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A VARIATIONAL APPROACH FOR ALMOST PERIODIC SOLUTIONS IN RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. To study the a.p. (almost periodic) solutions of retarded functional differential equations in the form $u''(t) = \int_{-r}^{0} D_1 f(u(t), u(t+\theta))d\theta + \int_{-r}^{0} D_2 f(u(t-\theta), u(t))d\theta + e(t)$, we introduce variational formalisms to characterize the a.p. solutions as a critical points of functionals defined on Banach spaces of a.p. functions. We obtain an existence result of weak a.p. solutions and a result of density of the a.p. forcing termes $e(.)$ for which the equation possesses usual a.p. solutions.

1. Introduction

From a function $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$, where $\mathbb{E}$ is a finite-dimensional real Euclidean space, and from $r \in (0, \infty)$ we consider the following (second order) retarded functional differential equation

$$u''(t) = \int_{-r}^{0} D_1 f(u(t), u(t+\theta))d\theta + \int_{-r}^{0} D_2 f(u(t-\theta), u(t))d\theta + e(t) \tag{1.1}$$

where D_j, $j = 1, 2$, denotes the partial gradient and where $e : \mathbb{R} \to \mathbb{E}$ is a forcing term.

We study the a.p. (almost periodic) solutions of (1.1) where e is an a.p. function.

A strong a.p. solution of (1.1) is a function $u : \mathbb{R} \to \mathbb{E}$ which is twice differentiable (in ordinary sense) with u, u' and u'' which are a.p. in the sense of Bohr [3, 6, 14]; the equality in (1.1) being satisfied for all $t \in \mathbb{R}$.

A weak a.p. solution of (1.1) is a function $u : \mathbb{R} \to \mathbb{E}$ which is a.p. in the sense of Besicovitch [5, 18], which possesses a first-order and a second-order gerneralized derivative; the equality in (1.1) means that the difference between the two members has a quadratic mean value equal to zero.

For the ordinary differential equations, this kind of weak a.p. solutions was considered in [8]. For neutral delay differential equations, this kind of weak a.p. solutions is considered in [4].

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Our approach uses a variational method. The a.p. solutions (strong or weak) of (1.1) are characterized as critical points of functionals in the form

$$u \mapsto \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\frac{1}{2} \left| u'(t) \right|^2 + \int_{-r}^{0} f(u(t), u(t+\theta)) d\theta + u(t).e(t) \right) dt$$

on Banach spaces of a.p. functions. And so (1.1) appears as an Euler-Lagrange equation.

Now we briefly describe the contents of the paper. After Section 2 devoted to precise our notations, in Section 3 we build a variational formalism to characterize the strong (also called usual) a.p. solutions of (1.1) (Theorem (3.3)), for which we can deduce a result on the structure of the set of strong a.p. solutions of (1.1) (Theorem (3.4)). In Section 4 we build a variational formalism to characterize the weak a.p. solutions of (1.1) (Theorem (4.5)), and to establish an existence result of weak a.p. solutions (Theorem (4.6)); we obtain also a result of the structure of the set of the weak a.p. solutions of (1.1).

In Section 5 we establish a result on the density of the a.p. forcing term for which (1.1) possesses a strong a.p. solutions (Theorem (5.3)); this result uses the weak a.p. solutions.

2. Notations

When $\mathbb{X}$ is a Banach space, $AP^0(\mathbb{X})$ denotes the space of the Bohr-a.p. functions from $\mathbb{R}$ in $\mathbb{X}$ [3, 6, 14]. It is a Banach space for the norm $\|u\|_\infty := \sup\{|u(t)| : t \in \mathbb{R}\}$. When $u \in AP^0(\mathbb{X})$, its mean value exists in $\mathbb{X}$:

$$\mathfrak{M}\{u\} = \mathfrak{M}_t\{u(t)\} := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)dt,$$

[3, 6, 14]. When $k \in \mathbb{N}$, $k \geqslant 1$, $AP^k(\mathbb{X})$ denotes the space of the $u \in \mathscr{C}^k(\mathbb{R}, \mathbb{X}) \cap AP^0(\mathbb{X})$ such that $u^j = \frac{d^j u}{dt^j} \in AP^0(\mathbb{X})$ for all $j = 1, \ldots, k$. It is a Banach space for the norm $\|u\|_{\mathscr{C}^k} := \|u\|_\infty + \sum_{1\leqslant j\leqslant k} \left\|u^j\right\|_\infty$.

$B^1(\mathbb{X})$ denotes the completion of $AP^0(\mathbb{X})$ with respect to the norm $\|u\|_{B^1} := \mathfrak{M}\{|u|\}$. It is a quotient space to transform the semi-norm $u \mapsto \mathfrak{M}\{|u|\}$ into a norm. When $\mathbb{X}$ is a Hilbert space, $B^2(\mathbb{X})$ denotes the completion of $AP^0(\mathbb{X})$ with respect to the norm $\|u\|_{B^2} := \mathfrak{M}\left\{|u|^2\right\}^{\frac{1}{2}}$. It is also a quotient space and it is a Hilbert space for the inner product $(u|v)_{B^2} := \mathfrak{M}\{(u|v)_{\mathbb{X}}\}$.

The generelized derivative of $u \in B^2(\mathbb{X})$ (when it exists) is $\nabla u \in B^2(\mathbb{X})$ such that $\mathfrak{M}_t\left\{\left|\nabla u(t) - \frac{1}{\tau}(u(t+\tau) - u(t))\right|^2\right\} \to 0 \ (\tau \to 0)$ [8, 12]. We consider $B^{1,2}(\mathbb{X}) := \left\{u \in B^2(\mathbb{X}) : \nabla u \in B^2(\mathbb{X})\right\}$ and $B^{2,2}(\mathbb{X}) := \left\{u \in B^{1,2}(\mathbb{X}) : \nabla^2 u := \nabla(\nabla u) \in B^2(\mathbb{X})\right\}$. They are Hilbert spaces for the respective norms

$$\|u\|_{B^{1,2}} := \left(\|u\|_{B^2}^2 + \|\nabla u\|_{B^2}^2\right)^{\frac{1}{2}}, \quad \|u\|_{B^{2,2}} := \left(\|u\|_{B^{1,2}}^2 + \left\|\nabla^2 u\right\|_{B^2}^2\right)^{\frac{1}{2}}.$$

When $u : \mathbb{R} \to \mathbb{E}$ is a continuous function, it is usual, in the theory of retarded functional differential equations, to consider, for all $t \in \mathbb{R}$, $u_t \in \mathscr{C}^0([-r,0],\mathbb{E})$ defined by $u_t(\theta) := u(t+\theta)$ for all $\theta \in [-r,0]$, [15].

When $u \in L^2_{loc}(\mathbb{R},\mathbb{E})$ (Lebesgue space), we denote by $\tilde{u} : \mathbb{R} \to L^2_{loc}([-r,0],\mathbb{E})$ the function defined by $\tilde{u}(t)(\theta) := u(t+\theta)$.

3. The Strong a.p. Solutions

We consider the following condition on f:

$$f \in \mathscr{C}^1(\mathbb{E} \times \mathbb{E}, \mathbb{R}). \tag{3.1}$$

LEMMA 3.1. *Under* (3.1) *we consider the mapping* $F_0 : \mathbb{E} \times \mathscr{C}^0([-r,0],\mathbb{E}) \to \mathbb{R}$ *defined by* $F_0(x,\psi) := \int_{-r}^0 f(x,\psi(\theta))d\theta$. *Then* F_0 *is of class* $\mathscr{C}^1$ *on* $\mathbb{E} \times \mathscr{C}^0([-r,0],\mathbb{E})$ *and* $DF_0(x,\psi)(y,\xi) = \int_{-r}^0 D_1 f(x,\psi).y d\theta + \int_{-r}^0 D_2 f(x,\psi(\theta)).\xi(\theta)d\theta$.

Proof. The following Nemytskii operator build on f :

$$\mathscr{N}_f^0 : \mathscr{C}^0([-r,0],\mathbb{E}) \times \mathscr{C}^0([-r,0],\mathbb{E}) \to \mathscr{C}^0([-r,0],\mathbb{E}),$$

$\mathscr{N}_f^0(\phi,\psi) := [\theta \mapsto f(\phi(\theta),\psi(\theta))]$, is of class $\mathscr{C}^1$ under (3.1), (see proposition 1 page 168, and proposition 2 page 170 in [1]).

The operator $A^0 : \mathbb{E} \times \mathscr{C}^0([-r,0],\mathbb{E}) \to \mathscr{C}^0([-r,0],\mathbb{E}) \times \mathscr{C}^0([-r,0],\mathbb{E})$ defined by $A^0(x,\psi) = (x,\psi)$ where the vector $x \in \mathbb{E}$ is considered as a (constant) continuous function, is a linear continuous and therefore A^0 is of class $\mathscr{C}^1$. The operator I^0 : $\mathscr{C}^0([-r,0],\mathbb{R}) \to \mathbb{R}$, $I^0(w) := \int_{-r}^0 w(t)dt$, is linear continuous and therefore it is of class $\mathscr{C}^1$.

Since $F_0 := I^0 \circ \mathscr{N}_f^0 \circ A^0$, F_0 is of class $\mathscr{C}^1$ as a composition of $\mathscr{C}^1$-mappings.

By using the chaine rule, we have

$$DF_0(x,\psi).(y,\xi) = I_0\left(D\mathscr{N}_f^0(A^0(x,\psi)).A^0(y,\xi)\right)$$

We know that $D\mathscr{N}_f^0(A^0(x,\psi)).A^0(y,\xi) = [\theta \mapsto D_1 f(x,\psi(\theta)).y + D_2 f(x,\psi(\theta)).\xi(\theta)]$, and so we obtain the announced formula.

LEMMA 3.2. *The operator* $S^0 : AP^0(\mathbb{E}) \to AP^0(\mathbb{R})$, *defined by*

$$S^0(u) := \left[t \mapsto \int_{-r}^0 f(u(t),u(t+\theta))d\theta\right],$$

is of class $\mathscr{C}^1$, *and*

$$DS^0(u)h = \left[t \mapsto \int_{-r}^0 D_1 f(u(t),u(t+\theta)).h(t)d\theta + \int_{-r}^0 D_2 f(u(t),u(t+\theta)).h(t+\theta)d\theta\right].$$

Proof. The Nemytskii operator defined on the mapping F_0 provided by Lemma (3.1),

$$\mathcal{N}_{F_0} : AP^0\left(\mathbb{E}\times\mathscr{C}^0([-r,0],\mathbb{E})\right) \equiv AP^0(\mathbb{E})\times AP^0\left(\mathscr{C}^0([-r,0],\mathbb{E})\right) \to AP^0(\mathbb{R}),$$

defined by

$$\mathcal{N}_{F_0}(u,\phi) := \left[t \mapsto F_0(u(t),\phi(t)) = \int_{-r}^{0} f(u(t),\phi(t)(\theta))d\theta\right]$$

is of class $\mathscr{C}^1$, since F_0 is of class $\mathscr{C}^1$. ([9], Corollary 5.3).

We introduce the operator $T^0 : AP^0(\mathbb{E}) \to AP^0\left(\mathscr{C}^0([-r,0],\mathbb{E})\right)$ by setting $T^0(u) := [t \mapsto u_t]$. Then T^0 is linear, T^0 is continuous since $\left\|T^0(u)\right\|_\infty = \|u\|_\infty$, and therefore T^0 is of class $\mathscr{C}^1$.

Since $S^0 = \mathcal{N}_{F_0} \circ (id, T^0)$, S^0 is of class $\mathscr{C}^1$ as a composition of $\mathscr{C}^1$-operators.

By using the chain rule we have $DS^0(u).h = D\mathcal{N}_{F_0}((id,T^0)(u)).D(id,T^0)(h) = D\mathcal{N}_{F_0}(u,\tilde{u}).(h,\tilde{h})$, and by using Lemma (3.1) we obtain

$$\begin{aligned}(DS^0(u).h)(t) &= \textstyle\int_{-r}^{0} D_1 f(u(t),\tilde{u}(t)(\theta)).h(t)d\theta + \int_{-r}^{0} D_2 f(u(t),\tilde{u}(t)(\theta)).\tilde{h}(t)(\theta)d\theta \\ &= \textstyle\int_{-r}^{0} D_1 f(u(t),u(t+\theta).h(t)d\theta + \int_{-r}^{0} D_2 f(u(t),u(t+\theta)).h(t+\theta)d\theta\end{aligned}$$

THEOREM 3.3. *Under* (3.1) *the functional* $J_0 : AP^1(\mathbb{E}) \to \mathbb{R}$*, defined by*

$$J_0(u) := \mathfrak{M}_t\left\{\frac{1}{2}\left|u'(t)\right|^2 + \int_{-r}^{0} f(u(t),u(t+\theta))d\theta + u(t).e(t)\right\},$$

is of class $\mathscr{C}^1$*, and when* $u \in AP^1(\mathbb{E})$ *we have* $DJ_0(u) = 0$ *if and only if* u *is a strong solution of* (1.1)

Proof. We consider the functional $Q_0 : AP^1(\mathbb{E}) \to \mathbb{R}$ defined by

$$Q_0(u) := \mathfrak{M}_t\left\{\frac{1}{2}\left|u'(t)\right|^2\right\}.$$

The mapping $q : \mathbb{E} \to \mathbb{R}$, $q(x) := \frac{1}{2}|x|^2 = \frac{1}{2}x.x$, is of class $\mathscr{C}^1$, therefore the Nemytskii operator $\mathcal{N}_q^0 : AP^0(\mathbb{E}) \to AP^0(\mathbb{R})$, $\mathcal{N}_q^0(\varphi) := \left[t \mapsto \frac{1}{2}|\varphi(t)|^2\right]$, is also of class $\mathscr{C}^1$, [7]. The operator $\frac{d}{dt} : AP^1(\mathbb{E}) \to AP^0(\mathbb{E})$, $\frac{d}{dt}(u) := u'$, is linear continuous, therefore it is of class $\mathscr{C}^1$. The functional $\mathfrak{M}^0 : AP^0(\mathbb{R}) \to R$, defined by $\mathfrak{M}^0(\varphi) := \mathfrak{M}_t\{\varphi(t)\}$, is linear continuous, therefore it is of class $\mathscr{C}^1$. Since $Q_0 = \mathfrak{M}^0 \circ \mathcal{N}_q^0 \circ \frac{d}{dt}$, Q_0 is of class $\mathscr{C}^1$ as composition of $\mathscr{C}^1$-mappings, and by using the chain rule we have

$$DQ_0(u).h = \mathfrak{M}_t\left\{u'(t).h'(t)\right\} \tag{3.2}$$

We consider the functional $\Phi_0 : AP^1(\mathbb{E}) \to \mathbb{R}$ defined by

$$\Phi_0(u) := \mathfrak{M}_t\left\{\int_{-r}^{0} f(u(t),u(t+\theta))d\theta\right\}.$$

We consider the operator $in_0 : AP^1(\mathbb{E}) \to AP^0(\mathbb{E})$, $in_0(u) := u$, which is linear continuous, and consequently in_0 is of class $\mathscr{C}^1$.

We note that we have Φ_0 is of class $\mathscr{C}^1$ as a composition of $\mathscr{C}^1$-mappings. By using Lemma (3.2) we obtain

$$\begin{aligned} D\Phi_0(u).h = \mathfrak{M}_t\{\int_{-r}^{0} D_1 f(u(t),u(t+\theta)).h(t)d\theta \\ + \int_{-r}^{0} D_2 f(u(t),u(t+\theta)).h(t+\theta)d\theta\} \end{aligned} \tag{3.3}$$

Now we want to improve this last formula.

Since $(t,\theta) \mapsto D_2 f(u(t),u(t+\theta)).h(t+\theta)$ is continuous on $\mathbb{R} \times [-r,0]$, it is Lebesgue-integrable and by using the Fubini theorem [2], we have

$$\begin{aligned} \frac{1}{2T}\int_{-T}^{T}\left(\int_{-r}^{0} D_2 f(u(t),u(t+\theta)).h(t+\theta)d\theta\right)dt \\ = \int_{-r}^{0}\left(\frac{1}{2T}\int_{-T}^{T} D_2 f(u(t),u(t+\theta)).h(t+\theta)dt\right)d\theta \end{aligned} \tag{3.4}$$

We set $g_T(\theta) := \frac{1}{2T}\int_{-T}^{T} D_2 f(u(t),u(t+\theta)).h(t+\theta)dt$. We know that, for all $\theta \in [-r,0]$,

$$\lim_{T\to\infty} g_T(\theta) = \mathfrak{M}_t\{D_2 f(u(t),u(t+\theta)).h(t+\theta)\}$$

since $t \mapsto D_2 f(u(t),u(t+\theta)).h(t+\theta)$ bellongs to $AP^0(\mathbb{R})$.

Furthermore, since $u,h \in AP^0(\mathbb{E})$, $\overline{u(\mathbb{R})}$ and $\overline{h(\mathbb{R})}$ are compact, [3, 6, 14], and since the mapping $(x,y,z) \mapsto D_2 f(x,y).z$ is continuous on the compact $\overline{u(\mathbb{R})} \times \overline{u(\mathbb{R})} \times \overline{h(\mathbb{R})}$, it is bounded, and consequently we have :

$$\sup_{\theta\in[-r,0]}\sup_{t\in\mathbb{R}} |D_2 f(u(t),u(t+\theta)).h(t+\theta)| := \sigma < \infty,$$

that implies $|g_T(\theta)| \leqslant \sigma$ for all $T > 0$, $\theta \in [-r,0]$. And so the assumptions of the dominated convergence theorem of Lebesgue are fulfilled, [2], and by using it we obtain

$$\lim_{T\to\infty}\int_{-r}^{0} g_T(\theta)d\theta = \int_{-r}^{0}\lim_{T\to\infty} g_T(\theta)d\theta,$$

and so by using (3.4) we obtain

$$\begin{aligned} &\textstyle\int_{-r}^{0}\mathfrak{M}_t\{D_2 f(u(t),u(t+\theta)).h(t+\theta)\}d\theta \\ &= \lim_{T\to\infty}\textstyle\int_{-r}^{0}\left(\frac{1}{2T}\int_{-T}^{T} D_2 f(u(t),u(t+\theta)).h(t+\theta)\right)d\theta \\ &= \lim_{T\to\infty}\textstyle\frac{1}{2T}\int_{-T}^{T}\left(\int_{-r}^{0} D_2 f(u(t),u(t+\theta)).h(t+\theta)d\theta\right)dt \end{aligned}$$

and so we have proven the following equality

$$\mathfrak{M}_t\left\{\int_{-r}^{0} D_2 f(u(t),u(t+\theta)).h(t+\theta)d\theta\right\} = \int_{-r}^{0}\mathfrak{M}_t\{D_2 f(u(t),u(t+\theta)).h(t+\theta)\}d\theta \tag{3.5}$$

By using a similar reasoning we obtain

$$\mathfrak{M}_t\left\{\int_{-r}^{0} D_2f(u(t-\theta),u(t)).h(t)d\theta\right\}=\int_{-r}^{0}\mathfrak{M}_t\left\{D_2f(u(t-\theta),u(t)).h(t)\right\}d\theta \quad (3.6)$$

Since the mean value is invariant by translation, [3, 6, 14], we have, for all $\theta\in[-r,0]$, the following equality

$$\mathfrak{M}_t\left\{D_2f(u(t),u(t+\theta)).h(t+\theta)\right\}=\mathfrak{M}_t\left\{D_2f(u(t-\theta),u(t)).h(t)\right\}.$$

By using it with (3.5) and (3.6) we obtain

$$\mathfrak{M}_t\left\{\int_{-r}^{0} D_2f(u(t),u(t+\theta)).h(t+\theta)d\theta\right\}=\mathfrak{M}_t\left\{\int_{-r}^{0} D_2f(u(t-\theta),u(t)).h(t)d\theta\right\}.$$

And by using this last equality in (3.3) we obtain

$$\begin{aligned} D\Phi_0(u).h =&\mathfrak{M}_t\{(\int_{-r}^{0} D_1f(u(t),u(t+\theta))d\theta \\ &+\int_{-r}^{0} D_2f(u(t-\theta),u(t))d\theta).h(t)\} \end{aligned} \quad (3.7)$$

We consider the functional $\Lambda_0 : AP^0(\mathbb{E})\to\mathbb{R}$, defined by $\Lambda_0(u):=\mathfrak{M}_t\left\{u(t).e(t)\right\}$. Note that Λ_0 is linear continuous and consequently it is of class $\mathscr{C}^1$ and we have

$$D\Lambda_0(u).h=\mathfrak{M}_t\left\{h(t).e(t)\right\}. \quad (3.8)$$

Since $J_0=Q_0+\Phi_0+\Lambda_0$, J_0 is of class $\mathscr{C}^1$ as a sum of three $\mathscr{C}^1$-functionals, and by using (3.2), (3.7) and (3.8) we obtain

$$\begin{aligned} DJ_0(u).h =&\mathfrak{M}_t\{u'(t).h'(t)+(\int_{-r}^{0} D_1f(u(t),u(t+\theta))d\theta \\ &+\int_{-r}^{0} D_2f(u(t-\theta),u(t))d\theta+e(t)).h(t)\} \end{aligned} \quad (3.9)$$

for all $u,h\in AP^1(\mathbb{E})$

We set $p(t):=\int_{-r}^{0} D_1f(u(t),u(t+\theta))d\theta+\int_{-r}^{0} D_2f(u(t-\theta),u(t))d\theta+e(t)$, and we have $p\in AP^0(\mathbb{E})$.

When $DJ_0(u)=0$ then by using (3.9) we have $\mathfrak{M}_t\left\{u'(t).h'(t)\right\}=-\mathfrak{M}_t\left\{p(t).h(t)\right\}$ for all $h\in AP^1(\mathbb{E})$ and by using the same reasoning that this one of the proof of Theorem 1 in [7] we obtain that $u\in AP^2(\mathbb{E})$ and $u''(t)=p(t)$, that is exactly (1.1).

Conversely, if u is a strong a.p. solution of (1.1), then we have $u''=p$ and so, for all $h\in AP^1(\mathbb{E})$, we have $DJ_0(u).h=\mathfrak{M}\left\{u'.h'+p.h\right\}=\mathfrak{M}\left\{\frac{d}{dt}(u'.h)\right\}=0$

THEOREM 3.4. *Under* (3.1)*, if we additionally assume that* f *is convex function, then the set of the strong a.p. solutions of* (1.1) *is a convex subset of* $AP^2(\mathbb{E})$.

Proof. When f is convex, it is easy to verify that J_0 is convex, $DJ_0 = 0$ is equivalent to $J_0 = \inf J_0(AP^1(\mathbb{E}))$, [10], and $\{u \in AP^1(\mathbb{E}) : J_0 = \inf J_0(AP^1(\mathbb{E}))\}$ is convex. And so $\{u \in AP^1(\mathbb{E}) : DJ_0 = 0\}$ is convex, and we obtain the conclusion by using Theorem (3.3).

A consequence of Theorem (3.4) is the following one: when $e = 0$, if (1.1) possesses a non-constant T_1-periodic solution u_1 and a non-constant T_2-periodic solution u_2 with $T_1/T_2 \notin \mathbb{Q}$ the $\frac{1}{2}u_1 + \frac{1}{2}u_2$ is a non-periodic a.p. solution of (1.1) since it is a convex combination of a.p. solution.

4. The Weak a.p. solutions

We begin this section by giving a precise definition of the notion of weak a.p. solution of (1.1). A weak a.p. solution of (1.1) is a function $u \in B^{2,2}(\mathbb{E})$ such that

$$\nabla^2 u = \int_{-r}^{0} D_1 f(u(t), u(t+\theta))d\theta + \int_{-r}^{0} D_2 f(u(t-\theta), u(t))d\theta + e(t),$$

this equality holding in $B^2(\mathbb{E})$.

We begin by establishing two lemmas which contain general propreties of the Besicovitch a.p. functions.

LEMMA 4.1. *Let $u \in B^2(\mathbb{E})$. Then the following equalities hold*

$$\begin{aligned}\mathfrak{M}_t\left\{\int_{-r}^{0} |u(t+\theta)|^2 d\theta\right\} &= \int_{-r}^{0} \mathfrak{M}_t\left\{|u(t+\theta)|^2\right\} d\theta \\ &= r.\mathfrak{M}_t\left\{|u(t)|^2\right\}\end{aligned}$$

Proof. Since $\mathfrak{M}_t\left\{|u(t)|^2\right\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |u(t)|^2 dt$ exists in $\mathbb{R}_+$, we have

$$M := \sup_{T\geqslant 1} \frac{1}{2T}\int_{-T}^{T} |u(t)|^2 dt < \infty.$$

For all $\theta \in [-r, 0]$ we have

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T} |u(t+\theta)|^2 dt &= \frac{1}{2T}\int_{-T+\theta}^{T+\theta} |u(s)|^2 ds \\ &\leqslant \frac{1}{2T}\int_{-T-r}^{T-r} |u(t)|^2 dt \\ &\leqslant \frac{1}{2T}\int_{-(T+r)}^{T+r} |u(t)|^2 dt \\ &= \frac{1.2(T+r)}{2T}.\frac{1}{2(T+r)}\int_{-(T+r)}^{T+r} |u(t)|^2 dt \\ &\leqslant (1+\frac{r}{T}).M \leqslant (1+r).M =: M_1,\end{aligned}$$

and so we have proven

$$\exists M_1 > 0, \forall \theta \in [-r,0], \forall T \geqslant 1, \frac{1}{2T}\int_{-T}^{T} |u(t+\theta)|^2\, dt \leqslant M_1 < \infty \tag{4.1}$$

For all $T \geqslant 1$ we define $\Phi_T : [-r,0] \to \mathbb{R}$ by setting $\Phi_T(\theta) := \frac{1}{2T}\int_{-T}^{T} |u(t+\theta)|^2\, dt$. Since $\Phi_T(\theta) = \frac{1}{2T}\int_{-T+\theta}^{T+\theta} |u(s)|^2\, ds$ we see that Φ_T is absolutely continuous on $[-r,0]$, and consequently we have $\Phi_T \in L^1([-r,0],\mathbb{R})$.

If $[a,b]$ is segment in $\mathbb{R}$, for all $\theta \in [-r,0]$, by using the Fubini theorem for the non negative mesurable functions [2], we have

$$\begin{aligned}
\int_{[a,b]\times[-r,0]} |u(t+\theta)|^2\, dt d\theta &= \int_{[-r,0]} \left(\int_{[a,b]} |u(t+\theta)|^2\, dt \right) d\theta \\
&= \int_{[-r,0]} \left(\int_{[a,b]+\theta} |u(s)|^2\, ds \right) d\theta \\
&\leqslant \int_{[-r,0]} \left(\int_{[a,b]+[-r,0]} |u(s)|^2\, ds \right) d\theta \\
&= r. \int_{[a,b]+[-r,0]} |u(s)|^2\, ds < \infty
\end{aligned}$$

since $|u|^2 \in L^1_{loc}(\mathbb{R},\mathbb{R}_+)$ and since $[a,b]+[-r,0]$ is compact. And so we have proven:

$$(t,\theta) \mapsto |u(t+\theta)|^2 \in L^1_{loc}(\mathbb{R}\times[-r,0],\mathbb{R}) \tag{4.2}$$

Then by using the Fubini theorem [2], for all $T > 0$ we obtain

$$\frac{1}{2T}\int_{-T}^{T} \left(\int_{-r}^{0} |u(t+\theta)|^2\, d\theta \right) dt = \int_{-r}^{0} \left(\frac{1}{2T}\int_{-T}^{T} |u(t+\theta)|^2\, dt \right) d\theta \tag{4.3}$$

Since $u \in B^2(\mathbb{E})$, we have $\lim_{T\to\infty} \Phi_T(\theta) = \mathfrak{M}_t\left\{|u(t+\theta)|^2\right\} = \mathfrak{M}_t\left\{|u(t)|^2\right\}$ since the mean value is invariant by translation, for all $\theta \in [-r,0]$. The constant M_1 is integrable on $[-r,0]$. And so by using (4.1), we can apply the dominated convergence theorem of Lebesgue to obtain $\int_{-r}^{0} \lim_{T\to\infty} \Phi_T(\theta) d\theta = \lim_{T\to\infty} \int_{-r}^{0} \Phi_T(\theta) d\theta$, that implie by using (4.3) that $\int_{-r}^{0} \mathfrak{M}_t\left\{|u(t+\theta)|^2\right\} d\theta = \mathfrak{M}_t\left\{\int_{-r}^{0} |u(t+\theta)|^2\, d\theta\right\}$. And since $\mathfrak{M}_t\left\{|u(t+\theta)|^2\right\} = \mathfrak{M}_t\left\{|u(t)|^2\right\}$ for all θ, we have also

$$\int_{-r}^{0} \mathfrak{M}_t\left\{|u(t+\theta)|^2\right\} d\theta = r.\mathfrak{M}_t\left\{|u(t)|^2\right\}.$$

LEMMA 4.2. *If $u \in B^2(\mathbb{E})$ then $\tilde{u} \in B^2\left(L^2([-r,0],\mathbb{E})\right)$ and we have*

$$\|\tilde{u}\|_{B^2\left(L^2([-r,0],\mathbb{E})\right)} = \sqrt{r}.\, \|u\|_{B^2(\mathbb{E})}$$

Proof. We fix $u \in B^2(\mathbb{E})$, and $\varepsilon > 0$. We can choose $q_\varepsilon \in AP^0(\mathbb{E})$ such that $\|u - q_\varepsilon\|_{B^2(\mathbb{E})} < \varepsilon$.

Since $L^2([-r,0],\mathbb{E})$ is separable, there exists a countable subset D in $L^2([-r,0],\mathbb{E})$ which is dense, and consequently the set $\{B(\varphi,\rho) : \varphi \in D, \rho \in \mathbb{Q} \cap (0,\infty)\}$ is a generator of the Borel σ-field of $L^2([-r,0],\mathbb{E})$, where

$$B(\varphi,\rho) := \left\{ \psi \in L^2([-r,0],\mathbb{E}) : \|\psi - \varphi\|_{L^2([-r,0],\mathbb{E})} < \rho \right\}.$$

We arbitrarily fix $\varphi \in D$ and $\rho \in \mathbb{Q} \cap (0,\infty)$, and we set

$$\alpha(t) := \int_{-r}^{0} |u(t+\theta) - \varphi(\theta)|^2 \, d\theta.$$

By using the same reasoning that this one used to establish (4.2) we obtain that $(t,\theta) \mapsto |u(t+\theta) - \varphi(\theta)|^2 \in L^1_{loc}(\mathbb{R} \times [-r,0],\mathbb{R})$ and consequently by using the Fubini theorem we know that $\alpha \in L^1_{loc}(\mathbb{R},\mathbb{R})$ and then we necessarily have α measurable.

We note that $t \in \tilde{u}^{-1}(B(\varphi,\rho))$ is equivalent to $t \in \alpha^{-1}([0,\rho^2[)$. Since α is measurable we have $\alpha^{-1}([0,\rho^2[) \in \mathscr{B}(\mathbb{R})$ and consequently $\tilde{u}^{-1}(B(\varphi,\rho)) \in \mathscr{B}(\mathbb{R})$, and so we have proven:

$$\tilde{u} \text{ is measurable from } (\mathbb{R},\mathscr{B}(\mathbb{R})) \text{ in } \left(L^2([-r,0],\mathbb{E}),\mathscr{B}\left(L^2([-r,0],\mathbb{E})\right)\right). \quad (4.4)$$

By using (4.2) we know that $(t,\theta) \mapsto |u(t+\theta)|^2 \in L^1_{loc}(\mathbb{R} \times [-r,0],\mathbb{R})$ and consequently, by using the Fubini theorem we obtain that $t \mapsto \int_{-r}^{0} |u(t+\theta)|^2 \, d\theta = \|\tilde{u}(t)\|^2_{L^2([-r,0],\mathbb{E})} \in L^1_{loc}(\mathbb{R},\mathbb{R})$.

Therefore we have obtained, [2] :

$$\tilde{u} \in L^2_{loc}\left(\mathbb{R}, L^2([-r,0],\mathbb{E})\right). \quad (4.5)$$

By using Lemma (4.1) whith $u - q_\varepsilon$ instead of u, we know that

$$\mathfrak{M}_t \left\{ \int_{-r}^{0} |u(t+\theta) - q_\varepsilon(t+\theta)|^2 d\theta \right\}$$

exists and that we have

$$\begin{aligned} \mathfrak{M}_t \left\{ \|\tilde{u}(t) - \tilde{q}(t)\|_{L^2([-r,0],\mathbb{E})} \right\} &= \mathfrak{M}_t \left\{ \int_{-r}^{0} |u(t+\theta) - q_\varepsilon(t+\theta)|^2 \, d\theta \right\} \\ &= r.\mathfrak{M}_t \left\{ |u(t) - q_\varepsilon(t)|^2 \right\} < r.\varepsilon^2. \end{aligned}$$

Since $\tilde{q}_\varepsilon \in AP^0\left(\mathscr{C}^0([-r,0],\mathbb{E})\right) \subset AP^0\left(L^2([-r,0],\mathbb{E})\right)$, when $\varepsilon \to 0$, we obtain that $\tilde{u} \in B^2\left(L^2([-r,0],\mathbb{E}),\mathbb{E}\right)$.

The relation between the norms of u and $\tilde{u}$ is a consequence of Lemma (4.1).

By modifying a function $u \in B^2(\mathbb{E})$ on a bounded interval of $\mathbb{R}$ we do not modify the (class of the) function u, and so we can ask to use $\tilde{u}(t)$, defined as the restiction of u on the interval $[t-r,t]$, possesses a meaning. Lemma (4.2) provides an answer to this question, since if $v \in B^2(\mathbb{E})$ is different of u, then we have $\tilde{u} \neq \tilde{v}$. And so the definition of $\tilde{u}$ is consistent, and the notion of weak a.p. solution is also consistent.

Now we introduce the following condition on f:

$$\begin{cases} \text{There exists } a \in (0,\infty) \text{ and } b \in \mathbb{R} \text{ such that} \\ |Df(x,y)| \leqslant a\,(|x|+|y|)+b \text{ for all } x,y \in \mathbb{E}. \end{cases} \tag{4.6}$$

LEMMA 4.3. *Under* (3.1) *and* (4.6)*, the operator* $S : B^2(\mathbb{E}) \to B^1(\mathbb{R})$ *defined by* $S(u) := \left[t \mapsto \int_{-r}^{0} f(u(t),u(t+\theta))d\theta\right]$ *is of class* $\mathscr{C}^1$ *and for all* $u,h \in B^2(\mathbb{E})$*, we have* $DS(u).h = \left[t \mapsto \int_{-r}^{0} D_1 f(u(t),u(t+\theta)).h(t) + D_2 f(u(t),u(t+\theta)).h(t+\theta)d\theta\right]$

Proof. The Nemytskii operator build on f, $\mathscr{N}_f : L^2([-r,0],\mathbb{E}) \times L^2([-r,0],\mathbb{E}) \to L^1([-r,0],\mathbb{R})$, $\mathscr{N}_f(\varphi,\psi) := [\theta \mapsto f(\varphi(\theta),\psi(\theta))]$, under (3.1) and (4.6) is of class $\mathscr{C}^1$, [11], and $D\mathscr{N}_f(\varphi,\psi).(\xi,\zeta) = [\theta \mapsto Df(\varphi(\theta),\psi(\theta)).(\xi(\theta),\zeta(\theta))]$

The operator $A : \mathbb{E} \times L^2([-r,0],\mathbb{E}) \to \left(L^2([-r,0],\mathbb{E})\right)^2$ defined by $A(x,\psi) := (x,\psi)$, where x is considered as a constant function, is linear continuous, therefore A is of class $\mathscr{C}^1$ and $DA(x,\psi) = A$.

The functional $I : L^1([-r,0],\mathbb{R}) \to \mathbb{R}$, $I(w) := \int_{-r}^{0} w(\theta)d\theta$, is linear continuous, therefore I is of class $\mathscr{C}^1$ and $DI(w) = I$.

We consider the mapping $F : \mathbb{E} \times L^2([-r,0],\mathbb{E}) \to \mathbb{R}$, defined by $F(x,\psi) := \int_{-r}^{0} f(x,\psi(\theta))d\theta$.

We note that $F = I \circ \mathscr{N}_f \circ A$, and so F is of class $\mathscr{C}^1$ as a composition of $\mathscr{C}^1$-mappings, and by using the chain rule we obtain, for all $x,y \in \mathbb{E}$ and for all $\psi,\xi \in L^2([-r,0],\mathbb{E})$, the following formula:

$$DF(x,\psi).(y,\xi) = \int_{-r}^{0} (D_1 f(x,\psi(\theta)).y + D_2 f(x,\psi(\theta)).\xi(\theta))\, d\theta.$$

Let $(y,\xi) \in \mathbb{E} \times L^2([-r,0],\mathbb{E})$ such that $\|(y,\xi)\| \leqslant 1$. Then we have

$$\begin{aligned} |DF(x,\psi).(y,\xi)| &\leqslant \int_{-r}^{0} |Df(x,\psi(\theta))|\,.\,|(y,\xi(\theta))|\,d\theta \\ &\leqslant \left(\int_{-r}^{0} |Df(x,\psi(\theta))|^2\, d\theta\right)^{\frac{1}{2}} . \left(\int_{-r}^{0} |(y,\xi(\theta))|^2\, d\theta\right)^{\frac{1}{2}} \end{aligned}$$

by using the Cauchy-Schwarz-Buniakovski inequality.

We note that

$$\int_{-r}^{0} |(y,\xi(\theta))|^2\, d\theta = \int_{-r}^{0} \left(|y|^2 + |\xi(\theta)|^2\right) d\theta = r|y|^2 + \int_{-r}^{0} |\xi(\theta)|^2\, d\theta \leqslant r_1 . \|(y,\xi)\|^2 \leqslant r_1,$$

where $r_1 := \max\{r,1\}$, and so we have:

$$
\begin{aligned}
|DF(x,\psi).(y,\xi)| &\leqslant \sqrt{r_1}.\left(\int_{-r}^{0} |Df(x,\psi(\theta))|^2\,d\theta\right)^{\frac{1}{2}} \\
&\leqslant \sqrt{r_1}.\left(\int_{-r}^{0} (a.|x|+a.|\psi(\theta)|+b)^2\,d\theta\right)^{\frac{1}{2}} \\
&= \sqrt{r_1}.\,\|a.|x|+a.|\psi|+|b|\|_{L^2([-r,0],\mathbb{E})} \\
&\leqslant \sqrt{r_1}.\left(a\,\||x|\|_{L^2([-r,0],\mathbb{R})}+a\,\||\psi|\|_{L^2([-r,0],\mathbb{R})}+\||b|\|_{L^2([-r,0],\mathbb{R})}\right).
\end{aligned}
$$

Since $\||x|\|_{L^2([-r,0],\mathbb{R})} = \sqrt{r}.|x|$, $\||b|\|_{L^2([-r,0],\mathbb{R})} = \sqrt{r}.|b|$ and $\||\psi|\|_{L^2([-r,0],\mathbb{R})} = \|\psi\|_{L^2([-r,0],\mathbb{E})}$, we have

$$
|DF(x,\psi).(y,\xi)| \leqslant a.\sqrt{r_1}.\sqrt{r}\left(|x|+\|\psi\|_{L^2([-r,0],\mathbb{E})}\right)+\sqrt{r_1}.\sqrt{r}.|b|.
$$

We set $a_1 := a.\sqrt{r_1}.\sqrt{r}$ and $b_1 := \sqrt{r_1}.\sqrt{r}\,|b|$ and so we obtain:

$$
|DF(x,\psi)| \leqslant a_1.\left(|x|+\|\psi\|_{L^2([-r,0],\mathbb{E})}\right)+b_1.
$$

And so the assumption of ([11], Theorem 2.6 page 14) are fulfilled and we can assert that $\mathscr{N}_F : B^2(\mathbb{E})\times B^2(L^2)\to B^1(\mathbb{R})$ is of class $\mathscr{C}^1$ and that we have, for all $u,h\in B^2(\mathbb{E})$ and for all $V,K\in L^2([-r,0],\mathbb{E})$, the following formula

$$
\begin{aligned}
D\mathscr{N}_F(u,V).(h,K) &= [t\mapsto DF(u(t),V(t)).(h(t),K(t)) \\
&= \int_{-r}^{0} (D_1f(u(t),V(t)(\theta)).h(t)+D_2f(u(t),V(t)(\theta)).K(t)(\theta))\,d\theta]
\end{aligned}
\tag{4.7}
$$

We consider the linear operator $T : B^2(\mathbb{E})\to B^2\left(L^2([-r,0],\mathbb{E})\right)$ defined by $T(u) := \tilde{u}$. By using Lemma (4.2) we know that T is continuous, and therefore T is of class $\mathscr{C}^1$ with $DT(u)=T$.

We note that we have $S=\mathscr{N}_f\circ(id,T)$, and so S is of class $\mathscr{C}^1$ as a composition of $\mathscr{C}^1$-operators, and by using the chain rule and (4.7) we obtain the announced formula.

LEMMA 4.4. *Under* (3.1) *and* (4.6)*, if u and h belong to $B^2(\mathbb{E})$ then the following equality holds :*

$$
\mathfrak{M}_t\left\{\int_{-r}^{0} D_2f(u(t),u(t+\theta)).h(t+\theta)d\theta\right\} = \mathfrak{M}_t\left\{\left(\int_{-r}^{0} D_2f(u(t-\theta),u(t))d\theta\right).h(t)\right\}
$$

Proof. By using a reasoning similar to this one used to establish (4.2) we obtain that $(t,\theta)\mapsto D_2f(u(t),u(t+\theta)).h(t+\theta)\in L^1_{loc}(\mathbb{R}\times[-r,0],\mathbb{R})$. And so we can use the Fubini theorem to obtain

$$
\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\left(\int_{-r}^{0} D_2f(u(t),u(t+\theta)).h(t+\theta)d\theta\right)dt \\
&\qquad = \int_{-r}^{0}\left(\frac{1}{2T}\int_{-T}^{T} D_2f(u(t),u(t+\theta)).h(t+\theta)dt\right)d\theta
\end{aligned}
\tag{4.8}
$$

for all $T\in(0,\infty)$.

For all $T \in [1,\infty)$ we introduce the function $g_T : [-r,0] \to \mathbb{R}$ defined by

$$g_T(\theta) := \frac{1}{2T}\int_{-T}^{T} D_2 f(u(t),u(t+\theta)).h(t+\theta)dt$$

Ever using the Fubini theorem we know that the g_T are borelian.

Since $t \mapsto D_2 f(u(t),u(t+\theta)).h(t+\theta) \in B^1(\mathbb{R})$ we know that the mean value exists in $\mathbb{R}$ and consequently we have

$$\lim_{T\to\infty} g_T(\theta) = \mathfrak{M}_t \{D_2 f(u(t),u(t+\theta)).h(t+\theta)\}$$

for all $\theta \in [-r,0]$.

Since $\mathfrak{M}_t \left\{|u(t)|^2\right\}$ exists in $\mathbb{R}$, we have $\sup_{t\geqslant 1}\left(\frac{1}{2T}\int_{-T}^{T}|u(t)|^2\,dt\right) =: M < \infty$. For all $\theta \in [-r,0]$ and, for all $T \geqslant 1+r$, we have

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T}|u(t+\theta)|^2\,dt &= \frac{1}{2T}\int_{-T+\theta}^{T+\theta}|u(s)|^2\,ds\\
&\leqslant \frac{1}{2T}\int_{-T+\theta}^{T-\theta}|u(s)|^2\,ds\\
&= \frac{2(T-\theta)}{2T}.\frac{1}{2(T-\theta)}.\int_{-(T-\theta)}^{T-\theta}|u(t)|^2\,dt\\
&\leqslant (1+r).M =: M_0
\end{aligned}$$

And so we have proven the following assertation

$$\begin{cases}\text{There exists } M_0 \in (0,\infty) \text{ such that, for all}\\ \theta \in [-r,0], \quad \sup_{T\geqslant 1+r}\frac{1}{2T}\int_{-T}^{T}|u(t+\theta)|^2\,dt \leqslant M_0.\end{cases} \tag{4.9}$$

Replacing u by h we similarly obtain the following assertation.

$$\begin{cases}\text{There exists } M_1 \in (0,\infty) \text{ such that, for all}\\ \theta \in [-r,0], \quad \sup_{T\geqslant 1+r}\frac{1}{2T}\int_{-T}^{T}|h(t+\theta)|^2\,dt \leqslant M_1.\end{cases} \tag{4.10}$$

By using the equivalence of the norms of $\mathbb{R}^2$ and the usual inequality $(A+B)^2 \leqslant 2(A^2+B^2)$ we obtain the existence of $a_2 \in (0,\infty)$ such that

$$\begin{aligned}
|D_2 f(u(t),u(t+\theta))|^2 &\leqslant \left(a_2\left[|u(t)|^2+|u(t+\theta)|^2\right]^{\frac{1}{2}}+b\right)^2\\
&\leqslant 2.\left(a_2|u(t)|^2+a_2|u(t+\theta)|^2+b^2\right)
\end{aligned}$$

that implies

$$\begin{aligned}
\left(\frac{1}{2T}\int_{-T}^{T}|D_2 f(u(t),u(t+\theta))|^2\,dt\right)^{\frac{1}{2}} &\leqslant \sqrt{2}(a_2.\frac{1}{2T}\int_{-T}^{T}|u(t)|^2\,dt\\
&\quad + a_2.\frac{1}{2T}\int_{-T}^{T}|u(t+\theta)|^2\,dt+b^2)^{\frac{1}{2}}\\
&\leqslant \sqrt{2}\left(a_2M_0+a_2M_0+b^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Then by setting $\gamma := \sqrt{2}\left(2a_2M_0+b^2\right)^{\frac{1}{2}} M_1^{\frac{1}{2}}$ we have proven the following assertation

$$\left(\frac{1}{2T}\int_{-T}^{T}|D_2f(u(t),u(t+\theta))|^2\,dt\right)^{\frac{1}{2}}.\left(\frac{1}{2T}\int_{-T}^{T}|h(t+\theta)|^2\,dt\right)^{\frac{1}{2}} \leqslant \gamma \qquad (4.11)$$

By using the Cauchy-Schwarz-Buniakovski inequality and (4.11) we obtain, for all $T \geqslant 1+r$ and for $\theta \in [-r,0]$,

$$|g_T(\theta)| \leqslant \frac{1}{2T}\int_{-T}^{T}|D_2f(u(t),u(t+\theta))|.|h(t+\theta)|\,dt \leqslant \sigma$$

Since the Lebesgue measure of $[-r,0]$ is finite, the constant σ is Lebesgue integrable in $[-r,0]$, and consequently the assumptions of the Lebesgue Dominated Convergence theroem are fulfilled and we can say :

$$\int_{-r}^{0}\lim_{T\to\infty} g_T(\theta)d\theta = \lim_{T\to\infty}\int_{-r}^{0} g_T(\theta)d\theta,$$

and we can conclude as in the proof of (3.5), (3.6), (3.7).

THEOREM 4.5. *We assume* (3.1) *and* (4.6) *fulfilled. Then the functional* $J : B^{1,2}(\mathbb{E}) \to \mathbb{R}$*, defined by*

$$J(u) := \mathfrak{M}_t\left\{\frac{1}{2}|\nabla u(t)|^2 + \int_{-r}^{0} f(u(t),u(t+\theta))d\theta + u(t).e(t)\right\},$$

is of class $\mathscr{C}^1$*. And when* $u \in B^{1,2}(\mathbb{E})$*, we have* $DJ(u).h = 0$ *if and only if* u *is a weak a.p. solution of* (1.1).

Proof. We consider the functional $Q : B^{1,2}(\mathbb{E}) \to \mathbb{R}$ defined by

$$Q(u) := \mathfrak{M}_t\left\{\frac{1}{2}|\nabla u(t)|^2\right\}.$$

We set $q(x) := \frac{1}{2}|x|^2$; $q : \mathbb{E} \to \mathbb{R}$ is a $\mathscr{C}^1$-function since $\mathbb{E}$ is euclidean. Since $Dq(x) = x$, q satisfies the condition of ([11], Theorem 2.6 page 14) to ensure that the Nemytskii operator $\mathscr{N}_q : B^2(\mathbb{E}) \to B^1(\mathbb{R})$ is of class $\mathscr{C}^1$ and $D\mathscr{N}_q(v).h = [t \mapsto v(t).h(t)]$ for all $v,h \in B^2(\mathbb{E})$. Since the derivation operator $\nabla : B^{1,2}(\mathbb{E}) \to B^2(\mathbb{E})$ is linear continuous, it is of class $\mathscr{C}^1$ and since the operator $\mathfrak{M} : B^1(\mathbb{R}) \to \mathbb{R}$, $\mathfrak{M}(v) := \mathfrak{M}_t\{v(t)\}$ is also linear continuous, it is of class $\mathscr{C}^1$. And so $Q := \mathfrak{M}\circ\mathscr{N}_q\circ\nabla$ is of class $\mathscr{C}^1$ as a composition of $\mathscr{C}^1$-mappings. Moreover by using the chain rule we have

$$DQ(u).h = \mathfrak{M}_t\{\nabla u(t).\nabla h(t)\} \qquad (4.12)$$

for all $u,h \in B^{1,2}(\mathbb{E})$.

We consider the functional $\Phi : B^{1,2}(\mathbb{E}) \to \mathbb{R}$ defined by

$$\Phi(u) := \mathfrak{M}_t\left\{\int_{-r}^{0} f(u(t),u(t+\theta))d\theta\right\}.$$

We note that the injection $in : B^{1,2}(\mathbb{E}) \to B^2(\mathbb{E})$, $in(u) := u$, is linear continuous and consequently it is of class $\mathscr{C}^1$. We note that $\Phi = \mathfrak{M} \circ S \circ in$, and by using Lemma (4.3) we know that S is of class $\mathscr{C}^1$. And so Φ is of class $\mathscr{C}^1$ as a composition of $\mathscr{C}^1$-mapping. Ever using Lemma (4.3) and the chain rule we obtain the following formula

$$\begin{aligned} D\Phi(u).h = \mathfrak{M}_t\{ &\int_{-r}^{0} D_1 f(u(t), u(t+\theta)).h(t)d\theta \\ &+ \int_{-r}^{0} D_2 f(u(t), u(t+\theta)).h(t+\theta)d\theta\}, \end{aligned}$$

and by using Lemma (4.4) we obtain

$$\begin{aligned} D\Phi(u).h = \mathfrak{M}_t\{ (&\int_{-r}^{0} D_1 f(u(t), u(t+\theta))d\theta \\ &+ \int_{-r}^{0} D_2 f(u(t-\theta), u(t))d\theta).h(t)\}. \end{aligned} \tag{4.13}$$

We consider the linear functional $\Lambda : B^{1,2}(\mathbb{E}) \to \mathbb{R}$ defined by

$$\Lambda(u) := \mathfrak{M}_t \{u(t).e(t)\},$$

and the linear functional $L : B^2(\mathbb{E}) \to \mathbb{R}$ defined by $L(u) := \mathfrak{M}_t \{u(t).e(t)\} = (u|e)_{B^2(\mathbb{E})}$. Since L is continuous (by using the Cauchy-Schwarz-Buniakovski inequality), $\Lambda := L \circ in$ is also continuous as a composition of continuous mappings, and consequently Λ is of class $\mathscr{C}^1$. Moreover, since $D\Lambda(u) = 1$ we obtain the following formula

$$D\Lambda(u).h = \mathfrak{M}_t \{h(t).e(t)\}, \text{ for all } u, h \in B^{1,2}(\mathbb{E}). \tag{4.14}$$

We note that $J = Q + \Phi + \Lambda$, and so J is of class $\mathscr{C}^1$ as a sum of $\mathscr{C}^1$-functionals. Moreover, by using (4.12), (4.13), (4.14), we obtain

$$\begin{aligned} DJ(u).h = &\mathfrak{M}_t\{\nabla u(t).\nabla h(t) + (\int_{-r}^{0} D_1 f(u(t), u(t+\theta))d\theta \\ &+ \int_{-r}^{0} D_2 f(u(t-\theta), u(t))d\theta + e(t)).h(t)\} \end{aligned} \tag{4.15}$$

for all $u, h \in B^{1,2}(E)$.

We set $p(t) := \int_{-r}^{0} [D_1 f(u(t), u(t+\theta)) + D_2 f(u(t-\theta), u(t))]d\theta + e(t)$ $(\in B^2(\mathbb{E}))$. And so the condition $DJ(u) = 0$ can be writen as $\mathfrak{M}_t \{\nabla u(t).\nabla h(t)\} = -\mathfrak{M}_t \{p(t).h(t)\}$ for all $h \in B^{1,2}(\mathbb{E})$. And so by using [8], this last condition implies that $\nabla u \in B^{1,2}(\mathbb{E})$, i.e. $u \in B^{2,2}(\mathbb{E})$, and $\nabla^2 u = p$ which exactly means that u is a weak a.p. solutions of (1.1).

Conversely, since $\mathfrak{M}\{\nabla v\} = 0$ for all $v \in B^{1,2}(\mathbb{R})$, we have $0 = \mathfrak{M}\{\nabla(\nabla u.h)\} = \mathfrak{M}\{\nabla^2 u.h)\} + \mathfrak{M}\{\nabla u.\nabla h\} = \mathfrak{M}\{p.h\} + \mathfrak{M}\{\nabla u.\nabla h\}$ for all $h \in B^{1,2}(\mathbb{E})$, that implies $DJ(u) = 0$.

Now we introduce an assumption of convexity :

$$f \text{ is a convex function on } \mathbb{E}\times\mathbb{E} \tag{4.16}$$

and an assumption of coerciveness :

$$\begin{cases} \text{There exists } c\in(0,\infty) \text{ and } d\in\mathbb{R} \text{ such that} \\ f(x,y)\geqslant c\,|x|^2+d \text{ for all } (x,y)\in\mathbb{E}\times\mathbb{E}. \end{cases} \tag{4.17}$$

THEOREM 4.6. *Under* (3.1)*,* (4.6)*,* (4.16)*,* (4.17)*, for all* $e\in B^2(\mathbb{E})$*, there exists* $u\in B^{2,2}(\mathbb{E})$ *which is a weak a.p. solution of* (1.1)*. Moreover the set of the weak a.p. solutions of* (1.1) *is a convex set.*

Proof. After Theorem (4.5) we know that the functional J is of class $\mathscr{C}^1$ on $B^{1,2}(\mathbb{E})$. By using (4.16) we deduce that J is a convex functional. Then J is weakly lower semi-continuous on the Hilbert space $B^{1,2}(\mathbb{E})$, [16]. From (4.17) we deduce that, for all $u\in B^{1,2}(\mathbb{E})$, we have

$$\begin{aligned} J(u) &\geqslant \frac{1}{2}\,\|\nabla u\|^2_{B^2(\mathbb{E})}+c\,\|u\|^2_{B^2(\mathbb{E})}-\|u\|_{B^2(\mathbb{E})}\cdot\|e\|_{B^2(\mathbb{E})} \\ &\geqslant c_1.\,\|u\|^2_{B^{1,2}(\mathbb{E})}-\|e\|_{B^2(\mathbb{E})}\cdot\|u\|_{B^{1,2}(\mathbb{E})} \end{aligned}$$

where $c_1:=\min\left\{\frac{1}{2},c\right\}\in(0,\infty)$. Consequently J is coercive, i.e. $J(u)\to\infty$ when $\|u\|^2_{B^{1,2}(\mathbb{E})}\to\infty$. Then, [10], we can assert that there exists $u\in B^{1,2}(\mathbb{E})$ such that $J(u)=\inf J\left(B^{1,2}(\mathbb{E})\right)$, and since J is of class $\mathscr{C}^1$ we have $DJ(u)=0$, and then, by using Theorem (4.5), we know that u is a weak a.p. solution of (1.1).

Ever using Theorem (4.5), we know that the set of the weak a.p. solutions of (1.1) is equal to the following set: $\left\{u\in B^{1,2}(\mathbb{E}):DJ(u)=0\right\}$, and since J is convex this last it is equal to the set $\left\{u\in B^{1,2}(\mathbb{E}):J(u)=\inf J\left(B^{1,2}(\mathbb{E})\right)\right\}$. Since J is convex this last set is a convex set. And so the set of the weak a.p. solutions of (1.1)is convex.

5. Density

LEMMA 5.1. *Under* (3.1) *and* (4.16) *we consider the operator* $\Gamma_1:B^2(\mathbb{E})\to B^2(E)$ *defined by*

$$\Gamma_1(u):=\left[t\mapsto\int_{-r}^{0}D_1f(u(t),u(t+\theta))d\theta\right].$$

Then Γ_1 *is continuous.*

Proof. Under (3.1) and (4.6) we know that we have $|D_1f(x,y)|\leqslant a\,(|x|+|y|)+b$ for all $x,y\in\mathbb{E}$. Then ([11], Theorem 2.5 page 9), the Nemytskii operator $\mathscr{N}_{D_1f}:L^2([-r,0],E)\times L^2([-r,0],E)\to L^2([-r,0],E)$, $\mathscr{N}_{D_1f}(\varphi,\psi):=[\theta\mapsto D_1f(\varphi(\theta),$

$\psi(\theta))]$, is continuous. We know that the operator A, $A(x,\psi) = (x,\psi)$, used in the proof of Lemma (4.3), is continuous from $\mathbb{E} \times L^2([-r,0],E)$ in $L^2([-r,0],E) \times L^2([-r,0],E)$. The functional I used in the proof of Lemma (4.3) is continuous.

We define $F_1 : \mathbb{E} \times L^2([-r,0],E) \to \mathbb{R}$ by setting

$$F_1(x,\psi) := I \circ \mathcal{N}_{D_1 f} \circ A(x,\psi) = \int_{-r}^{0} D_1 f(u(t), u(t+\theta))d\theta.$$

Then F_1 is continuous as a composition of continuous mappings.

For all $x \in \mathbb{E}$ and $\psi \in L^2([-r,0],E)$ we have

$$\begin{aligned}
|F_1(x,\psi)| &\leqslant \int_{-r}^{0} |D_1 f(x, \psi(\theta))|\, d\theta \\
&\leqslant \int_{-r}^{0} (a|x| + a|\psi(\theta)| + b)\, d\theta \\
&= r.a.\, |x| + a. \int_{-r}^{0} |\psi(\theta)|\, d\theta + r.b \\
&\leqslant r.a.\, |x| + a.\sqrt{r}\, \|\psi\|_{L^2([-r,0],E)} + r.b \\
&\leqslant a_3. \left(|x| + \|\psi\|_{L^2([-r,0],E)} \right) + r.b,
\end{aligned}$$

where $a_3 := a.\max\{r, \sqrt{r}\}$. And so the assumptions of ([18], Remark 2.7 page 54) are fulfilled that ensure that the Nemytskii operator

$$\mathcal{N}_{F_1} : B^2(\mathbb{E}) \times B^2\left(L^2([-r,0],E)\right) \to B^2(\mathbb{E}),$$

$$\mathcal{N}_{F_1}(u,\xi) := \left[t \mapsto F_1(u(t), \xi(t)) = \int_{-r}^{0} D_1 f(u(t), \xi(t)(\theta)) d\theta \right]$$

is continuous.

We note that $\Gamma_1 = \mathcal{N}_{F_1} \circ (id, T)$, where $T(u) = \tilde{u}$, and so Γ_1 is continuous as a composition of continuous mappings.

LEMMA 5.2. *Under* (3.1) *and* (4.6) *we consider the operator* $\Gamma_2 : B^2(\mathbb{E}) \to B^2(\mathbb{E})$ *defined by*

$$\Gamma_2(u) := \left[t \mapsto \int_{-r}^{0} D_2 f(u(t-\theta), u(t)) d\theta \right].$$

Then Γ_2 *is continuous.*

Proof. By a reasoning similar to this one used in Lemma (5.1), the Nemytskii operator $\mathcal{N}_{D_2 f} : L^2([-r,0],E) \times L^2([-r,0],E) \to L^2([-r,0],E)$, $\mathcal{N}_{D_2 f}(\varphi,\psi) := [\theta \mapsto D_2 f(\varphi(\theta), \psi(\theta))]$, is continuous.

We introduce the operator $A_1 : L^2([-r,0],E) \times \mathbb{E} \to L^2([-r,0],E) L^2([-r,0],E)$, $A_1(\varphi,y) := [\theta(\varphi(\theta),y)]$. A_1 is linear continuous.

We consider also the functional I like in the proof of Lemma (5.1).

We define $F_2 : L^2([-r,0],E) \times \mathbb{E} \to \mathbb{R}$ by setting

$$F_2(\varphi,y) := I \circ \mathscr{N}_{D_2F} \circ A_1(\varphi,y) = \int_{-r}^{0} D_2 f(\varphi(\theta),y)d\theta.$$

And also F_2 is continuous as composition of continuous functions. Like in the proof of Lemma (5.1) we establish that

$$|F_2(\varphi,y)| \leqslant a_3 \left(\|\varphi\|_{L^2([-r,0],E)} + |y| \right) + rb.$$

And so by using ([18], Remark 2.7 page 54) we know that the Nemytskii operator

$$\mathscr{N}_{F_2} : B^2\left(L^2([-r,0],E)\right) \times B^2(\mathbb{E}) \to B^2(\mathbb{E}),$$

$$\mathscr{N}_{F_2}(\xi,u) := \left[t \mapsto \int_{-r}^{0} D_2 f(\xi(t)(\theta),u(t))d\theta \right],$$

is continuous.

For all $u \in B^2(\mathbb{E})$ and for all $t \in \mathbb{R}$, we denote by $\tilde{u}(t) := [\theta \mapsto u(t-\theta)] \in L^2([-r,0],E)$. Proceeding like in Lemma (4.2) we can establish that $\tilde{u} \in B^2(L^2([-r,0],E))$ and that $\|\tilde{u}\|_{B^2\left(L^2([-r,0],E)\right)} = \sqrt{r}.\,\|u\|_{B^2(\mathbb{E})}$. And so the operator

$$T_1 : B^2(\mathbb{E}) \to B^2\left(L^2([-r,0],E)\right), \quad T_1(u) := \tilde{u},$$

is linear continuous.

We note that $\Gamma_2 = \mathscr{N}_{D_2 f} \circ (T_1, id)$ that permits us to say that Γ_2 is continuous as a composition of continuous mappings.

THEOREM 5.3. *Under* (3.1), (4.6), (4.16), (4.17), *for all* $e \in AP^0(\mathbb{E})$, *and for all* $\varepsilon \in (0,\infty)$, *there exists* $e_\varepsilon \in AP^0(\mathbb{E})$ *such that* $\|e - e_\varepsilon\|_{B^2(\mathbb{E})} \leqslant \varepsilon$ *and such that there exists* $u_\varepsilon \in AP^2(\mathbb{E})$ *wich is a strong a.p. solution of*

$$u_\varepsilon''(t) = \int_{-r}^{0} D_1 f(u(t),u(t+\theta))d\theta + \int_{-r}^{0} D_2 f(u(t-\theta),u(t))d\theta + e_\varepsilon(t).$$

Proof. We set $\Gamma := \Gamma_1 + \Gamma_2$ where Γ_1 comes from Lemma (5.1) and Γ_2 comes from Lemma (5.2). We consider the operator $\mathscr{T} : B^{2,2}(\mathbb{E}) \to B^2(\mathbb{E})$, $\mathscr{T}(u) := \nabla^2(u) - \Gamma(u)$.The operator $\nabla^2 : B^{2,2}(\mathbb{E}) \to B^2(\mathbb{E})$ is linear continuous and by using Lemma (5.1) and Lemma (5.2), we see that $\mathscr{T}$ is continuous.

By using Theorem (4.6) we know that $\mathscr{T}(B^{2,2}(\mathbb{E})) = B^2(\mathbb{E})$ and consequently we have $AP^0(\mathbb{E}) \subset \mathscr{T}(B^{2,2}(\mathbb{E}))$. Since AP^2 is dense in $B^{2,2}(\mathbb{E})$ and since $\mathscr{T}$ is continuous, for all $e \in AP^0(\mathbb{E})$ and for all $\varepsilon \in (0,\infty)$, we obtain that there exists $u_\varepsilon \in AP^2$ such that $\|\mathscr{T}(u_\varepsilon) - e\|_{B^2(\mathbb{E})} < \varepsilon$.

By proceeding like in the proof of Lemma (3.2), we obtain that $\Gamma_1(u_\varepsilon)$ and $\Gamma_2(u_\varepsilon)$ belong to $AP^0(\mathbb{E})$. Since $\nabla^2(u_\varepsilon) = u_\varepsilon''$, [8], we obtain that $\mathscr{T}(u_\varepsilon) \in AP^0(\mathbb{E})$. We set $e_\varepsilon := \mathscr{T}(u_\varepsilon)$, and so e_ε satisfies the announced conditions.

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