x,y)|_1 &+ \int_0^a\int_0^b |g(x,y,s,t,0,0,0)|_1 dtds + \int_0^a\int_0^b r(x,y,s,t)|u(s,t)|_1 dtds \\
&+ \int_0^a\int_0^b r_1(x,y,s,t)|u(s,t)|_1 dtds + \int_0^a\int_0^b r_2(x,y,s,t)|u(s,t)|_1 dtds \\
\leqslant \beta\exp(\lambda(x+y)) &+ |u|_E \int_0^a\int_0^b [r(x,y,s,t) \\
&+ r_1(x,y,s,t) + r_2(x,y,s,t)]\exp(\lambda(s+t))dtds \\
\leqslant [\beta+M\alpha]&\exp(\lambda(x+y)). \qquad (2.14)
\end{aligned}$$

From (2.14), it follows that $Tu\in E$. This proves that the operator T maps E into itself.

Next, we verify that the operator T is a contraction map. Let $u(x,y), v(x,y)\in E$. From (2.12), (2.13) and using the hypotheses, we have

$$
\begin{aligned}
&|(Tu)(x,y)-(Tv)(x,y)|_1 \\
&\quad\leqslant \int_0^a\int_0^b |g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)) \\
&\qquad\qquad -g(x,y,s,t,v(s,t),D_1v(s,t),D_2v(s,t))|dtds \\
&\qquad +\int_0^a\int_0^b |D_1g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)) \\
&\qquad\qquad -D_1g(x,y,s,t,v(s,t),D_1v(s,t),D_2v(s,t))|dtds \\
&\qquad +\int_0^a\int_0^b |D_2g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)) \\
&\qquad\qquad -D_2g(x,y,s,t,v(s,t),D_1v(s,t),D_2v(s,t))|dtds \\
&\quad\leqslant \int_0^a\int_0^b r(x,y,s,t)|u(s,t)-v(s,t)|_1 dtds+\int_0^a\int_0^b r_1(x,y,s,t)|u(s,t)-v(s,t)|_1 dtds \\
&\qquad +\int_0^a\int_0^b r_2(x,y,s,t)|u(s,t)-v(s,t)|_1 dtds \\
&\quad\leqslant |u-v|_E\int_0^a\int_0^b [r(x,y,s,t) \\
&\qquad +r_1(x,y,s,t)+r_2(x,y,s,t)]\exp(\lambda(s+t))dtds \\
&\quad\leqslant |u-v|_E\alpha\exp(\lambda(x+y)). \qquad (2.15)
\end{aligned}
$$

From (2.15), we obtain

$$|Tu-Tv|_E\leqslant\alpha|u-v|_E.$$

Since $\alpha<1$, it follows from Banach fixed point theorem (see [4, p.37]) that T has a unique fixed point in E. The fixed point of T is however a solution of equation (1.2). The proof is complete.

REMARK 1. We note that the norm $|.|_E$ defined by (2.3) is a variant of Bielecki's norm [2], first used in 1956 for proving global existence and uniqueness of solutions of ordinary differential equations.

The next result deals with the uniqueness of solutions of equation (1.2) in R without existence part.

THEOREM 2. *Suppose that the function g in equation (1.2) and its derivatives D_ig for $i=1,2$ satisfy the conditions (2.8) and (2.9) with $r(x,y,s,t)=c(x,y)h(s,t)$,*

$r_i(x,y,s,t) = c(x,y)h_i(s,t)$ for $i = 1,2$, where $c,h,h_i \in C(\Delta, R_+)$ and

$$d_1 = \int_0^a \int_0^b [h(s,t) + h_1(s,t) + h_2(s,t)]c(s,t)dtds < 1. \tag{2.16}$$

Then the equation (1.2) has at most one solution in R on Δ.

Proof. Let $u(x,y)$ and $v(x,y)$ be two solutions of equation (1.2). Then from the hypotheses, we have

$$\begin{aligned}
|u(x,y) - v(x,y)|_1 \leqslant & \int_0^a \int_0^b |g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)) \\
& \qquad - g(x,y,s,t,v(s,t),D_1v(s,t),D_2v(s,t))|dtds \\
& + \int_0^a \int_0^b |D_1g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)) \\
& \qquad - D_1g(x,y,s,t,v(s,t),D_1v(s,t),D_2v(s,t))|dtds \\
& + \int_0^a \int_0^b |D_2g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)) \\
& \qquad - D_2g(x,y,s,t,v(s,t),D_1v(s,t),D_2v(s,t))|dtds \\
\leqslant & \int_0^a \int_0^b c(x,y)h(s,t)|u(s,t) - v(s,t)|_1 dtds + \int_0^a \int_0^b c(x,y)h_1(s,t)|u(s,t) - v(s,t)|_1 dtds \\
& + \int_0^a \int_0^b c(x,y)h_2(s,t)|u(s,t) - v(s,t)|_1 dtds \\
= & \, c(x,y) \int_0^a \int_0^b [h(s,t) + h_1(s,t) + h_2(s,t)]|u(s,t) - v(s,t)|_1 dtds.
\end{aligned} \tag{2.17}$$

Now, an application of Lemma (when $p(x,y) = 0$) to (2.17) yields $|u(x,y) - v(x,y)|_1 \leqslant 0$, and hence $u(x,y) = v(x,y)$, which proves the uniqueness of solutions of equation (1.2) on Δ.

3. Estimates on the solutions

In this section, we obtain estimates on the solutions of equation (1.2) under some suitable conditions on the functions involved therein.

The following theorem concerning the estimate on the solution of equation (1.2) holds.

THEOREM 3. *Suppose that the function g in equation (1.2) and its derivatives $D_i g$ for $i = 1,2$ satisfy the conditions*

$$|g(x,y,s,t,u,v,w)| \leqslant k(x,y)e(s,t)[|u|+|v|+|w|], \tag{3.1}$$

and

$$|D_i g(x,y,s,t,u,v,w)| \leqslant k(x,y)e_i(s,t)[|u|+|v|+|w|], \tag{3.2}$$

where $k, e, e_i \in C(\Delta, R_+)$ and

$$d_2 = \int_0^a \int_0^b [e(s,t)+e_1(s,t)+e_2(s,t)]k(s,t)dtds < 1. \tag{3.3}$$

Then for every solution $u \in C(\Delta, R)$ of equation (1.2), we have the estimate

$$|u(x,y)|_1 \leqslant |f(x,y)|_1 + k(x,y) \times \left\{ \frac{1}{1-d_2} \int_0^a \int_0^b [e(s,t)+e_1(s,t)+e_2(s,t)]|f(s,t)|_1 dtds \right\}, \tag{3.4}$$

for $(x,y) \in \Delta$.

Proof. Let $u \in C(\Delta, R)$ be a solution of equation (1.2). Then from the hypotheses, we have

$$\begin{aligned}
|u(x,y)|_1 \leqslant\ & |f(x,y)| + \int_0^a \int_0^b |g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t))|dtds \\
& + |D_1 f(x,y)| + \int_0^a \int_0^b |D_1 g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t))|dtds \\
& + |D_2 f(x,y)| + \int_0^a \int_0^b |D_2 g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t))|dtds \\
\leqslant\ & |f(x,y)|_1 + \int_0^a \int_0^b k(x,y)e(s,t)|u(s,t)|_1 dtds \\
& + \int_0^a \int_0^b k(x,y)e_1(s,t)|u(s,t)|_1 dtds + \int_0^a \int_0^b k(x,y)e_2(s,t)|u(s,t)|_1 dtds \\
=\ & |f(x,y)|_1 + k(x,y) \int_0^a \int_0^b [e(s,t)+e_1(s,t)+e_2(s,t)]|u(s,t)|_1 dtds.
\end{aligned} \tag{3.5}$$

Now, an application of Lemma to (3.5) gives the desired estimate in (3.4).

REMARK 2. We note that the estimate obtained in (3.4) yields not only the bound on the solution u of equation (1.2), but also the bound on their derivatives $D_i u$ for $i = 1,2$.

Next, we shall obtain the estimate on the solution of equation (1.2) assuming that the function g and its derivatives $D_i g$ for $i = 1,2$ satisfy Lipschitz type conditions.

THEOREM 4. *Suppose that the function g in equation (1.2) and its derivatives $D_i g$ for $i = 1,2$ satisfy the conditions in Theorem 2 and the condition (2.16) holds. If $u \in C(\Delta, R)$ is any solution of equation (1.2) on Δ, then*

$$|u(x,y) - f(x,y)|_1 \leqslant Q(x,y) + c(x,y) \times \left\{ \frac{1}{1-d_1} \int_0^a \int_0^b [h(s,t) + h_1(s,t) + h_2(s,t)] Q(s,t)\, dt\, ds \right\}, \tag{3.6}$$

for $(x,y) \in \Delta$, where

$$Q(x,y) = \int_0^a \int_0^b |g(x,y,\sigma,\tau,f(\sigma,\tau),D_1 f(\sigma,\tau),D_2 f(\sigma,\tau))|_1\, d\tau\, d\sigma, \tag{3.7}$$

for $(x,y) \in \Delta$.

Proof. Since $u(x,y)$ is a solution of equation (1.2), by using the hypotheses, we have

$$\begin{aligned}
|u(x,y) - f(x,y)|_1 \leqslant & \int_0^a \int_0^b |g(x,y,s,t,u(s,t),D_1 u(s,t),D_2 u(s,t)) \\
& \qquad - g(x,y,s,t,f(s,t),D_1 f(s,t),D_2 f(s,t))|\, dt\, ds \\
& + \int_0^a \int_0^b |g(x,y,s,t,f(s,t),D_1 f(s,t),D_2 f(s,t))|\, dt\, ds \\
& + \int_0^a \int_0^b |D_1 g(x,y,s,t,u(s,t),D_1 u(s,t),D_2 u(s,t)) \\
& \qquad - D_1 g(x,y,s,t,f(s,t),D_1 f(s,t),D_2 f(s,t))|\, dt\, ds \\
& + \int_0^a \int_0^b |D_1 g(x,y,s,t,f(s,t),D_1 f(s,t),D_2 f(s,t))|\, dt\, ds \\
& + \int_0^a \int_0^b |D_2 g(x,y,s,t,u(s,t),D_1 u(s,t),D_2 u(s,t)) \\
& \qquad - D_2 g(x,y,s,t,f(s,t),D_1 f(s,t),D_2 f(s,t))|\, dt\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^a\int_0^b |D_2 g(x,y,s,t,f(s,t),D_1 f(s,t),D_2 f(s,t))|dtds\\
&\leqslant Q(x,y)+\int_0^a\int_0^b c(x,y)h(s,t)|u(s,t)-f(s,t)|_1 dtds\\
&+\int_0^a\int_0^b c(x,y)h_1(s,t)|u(s,t)-f(s,t)|_1 dtds\\
&+\int_0^a\int_0^b c(x,y)h_2(s,t)|u(s,t)-f(s,t)|_1 dtds\\
&= Q(x,y)+c(x,y)\int_0^a\int_0^b [h(s,t)+h_1(s,t)+h_2(s,t)]|u(s,t)-f(s,t)|_1 dtds.
\end{aligned}
\tag{3.8}
$$

Now, an application of Lemma to (3.8) yields (3.6).

We next consider the equation (1.2) and the following Fredholm type integral equation

$$
z(x,y)=F(x,y)+\int_0^a\int_0^b G(x,y,s,t,z(s,t),D_1 z(s,t),D_2 z(s,t))dtds, \tag{3.9}
$$

for $(x,y)\in\Delta$, where $F\in C(\Delta,R)$, $G\in C(\Delta^2\times R^3,R)$ and $D_iF\in C(\Delta,R)$, $D_iG\in C(\Delta^2\times R^3,R)$ for $i=1,2$.

The following theorem holds.

THEOREM 5. *Suppose that the function g in equation (1.2) and its derivatives D_ig for $i=1,2$ satisfy the conditions as in Theorem 4 and the condition (2.16) holds. Then for every given solution $z\in C(\Delta,R)$ of equation (3.9) and any solution $u\in C(\Delta,R)$ of equation (1.2), we have the estimate*

$$
\begin{aligned}
|u(x,y)-z(x,y)|_1 &\leqslant [|f(x,y)-F(x,y)|_1+M(x,y)]+c(x,y)\\
&\times\left\{\frac{1}{1-d_1}\int_0^a\int_0^b [h(s,t)+h_1(s,t)+h_2(s,t)][|f(s,t)-F(s,t)|_1+M(s,t)]dtds\right\},
\end{aligned}
\tag{3.10}
$$

for $(x,y)\in\Delta$, where

$$
\begin{aligned}
M(x,y)=\int_0^a\int_0^b &|g(x,y,\sigma,\tau,z(\sigma,\tau),D_1 z(\sigma,\tau),D_2 z(\sigma,\tau))\\
&-G(x,y,\sigma,\tau,z(\sigma,\tau),D_1 z(\sigma,\tau),D_2 z(\sigma,\tau))|_1 d\tau d\sigma,
\end{aligned}
\tag{3.11}
$$

for $(x,y)\in\Delta$.

Proof. Using the facts that $u(x,y)$ and $z(x,y)$ are respectively the solutions of equations (1.2) and (3.9) and hypotheses, we have

$$\begin{aligned}
|u(x,y)-z(x,y)|_1 \leqslant |f(x,y)-F(x,y)|_1 + \int_0^a\int_0^b &|g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)) \\
&-g(x,y,s,t,z(s,t),D_1z(s,t),D_2z(s,t))|dtds \\
+\int_0^a\int_0^b |g(x,y,s,t,z(s,t),D_1z(s,t),D_2z(s,t)) & \\
-G(x,y,s,t,z(s,t),D_1z(s,t),D_2z(s,t))|dtds & \\
+\int_0^a\int_0^b |D_1g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)) & \\
-D_1g(x,y,s,t,z(s,t),D_1z(s,t),D_2z(s,t))|dtds & \\
+\int_0^a\int_0^b |D_1g(x,y,s,t,z(s,t),D_1z(s,t),D_2z(s,t)) & \\
-D_1G(x,y,s,t,z(s,t),D_1z(s,t),D_2z(s,t))|dtds & \\
+\int_0^a\int_0^b |D_2g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)) & \\
-D_2g(x,y,s,t,z(s,t),D_1z(s,t),D_2z(s,t))|dtds & \\
+\int_0^a\int_0^b |D_2g(x,y,s,t,z(s,t),D_1z(s,t),D_2z(s,t)) & \\
-D_2G(x,y,s,t,z(s,t),D_1z(s,t),D_2z(s,t))|dtds & \\
\leqslant [|f(x,y)-F(x,y)|_1 + M(x,y)] + c(x,y) & \\
\times \int_0^a\int_0^b [h(s,t)+h_1(s,t)+h_2(s,t)]|u(s,t)-z(s,t)|_1 dtds. & \qquad (3.12)
\end{aligned}$$

Now, an application of Lemma to (3.12) yields (3.10).

REMARK 3. We note that, one can formulate results on the continuous dependence of solutions of equation (1.2) by closely looking at the corresponding results recently given in [6,7]. Furthermore, the idea used in this paper can be very easily extended to study the version of equation (1.2) involving functions of more than two variables. Moreover, the results established in Theorems 1-5 can be extended for equations of the form (1.2) when the function g is of the form

$$g\left(x,y,s,t,u(s,t),\frac{\partial^n u(s,t)}{\partial s^n},\frac{\partial^m u(s,t)}{\partial t^m}\right)$$

or

$$g\left(x,y,s,t,u(s,t),\frac{\partial^n u(s,t)}{\partial s^n},\frac{\partial^m u(s,t)}{\partial t^m},\frac{\partial^{n+m} u(s,t)}{\partial s^n \partial t^m}\right)$$

or

$$g\left(x,y,s,t,u(s,t),\frac{\partial u(s,t)}{\partial s},\ldots,\frac{\partial^n u(s,t)}{\partial s^n},\frac{\partial u(s,t)}{\partial t},\ldots,\frac{\partial^m u(s,t)}{\partial t^m}\right)$$

or

$$g\left(x,y,s,t,u(s,t),\frac{\partial u(s,t)}{\partial s},\ldots,\frac{\partial^n u(s,t)}{\partial s^n},\frac{\partial u(s,t)}{\partial t},\ldots,\frac{\partial^m u(s,t)}{\partial t^m},\right.$$

$$\left.\frac{\partial^{n+1} u(s,t)}{\partial s^n\partial t},\ldots,\frac{\partial^{n+m-1} u(s,t)}{\partial s^n\partial t^{m-1}},\frac{\partial^{1+m} u(s,t)}{\partial s\partial t^m},\ldots,\frac{\partial^{n-1+m} u(s,t)}{\partial s^{n-1}\partial t^m},\frac{\partial^{n+m} u(s,t)}{\partial s^n\partial t^m}\right),$$

under some suitable conditions. Naturally, these considerations will make the analysis more complicated and we leave it to the reader to fill in where needed.

4. Applications

The generality of the integral equation (1.2) allow us to obtain results similar to the ones given above, concerning the following integral equation

$$u(x,y) = f(x,y) + \int_0^a\int_0^b K(x,y,s,t)h(s,t,u(s,t),D_1u(s,t),D_2u(s,t))dtds, \tag{4.1}$$

where $f \in C(\Delta,R), K \in C(\Delta^2,R), h \in C(\Delta\times R^3,R)$ and assume that $D_if \in C(\Delta,R), D_iK \in C(\Delta^2,R)$. In this section we present a result on the existence of a unique solution of equation (4.1), relaxing the assumptions posed on the derivatives of a general function g in (1.2) by using the Banach fixed point theorem coupled with supremum norm. One can formulate other results given above for the equation (4.1) by applying the ideas used in Theorems 2-5 with relaxed assumptions.

THEOREM 6. *Suppose that*

(i) $f,\ D_if \in C(\Delta,R)$ *for* $i=1,2$,

(ii) $h \in C(\Delta\times R^3,R),\ h(s,t,0,0,0)=0$ *and there is a constant* $L>0$ *such that*

$$|h(s,t,u,v,w)-h(s,t,\bar{u},\bar{v},\bar{w})| \leqslant L[|u-\bar{u}|+|v-\bar{v}|+|w-\bar{w}|], \tag{4.2}$$

(iii) $K,\ D_iK \in C(\Delta^2,R)$ *for* $i=1,2$ *and*

$$L\int_0^a\int_0^b |K(x,y,s,t)|_1 dtds \leqslant \alpha < 1. \tag{4.3}$$

Then the equation (4.1) has a unique solution $u(x,y)$ *on* Δ.

Proof. Let B be a Banach space of bounded continuous functions $u:\Delta\to R$ which are continuously differentiable with respect to x and y on Δ with supremum norm $\|.\|$, where $\|u\| = \sup_{(x,y)\in\Delta} |u(x,y)|_1$. Let $u(x,y)\in B$ and define the operator F by

$$(Fu)(x,y) = f(x,y) + \int_0^a\int_0^b K(x,y,s,t)h(s,t,u(s,t),D_1u(s,t),D_2u(s,t))dtds. \tag{4.4}$$

Differentiating both sides of (4.4) partially with respect to x and y, we have

$$D_i(Fu)(x,y) = D_i f(x,y) + \int_0^a \int_0^b D_i K(x,y,s,t) h(s,t,u(s,t),D_1 u(s,t),D_2 u(s,t)) dt ds, \tag{4.5}$$

for $i = 1,2$. We shall show that F is a contraction map. Therefore, a fixed point of F is a solution of equation (4.1). From the assumptions, it follows that $Fu, D_i(Fu)$ $(i = 1,2)$ are continuous on Δ and

$$\begin{aligned} |(Fu)(x,y)|_1 &\leqslant |f(x,y)|_1 + \int_0^a \int_0^b |K(x,y,s,t)|_1 |h(s,t,u(s,t),D_1 u(s,t),D_2 u(s,t)) \\ &\qquad - h(s,t,0,0,0)| dt ds \\ &\leqslant |f(x,y)|_1 + L \int_0^a \int_0^b |K(x,y,s,t)|_1 |u(s,t)|_1 dt ds \\ &\leqslant |f(x,y)|_1 + L\|u\| \int_0^a \int_0^b |K(x,y,s,t)|_1 dt ds < \infty. \end{aligned} \tag{4.6}$$

Here, we have used the fact that $|f(x,y)|_1$ is bounded, since $f, D_i f \in C(\Delta, R)$ and the condition (4.3). This proves that the operator F maps B into itself.

Let $u(x,y), v(x,y) \in B$. From (4.4), (4.5) and the hypotheses, we have

$$\begin{aligned} |(Fu)(x,y) - (Fv)(x,y)|_1 &\leqslant \int_0^a \int_0^b |K(x,y,s,t)|_1 |h(s,t,u(s,t),D_1 u(s,t),D_2 u(s,t)) \\ &\qquad - h(s,t,v(s,t),D_1 v(s,t),D_2 v(s,t))| dt ds \\ &\leqslant L \int_0^a \int_0^b |K(x,y,s,t)|_1 |u(s,t) - v(s,t)|_1 dt ds \\ &\leqslant L\|u - v\| \int_0^a \int_0^b |K(x,y,s,t)|_1 dt ds \\ &\leqslant \alpha \|u - v\|. \end{aligned} \tag{4.7}$$

From (4.7) and since $\alpha < 1$, it follows that F is a contraction mapping, which proves that the equation (4.1) has a unique solution on Δ.

REMARK 4. We note that the equation (4.1) includes as a special case, the study of the following important integral equation

$$u(x,y) = f(x,y) + \int_0^a \int_0^b K(x,y,s,t) h(s,t,u(s,t)) dt ds, \tag{4.8}$$

which may be considered as a two independent variable generalization of the well known Hammerstein type integral equation studied by many authors in the literature, see [6, section 1.5] and the references cited therein.

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