Differential
Equations
& Applications
Volume 1, Number 1 (2009), 41–65

EXISTENCE RESULTS FOR SEMILINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS

S. K. NTOUYAS AND D. O'REGAN

(*communicated by E. Hernandez*)

Abstract. In this paper we prove existence results for first and second order semilinear neutral functional differential inclusions with finite or infinite delay in Banach spaces and nonlocal conditions, via a Nonlinear Alternative for condensing maps. Our theory makes use of analytic semigroups and fractional powers of closed operators, integrated semigroups and cosine families.

1. Introduction

Recently in [17] we proved existence results for first and second order semilinear neutral functional differential inclusions in a real Banach space, with nonlocal conditions. More precisely, we studied first order initial value problems for semilinear neutral functional differential inclusions with nonlocal conditions of the form,

$$(d/dt)[y(t)-f(t,y_t)] \in Ay(t)+F(t,y_t), \quad \text{a.e. } t \in J=[0,T] \tag{1.1}$$

$$y(t)+h_t(y)=\phi(t),\ t\in[-r,0] \tag{1.2}$$

where $f: J\times\mathscr{D}\to E,\ F: J\times\mathscr{D}\longrightarrow\mathscr{P}(E)$ is a multivalued map, $h_t\in\mathscr{D},\ \phi\in\mathscr{D}$, $\mathscr{D}=\{\psi:[-r,0]\to E|\psi$ is continuous$\}$, A is the infinitesimal generator of a strongly continuous semigroup $\mathscr{S}(t)$, $t\geqslant 0$ and E a separable real Banach space with the norm $\|\cdot\|$.

The method used in [17] consists in applying the Leray-Schauder Nonlinear Alternative for compact maps. Among other assumptions in [17] we suppose the following conditions hold:

(*i*) the map $H: C([-r,T],E)\to C([0,T],E)$, given by $H(y)(t)=f(t,y_t)$ for $t\in[0,T]$, is continuous and completely continuous;

(*ii*) for each $t\in[-r,0]$, $h_t: C([-r,T],E)\to C([-r,0],E)$ is completely continuous;

(*iii*) given $\varepsilon>0$, then for any bounded subset D of $C([-r,T],E)$ there exists a $\delta>0$ with $\|(\mathscr{S}(\theta)-I)h_0(y)\|<\varepsilon$ for all $y\in D$ and $\theta\in[0,\delta]$ and $\|h_t(y)-h_s(y)\|<\varepsilon$ for all $y\in D$ and $t,s\in[-r,0]$ with $|t-s|<\delta$.

Mathematics subject classification (2000): 34A60,34G20, 93B05.
Keywords and phrases: Semilinear differential inclusions, nonlocal conditions, analytic semigroups, cosine functions, integrated semigroups, fixed point, infinite delay.

Paper DEA-01-03

It is worth noting here that (*ii*) implies $\{h_0(y) : y \in D\}$ is relatively compact so given $\varepsilon > 0$ there exists $\delta > 0$ with $\|(\mathscr{S}(\theta) - I)h_0(y)\| < \varepsilon$ for all $y \in D$ and $\theta \in [0, \delta]$.

We emphasize that (*i*) and (*ii*) are very strong conditions when E is infinite dimensional so sublinear or linear f cannot be discussed (these automatically fit within the results of this paper).

In this paper we consider the following semilinear neutral functional differential inclusions with nonlocal conditions

$$(d/dt)[y(t) - f(t, y_t)] \in Ay(t) + F(t, y_t), \quad \text{a.e. } t \in J = [0, T] \tag{1.3}$$

$$y_0(t) = \phi(t) + q(y_{t_1}, \dots, y_{t_n})(t), \text{ for } t \in [-r, 0] \tag{1.4}$$

where f, F, A, ϕ are as in problem (1.1)–(1.2), $0 < t_1 < t_2 < \ldots < t_n \leqslant T$ and $q : \mathscr{D}^n \to \mathscr{D}$.

For any continuous function y defined on the interval $[-r, T]$ and any $t \in J$, we denote by y_t the element of $\mathscr{D}$ defined by $y_t(\theta) = y(t + \theta)$, $\theta \in [-r, 0]$. For $\psi \in \mathscr{D}$ the norm of ψ is defined by

$$\|\psi\|_{\mathscr{D}} = \sup\{\|\psi(\theta)\| : \theta \in [-r, 0]\}.$$

We also let

$$\|u\| = \sup\{\|u(t)\| : t \in [-r, T]\} \quad \text{for} \quad u \in C([-r, T], E).$$

In this paper we present some new existence results by applying the Nonlinear Alternative for contractive maps, deleting the conditions (*i*)–(*iii*) above and replacing with Lipschitz conditions on f and q.

Thus in Section 2, we study the problem (1.3)–(1.4) and in Section 3 we consider a general form for the problem (1.3)–(1.4) where $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator. In Section 4 we consider first order semilinear neutral functional differential inclusions with infinite delay. Finally, in Section 5 we study second order initial value problems for semilinear neutral functional differential inclusions with nonlocal conditions of the form

$$(d/dt)[y'(t) - f(t, y_t)] \in Ay(t) + F(t, y_t), \; t \in J := [0, T], \tag{1.5}$$

$$y_0(t) = \phi(t) + q(y_{t_1}, \dots, y_{t_n})(t), \text{ for } t \in [-r, 0], \quad y'(0^+) + h(y) = \eta, \tag{1.6}$$

where A is the infinitesimal generator of a family of cosine operators $\{C(t) : t \geqslant 0\}$, $\eta \in E$, f, F, ϕ, q are as in problem (1.3)–(1.4) and $h : C([-r, T], E) \to E$ is continuous.

Nonlocal conditions for evolution equations were initiated by Byszewski. We refer the reader to [5] and the references cited therein for a motivation regarding nonlocal initial conditions. The nonlocal conditions can be applied in physics and is more natural than the classical initial condition $y(0) = y_0$.

IVPs (1.3)–(1.4) and (1.5)-(1.6) were studied in the literature under growth conditions on F, or by assuming the existence of a maximal solution to appropriate problems; see [3, 17] and the references cited therein. A special case of the IVP (1.3)–(1.4) when F is single-valued was studied in [13] by using Banach contraction principle.

We will use in the sequel the following form of the Nonlinear Alternative for contractive maps [19, Corollary 3.8].

THEOREM 1.1. *Let X be a Banach space, and D a bounded neighborhood of $0 \in X$. Let $Z_1 : X \to \mathscr{P}_{cp,c}(X)$ (here $\mathscr{P}_{cp,c}(X)$ denotes the family of all nonempty, compact and convex subsets of X) and $Z_2 : \bar{D} \to \mathscr{P}_{cp,c}(X)$ two multi-valued operators satisfying*

(a) *Z_1 is contraction, and*

(b) *Z_2 is u.s.c and compact.*

Then, if $G = Z_1 + Z_2$, either

(i) *G has a fixed point in $\bar{D}$ or*

(ii) *there is a point $u \in \partial D$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.*

2. First order semilinear neutral functional differential inclusions with nonlocal conditions

Let E be a separable Banach space with norm $\|\cdot\|$, $B(E)$ be the Banach space of linear bounded operators and $A : D(A) \to E$ will be the infinitesimal generator of an analytic semigroup, $\mathscr{S}(t)$, $t \geqslant 0$, of bounded linear operators on E. For the theory of strongly continuous semigroups, we refer the reader to Pazy [18]. If $\mathscr{S}(t)$, $t \geqslant 0$, is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leqslant 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in E, and the expression $\|x\|_\alpha = \|(-A)^\alpha x\|, \quad x \in D(-A)^\alpha$ defines a norm on $D(-A)^\alpha$. Hereafter we denote by E_α the Banach space $D(-A)^\alpha$ normed with $\|\cdot\|_\alpha$. Then for each $0 < \alpha \leqslant 1$, E_α is a Banach space, and $E_\alpha \hookrightarrow E_\beta$ for $0 < \beta \leqslant \alpha \leqslant 1$ and the imbedding is compact whenever the resolvent operator of A is compact. Also for every $0 < \alpha \leqslant 1$ there exists $C_\alpha > 0$ such that $\|(-A)^\alpha \mathscr{S}(t)\| \leqslant \dfrac{C_\alpha}{t^\alpha}, \quad 0 < t \leqslant T$.

Let us start by introducing the concept of mild solution for the problem (1.3)–(1.4).

DEFINITION 2.1. A function $y \in C([-r,T],E)$ is said to be a mild solution of (1.3)–(1.4) if $y_0(t) = \phi(t) + q(y_{t_1},\dots,y_{t_n})(t)$ for $t \in [-r,0]$ and there exists $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$ a.e on J and

$$\begin{aligned} y(t) &= \mathscr{S}(t)[\phi(0) + q(y_{t_1},\dots,y_{t_n})(0) - f(0,y_0)] + f(t,y_t) \\ &\quad + \int_0^t A\mathscr{S}(t-s) f(s,y_s)\,ds + \int_0^t \mathscr{S}(t-s)v(s)\,ds,\ t \in J. \end{aligned}$$

For the multivalued map F and for each $y \in C([-r,T],E)$, we define $S_{F,y}$ by

$$S_{F,y} = \{v \in L^1(J,E) : v(t) \in F(t,y_t) \text{ for a.e. } t \in J\}.$$

Our first existence result for the IVP (1.3)–(1.4) is the following.

THEOREM 2.1. *Assume that:*

(2.1.1) $A : D(A) \subset E \to E$ *is the infinitesimal generator of an analytic semigroup* $\mathscr{S}(t), t \geqslant 0$, *of bounded linear operators on* E. *Assume that* $0 \in \rho(A)$, $\mathscr{S}(t)$ *is compact for* $t > 0$, *and there exist constants* $M \geqslant 1$ *and* C, *depending on* β, $0 < \beta < 1$, *which we denote as* $C_{1-\beta}$ *such that*

$$\|\mathscr{S}(t)\|_{B(E)} \leqslant M \quad \text{and} \quad \|(-A)^{1-\beta}\mathscr{S}(t)\| \leqslant \frac{C_{1-\beta}}{t^{1-\beta}}, \text{ for all } t > 0;$$

(2.1.2) (i) *function* $q : \mathscr{D}^n \to \mathscr{D}$ *is continuous and there exists* $Q > 0$ *such that*

$$\|q(\phi_1, \dots, \phi_n)\|_{\mathscr{D}} \leqslant Q,$$

for all $\phi_i \in \mathscr{D}$, $i = 1, \dots, n$;
(ii) *there exist constants* $L_i(q)$, $i = 1, \dots, n$ *such that*

$$\|q(\psi_1, \dots, \psi_n) - q(\phi_1, \dots, \phi_n)\| \leqslant \sum_{i=1}^{n} L_i(q)\|\psi_i - \phi_i\|_{\mathscr{D}}, \text{ for all } \psi_i, \phi_i \in \mathscr{D};$$

(2.1.3) *there exist constants* $0 < \beta < 1$, c_1, c_2, L_f *such that* f *is* E_β*-valued,* $(-A)^\beta f$ *is continuous, and*

(i) $\|(-A)^\beta f(t,x)\| \leqslant c_1\|x\|_{\mathscr{D}} + c_2$, $(t,x) \in J \times \mathscr{D}$,

(ii) $\|(-A)^\beta f(t,x_1) - (-A)^\beta f(t,x_2)\| \leqslant L_f\|x_1 - x_2\|_{\mathscr{D}}$, $(t,x_i) \in J \times \mathscr{D}$, $i = 1,2$, *with* $c_1\|(-A)^{-\beta}\| < 1$ *and*

$$L_0 := M\sum_{i=1}^{n} L_i(q) + L_f(M+1)\|(-A)^{-\beta}\| + L_f\frac{C_{1-\beta}T^\beta}{\beta} < 1;$$

(2.1.4) $F : J \times \mathscr{D} \to \mathscr{P}_{cp,c}(E)$ *is a* L^1*-Carathéodory multivalued map; (that is,*

(i) $t \mapsto F(t,x)$ *is measurable for each* $x \in \mathscr{D}$,

(ii) $x \mapsto F(t,x)$ *is upper semi-continuous for almost all* $t \in J$, *and*

(iii) *for each real number* $\rho > 0$, *there exists a function* $\varphi_\rho \in L^1(J, \mathbb{R}^+)$ *such that* $\|F(t,u)\| := \sup\{\|v\| : v \in F(t,u)\} \leqslant \varphi_\rho(t)$, *a.e.* $t \in J$ *for all* $u \in \mathscr{D}$ *with* $\|u\|_{\mathscr{D}} \leqslant \rho$.*)*

(2.1.5) *there exist a* L^1*-Carathéodory function* $g : J \times [0,\infty) \to [0,\infty)$ *such that*

$$\|F(t,u)\| \leqslant g(t, \|u\|_{\mathscr{D}}) \quad \text{for almost all} \quad t \in J \text{ and all } u \in \mathscr{D};$$

(2.1.6) $g(t,\varphi)$ *is nondecreasing in* φ *for a.e.* $t \in J$;

(2.1.7) *the problem*

$$\begin{aligned} u'(t) &= bK_2 g(t,u(t)), \quad a.e.\ t \in J, \\ u(0) &= bK_0, \end{aligned}$$

where

$$K_0 = \Lambda(1 - c_1\|(-A)^{-\beta}\|)^{-1}, \quad K_1 = C_{1-\beta}c_1(1 - c_1\|(-A)^{-\beta}\|)^{-1},$$
$$K_1 T^\beta \beta^{-1} < 1, \quad K_2 = M(1 - c_1\|(-A)^{-\beta}\|)^{-1},$$
$$\Lambda = M\|\phi\|_{\mathscr{D}}\left\{1 + c_1\|(-A)^{-\beta}\|\right\} + MQ + M\|(-A)^{-\beta}\|(Q + c_2)$$
$$+ c_2\|(-A)^{-\beta}\| + \frac{C_{1-\beta}c_2T^\beta}{\beta},$$
$$b = e^{K_1^m(\Gamma(\beta))^m T^{m\beta}/\Gamma(m\beta)} \sum_{j=0}^{m-1}\left(\frac{K_1T^\beta}{\beta}\right)^j, \ \Gamma \textit{ is the Gamma function,}$$

and m is the first integer such that $m\beta > 1$, has a maximal solution $\rho(t)$.

Then the IVP (1.3)–(1.4) *has at least one mild solution on* $[-r, T]$.

REMARK 2.1. Of course (2.1.3) (i) follows from (2.1.3) (ii) with $c_1 = L_f$ and $c_2 = \max_{t\in J}\|(-A)^\beta f(t,0)\|$. However there might be a better c_1 and c_2 and the best choice is needed in (2.1.7).

Proof. Consider the operator $\mathscr{N} : C([-r,T],E) \longrightarrow \mathscr{P}(C([-r,T],E))$ defined by:

$$\mathscr{N}(y) = \left\{ \begin{array}{l} h \in C([-r,T],E): \\ h(t) = \begin{cases} \phi(t) + q(y_{t_1},\ldots,y_{t_n})(t), & t \in [-r,0], \\ \mathscr{S}(t)[\phi(0) + q(y_{t_1},\ldots,y_{t_n})(0) \\ -f(0,y_0)] + f(t,y_t) \\ +\int_0^t A\mathscr{S}(t-s)f(s,y_s)ds + \int_0^t \mathscr{S}(t-s)v(s)ds, \, t \in J, \end{cases} \end{array} \right\}$$

where $v \in S_{F,y}$. Now, we define two operators as follows. $\mathscr{A} : C([-r,T],E) \longrightarrow C([-r,T],E)$ by

$$\mathscr{A}(y)(t) = \begin{cases} q(y_{t_1},\ldots,y_{t_n})(t), & t \in [-r,0], \\ \mathscr{S}(t)[-f(0,y_0) + q(y_{t_1},\ldots,y_{t_n})(0)] + f(t,y_t) \\ +\int_0^t A\mathscr{S}(t-s)f(s,y_s)ds, & t \in J, \end{cases} \tag{2.1}$$

and the multi-valued operator $\mathscr{B} : C([-r,T],E) \longrightarrow \mathscr{P}(C([-r,T],E))$ by

$$\mathscr{B}(y) = \left\{ \begin{array}{l} h \in C([-r,T],E): \\ h(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \mathscr{S}(t)\phi(0) + \int_0^t \mathscr{S}(t-s)v(s)ds, \, t \in J \end{cases} \end{array} \right\}. \tag{2.2}$$

Then $\mathscr{N} := \mathscr{A} + \mathscr{B}$. We shall show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 1.1 on $C([-r,T],E)$. For better readability, we break the proof into a sequence of steps.

Step 1. We show that $\mathscr{A}$ *is a contraction on* $C([-r,T],E)$. Let $x,y\in C([-r,T],E)$. Then

$$\begin{aligned}\|\mathscr{A}(x)(t)-\mathscr{A}(y)(t)\| &\leqslant \|(-A)^{-\beta}\|L_f \max_{0\leqslant s\leqslant t}\|x_s-y_s\|_{\mathscr{D}}+M\sum_{i=1}^{n}L_i(q)\|x_{t_i}-y_{t_i}\|_{\mathscr{D}}\\ &\quad +ML_f\|(-A)^{-\beta}\|\|x-y\|+\int_0^t\frac{C_{1-\beta}}{(t-s)^{1-\beta}}\,ds\,L_f\max_{0\leqslant s\leqslant t}\|x_s-y_s\|_{\mathscr{D}}\\ &\leqslant\left[M\sum_{i=1}^{n}L_i(q)+L_f(M+1)\|(-A)^{-\beta}\|+L_f\frac{C_{1-\beta}T^{\beta}}{\beta}\right]\|x-y\|.\end{aligned}$$

Taking the supremum over t gives,

$$\|\mathscr{A}(x)-\mathscr{A}(y)\|\leqslant L_0\|x-y\|.$$

This shows that $\mathscr{A}$ is a contraction, since $L_0<1$.

Step 2. $\mathscr{B}$ has a *closed graph* (and therefore has closed values); see [17, Theorem 3.2, Step 4]. Also the multi-valued operator $\mathscr{B}$ is *completely continuous on* $C([-r,T],E)$. This was proved in [17, Theorem 3.2, Steps 2,3]. As a result $\mathscr{B}$ is compact valued.

Therefore the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 1.1 and hence an application of it yields that either condition (*i*) or condition (*ii*) holds. We show that the conclusion (ii) is not possible.

If $y\in\lambda\mathscr{A}(y)+\lambda\mathscr{B}(y)$ for $\lambda\in(0,1)$, then there exists $v\in S_{F,y}$ such that

$$\begin{aligned}y(t)&=\lambda\mathscr{S}(t)[\phi(0)+q(y_{t_1},\dots,y_{t_n})(0)-f(0,y_0)]+\lambda f(t,y_t)\\ &\quad+\lambda\int_0^t A\mathscr{S}(t-s)f(s,y_s)ds+\lambda\int_0^t\mathscr{S}(t-s)v(s)ds,\ t\in J.\end{aligned}\tag{2.3}$$

Then

$$\begin{aligned}\|y(t)\|&\leqslant M(\|\phi\|_{\mathscr{D}}+Q)+M\|(-A)^{-\beta}\|[c_1(\|\phi\|_{\mathscr{D}}+Q)+c_2]+\|(-A)^{-\beta}\|[c_1\|y_t\|_{\mathscr{D}}+c_2]\\ &\quad+\int_0^t\frac{C_{1-\beta}c_1\|y_s\|_{\mathscr{D}}}{(t-s)^{1-\beta}}ds+\frac{C_{1-\beta}c_2T^{\beta}}{\beta}+M\int_0^t g(s,\|y_s\|_{\mathscr{D}})ds\\ &\leqslant\Lambda+c_1\|(-A)^{-\beta}\|\|y_t\|_{\mathscr{D}}+\int_0^t\frac{C_{1-\beta}c_1\|y_s\|_{\mathscr{D}}}{(t-s)^{1-\beta}}ds+M\int_0^t g(s,\|y_s\|_{\mathscr{D}})ds,\quad t\in J.\end{aligned}$$

Put $w(t)=\max\{\|y(s)\|:-r\leqslant s\leqslant t\}$, $t\in J$. Then $\|y_t\|_{\mathscr{D}}\leqslant w(t)$ for all $t\in J$. Fix $t\in J$ and note there is a point $t^*\in[-r,t]$ such that $w(t)=\|y(t^*)\|$. If $t^*\in[-r,0]$ then $w(t)\leqslant\|\phi\|_{\mathscr{D}}+Q$. If $t\in[0,T]$ then by the previous inequality we have

$$\begin{aligned}w(t)&=\|y(t^*)\|\\ &\leqslant\Lambda+c_1\|(-A)^{-\beta}\|w(t)+C_{1-\beta}c_1\int_0^t\frac{w(s)}{(t-s)^{1-\beta}}ds+M\int_0^t g(s,w(s))\,ds,\end{aligned}$$

or

$$
\begin{aligned}
w(t) &\leqslant \frac{1}{1-c_1\|(-A)^{-\beta}\|}\left\{\Lambda + C_{1-\beta}c_1\int_0^t \frac{w(s)}{(t-s)^{1-\beta}}\,ds + M\int_0^t g(s,w(s))\,ds\right\}\\
&\leqslant K_0 + K_1\int_0^t \frac{w(s)}{(t-s)^{1-\beta}}\,ds + K_2\int_0^t g(s,w(s))\,ds, \quad t\in J.
\end{aligned}
$$

From [17, Lemma 2.3] we have

$$
w(t) \leqslant b\left(K_0 + K_2\int_0^t g(s,w(s))\,ds\right) := \zeta(t), \quad t\in J.
$$

Then we have $w(t)\leqslant \zeta(t)$ for all $t\in J,\ \zeta(0) = bK_0$ and

$$
\zeta'(t) = bK_2 g(t,w(t)), \text{ a.e. } t\in J.
$$

Using the nondecreasing character of g we get

$$
\zeta'(t) \leqslant bK_2 g(t,\zeta(t)),\ t\in J.
$$

This implies that ([16] Theorem 1.10.2) $\zeta(t)\leqslant \rho(t)$ for $t\in J$, and hence $w(t)\leqslant b_0 = \sup_{t\in J}\rho(t)$. Thus $\sup\{\|y(t)\| : -r\leqslant t\leqslant T\}\leqslant b_0' := \max\{\|\phi\|_{\mathscr{D}} + Q, b_0\}$, where b_0 depends only on T and on the function ρ. So now we can take a large enough ball so that (*ii*) does not occur. Hence the conclusion (*i*) holds and consequently the initial value problem (1.3)–(1.4) has a solution y on $[-r,T]$. This completes the proof.

REMARK 2.2. We would like point out that the condition $L_0 < 1$ can be deleted if we use the well-known Bielecki's renorming method. Of course assumption (2.1.3) has to be adjusted slightly for the new norm.

EXAMPLE 2.1. As an example of a function F satisfying the conditions (2.1.5)–(2.1.7) of Theorem 2.1 we assume that:

(h) there exist a continuous non-decreasing function $\psi : [0,\infty)\longrightarrow(0,\infty),\ p\in L^1(J,\mathbb{R}_+)$ such that $\|F(t,u)\|\leqslant p(t)\psi(\|u\|_{\mathscr{D}})$ for each $(t,u)\in J\times\mathscr{D}$ with

$$
bK_2\int_0^T p(s)\,ds < \int_{bK_0}^{\infty}\frac{ds}{\psi(s)},
$$

where b, K_0, K_2 are defined in Theorem 2.1.

Notice (2.1.5)– (2.1.7) follow if we take $g(t,u) = p(t)\psi(u)$ and the fact that the maximal solution in (2.1.7) is

$$
\rho(t) = I^{-1}\left(bK_2\int_0^t p(s)ds\right) \quad \text{where} \quad I(z) = \int_{bK_0}^{z}\frac{du}{\psi(u)}.
$$

Next we present another existence result which is natural from an application point of view.

THEOREM 2.2. *Assume that the conditions* (2.1.1)–(2.1.4) *hold. In addition we suppose that the following condition is satisfied:*

(2.2.1) *there exist a continuous non-decreasing function* $\psi : [0,\infty) \longrightarrow (0,\infty)$, $p \in L^1(J,\mathbb{R}_+)$ *such that*

$$\|F(t,u)\| := \sup\{\|v\| : v \in F(t,u)\} \leqslant p(t)\psi(\|u\|_{\mathscr{D}}) \text{ for each } (t,u) \in J \times \mathscr{D}$$

and there exists a constant $M_* > 0$ *with*

$$\left(1 - K_1 \frac{T^\beta}{\beta}\right) M_* \Big/ \left(K_0 + K_2 \psi(M_*) \int_0^T p(s)ds\right) > 1,$$

where K_0, K_1, K_2 *are as in Theorem* 2.1.

Then the initial value problem (1.3)–(1.4) *has at least one mild solution on* $[-r,T]$.

Proof. As in Theorem 2.1 consider the operators $\mathscr{N}$, $\mathscr{A}$ and $\mathscr{B}$. As in Theorem 2.1 we can prove that $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 1.1. To show that the conclusion (*ii*) is not possible we procced as follows:

From equation (2.3) as in Theorem 2.1 we get

$$w(t) \leqslant K_0 + K_1 \int_0^t \frac{w(s)}{(t-s)^{1-\beta}} ds + K_2 \int_0^t p(s)\psi(w(s))ds, \quad t \in J.$$

Consequently

$$\left(1 - K_1 \frac{T^\beta}{\beta}\right) \|w\| \Big/ \left(K_0 + K_2 \psi(\|w\|) \int_0^T p(s)ds\right) \leqslant 1. \tag{2.4}$$

If condition (*ii*) of Theorem 1.1 holds, then there exists $\lambda \in (0,1)$ and $y \in \partial D$ with $y = \lambda N(y)$. Then, y is a solution of (2.3) with $\|y\| = M_*$. Now, (2.4) implies

$$\left(1 - K_1 \frac{T^\beta}{\beta}\right) M_* \Big/ \left(K_0 + K_2 \psi(M_*) \int_0^T p(s)ds\right) \leqslant 1,$$

which contradicts (2.2.1). Hence, $\mathscr{N}$ has a fixed point in $[-r,T]$ by Theorem 1.1, and consequently the initial value problem (1.3)–(1.4) has a solution. This completes the proof.

REMARK 2.3. If ψ satisfies a sublinear condition or more generally

$$\lim_{\xi \to \infty} \frac{\xi}{K_0 + \psi(\xi) K_2 \int_0^T p(s)\, ds} > 1 - K_1 \frac{T^\beta}{\beta}$$

then the existence of M_* in (2.2.1) is guaranteed.

Next, we study the case where F is not necessarily convex valued. Our approach here is based on the Leray-Schauder Alternative for single valued maps combined with a selection theorem due to Bressan and Colombo [4] for lower semicontinuous multi-valued operators with decomposable values.

THEOREM 2.3. *Suppose that:*

(2.3.1) $F : J \times \mathscr{D} \longrightarrow \mathscr{P}(E)$ *is a nonempty, compact-valued, multivalued map such that:*
a) $(t,u) \mapsto F(t,u)$ *is* $\mathscr{L} \otimes \mathscr{B}$ *measurable;*
b) $u \mapsto F(t,u)$ *is lower semi-continuous for a.e.* $t \in J$;

(2.3.2) *for each* $\rho > 0$, *there exists a function* $\varphi_\rho \in L^1(J,\mathbb{R}^+)$ *such that*

$$\|F(t,u)\| \leqslant \varphi_\rho(t) \text{ for a.e. } t \in J \text{ and for } u \in \mathscr{D} \text{ with } \|u\|_{\mathscr{D}} \leqslant \rho.$$

In addition suppose (2.1.1)–(2.1.3), (2.1.5)–(2.1.7) *are satisfied. Then the initial value problem* (1.3)–(1.4) *has at least one solution on* $[-r,T]$.

Proof. Assumptions (2.3.1) and (2.3.2) imply that F is of lower semicontinuous type. Then there exists ([4]) a continuous function $p : C([-r,T],E) \to L^1(J,E)$ such that $p(y) \in \Phi(y)$ for all $y \in C([-r,T],E)$, where Φ is the Nemitsky operator defined by

$$\Phi(y) = \{w \in L^1(J,E) : w(t) \in F(t,y_t) \text{ for a.e. } t \in J\}.$$

Consider the problem

$$(d/dt)[y(t) - f(t,y_t)] - Ay(t) = p(y)(t), \quad t \in J, \tag{2.5}$$

$$y_0(t) = \phi(t) + q(y_{t_1},\dots,y_{t_n})(t), \quad \text{for } t \in [-r,0]. \tag{2.6}$$

It is obvious that if $y \in C([-r,T],E)$ is a solution of the problem (2.5)–(2.6), then y is a solution to the problem (1.3)–(1.4).

Consider the operator $N' : C([-r,T],E) \to C([-r,T],E)$ defined by:

$$N'(y)(t) := \begin{cases} \phi(t) + q(y_{t_1},\dots,y_{t_n})(t), & \text{if } t \in [-r,0] \\ \mathscr{S}(t)[\phi(0) + q(y_{t_1},\dots,y_{t_n})(0) - f(0,y_0)] + f(t,y_t) & \\ + \int_0^t A\mathscr{S}(t-s)f(s,y_s)ds + \int_0^t \mathscr{S}(t-s)p(y)(s)ds, \ t \in J. & \end{cases}$$

Now, we define two operators $A', B' : C([-r,T],E) \longrightarrow C([-r,T],E)$ as follows:

$$A'(y)(t) = \begin{cases} q(y_{t_1},\dots,y_{t_n})(t), & t \in [-r,0], \\ \mathscr{S}(t)[-f(0,y_0) + q(y_{t_1},\dots,y_{t_n})(0)] + f(t,y_t) & \\ + \int_0^t A\mathscr{S}(t-s)f(s,y_s)ds, & t \in J, \end{cases}$$

and

$$B'(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \mathscr{S}(t)\phi(0) + \int_0^t \mathscr{S}(t-s)p(y)ds, & t \in J, \end{cases}$$

Now $A', B' : C(J,E) \longrightarrow C(J,E)$ *are continuous.* Also the argument in Theorem 2.1 guarantees that A' and B' satisfy all the conditions of the Nonlinear Alternative for contractive maps in the single valued setting [10] and hence the problem (1.3)–(1.4) has a mild solution.

3. Semilinear neutral functional differential inclusions with nondense domain and nonlocal conditions

Recently in [1] the authors consider the following general class of nonlinear neutral functional differential equations with infinite delay

$$(d/dt)[x(t) - f(t,x_t)] = A[x(t) - f(t,x_t)] + F(t,x_t), \quad t \geqslant 0 \tag{3.1}$$

$$x_0 = \phi \in \mathscr{F} \tag{3.2}$$

where the operator A is nondensely defined, $f, F : [0,\infty) \times \mathscr{F} \to E$ and $\mathscr{F}$ is the phase space of functions mapping $(-\infty, 0]$ into E. There are many examples where evolution equations are nondensely defined. For example, when we look at a one-dimensional heat equation with Dirichlet conditions on $[0,1]$ and consider $A = \dfrac{\partial^2}{\partial x^2}$ in $C([0,1],\mathbb{R})$ in order to measure the solutions in the sup-norm, then the domain,

$$D(A) = \{\phi \in C^2([0,1],\mathbb{R}) : \phi(0) = \phi(1) = 0\},$$

is not dense in $C([0,1],\mathbb{R})$ with the sup-norm. See [6] for more examples and remarks concerning nondensely defined operators.

In this section we consider the following first order semilinear neutral functional differential inclusion with nonlocal conditions and finite delay $r > 0$,

$$(d/dt)[y(t) - f(t,y_t)] \in A[y(t) - f(t,y_t)] + F(t,y_t), \text{ a.e. } t \in J, \tag{3.3}$$

$$y_0(t) = \phi(t) + q(y_{t_1}, \dots, y_{t_n})(t), \text{ for } t \in [-r, 0], \tag{3.4}$$

where f, F, ϕ are as in problem (1.3)–(1.4) and A is nondensely defined. We give an existence result by assuming Lipschitz and growth conditions. The basic tool for this study is the theory of integrated semigroups. For details on integrated semigroups we refer to [2, 15].

Let $(\mathscr{S}(t))_{t\geqslant 0}$, be the integrated semigroup generated by A. We note that, since A satisfies the Hille-Yosida condition, $\|\mathscr{S}'(t)\|_{B(E)} \leqslant Me^{\omega t}$, $t \geqslant 0$, where M and ω are from the Hille-Yosida condition (see [15]).

In the sequel, we give some results for the existence of solutions of the following problem:

$$y'(t) = Ay(t) + g(t), \; t \geqslant 0, \tag{3.5}$$

$$y(0) = y_0 \in E, \tag{3.6}$$

where A satisfies the Hille-Yosida condition, without being densely defined.

THEOREM 3.1. [15]. *Let $g : [0,b] \to E$ be a continuous function. Then for $y_0 \in \overline{D(A)}$, there exists a unique continuous function $y : [0,b] \to E$ such that*

(*i*) $\int_0^t y(s)ds \in D(A)$ *for* $t \in [0,b]$,

(*ii*) $y(t) = y_0 + A\int_0^t y(s)ds + \int_0^t g(s)ds, \quad t \in [0,b]$,

(*iii*) $\|y(t)\| \leqslant Me^{\omega t}\left(\|y_0\| + \int_0^t e^{-\omega s}\|g(s)\|ds\right), \quad t \in [0,b]$.

Moreover, y satisfies the following variation of constant formula:

$$y(t) = \mathscr{S}'(t)y_0 + \frac{d}{dt}\int_0^t \mathscr{S}(t-s)g(s)ds, \quad t \geqslant 0. \tag{3.7}$$

THEOREM 3.2. [20]. *Let $A : D(A) \subset X \to X$ be a linear operator satisfying the Hille-Yosida condition, $\{\mathscr{S}(t)\}_{t\geqslant 0}$ be the integrated semigroup generated by A and $f : [0,T] \to X, T > 0$ be a Bohner-integrable function. Then the function $K : [0,T] \to X$ defined by $K(t) = \int_0^t \mathscr{S}(t-s)f(s)ds$ is continuously differentiable on $[0,T]$ and satisfies that, for $\lambda > \omega$ and $t \in [0,T]$,*

$$R(\lambda,A)K'(t) = \int_0^t \mathscr{S}'(t-s)R(\lambda,A)f(s)ds.$$

Let $B_\lambda = \lambda R(\lambda,A) := \lambda(\lambda I - A)^{-1}$. Then ([15]) for all $x \in \overline{D(A)}, B_\lambda x \to x$ as $\lambda \to \infty$. Also from the Hille-Yosida condition (with $n = 1$) it easy to see that $\lim\limits_{\lambda\to\infty} \|B_\lambda x\| \leqslant M\|x\|$, since

$$\|B_\lambda\| = \|\lambda(\lambda I - A)^{-1}\| \leqslant \frac{M\lambda}{\lambda - \omega}.$$

Thus $\lim\limits_{\lambda\to\infty} \|B_\lambda\| \leqslant M$. Also if y satisfies (3.7), then

$$y(t) = \mathscr{S}'(t)y_0 + \lim_{\lambda\to\infty}\int_0^t \mathscr{S}'(t-s)B_\lambda g(s)ds, \quad t \geqslant 0. \tag{3.8}$$

We define what we mean by an integral solution of the IVP (3.3)–(3.4).

DEFINITION 3.1. We say that $y : [-r,T] \to E$ is an integral solution of (3.3)–(3.4) if

(*i*) $y \in C([-r,T],E)$,

(*ii*) $\int_0^t [y(s) - f(s,y_s)]ds \in D(A)$ for $t \in J$,

(iii) there exists a function $v \in L^1(J,E)$, such that $v(t) \in F(t,y_t)$ a.e. in J and

$$y(t) = \mathscr{S}'(t)[\phi(0) + q(y_{t_1},\dots,y_{t_n})(0) - f(0,y_0)] + f(t,y_t) + \frac{d}{dt}\int_0^t \mathscr{S}(t-s)v(s)ds$$

and $y_0(t) = \phi(t) + q(y_{t_1},\dots,y_{t_n})(t)$ for $t \in [-r,0]$.

THEOREM 3.3. *Assume that* (2.1.2), (2.1.4)–(2.1.6) *hold and in addition suppose that the following conditions are satisfied:*

(3.3.1) *A satisfies the Hille-Yosida condition (then there exist $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that $\|\mathscr{S}'(t)\| \leqslant Me^{\omega t}, t \geqslant 0$);*

(3.3.2) *the operator $\mathscr{S}'(t)$ is compact in $\overline{D(A)}$ whenever $t > 0$.*

(3.3.3) $\phi(0) + q(\psi_{t_1},\dots,\psi_{t_n})(0) - f(0,\psi_0) \in \overline{D(A)}$ *for all* $\psi_0,\psi_1,\dots,\psi_n \in C([-r,0],E)$;

(3.3.4) *there exist constants $0 < c_1 < 1, c_2 \geqslant 0, \ell \geqslant 0$ such that*

(i) $\|f(t,x)\| \leqslant c_1\|x\|_{\mathscr{D}} + c_2, \quad (t,x) \in J \times \mathscr{D}$;

(ii) $\|f(t,x_1) - f(t,x_2)\| \leqslant \ell\|x_1 - x_2\|_{\mathscr{D}}$, $(t,x_i) \in J \times \mathscr{D}$, $i = 1,2$, *with*

$$M^*\sum_{i=1}^n L_i(q) + \ell(M^*+1) < 1, \quad M^* = M\max\{e^{\omega T},1\};$$

(3.3.5) *the problem*

$$\begin{aligned} u'(t) &= \frac{MM^*}{1-c_1}e^{-\omega t}g(t,u(t)), \quad a.e.\, t \in J,\\ u(0) &= \frac{M^*}{1-c_1}\left[(1+c_1)(\|\phi\|_{\mathscr{D}} + Q) + c_2 + \frac{c_2}{M^*}\right], \end{aligned}$$

has a maximal solution $\rho(t)$. (Here ω is the constant from (3.3.1)*).*

Then the IVP (3.3)–(3.4) *has at least one integral solution on* $[-r,T]$.

Proof. Consider the operator $\mathscr{N}_1 : C([-r,T],E) \to \mathscr{P}(C([-r,T],E))$ defined by

$$\mathscr{N}_1(y) := \left\{ \begin{array}{l} h \in C([-r,T],E) : \\ h(t) = \begin{cases} \phi(t) + q(y_{t_1},\dots,y_{t_n})(t), & \text{if } t \in [-r,0],\\ \mathscr{S}'(t)[\phi(0) + q(y_{t_1},\dots,y_{t_n})(0) - f(0,y_0)] & \\ +f(t,y_t) + \dfrac{d}{dt}\displaystyle\int_0^t \mathscr{S}(t-s)v(s)ds, & \text{if } t \in J, \end{cases} \end{array} \right\}$$

where $v \in S_{F,y}$. Now, we define two operators as follows. $\mathscr{A}_1 : C([-r,T],E) \longrightarrow C([-r,T],E)$ by

$$\mathscr{A}_1(y)(t) = \begin{cases} q(y_{t_1},\dots,y_{t_n})(t), & \text{if } t \in [-r,0],\\ \mathscr{S}'(t)[-f(0,y_0) + q(y_{t_1},\dots,y_{t_n})(0)] + f(t,y_t), & \text{if } t \in J, \end{cases}$$

and the multi-valued operator $\mathscr{B}_1 : C([-r,T],E) \longrightarrow \mathscr{P}(C([-r,T],E))$ by

$$\mathscr{B}_1(y) = \left\{ \begin{array}{l} h \in C([-r,T],E) : \\ h(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \mathscr{S}'(t)\phi(0) + \dfrac{d}{dt}\displaystyle\int_0^t \mathscr{S}(t-s)v(s)ds, & t \in J, \end{cases} \end{array} \right\}$$

We shall show that $\mathscr{N}_1 := \mathscr{A}_1 + \mathscr{B}_1$ has a fixed point. The proof is given in several steps.

Step 1: We show that $\mathscr{A}_1$ *is a contraction on* $C([-r,T],E)$. Let $x,y \in C([-r,T],E)$. Then

$$\begin{aligned} \|\mathscr{A}_1(x)(t) - \mathscr{A}_1(y)(t)\| &\leqslant \ell \max_{0\leqslant s\leqslant t} \|x_s - y_s\|_{\mathscr{D}} + M^* \sum_{i=1}^n L_i(q)\|x_{t_i} - y_{t_i}\|_{\mathscr{D}} + M^*\ell\|x-y\| \\ &\leqslant \left[M^* \sum_{i=1}^n L_i(q) + \ell(M^*+1)\right] \|x-y\|. \end{aligned}$$

Taking the supremum over t gives,

$$\|\mathscr{A}_1(x) - \mathscr{A}_1(y)\| \leqslant \left[M^* \sum_{i=1}^n L_i(q) + \ell(M^*+1)\right] \|x-y\|.$$

This shows that $\mathscr{A}_1$ is a contraction, since $M^* \sum_{i=1}^n L_i(q) + \ell(M^*+1) < 1$.

Step 2: $\mathscr{B}_1$ has a *closed graph* (and therefore has closed values); see [17, Theorem 4.6, Step 4]. Also [17, Theorem 4.6] guarantees that the operator $\mathscr{B}_1$ is completely continuous on $C([-r,T],E)$. As a result $\mathscr{B}$ is compact valued.

Therefore the operators $\mathscr{A}_1$ and $\mathscr{B}_1$ satisfy all the conditions of Theorem 1.1 and hence an application of it yields that either condition (*i*) or condition (*ii*) holds. We show that the conclusion (*ii*) is not possible.

If $y \in \sigma\mathscr{A}_1(y) + \sigma\mathscr{B}_1(y)$ for $\sigma \in (0,1)$, then

$$\begin{aligned} y(t) &= \sigma\mathscr{S}'(t)[\phi(0) + q(y_{t_1},\ldots,y_{t_n})(0) - f(0,y_0)] + \sigma f(t,y_t) \\ &\quad + \sigma\frac{d}{dt}\int_0^t \mathscr{S}(t-s)v(s)ds,\ t \in J. \end{aligned} \tag{3.9}$$

From Theorem 3.2 we get

$$\begin{aligned} \left\|B_\lambda \frac{d}{dt}\int_0^t \mathscr{S}(t-s)v(s)ds\right\| &= \left\|\int_0^t \mathscr{S}'(t-s)B_\lambda v(s)ds\right\| \\ &\leqslant Me^{\omega T}\int_0^t e^{-\omega s}\|B_\lambda\|\|v(s)\|ds \\ &\leqslant M^*\int_0^t e^{-\omega s}\|B_\lambda\|\|v(s)\|ds. \end{aligned}$$

Letting $\lambda \to +\infty$, we obtain that

$$\lim_{\lambda\to+\infty}\left\|\int_0^t \mathscr{S}'(t-s)B_\lambda v(s)ds\right\| \leqslant MM^*\int_0^t e^{-\omega s}\|v(s)\|ds.$$

Thus

$$\|y(t)\| \leqslant M^*[(1+c_1)(\|\phi\|_{\mathscr{D}}+Q)+c_2]+c_1\|y_t\|_{\mathscr{D}}+c_2 \\ +MM^*\int_0^t e^{-\omega s}g(s,\|y_s\|_{\mathscr{D}})ds,\ t\in J.$$

We consider the function μ defined by $\mu(t) := \sup\{\|y(s)\| : -r \leqslant s \leqslant t\}$, $t \in [0,T]$. Let $t^* \in [-r,t]$ be such that $\mu(t) = \|y(t^*)\|$. If $t^* \in [0,T]$, then by the previous inequality, we have for $t \in [0,T]$,

$$(1-c_1)\mu(t) \leqslant M^*[(1+c_1)(\|\phi\|_{\mathscr{D}}+Q)+c_2]+c_2+MM^*\int_0^t e^{-\omega s}g(s,\mu(s))ds,$$

or

$$\mu(t) \leqslant \frac{M^*}{1-c_1}\left[(1+c_1)(\|\phi\|_{\mathscr{D}}+Q)+c_2+\frac{c_2}{M^*}+M\int_0^t e^{-\omega s}g(s,\mu(s))ds\right],\ t\in J.$$

If $t^* \in [-r,0]$ then $\mu(t) \leqslant \|\phi\|_{\mathscr{D}}+Q$ and the inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$v(0) = \frac{M^*}{1-c_1}\left[(1+c_1)(\|\phi\|_{\mathscr{D}}+Q)+c_2+\frac{c_2}{M^*}\right],\quad \mu(t)\leqslant v(t),\ t\in J$$

and

$$v'(t) = \frac{MM^*}{1-c_1}e^{-\omega t}g(t,\mu(t)) \\ \leqslant \frac{MM^*}{1-c_1}e^{-\omega t}g(t,v(t)),\ t\in[0,T].$$

This implies that ([16] Theorem 1.10.2) $v(t) \leqslant \rho(t)$ for $t \in J$, and hence $\|y(t)\| \leqslant b_2' = \sup_{t\in[-r,T]}\rho(t)$, $t \in [-r,0]$ where b_2' depends only on T and on the function ρ. So now we can take a large enough ball so that (ii) does not occur. Hence the conclusion (i) holds and consequently the initial value problem (3.3)-(3.4) has a solution y on $[-r,T]$. This completes the proof.

EXAMPLE 3.1. As an example of a function F satisfying the conditions (2.1.5), (2.1.6) of Theorem 2.1 and (3.3.5) of Theorem 3.3 we assume that:

(h1) there exist a continuous non-decreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$, $p \in L^1(J,\mathbb{R}_+)$ such that $\|F(t,u)\| \leqslant p(t)\psi(\|u\|_{\mathscr{D}})$ for each $(t,u) \in J\times\mathscr{D}$ with

$$\int_0^T \gamma(s)\,ds < \int_c^\infty \frac{ds}{\psi(s)},$$

where

$$\gamma(t) = \frac{MM^* e^{-\omega t} p(t)}{1-c_1} \quad \text{and} \quad c = \frac{M^*}{1-c_1}\left[(1+c_1)(\|\phi\|_{\mathscr{D}}+Q)+c_2+\frac{c_2}{M^*}\right].$$

The ideas in the proof of Theorem 2.2 immediately yield the following result.

THEOREM 3.4. *Assume that the conditions* (2.1.2), (2.1.4), (3.3.1)–(3.3.4) *hold. In addition we suppose that the following condition is satisfied:*

(3.4.1) *there exist a continuous non-decreasing function* $\psi : [0,\infty) \longrightarrow (0,\infty)$, $p \in L^1(J,\mathbb{R}_+)$ *such that*

$$\|F(t,u)\| \leqslant p(t)\psi(\|u\|_{\mathscr{D}}) \text{ for each } (t,u) \in J\times\mathscr{D}$$

and there exists a constant $M_* > 0$ *with*

$$\frac{(1-c_1)M_*}{M^*(1+c_1)\|\phi\|_{\mathscr{D}}+M^*(c_2+Q)+c_2+MM^*\psi(M_*)\displaystyle\int_0^T e^{-\omega s}p(s)\,ds} > 1.$$

Then the IVP (3.3)–(3.4) *has at least one integral solution on* $[-r,T]$.

4. Semilinear evolution differential inclusions with infinite delay

In this section we are interested in existence results for neutral functional differential inclusions with infinite delay of the form

$$(d/dt)[y(t)-f(t,y_t)] \in Ay(t)+F(t,y_t),\ \ t\in J:=[0,T], \tag{4.1}$$

$$y_0(t) = \phi(t)+q(y_{t_1},\dots,y_{t_n})(t),\ \text{for } t\in(-\infty,0], \tag{4.2}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\mathscr{S}(t), t\geqslant 0$ on a Banach space E, $F : J\times\mathscr{F}\to\mathscr{P}(E)$ is a bounded, closed, convex-valued multivalued map, $f : J\times\mathscr{F}\to E$ and $y_t : (-\infty,0]\to E$, $y_t(\theta) = y(t+\theta), \theta\leqslant 0$ belongs to some abstract phase space $\mathscr{F}$, that is, a linear space of functions mapping $(-\infty,0]$ into E endowed with a seminorm $\|\cdot\|_{\mathscr{F}}$ in $\mathscr{F}$, $0<t_1<t_2<\dots<t_n\leqslant T$ and $q : \mathscr{F}^n\to\mathscr{F}$.

Let $\mathscr{F}_T$ be the space of all functions $y : (-\infty,T]\to E$ such that $y_0\in\mathscr{F}$ and the restriction $y : J\to E$ is continuous. Let $\|\cdot\|_T$ be the seminorm in $\mathscr{F}_T$ defined by

$$\|y\|_T = \|y_0\|_{\mathscr{F}}+\sup\{\|y(s)\| : 0\leqslant s\leqslant T\},\ y\in\mathscr{F}_T.$$

We will employ an axiomatic definition of the phase space $\mathscr{F}$ similar to those introduced by Hale and Kato [11] (see also the book of Hino *et al* [14]). We will assume that $\mathscr{F}$ satisfies the following axioms:

(A) If $y:(-\infty,T]\to E, T>0$ is such that $y|_{[0,T]}$ is continuous and $y_0\in\mathscr{F}$, then for every t in $[0,T)$ the following conditions hold:

(*i*) y_t is in $\mathscr{F}$;

(*ii*) $\|y(t)\|\leqslant H\|y_t\|_{\mathscr{F}}$;

(*iii*) $\|y_t\|_{\mathscr{F}}\leqslant K(t)\sup\{\|y(s)\|:0\leqslant s\leqslant t\}+M(t)\|y_0\|_{\mathscr{F}}$,

where $H\geqslant 0$ is a constant, $K:[0,\infty)\to[0,\infty)$ is continuous, $M:[0,\infty)\to[0,\infty)$ is locally bounded and H,K,M are independent of $y(\cdot)$.

Denote

$$K_T=\sup\{K(t),t\in[0,T]\},\quad M_T=\sup\{M(t),t\in[0,T]\}.$$

(A-1) For the function y in (A), the mapping $t\in[0,T)\to y_t\in\mathscr{F}$ is continuous on $[0,T)$.

(A-2) The space $\mathscr{F}$ is complete.

In the statements that follow we consider conditions (2.1.2), (2.1.4), (2.1.5) with $\mathscr{F}$ instead of $\mathscr{D}$. Also for the remainder of this section we will have an extra assumption, namely we will assume $(\mathscr{F},\|.\|_{\mathscr{F}})$ is a normed space.

THEOREM 4.1. *Assume that* (2.1.1), (2.1.2), (2.1.4), (2.1.5), *and* (2.1.6) *hold. In addition we suppose that*

(4.1.1) *there exist constants* $0<\beta<1, c_1, c_2, L_f$ *such that* f *is* E_β*-valued,* $(-A)^\beta f$ *is continuous, and*

(i) $\|(-A)^\beta f(t,x)\|\leqslant c_1\|x\|_{\mathscr{F}}+c_2$, $(t,x)\in J\times\mathscr{F}$,

(ii) $\|(-A)^\beta f(t,x_1)-(-A)^\beta f(t,x_2)\|\leqslant L_f\|x_1-x_2\|_{\mathscr{F}}$, $(t,x_i)\in J\times\mathscr{F}$, $i=1,2$,

(4.1.2) *the problem*

$$\begin{aligned} u'(t)&=b'K_2'g(t,u(t)),\quad a.e.\, t\in J,\\ u(0)&=b'K_0',\end{aligned}$$

where

$$K_0'=[M_T\|\phi\|_{\mathscr{F}}+K_T\Lambda'](1-c_1\|(-A)^{-\beta}\|)^{-1},$$

$$K_1'=K_TC_{1-\beta}c_1(1-c_1\|(-A)^{-\beta}\|)^{-1},\quad K_1'T^\beta\beta^{-1}<1$$

$$K_2'=K_TM(1-c_1\|(-A)^{-\beta}\|)^{-1},\quad b'=e^{K_1'^m(\Gamma(\beta))^mT^{m\beta}/\Gamma(m\beta)}\sum_{j=0}^{m-1}\left(\frac{K_1'T^\beta}{\beta}\right)^j,$$

$$\Lambda'=M\|\phi\|_{\mathscr{F}}\left\{H+c_1\|(-A)^{-\beta}\|\right\}+MQ(1+\|(-A)^{-\beta}\|)$$
$$+c_2\|(-A)^{-\beta}\|\{M+1\}+\frac{C_{1-\beta}c_2T^\beta}{\beta},$$

has a maximal solution $r(t)$;

(4.1.3) *assume that*

$$\Theta = \max\left\{K_T\Lambda_1 + M_T^2\sum_{i=1}^{n} L_i(q), \Lambda_2\sum_{i=1}^{n} L_i(q)\right\} < 1,$$

where

$$\begin{aligned}\Lambda_1 &= M_T\left[\|(-A)^{-\beta}\|L_f + MH\sum_{i=1}^{n} L_i(q) + L_f\frac{C_{1-\beta}T^{\beta}}{\beta}\right] + ML_f\|(-A)^{-\beta}\|,\\ \Lambda_2 &= K_T\left[\|(-A)^{-\beta}\|L_f + MH\sum_{i=1}^{n} L_i(q) + L_f\frac{C_{1-\beta}T^{\beta}}{\beta}\right].\end{aligned}$$

Then the IVP (4.1)–(4.2) *has at least one mild solution on* $(-\infty, T]$.

Proof. Consider the operator $\mathscr{N}' : \mathscr{F}_T \longrightarrow \mathscr{P}(\mathscr{F}_T)$ defined by:

$$\mathscr{N}'(y) = \left\{\begin{array}{l} h \in C((-\infty, T], E) : \\ h(t) = \begin{cases} \phi(t) + q(y_{t_1}, \dots, y_{t_n})(t), & t \in (-\infty, 0], \\ \mathscr{S}(t)[\phi(0) + q(y_{t_1}, \dots, y_{t_n})(0) - f(0, y_0)] & \\ +f(t, y_t) + \int_0^t A\mathscr{S}(t-s)f(s, y_s)ds & \\ +\int_0^t \mathscr{S}(t-s)v(s)ds, & t \in J, \end{cases}\end{array}\right\}$$

where $v \in S_{F,y}$. Now, we define two operators as follows. $\mathscr{A}' : \mathscr{F}_T \longrightarrow \mathscr{F}_T$ by

$$\mathscr{A}'(y)(t) = \begin{cases} \phi(t) + q(y_{t_1}, \dots, y_{t_n})(t), & t \in (-\infty, 0], \\ \mathscr{S}(t)[\phi(0) - f(0, y_0) + q(y_{t_1}, \dots, y_{t_n})(0)] + f(t, y_t) & \\ + \int_0^t A\mathscr{S}(t-s)f(s, y_s)ds, & t \in J, \end{cases} \tag{4.3}$$

and the multi-valued operator $\mathscr{B}' : \mathscr{F}_T \longrightarrow \mathscr{P}(\mathscr{F}_T)$ by

$$\mathscr{B}'(y) = \left\{h \in C([-r, T], E) : h(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \int_0^t \mathscr{S}(t-s)v(s)ds, & t \in J, \end{cases}\right\} \tag{4.4}$$

Then $\mathscr{N}' = \mathscr{A}' + \mathscr{B}'$. We shall show that the operators $\mathscr{A}'$ and $\mathscr{B}'$ satisfy all the conditions of Theorem 1.1. For better readability, we break the proof into a sequence of steps.

Step 1: We show that $\mathscr{A}'$ *is a contraction on* $\mathscr{F}_T$. Let $x, y \in \mathscr{F}_T$. Then

$$\begin{aligned}
&\|\mathscr{A}'(x)(t) - \mathscr{A}'(y)(t)\| \\
&\leqslant ML_f\|(-A)^{-\beta}\|\|x_0 - y_0\|_{\mathscr{F}} + MH\sum_{i=1}^{n} L_i(q)\|x_{t_i} - y_{t_i}\|_{\mathscr{F}} \\
&\quad + \|(-A)^{-\beta}\|L_f\|x_t - y_t\|_{\mathscr{F}} + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}\,ds\,L_f \max_{0\leqslant s\leqslant t}\|x_s - y_s\|_{\mathscr{F}} \\
&\leqslant \left[\|(-A)^{-\beta}\|L_f + MH\sum_{i=1}^{n} L_i(q) + L_f\frac{C_{1-\beta}T^{\beta}}{\beta}\right][K_T\|x-y\|_T + M_T\|x_0 - y_0\|_{\mathscr{F}}] \\
&\quad + ML_f\|(-A)^{-\beta}\|\|x_0 - y_0\|_{\mathscr{F}} \\
&\leqslant \Lambda_1\|x_0 - y_0\|_{\mathscr{F}} + \Lambda_2\|x-y\|_T.
\end{aligned}$$

On the other hand for $t \in [-r, 0]$ we have $\mathscr{A}'(y)(t) = \phi(t) + q(y_{t_1}, \ldots, y_{t_n})(t)$ from which we deduce

$$\|(\mathscr{A}'(x))_0 - (\mathscr{A}'(y))_0\| \leqslant M_T\sum_{i=1}^{n} L_i(q)\|x_0 - y_0\|_{\mathscr{F}} + K_T\sum_{i=1}^{n} L_i(q)\|x-y\|_T.$$

The above relations and assumption (4.1.3) show that $\mathscr{A}'$ is a contraction.

Step 2: Theorem 2.1 guarantees that the multi-valued operator $\mathscr{B}'$ has compact convex values and it is completely continuous.

Therefore the operators $\mathscr{A}'$ and $\mathscr{B}'$ satisfy all the conditions of Theorem 1.1 and hence an application of it yields that either condition (*i*) or condition (*ii*) holds. We show that the conclusion (*ii*) is not possible. If $y \in \lambda\mathscr{A}'(y) + \lambda\mathscr{B}'(y)$ for $\lambda \in (0,1)$, then there exists $v \in S_{F,y}$ such that

$$\begin{aligned}
y(t) &= \lambda\mathscr{S}(t)[\phi(0) + q(y_{t_1}, \ldots, y_{t_n})(0) - f(0, y_0)] + \lambda f(t, y_t) \\
&\quad + \lambda\int_0^t A\mathscr{S}(t-s)f(s, y_s)ds + \lambda\int_0^t \mathscr{S}(t-s)v(s)ds,\ t \in J.
\end{aligned} \tag{4.5}$$

Then

$$\begin{aligned}
\|y(t)\| &\leqslant M(H\|\phi\|_{\mathscr{F}} + Q) + M\|(-A)^{-\beta}\|[c_1(\|\phi\|_{\mathscr{F}} + Q) + c_2] \\
&\quad + \|(-A)^{-\beta}\|[c_1\|y_t\|_{\mathscr{F}} + c_2] \\
&\quad + C_{1-\beta}c_1\int_0^t \frac{\|y_s\|_{\mathscr{F}}}{(t-s)^{1-\beta}}\,ds + \frac{C_{1-\beta}c_2T^{\beta}}{\beta} + M\int_0^t g(s, \|y_s\|_{\mathscr{F}})ds \\
&\leqslant \Lambda' + c_1\|(-A)^{-\beta}\|\omega(t) + C_{1-\beta}c_1\int_0^t \frac{\omega(s)}{(t-s)^{1-\beta}}\,ds \\
&\quad + M\int_0^t g(s, \omega(s))ds, \quad t \in J,
\end{aligned}$$

where the inequality

$$\|y_t\|_{\mathscr{F}} \leqslant K_T \sup\{\|y(s)\| : 0 \leqslant s \leqslant t\} + M_T\|\phi\|_{\mathscr{F}} := \omega(t)$$

is used. Since $\omega(\cdot)$ is nondecreasing, it follows that

$$\sup\{\|y(s)\| : 0 \leqslant s \leqslant t\} \leqslant \Lambda' + c_1\|(-A)^{-\beta}\|\omega(t) \\ +C_{1-\beta}c_1 \int_0^t \frac{\omega(s)}{(t-s)^{1-\beta}}\,ds + M\int_0^t g(s,\omega(s))ds, \quad t \in J.$$

Employing the last inequality and the definition of $\omega(\cdot)$ we get

$$\begin{aligned}\omega(t) &= M_T\|\phi\|_{\mathscr{F}} + K_T \sup\{\|y(s)\| : 0 \leqslant s \leqslant t\} \\ &\leqslant M_T\|\phi\|_{\mathscr{F}} + K_T\Lambda' + K_T c_1\|(-A)^{-\beta}\|\omega(t) \\ &\quad +K_T C_{1-\beta}c_1 \int_0^t \frac{\omega(s)}{(t-s)^{1-\beta}}\,ds + MK_T\int_0^t g(s,\omega(s))ds, \quad t \in J,\end{aligned}$$

and hence

$$\begin{aligned}\omega(t) &\leqslant \frac{1}{1-K_T c_1\|(-A)^{-\beta}\|}\Big\{M_T\|\phi\|_{\mathscr{F}} + K_T\Lambda' + K_T C_{1-\beta}c_1\int_0^t \frac{\omega(s)}{(t-s)^{1-\beta}}\,ds \\ &\quad +MK_T\int_0^t g(s,\omega(s))\,ds\Big\} \\ &\leqslant K_0' + K_1'\int_0^t \frac{\omega(s)}{(t-s)^{1-\beta}}\,ds + K_2'\int_0^t g(s,\omega(s))\,ds, \quad t \in J.\end{aligned}$$

From [17, Lemma 2.3] we have

$$\omega(t) \leqslant b\left(K_0' + K_2'\int_0^t g(s,\omega(s))\,ds\right), \quad t \in J.$$

Proceed as in Theorem 2.1 to get that $\|y\| \leqslant b_0'$. So now we can take a large enough ball so that (*ii*) does not occur. Hence the conclusion (*i*) holds and consequently the initial value problem (1.3)–(1.4) has a solution y on $(-\infty, T]$. This completes the proof.

EXAMPLE 4.1. As an example of a function F satisfying the conditions (2.1.5), (2.1.6) of Theorem 2.1 and (4.1.2) of Theorem 4.1 we assume that:

(h2) there exist a continuous non-decreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$, $p \in L^1(J,\mathbb{R}_+)$ such that $\|F(t,u)\| \leqslant p(t)\psi(\|u\|_{\mathscr{D}})$ for each $(t,u) \in J \times \mathscr{D}$ with

$$bK_2'\int_0^T p(s)\,ds < \int_{bK_{0'}}^{\infty} \frac{ds}{\psi(s)},$$

where b, K_0', K_2' are defined in Theorem 4.1.

THEOREM 4.2. *Assume that the conditions* (2.1.1)*,* (2.1.2)*,* (2.1.4)*,* (4.1.1) *and* (4.1.3) *hold. In addition we suppose that the following condition is satisfied:*

(4.2.1) *there exist a continuous non-decreasing function* $\psi : [0,\infty) \longrightarrow (0,\infty)$, $p \in L^1(J,\mathbb{R}_+)$ *such that*

$$\|F(t,u)\| \leqslant p(t)\psi(\|u\|_{\mathscr{F}}) \text{ for each } (t,u) \in J \times \mathscr{D}$$

and there exists a constant $M_* > 0$ *with*

$$\left(1 - K_1' \frac{T^\beta}{\beta}\right) M_* \Big/ \left(K_0' + K_2' \psi(M_*) \int_0^T p(s)ds\right) > 1,$$

where K_0', K_1', K_2' *are as in Theorem* 4.1.

Then the IVP (3.3)–(3.4) *has at least one mild solution on* $(-\infty, T]$.

5. Second order semilinear neutral functional differential inclusions with nonlocal conditions

In this section we study the problem (1.5)–(1.6). Some preliminaries facts are necessary.

We say that a family $\{C(t) \mid t \in \mathbb{R}\}$ of operators in $B(E)$ is a *strongly continuous cosine family* if

(i) $C(0) = I$,

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$, for all $s,t \in \mathbb{R}$,

(iii) the map $t \mapsto C(t)(x)$ is strongly continuous, for each $x \in E$.

The strongly continuous sine family $\{S(t) \mid t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) \mid t \in \mathbb{R}\}$, is defined by

$$S(t)(x) = \int_0^t C(s)(x)\,ds, \quad x \in E,\ t \in \mathbb{R}. \tag{5.1}$$

The infinitesimal generator $A : E \to E$ of a cosine family $\{C(t) \mid t \in \mathbb{R}\}$ is defined by

$$A(x) = \frac{d^2}{dt^2} C(t)(x)\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [8], Heikkila and Lakshmikantham [12] and Fattorini [7] and the papers [21] (in particular Proposition 2.2) and [22].

PROPOSITION 5.1. [21] *Let* $C(t), t \in \mathbb{R}$ *be a strongly continuous cosine family in* E. *Then:*

(i) *there exist constants* $M_1 \geqslant 1$ *and* $\omega \geqslant 0$ *such that* $|C(t)| \leqslant M_1 e^{\omega|t|}$ *for all* $t \in \mathbb{R}$;

(ii) $|S(t_1) - S(t_2)| \leqslant M_1 \left| \int_{t_2}^{t_1} e^{\omega|s|} ds \right|$ *for all* $t_1, t_2 \in \mathbb{R}$.

DEFINITION 5.1. A function $y \in C([-r,T],E)$ is said to be a mild solution of (1.5)-(1.6) if $y_0(t) = \phi(t) + q(y_{t_1},\dots,y_{t_n})(t)$, for $t \in [-r,0]$, $\quad y'(0^+) + h(y) = \eta$ and there exists $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$ a.e. on J and

$$
\begin{aligned}
y(t) &= C(t)[\phi(0) + q(y_{t_1},\dots,y_{t_n})(0)] + S(t)[\eta - h(y) - f(0,y_0)] \\
&\quad + \int_0^t C(t-s)f(s,y_s)ds + \int_0^t S(t-s)v(s)ds, \ t \in J.
\end{aligned}
$$

THEOREM 5.1. *Assume* (2.1.2)*,* (2.1.4)*,* (2.1.5)*,* (2.1.6)*, and the conditions*

(5.1.1) (*i*) *the function* $h : C([-r,T],E) \to E$ *is continuous and there exists a constant* $Q_1 > 0$ *such that* $\|h(y)\| \leqslant Q_1$, *for all* $y \in C([-r,T],E)$;
(*ii*) *there exists a constant* $k_1 > 0$ *such that*

$$\|h(x) - h(y)\| \leqslant k_1\|x - y\|, \textit{ for all } x,y \in C([-r,T],E)$$

(5.1.2) *for any* $y \in C([-r,T],E)$ *there exists a* $\delta > 0$ *such that*

(*a*) *the map* $t \mapsto C(t)[\phi(0) + q(y_{t_1},\dots,y_{t_n})(0)]$ *is continuously differentiable on* $(0,\delta)$, *and*

(*b*) *the map* $s \mapsto f(s,y_s)$ *is continuously differentiable a.e. on* $(0,\delta)$;

(5.1.3) *there exist constants* $0 < c_1 < 1, c_2 \geqslant 0, \ell \geqslant 0$ *such that*

(*i*) $\|f(t,x)\| \leqslant c_1\|x\|_{\mathscr{D}} + c_2, \quad (t,x) \in J \times \mathscr{D}$;

(*ii*) $\|f(t,x_1) - f(t,x_2)\| \leqslant \ell\|x_1 - x_2\|_{\mathscr{D}}, \ (t,x_i) \in J \times \mathscr{D}, \ i = 1,2$;

(5.1.4) $A : D(A) \subset E \to E$ *is the infinitesimal generator of a strongly continuous cosine family* $\{C(t) : t \in J\}$, *and there exist constants* $N_1 \geqslant 1$, *and* $N_2 \geqslant 0$ *such that* $\|C(t)\|_{B(E)} \leqslant N_1$, $\|S(t)\|_{B(E)} \leqslant N_2$ *for all* $t \in [0,T]$;

(5.1.5) *for each bounded* $B \subseteq C([-r,T],E)$, *and* $t \in J$ *the set*

$$\left\{ \int_0^t S(t-s)v(s)ds, v \in S_{F,B} \right\}$$

is relatively compact in E, *where* $S_{F,B} = \cup\{S_{F,y} : y \in B\}$;

(5.1.6) $L_1 := N_1\left(\ell T + \sum_{i=1}^n L_i(q)\right) + N_2(k_1 + \ell) < 1$,

(5.1.7) *the problem*

$$
\begin{aligned}
u'(t) &= N_1c_1u(t) + N_2g(t,u(t)), \quad a.e.\ t \in J, \\
u(0) &= C_1,
\end{aligned}
$$

where $C_1 = N_1[\|\phi\|_{\mathscr{D}} + Q] + N_2[\|\eta\| + Q_1 + c_1(\|\phi\|_{\mathscr{D}} + Q) + c_2] + N_1c_2T$, *has a maximal solution* $\rho(t)$,

are satisfied. Then the problem (1.5)–(1.6) *has at least one mild solution on* $[-r,T]$.

Proof. Consider the multivalued map $\mathscr{N}_2 : C([-r,T],E) \longrightarrow \mathscr{P}(C([-r,T],E))$ defined by

$$\mathscr{N}_2(y) := \left\{ \begin{array}{l} g \in C([-r,T],E) : \\ g(t) = \begin{cases} \phi(t) + q(y_{t_1},\ldots,y_{t_n})(t), & \text{if } t \in [-r,0] \\ C(t)[\phi(0)+q(y_{t_1},\ldots,y_{t_n})(0)] \\ +S(t)[\eta - h(y) - f(0,y_0)] \\ +\int_0^t C(t-s)f(s,y_s)ds \\ +\int_0^t S(t-s)v(s)ds, & \text{if } t \in [0,T] \end{cases} \end{array} \right\}$$

where $v \in S_{F,y}$. We shall show that $\mathscr{N}_2$ has a fixed point. Now, we define two operators $\mathscr{A}_2 : C([-r,T],E) \longrightarrow C([-r,T],E)$ by

$$\mathscr{A}_2(y)(t) = \begin{cases} q(y_{t_1},\ldots,y_{t_n})(t), & \text{if } t \in [-r,0] \\ C(t)q(y_{t_1},\ldots,y_{t_n})(0) \\ S(t)[-h(y)-f(0,y_0)] + \int_0^t C(t-s)f(s,y_s)ds, & \text{if } t \in J \end{cases}$$

and the multivalued map $\mathscr{B}_2 : C([-r,T],E) \longrightarrow \mathscr{P}(C([-r,T],E))$ by

$$\mathscr{B}_2(y) := \left\{ \begin{array}{l} g \in C([-r,T],E) : \\ g(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0] \\ C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)v(s)ds, & \text{if } t \in J \end{cases} \end{array} \right\}$$

where $v \in S_{F,y}$. We shall show that $\mathscr{N}_2 := \mathscr{A}_2 + \mathscr{B}_2$ has a fixed point, by showing that the operators $\mathscr{A}_2$ and $\mathscr{B}_2$ satisfy all the conditions of Theorem 1.1 on $C([-r,T],E)$. We break the proof into a sequence of steps.

Step 1: We show that $\mathscr{A}_2$ *is a contraction on* $C([-r,T],E)$. Let $x,y \in C([-r,T],E)$. Then

$$\begin{aligned} \|\mathscr{A}_2(x)(t) - \mathscr{A}_2(y)(t)\| &\leqslant N_1 \sum_{i=1}^n L_i(q)\|x-y\| + N_2k_1\|x-y\| \\ &\quad + N_2\ell\|x-y\| + N_1\ell T \max_{0\leqslant s\leqslant t} \|x_s - y_s\|_{\mathscr{D}} \\ &\leqslant \left[N_1 \sum_{i=1}^n L_i(q) + N_2k_1 + N_2\ell + N_1\ell T \right] \|x-y\|. \end{aligned}$$

Taking the supremum over t gives,

$$\|\mathscr{A}_2(x) - \mathscr{A}_2(y)\| \leqslant \left[N_1 \sum_{i=1}^n L_i(q) + N_1\ell T \right] \|x-y\|.$$

This shows that $\mathscr{A}_2$ is a contraction, since $L_1 < 1$.

Step 2: $\mathscr{B}_2$ has a *closed graph* (and therefore has closed values); see [17, Theorem 5.2, Step 4]. Moreover the operator $\mathscr{B}_2$ is completely continuous on $C([-r,T],E)$, as [17, Theorem 5.2] guarantees. As a result $\mathscr{B}_2$ is compact valued.

Therefore the operators $\mathscr{A}_2$ and $\mathscr{B}_2$ satisfy all the conditions of Theorem 1.1 and hence an application of it yields that either condition (*i*) or condition (*ii*) holds. We show that the conclusion (*ii*) is not possible.

If $y \in \lambda\mathscr{A}_2(y) + \lambda\mathscr{B}_2(y)$ for $\lambda \in (0,1)$, then there exists $v \in S_{F,y}$ such that

$$\begin{aligned} y(t) &= \lambda C(t)[\phi(0) + q(y_{t_1},\ldots,y_{t_n})(0)] + \lambda S(t)[\eta - h(y) - f(0,y_0)] \\ &\quad + \lambda\int_0^t C(t-s)f(s,y_s)ds + \lambda\int_0^t S(t-s)v(s)ds, \;\; t \in J. \end{aligned} \tag{5.2}$$

This implies by our assumptions that for each $t \in J$ we have

$$\begin{aligned} \|y(t)\| &\leqslant N_1[\|\phi\|_{\mathscr{D}} + Q] + N_2[\|\eta\| + Q_1 + c_1(\|\phi\|_{\mathscr{D}} + Q) + c_2] \\ &\quad + N_1\int_0^t (c_1\|y_s\|_{\mathscr{D}} + c_2)ds + N_2\int_0^t g(s,\|y_s\|_{\mathscr{D}})ds. \end{aligned}$$

We consider the function μ defined by $\mu(t) = \sup\{\|y(s)\| : -r \leqslant s \leqslant t\},\;\; 0 \leqslant t \leqslant T$. Let $t^* \in [-r,t]$ be such that $\mu(t) = \|y(t^*)\|$. If $t^* \in J$, by the previous inequality we have for $t \in J$

$$\begin{aligned} \mu(t) &\leqslant N_1[\|\phi\|_{\mathscr{D}} + Q] + N_2[\|\eta\| + Q_1 + c_1(\|\phi\|_{\mathscr{D}} + Q) + c_2] \\ &\quad + N_1c_1\int_0^t \mu(s)ds + N_1c_2T + N_2\int_0^t g(s,\mu(s))ds \\ &\leqslant C_1 + N_1c_1\int_0^t \mu(s)ds + N_2\int_0^t g(s,\mu(s))ds. \end{aligned}$$

If $t^* \in J_0$ then $\mu(t) \leqslant \|\phi\|_{\mathscr{D}} + Q$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $\gamma(t)$. Then we have

$$\gamma(0) = C_1, \quad \mu(t) \leqslant \gamma(t),\; t \in J$$

and

$$\begin{aligned} \gamma'(t) &= N_1c_1\mu(t) + N_2g(t,\mu(t)) \\ &\leqslant N_1c_1v(t) + N_2g(t,\gamma(t)),\; t \in [0,T]. \end{aligned}$$

This implies that ([16] Theorem 1.10.2) $\gamma(t) \leqslant \rho(t)$ for $t \in J$, and hence $\|y(t)\| \leqslant b_1' = \sup_{t\in[-r,T]}\rho(t)$, where b_1' depends only on T and on the function ρ. So now we can take a large enough ball so that (*ii*) does not occur. Hence the conclusion (*i*) holds and consequently the initial value problem (1.5)-(1.6) has a solution y on $[-r,T]$. This completes the proof.

EXAMPLE 5.1. As an example of a function F satisfying the conditions (2.1.5), (2.1.6) of Theorem 2.1 and (5.1.5) of Theorem 5.1 we assume that:

(h) there exist a continuous non-decreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$, $p \in L^1(J,\mathbb{R}_+)$ such that $\|F(t,u)\| \leqslant p(t)\psi(\|u\|_{\mathscr{D}})$ for each $(t,u) \in J \times \mathscr{D}$ with

$$\int_0^T \gamma_1(s)\,ds < \int_{C_1}^{\infty} \frac{ds}{s+\psi(s)},$$

where $\gamma_1(t) = \max\{N_1 c_1, N_2 p(t)\}$ and C_1 as defined in Theorem 5.1.

THEOREM 5.2. *Assume that the conditions* (2.1.2), (2.1.4), (5.1.1) *hold. In addition we suppose that the following condition is satisfied:*

(5.2.1) *there exist a continuous non-decreasing function* $\psi : [0,\infty) \longrightarrow (0,\infty)$, $p \in L^1(J,\mathbb{R}_+)$ *such that*

$$\|F(t,u)\| \leqslant p(t)\psi(\|u\|_{\mathscr{D}}) \text{ for each } (t,u) \in J \times \mathscr{D}$$

and there exists $M_{**} > 0$ *such that*

$$(1 - Tc_1N_1)M_{**} \Big/ \left((C_1 + N_2)\psi(M_{**}) \int_0^T p_1(s)ds \right) > 1,$$

where C_1 *as defined in Theorem* 5.1.

Then the problem (1.5)–(1.6) *has at least one mild solution on* $[-r,T]$.

Acknowledgement. The authors thank the reviewers for their important comments and suggestions on this paper.

REFERENCES

[1] M. ADIMY, H. BOUZAHIR AND K. EZZINBI, *Existence and stability for some partial functional differential equations with infinite delay*, J. Math. Anal. Appl. **294** (2004), 438-461.

[2] W. ARENDT, *Vector valued Laplace transforms and Cauchy problems*, Israel J. Math., **59** (1987), 327-352.

[3] M. BENCHOHRA AND S. K. NTOUYAS, *Neutral functional differential and integrodifferential inclusions in Banach spaces*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat., **4**, 8 (2001), 767-782.

[4] A. BRESSAN AND G. COLOMBO, *Extensions and selections of maps with decomposable values*, Studia Math., **90** (1988), 69-86.

[5] L. BYSZEWSKI, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl., **162** (1991), 494-505.

[6] G. DA PRATO AND E. SINESTRARI, *Differential operators with non-dense domains*, Ann. Scuola. Norm. Sup. Pisa Sci., **14** (1987), 285-344.

[7] H. O. FATTORINI, *Second Order Linear Differential Equations in Banach Spaces*, North-Holland Mathematics Studies, Vol. 108, North-Holland, Amsterdam, 1985.

[8] J. A. GOLDSTEIN, *Semigroups of Linear Operators and Applications*, Oxford Univ. Press, New York, 1985.

[9] L. GORNIEWICZ, S. K. NTOUYAS AND D. O'REGAN, *Existence and controllability results for first and second order functional semilinear differential inclusions with nonlocal conditions*, Numer. Funct. Anal. Optim., **28** (2007), 53-82.

[10] A. GRANAS AND J. DUGUNDJI, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[11] J.K.HALE, J.KATO, *Phase space for retarded equations with infinite delay*, Funkcial. Ekvac., **21** (1978), 11-41.

[12] S. HEIKKILA AND V. LAKSHMIKANTHAM, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker, New York, 1994.

[13] E. HERNANDEZ, *Existence results for partial neutral functional differential equations with nonlocal conditions*, Dynam. Systems Appl., **11** (2002), 241–252.

[14] Y. HINO, S. MURAKAMI AND T. NAITO, *Functional Differentia Equations with Infinite Delay*, Lect. Notes in Math., **1473**, Springer-Verlag, Berlin, 1991.

[15] H. KELLERMAN AND M. HIEBER, INTEGRATED SEMIGROUPS, *J. Funct. Anal.*, **84** (1989), 160-180.

[16] V. LAKSHMIKANTHAM AND S. LEELA, *Differential and Integral Inequalities*, vol. I, Academic Press, New York, 1969.

[17] S. K. NTOUYAS AND D. O'REGAN, *Existence and controllability results for semilinear neutral functional differential inclusions with nonlocal conditions*, Dissc. Math. Differ. Incl., **27** (2007), 213-264.

[18] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[19] W.V. PETRYSHYN AND P.M. FITZPATRICK, *A degree theory, fixed point theorems, and mapping theorems for multivalued noncompact maps*, Trans. Amer. Math. Soc., **194** (1974), 1-25.

[20] H. R. THIEME, *Integrated semigroups and integrated solutions to abstract Cauchy problems*, J. Math. Anal. Appl., **152** (1990), 416-447.

[21] C. TRAVIS AND G. WEBB, *Cosine families and abstract nonlinear second order differential equations*, Acta Math. Hungar. **32** (1978), 75–96.

[22] C. TRAVIS AND G. WEBB, *An abstract second order semilinear Volterra integrodifferential equation*, SIAM J. Math. Anal., **10** (1979), 412–424.

(Received August 13, 2008)

S. K. Ntouyas
Department of Mathematics
University of Ioannina
451 10 Ioannina
Greece
e-mail: sntouyas@uoi.gr

D. O'Regan
Department of Mathematics
National University of Ireland
Galway
Ireland
e-mail: donal.oregan@unigalway.ie