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TWO–POINT OSCILLATIONS IN SECOND–ORDER LINEAR DIFFERENTIAL EQUATIONS

MERVAN PAŠIĆ AND JAMES S. W. WONG

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Abstract. A second-order linear differential equation (P): $y'' + f(x)y = 0$, $x \in I$, where $I = (0,1)$ and $f \in C(I)$, is said to be two-point oscillatory on I, if all its nontrivial solutions $y \in C(\overline{I}) \cap C^2(I)$, oscillate both at $x = 0$ and $x = 1$, i.e. having sequences of infinite zeros converging to $x = 0$ and $x = 1$. It necessarily implies that all solutions $y(x)$ of (P) must satisfy the Dirichlet boundary conditions and that $f(x)$ must be singular at both end points of $\overline{I}$. We first describe a class of two-point oscillatory equations of (P). Secondly, we prove that (P) is two-point oscillatory if $f(x)$ satisfies certain Hartman-Wintner type asymptotic conditions. Furthermore, we study the arclength of the graph $G(y)$ of solutions curve $y(x)$ on I. Two-point oscillatory equation (P) is said to be two-point rectifiable (unrectifiable) oscillatory if the arclengths of all solutions are finite (infinite). We give conditions on $f(x)$ which imply (P) is two-point rectifiable (unrectifiable) oscillatory. When (P) is two-point unrectifiable oscillatory, we determine the fractal dimension of its solution curves for a special class of $f(x)$ similar to the Euler type equations when $f(x)$ is only singular at one end point of I. Finally, the preceding results motivate a study on two-sided oscillations of (P) at an interior point of $\overline{I}$.

1. Introduction

Let $I = (0,1)$ be the unit interval in $\mathbf{R}$ and let $f \in C(I)$. Let $y = y(x)$ be a real function defined on the interval $\overline{I} = [0,1]$ and smooth enough on I, that is, $y \in C(\overline{I}) \cap C^2(I)$. Let $G(y)$ denote the graph of $y(x)$ defined as usual by $G(y) = \{(x, y(x)) : 0 \leqslant x \leqslant 1\} \subseteq \mathbf{R}^2$. A function $y(x)$ is said to be *oscillatory* (respectively *nonoscillatory*) on an interval $J \subseteq \mathbf{R}$ if it has an infinite (respectively a finite) number of zeros on J. A linear differential equation $y'' + f(x)y = 0$ is said to be *oscillatory* (respectively *nonoscillatory*) on J if all its nontrivial solutions are oscillatory (respectively nonoscillatory) on J. If an interval $J \subseteq \mathbf{R}$ is infinite or if J is finite and $0 \in \overline{J}$, then the famous Euler linear differential equation $y'' + \lambda x^{-2} y = 0$ is oscillatory (respectively nonoscillatory) on J provided $\lambda > 1/4$ (respectively $\lambda \leqslant 1/4$), see for instance [22]. This kind of results was generalized to several class of linear and nonlinear ordinary differential equations on infinite intervals, with the help of several methods like the Sturm comparison principle, the transformation to Ricati equation, etc.. See for instance [7], [22], and references therein.

In the paper, we study the so-called 2-point oscillations of real functions and linear differential equations on the finite interval I, introduced in the following way.

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DEFINITION 1.1. A function $y(x)$ is said to be 2*-point* oscillatory on the interval I if:

(i) for any closed interval $J \subseteq I$, $0 \notin J$ and $1 \notin J$, $y(x)$ is nonoscillatory on J,

(ii) there is a decreasing sequence $a_k \in I$ and an increasing sequence $b_k \in I$ of consecutive zeros of $y(x)$ such that $a_k \searrow 0$ and $b_k \nearrow 1$.

Let $T > 0$ and let $W = W(t)$ be a T-periodic and smooth real function with $W(t_0) = 0$ for some $t_0 \in \mathbf{R}$. As a basic class of 2-point oscillatory functions on I can be taken $y(x) = p(x)W(q(x))$, where $p, q \in C^2(I)$, $|p(x)| > 0$ in I, $p(0+) = p(1-) = 0$, and $|q(0+)| = |q(1-)| = \infty$. For instance, for $\alpha > 0$, $\beta > 0$, $\rho > 0$, and $W(t) = \sin t$ or $W(t) = \cos t$, the functions $y(x) = (x - x^2)^{\alpha} W((x - x^2)^{-\beta})$ and $y(x) = [x \ln(1/x)]^{\alpha} W(\rho \ln\ln(1/x))$ are 2-point oscillatory on I.

Using the proto-type of 2-point oscillatory functions introduced above, we can consider 2-point oscillations in second-order linear differential equations on the finite interval I.

DEFINITION 1.2. A linear differential equation $y'' + f(x)y = 0$ is said to be 2*-point* oscillatory on I if all its non-trivial solutions $y(x)$ are 2-point oscillatory on I.

The condition (ii) from Definition 1.1 necessarily implies that all solutions $y(x)$ of a 2-point oscillatory linear differential equation $y'' + f(x)y = 0$ on I satisfy the Dirichlet boundary conditions on I, and hence, we can always to assume $y(0) = y(1) = 0$. As the pre-model-equation for 2-point oscillations on I, we consider the so-called Riemann-Weber version of the Euler differential equation,

$$y'' + \frac{1}{x^2}\left(\frac{1}{4} + \frac{\lambda}{|\ln x|^2}\right)y = 0, \quad x \in I, \tag{1}$$

where $\lambda > 1/4$. This equation plays an important role in the theory of nonlinear oscillations of Euler type equations: see for instance Sugie and Hara [20], Sugie and Kita [21], and Wong [24]. The general solution of (1) is explicitly given by $y(x) = c_1 y_1(x) + c_2 y_2(x)$, where

$$y_1(x) = [x \ln(1/x)]^{1/2} \cos(\rho \ln\ln(1/x)),\ y_2(x) = [x \ln(1/x)]^{1/2} \sin(\rho \ln\ln(1/x)),$$

and $\rho = (\lambda - 1/4)^{1/2}$. Obviously, the functions $y_1(x)$ and $y_2(x)$ are 2-point oscillatory on I, and hence, the equation (1) is 2-point oscillatory on I. Moreover, in Section 2 below, we will present a more systematic way to establish 2-point oscillations of (1). We shall also discuss other model-equations for 2-point oscillations with $f(x)$ involving polynomial or exponential functions which are singular at both end points of I.

In the sequel, some essential results on 2-point oscillations on I will be proved by using the following 2-point version of well known Sturm comparison principle.

LEMMA 1.3. *Let $f, g \in C(I)$ and let equation $y'' + f(x)y = 0$ be 2-point oscillatory on I. If $f(x) \leqslant g(x)$ near $x = 0$ and $x = 1$, then equation $y'' + g(x)y = 0$ is 2-point oscillatory on I too.*

Now, by means of Lemma 1.3 we can obtain a large class of second-order linear differential equations which are 2-point oscillatory on I.

THEOREM 1.4. *Let $\lambda > 1/4$ and let $f \in C(I)$ such that*

$$f(x) \geqslant \frac{1}{x^2}\left(\frac{1}{4} + \frac{\lambda}{|\ln x|^2}\right) \quad near \quad x = 0 \quad and \quad x = 1. \tag{2}$$

Then equation $y'' + f(x)y = 0$ is 2-point oscillatory on I.

In view of Theorem 1.4, it is natural to seek conditions on $f(x)$ for 2-point oscillations of equation $y'' + f(x)y = 0$ on I when (2) is not satisfied.

THEOREM 1.5. *Let $f \in C(I)$ such that*

$$f(x) \leqslant \frac{1}{4x^2}\left(1 + \frac{1}{|\ln x|^2}\right) \quad near \quad x = 0 \quad and \quad x = 1. \tag{3}$$

Then equation $y'' + f(x)y = 0$ is not 2-point oscillatory on I.

The proof of Theorem 1.5 follows also from Sturm comparison theorem and the fact that equation (1) is nonoscillatory for $\lambda = 1/4$. It is because the general solution $y(x) = c_1y_1(x) + c_2y_2(x)$ of equation (1) for $\lambda = 1/4$ is determined by the functions $y_1(x) = x^{-1/2}\ln(1/x)$ and $y_2(x) = x^{-1/2}\ln(1/x)\ln\ln(1/x)$ which are nonoscillatory at $x = 0$ and $x = 1$, even $y_1(0) = y_2(0) = y_1(1) = y_2(1) = 0$.

There are many classes of linear differential equations which are not 2-point oscillatory on the interval I. Therefore, it is helpful to have a necessary condition for 2-point oscillations of a linear differential equation on I, which again follows from the Sturm's comparison principle.

THEOREM 1.6. *Let $f \in C(I)$ and let equation $y'' + f(x)y = 0$ be 2-point oscillatory on the interval I. Then $|f(0+)| = |f(1-)| = +\infty$.*

Consequently, the Euler linear differential equation $y'' + \lambda x^{-2}y = 0$, $\lambda > 1/4$, as well as its generalization, $y'' + \lambda x^{-\alpha}y = 0$, $x \in I$, where $\lambda > 0$ and $\alpha > 2$, are not 2-point oscillatory on I. Also, the so called chirp-equation $y'' + x^{-2}\big(\delta^2x^{-2\delta} + (1 - \delta^2)/4\big)y = 0$ as well as the equation $y'' + x^{-4}\big(e^{2/x} - 1/4\big)y = 0$ with exponential term in its coefficient, are also not 2-point oscillatory on I. These kinds of equations have been recently studied in [12], [13], [14], and [25].

In Section 2, we present a method by which the functions $y_1(x) = p(x)\cos q(x)$ and $y_2(x) = p(x)\sin q(x)$ via general solution formula $y(x) = c_1y_1(x) + c_2y_2(x)$ produce an important class of 2-point oscillatory equations $y'' + f(x)y = 0$ on I, where $f(x)$ is explicitly expressed in terms of given functions $p(x)$ and $q(x)$. In Section 3, in a more general setting, we explore some asymptotic conditions of Hartman-Wintner type on the coefficient $f(x)$ such that a second-order linear differential equation $y'' + f(x)y = 0$ is 2-point oscillatory on I. As a consequence of these results, we prove that equation

$$y'' + \frac{c(x)}{(x - x^2)^{\sigma}}y = 0, \quad x \in I, \tag{4}$$

is 2-point oscillatory on I if $\sigma > 2$, where $c(x)$ is a smooth and positive function on $\bar{I}$. Hence, besides equation (1), the equation (4) is also a model-equation for 2-point oscillations on the interval I.

The geometric structure of all solutions of equation (4) is rather rich. More precisely, since the arc-length of the graph $G(y) = \{(x, y(x)) : 0 \leqslant x \leqslant 1\}$ of a 2-point oscillatory function $y(x)$ on I may be finite, infinite or of fractal type, the so-called 2-point rectifiable, unrectifiable, and fractal oscillations of $y(x)$ on I are introduced and studied respectively in Section 4, Section 5, and Section 6. As a consequence of these results, we observe that such three kinds of 2-point oscillations of equation (4) depend only on parameter σ in this way: equation (4) is 2-point rectifiable oscillatory on I if $\sigma \in (2,4)$, 2-point unrectifiable oscillatory on I if $\sigma \geqslant 4$, and 2-point fractal oscillatory on I if $\sigma > 4$. Since equation (1) is only 2-point rectifiable oscillatory on I, the equation (4) will be taken as a principal model-equation for all remaining discussion in the paper.

According to the above results, in Section 8 we propose a study on the oscillations of equation (P): $y'' + f(x)y = 0$ near an interior point x_0 of $\bar{I}$, where x_0 is a singular point of $f(x)$. Analogously to 2-point oscillations, it provides corresponding results on the so-called 2-sided rectifiable, unrectifiable, and fractal oscillations of equation (P) at x_0.

When equation (4) is fractal oscillatory, we determine the box dimension of its solution curves to be $3/2 - 2/\sigma$, similar to the simpler case of equation (P) of Euler type, see [13]. By box dimension, we refer to Minkowski-Bouligand dimension see [6] and [11]. Fractal oscillations have been recently studied in nonlinear equations: in half-linear equation — see [15], in Liénard equation — see [16], and Emden-Fowler equation — see [26], [27]. Geometric measure theory has been successfully applied to prove smoothness of weak solutions of Navier-Stokes equations see [3], [9], and [19].

2. Existence of a class of 2-point oscillatory equations on I

In this section, we give the existence of a large class of second-order linear differential equations which are 2-point oscillatory on the interval I and which contain, as a particular case, our pre-model-equation (1). That class of equations will be determined by a given real function $q = q(x)$ which satisfies the following structural conditions:

$$q \in C^3(I), \tag{5}$$

$$|q(0+)| = |q(1-)| = +\infty \text{ and } |q'(0+)| = |q'(1-)| = +\infty, \tag{6}$$

$$q'(x) < 0 \text{ for all } x \in I \text{ and } S(q') \in C(I). \tag{7}$$

Here $S(q')(x)$ denotes as usual the Schwarzian derivative of $q(x)$ defined by

$$S(q')(x) = \frac{q'''(x)}{q'(x)} - \frac{3}{2}\left[\frac{q''(x)}{q'(x)}\right]^2, \; x \in I.$$

We mention that in the theory of chaos, the Schwarzian derivative $S(q')(x)$ plays an important role to determine the chaotic behaviour of a discrete iteration equation $x_{n+1} = q(x_n)$ when $n \to \infty$, see for instance [2] and [18].

With the help of the general solution $y(x) = c_1y_1(x) + c_2y_2(x)$ which is explicitly given in terms of the function $q(x)$ by the formula:

$$y_1(x) = |q'(x)|^{-\frac{1}{2}} \cos q(x) \quad \text{and} \quad y_2(x) = |q'(x)|^{-\frac{1}{2}} \sin q(x), \tag{8}$$

one can form the following class of second-order linear differential equations on I:

$$y'' + \left[\frac{1}{2}S(q')(x) + (q')^2(x)\right]y = 0,\ x \in I. \tag{9}$$

In the first main result of this section, the structural conditions (5), (6), and (7) ensure the existence of 2-point oscillations of equation (9).

THEOREM 2.1. *Let $q(x)$ satisfy the conditions* (5), (6), *and* (7). *Then equation* (9) *is* 2*-point oscillatory on I.*

In order to prove this theorem, it is enough to show that (8) is the general solution of (9) and that the conditions (5), (6), and (7) imply 2-point oscillations of general solution (8). It is an elementary procedure and we leave it to the reader.

We are able now to verify 2-point oscillations in equation (1) in a different way than in Section 1, by showing that (1) is a particular case of equation (9) where the corresponding function $q(x)$ satisfies the conditions (5), (6), and (7).

EXAMPLE 2.2. Let $q(x) = \rho \ln\ln(1/x)$ and $\rho = (\lambda - 1/4)^{1/2}$. It is easy to check that for all $x \in I$, $q'(x) = -\rho/[x\ln(1/x)] < 0$ and $q' \in C^2(I)$, which implies that $q(x)$ satisfies the conditions (5) and (6). Also,

$$S(q')(x) = \frac{1+\ln^2(1/x)}{2x^2\ln^2(1/x)} \in C(I) \quad \text{and} \quad \frac{1}{x^2}\left(\frac{1}{4} + \frac{\lambda}{|\ln x|^2}\right) = \frac{1}{2}S(q')(x) + (q')^2(x).$$

It shows that $q(x)$ satisfies the condition (7) and that equation (1) is a particular case of (9). Hence by Theorem 2.1, the equation (1) is 2-point oscillatory on I. $\square$

The condition (7) implies the existence and continuity of $f(x) = \frac{1}{2}S(q')(x) + (q'(x))^2$. In the following example we give a very simple function $q(x)$ which satisfies the conditions (5) and (6) but does not satisfy the condition (7), and so $f \notin C(I)$.

EXAMPLE 2.3. Let $q(x) = 1/(x-x^2)$. Since $q'(x) = (2x-1)/(x-x^2)^2$ and $q' \in C^2(I)$, it is clear that $q(x)$ satisfies the conditions (5) and (6). Also, since $q'(1/2) = 0$ and $S(q')(x) = -6/(1-2x)^2$, we have obviously that $S(q')$ is singular at $x = 1/2$ and thus the condition (7) is not satisfied. $\square$

An example for the function $q(x)$ which satisfies the conditions (5), (6), and (7), is the following.

EXAMPLE 2.4. Let $q(x) = (1-2x)/(x-x^2)^\beta$, $\beta > 0$. Then $q \in C^3(I)$ and:

$$q'(x) = -\frac{Q(x)}{(x-x^2)^{\beta+1}} < 0 \quad \text{and} \quad S(q')(x) = \frac{P_6(x,\beta)}{(x-x^2)^2Q^2(x)}, \tag{10}$$

where $Q(x) = 2(2\beta-1)x^2 - 2(2\beta-1)x + \beta$ and $P_6(x,\beta)$ is a polynomial function in variables x and β of the 6^{th} degree. Therefore, the function $q(x)$ satisfies the conditions (5), (6), and (7), and for such a choice of $q(x)$, equation (9) is 2-point oscillatory on I by Theorem 2.1. For all $\beta > 0$, we have

$$0 < \min\{\beta, 1/4\} \leqslant Q(x) \leqslant \max\{\beta, 1/4\} \text{ for all } x \in \bar{I} \tag{11}$$

and $Q(x)$ is decreasing near $x = 0$ and increasing near $x = 1$ when $\beta > 1/2$. We therefore have the following estimates for $|q'(x)|^{-1}$ which will be frequently used in Section 5 and Section 6 below:

$$\begin{cases} c_1(x-x^2)^{\beta+1} \leqslant |q'(x)|^{-1} \leqslant c_2(x-x^2)^{\beta+1} \text{ for all } x \in I \text{ and } \beta > 0, \\ |q'(x)|^{-1} \text{ is increasing near } x = 0 \text{ and decreasing near } x = 1, \text{ if } \beta > 1/2, \end{cases} \tag{12}$$

where c_1, c_2 are positive constants. Next, we obviously have $S(q')(x) + 2(q')^2(x) = P_6(x,\beta)(x-x^2)^{-2}Q^{-2}(x) + 2Q^2(x)(x-x^2)^{-2\beta-2}$. Since for all $\beta > 0$ the functions $P_6(x,\beta)$ and $Q(x)$ are bounded on I from below and above, from previous equality easily follows

$$\left[\frac{1}{2}S(q')(x) + (q')^2(x)\right] \sim (x-x^2)^{-2\beta-2} \text{ near } x = 0 \text{ and } x = 1. \ \square \tag{13}$$

Next, by using the Sturm's comparison principle, Theorem 2.1 can be extended to a general class of linear differential equations.

THEOREM 2.5. *Let $f \in C(I)$ and let there be a real function $q(x)$ satisfying* (5), (6), *and* (7) *such that*

$$f(x) \geqslant \tfrac{1}{2}S(q')(x) + (q')^2(x) \text{ near } x = 0 \text{ and } x = 1.$$

Then equation $y'' + f(x)y = 0$ is 2-point oscillatory on I.

As a consequence of Theorem 2.5, besides equation (9), one can explore some additional class of model-equations which are 2-point oscillatory on I. It is equation (4). With the help of Example 2.4, one can find a function $c(x)$ for which equation (4) equals to equation (9), where $q(x) = (1-2x)/(x-x^2)^{\beta}$, $\beta > 0$, and $\sigma = 2\beta + 2$. In general, since the coefficient $c(x)$ from (4) is an arbitrarily given positive continuous function on $\bar{I}$, it would be difficult to find a suitable $q(x)$ so that equation (9) becomes (4).

COROLLARY 2.6. *If $\sigma > 2$, then equation* (4) *is 2-point oscillatory on I.*

Proof. Let $g(x)$ be a function defined by $g(x) = c(x)/(x-x^2)^{\sigma}$, $x \in I$. Since $c(x)$ is a positive and continuous function on $\bar{I}$, there is a constant $m > 0$ such that $m \leqslant c(x)$ for all $x \in \bar{I}$, and so, since $\sigma > 2$ there is a $\beta_1 > 0$ such that $2\beta_1 + 2 < \sigma$ and

$$\frac{m}{(x-x^2)^{2\beta_1+2}} \leqslant \frac{c(x)}{(x-x^2)^{\sigma}} = g(x), \ x \in I. \tag{14}$$

Next, for $q(x) = (1-2x)/(x-x^2)^{\beta}$, $x \in I$, and for any $\beta > 0$ such that $\beta < \beta_1$, let $f(x)$ be a function defined by $f(x) = \left[\frac{1}{2}S(q')(x) + (q')^2(x)\right]$, $x \in I$. From Example 2.4 we know that $f(x)$ is a continuous function on I and that equation $y'' + f(x)y = 0$ is 2-point oscillatory on I. By (13) and (14), near $x = 0$ and $x = 1$, we have:

$$f(x) = \left[\frac{1}{2}S(q')(x) + (q')^2(x)\right] \leqslant \frac{m}{(x-x^2)^{2\beta+2}} \leqslant \frac{m}{(x-x^2)^{2\beta_1+2}} \leqslant g(x).$$

Hence, by using Lemma 1.3 we conclude that equation (4) is 2-point oscillatory on I. $\square$

Now, we present a particular case of equation (9), where the singular term is of exponential type.

EXAMPLE 2.7. Let $c(x)$ be a continuous function on $\bar{I}$ such that $c(x) \geqslant 1$ for all $x \in I$. We consider the equation

$$y''(x) + c(x)e^{\frac{4}{x-x^2}}y(x) = 0, \ x \in I. \tag{15}$$

This equation is a particular case of (9) when $q(x) = (1-2x)e^{1/(x-x^2)}$, $x \in I$. Also, it can be shown that for such a choice of $q(x)$, all conditions of Theorem 2.5 are satisfied and hence, the equation (15) is 2-point oscillatory on I. Indeed, $q \in C^3(I)$ and

$$f(x) = c(x)e^{\frac{4}{x-x^2}} \geqslant e^{\frac{4}{x-x^2}} \geqslant e^{\frac{4}{x-x^2}} \frac{e^{\frac{-2}{x-x^2}}}{(x-x^2)^4} = \frac{e^{\frac{2}{x-x^2}}}{(x-x^2)^4}$$

$$\geqslant \frac{Q(x)e^{\frac{2}{x-x^2}}}{(x-x^2)^4} = \frac{1}{2}S(q')(x) + (q')^2(x),$$

where $Q(x)$ is a continuous function on $\bar{I}$ such that

$$Q(x) = \frac{1}{4[1+2(x^2-x)(x^2-x+2)]^2}[e^{\frac{-2}{x-x^2}}P_{16}(x) + Q_{16}(x)],$$

and $P_{16}(x)$ and $Q_{16}(x)$ are two suitable polynomial functions. Also,

$$q'(x) = -\frac{(1-2x)^2 + 2x^2(1-x)^2}{(x-x^2)^2}e^{\frac{1}{x-x^2}} < 0. \ \square$$

3. Hartman-Wintner type asymptotic conditions

The equations (1), (4), and (9) have been proposed in the previous sections as the model-equations for 2-point oscillations on the interval I. It has been based on Theorem 2.1 and Theorem 2.5. In this section, we study 2-point oscillations on I in the case of second-order linear differential equations in a general form $y''+f(x)y=0$, where the coefficient $f(x)$ satisfies the Hartman-Wintner asymptotic condition on I:

$$f^{-\frac{1}{4}}(f^{-\frac{1}{4}})'' \in L^1(I). \tag{16}$$

The main properties for such a class of functions $f(x)$ satisfying (16) are the non-integrability of $f^{1/2}(x)$ on I and the regular asymptotic behaviour of $f^{-3/2}(x)f'(x)$ near $x=0$ and $x=1$ as follows.

LEMMA 3.1. *Let $f \in C^2(I)$, $f(x)>0$ on I, $f(0+)=f(1-)=\infty$, and let $f(x)$ satisfy the Hartman-Wintner condition* (16). *Then we have:*

$$\int_0^{1/2} f^{\frac{1}{2}}(x)dx = \int_{1/2}^1 f^{\frac{1}{2}}(x)dx = \infty, \tag{17}$$

and

$$\lim_{x\to 0+} f^{-\frac{3}{2}}f'(x) = \lim_{x\to 1-} f^{-\frac{3}{2}}f'(x) = 0. \tag{18}$$

Proof. In order to prove (17), it is enough to use the following technical result, which will be proved in Appendix of the paper.

PROPOSITION 3.2. *Let $F=F(x)$ be a real function such that $F\in C^2(I)$, $F(x)>0$ on I, and $F(0)=F(1)=0$. If $A>1$ and $F^{A-1}F''\in L^1(I)$, then $F^{-A}\notin L^1(0,\frac{1}{2})$, $F^{-A}\notin L^1(\frac{1}{2},1)$, and $\lim_{x\to 0}F^{A-1}F'(x)=\lim_{x\to 1}F^{A-1}F'(x)=0$.*

Now, the desired properties (17) and (18) easily follow from Proposition 3.2 by putting $F(x)=f^{-1/4}(x)$. $\square$

Now, we are able to state the main result of this section, which generalizes results obtained in previous sections.

THEOREM 3.3. *Let $f\in C^2(I)$, $f(x)>0$ on I, $f(0+)=f(1-)=\infty$, and let $f(x)$ satisfy the Hartman-Wintner condition* (16). *Then equation $y''+f(x)y=0$ is 2-point oscillatory on I.*

Proof. Using Liouville transformation $u(s)=y(x)f^{\frac{1}{4}}(x)$ and $s=\int_x^{1/2} f^{\frac{1}{2}}(\xi)d\xi$, we transform equation $y''+f(x)y=0$ on $(0,1/2]$ to the equation

$$\ddot{u}(s)+(1+\phi(s))u(s)=0,\ s\geqslant 0, \tag{19}$$

where "dot" denotes differentiation with respect to s and

$$\phi(s)=\frac{5}{16}f^{-3}f'^2-\frac{1}{4}f^{-2}f''\in L^1[0,\infty).$$

Applying Hartman-Wintner theorem (see Hartman [7, Corollary 8.1, p. 371]) to equation (19), we obtain

$$y(x)=f^{-\frac{1}{4}}(x)\big[c_1\cos(\int_x^{1/2} f^{\frac{1}{2}}(\xi)d\xi)+c_2\sin(\int_x^{1/2} f^{\frac{1}{2}}(\xi)d\xi)+o(1)\big] \tag{20}$$

and

$$y'(x) = f^{\frac{1}{4}}(x)\big[c_2\cos(\int_x^{1/2} f^{\frac{1}{2}}(\xi)d\xi) - c_1\sin(\int_x^{1/2} f^{\frac{1}{2}}(\xi)d\xi) + o(1)\big] \tag{21}$$

near $x = 0$. Similarly, we use $s = \int_{1/2}^{x} f^{\frac{1}{2}}(\xi)d\xi$ in the above to obtain equation (19) and

$$y(x) = f^{-\frac{1}{4}}(x)\big[c_1\cos(\int_{1/2}^{x} f^{\frac{1}{2}}(\xi)d\xi) + c_2\sin(\int_{1/2}^{x} f^{\frac{1}{2}}(\xi)d\xi) + o(1)\big] \tag{22}$$

and

$$y'(x) = f^{\frac{1}{4}}(x)\big[c_2\cos(\int_{1/2}^{x} f^{\frac{1}{2}}(\xi)d\xi) - c_1\sin(\int_{1/2}^{x} f^{\frac{1}{2}}(\xi)d\xi) + o(1)\big] \tag{23}$$

near $x = 1$. Now, since $f(0+) = f(1-) = \infty$, so by (17), (20), and (22), we deduce that $y'' + f(x)y = 0$ is 2-point oscillatory on I, which proves the theorem. □

With the help of Theorem 3.3, one can establish 2-point oscillations of equation (4) in a different way than the one presented in Corollary 2.6.

EXAMPLE 3.4. Let $c(x)$ be a continuous and smooth function on $\bar{I}$ such that $c(x) > 0$ for all $x \in \bar{I}$ and let $\sigma > 2$. It is easy to check that the function $f(x) = c(x)/(x-x^2)^{\sigma}$ satisfies: $f \in C^2(I)$, $f(x) > 0$ on I, $f(0+) = f(1-) = \infty$, and

$$f^{-\frac{1}{4}}(f^{-\frac{1}{4}})'' = Q(x)(x-x^2)^{-2+\frac{\sigma}{2}} \in L^1(I),$$

where $Q(x)$ is a smooth and bounded function on I, that is,

$$Q(x) = -(x-x^2)^2\Big[\frac{c''(x)}{4c^{3/2}(x)} - \frac{5c'^2(x)}{16c^{5/2}(x)}\Big] - (x-x^2)\Big[\frac{\sigma(1-2x)c'(x)}{8c^{3/2}(x)} + \frac{\sigma}{2c^{1/4}(x)}\Big]$$

$$+\frac{\sigma(\sigma-1/4)(1-2x)^2}{4c^{1/4}(x)}.$$

Now, by the use of Theorem 3.3, we can conclude once again that equation (4) is 2-point oscillatory on I. □

Now, we are able to derive some important consequences of the asymptotic formulas (20), (21), (22), and (23), which will be frequently used in the following sections. The first one is about the a priori estimates of $y(x)$ near $x = 0$ and $x = 1$ and the second one is about the stationary points of $y(x)$, where $y(x)$ is a solution of equation $y'' + f(x)y = 0$.

COROLLARY 3.5. *Let $f \in C^2(I)$, $f(x) > 0$ on I, $f(0+) = f(1-) = \infty$, and let $f(x)$ satisfy the Hartman-Wintner condition* (16). *Then for all solutions $y(x)$ of equation $y'' + f(x)y = 0$ we have:*

$$|y(x)| \leqslant cf^{-\frac{1}{4}}(x) \ \ and \ \ |y'(x)| \leqslant cf^{\frac{1}{4}}(x) \quad near\ x = 0\ and\ x = 1. \tag{24}$$

Furthermore, let $s_k, t_k \in I$ *be two sequences of consecutive zeros of* $y'(x)$ *such that* $s_k \searrow 0$ *and* $t_k \nearrow 1$*. Then there are two constants* $c_0 >$ *and* $c_1 > 0$ *and a* $k_0 \in \mathbf{N}$ *such that for all* $k \geqslant k_0$ *there hold true:*

$$|y(s_k)| \geqslant c_0 f^{-\frac{1}{4}}(s_k) \text{ and } |y(t_k)| \geqslant c_0 f^{-\frac{1}{4}}(t_k), \tag{25}$$

and

$$\int_{s_{k+1}}^{s_k} f^{\frac{1}{2}}(\xi)d\xi \leqslant c_1\pi \text{ and } \int_{t_k}^{t_{k+1}} f^{\frac{1}{2}}(\xi)d\xi \leqslant c_1\pi. \tag{26}$$

Proof. It is clear that the desired statement (24) immediately follows from (20), (21), (22), and (23). Next, we can rewrite (20), (21) as follows (see Hartman [7, p. 371]):

$$y(x) = Af^{-\frac{1}{4}}(x)\cos\Big(\int_x^{1/2} f^{\frac{1}{2}}(\xi)d\xi + B + o(1)\Big), \tag{27}$$

and

$$y'(x) = -Af^{\frac{1}{4}}(x)\sin\Big(\int_x^{1/2} f^{\frac{1}{2}}(\xi)d\xi + B + o(1)\Big) \tag{28}$$

near $x = 0$. So, for $x = s_k$ where $y'(s_k) = 0$ and $\lim_{k\to\infty} s_k = 0$, we obtain from (28)

$$\int_{s_k}^{1/2} f^{\frac{1}{2}}(\xi)d\xi + B + o(1) = n(k)\pi, \tag{29}$$

where $n(k)$ is an integer which increases by 1 as k is increased by 1. Using (29) in (27) we obtain $|y(s_k)| \geqslant c_0 f^{-\frac{1}{4}}(s_k)$, $k \geqslant k_0$, where c_0 is any positive constant less than $|A|$ for sufficiently large k_0. A similar argument applies to the sequence $t_k \in I$, where $y'(t_k) = 0$ and $\lim_{k\to\infty} t_k = 1$. This proves (25). Next, from (29), we deduce that

$$\int_{s_k}^{1/2} f^{\frac{1}{2}}(\xi)d\xi + o(1) = \pi,$$

so for $k \geqslant k_0$, k_0 sufficiently large, we have (26), where c_1 is any real number greater than 1. Likewise, a similar argument applies to the sequence $t_k \in I$ in (26). $\square$

REMARK 3.6. Let $S(f)$ denote the Schwarzian derivative of $\int_x^{1/2} f^{\frac{1}{2}}(\xi)d\xi$ or $\int_{1/2}^x f^{\frac{1}{2}}(\xi)d\xi$, which is defined by

$$S(f)(x) = \frac{f''(x)}{f(x)} - \frac{3}{2}\Big[\frac{f'(x)}{f(x)}\Big]^2, \; x \in I.$$

In equation (9), let $f(x) = q'^2(x)$ where $q(x)$ satisfies (5), (6), and (7). Note that $f \in C^2(I)$, $f(x) > 0$, and

$$\frac{1}{2}S(q')(x) = \frac{1}{2}S(f^{\frac{1}{2}})(x) = f^{\frac{1}{4}}(x)(f^{-\frac{1}{4}}(x))''. \tag{30}$$

Using Liouville transformation $u(s) = y(x)f^{\frac{1}{4}}(x)$ and $s = \int_x^{1/2} f^{\frac{1}{2}}(\xi)d\xi$ or $s = \int_{1/2}^x f^{\frac{1}{2}}(\xi)d\xi$ and the identity (30), we can transform equation (9) to $\ddot{u}(s) + u(s) = 0$. Condition (6) implies that $f(0+) = f(1-) = \infty$, so $y(x) = u(s)f^{-\frac{1}{4}}(x)$ shows that equation (9) is 2-point oscillatory on I.

On the other hand, we can write equation $y'' + f(x)y = 0$ as a perturbation of equation (9) as below:

$$y''(x) + \big[\big(\frac{1}{2}S(f^{\frac{1}{2}})(x) + f(x)\big) - \frac{1}{2}S(f^{\frac{1}{2}})(x)\big]y(x) = 0. \tag{31}$$

Observe that Liouville transformation now transforms (31) into $\ddot{u}(s) + (1+\phi(s))u(s) = 0$, where

$$-\phi(s) = \frac{1}{2f(x)}S(f^{\frac{1}{2}})(x) = f^{-\frac{3}{4}}(x)(f^{-\frac{1}{4}}(x))''.$$

By Hartman-Wintner Theorem (see Hartman [7, Corollary 8.1]), $y''(x) + f(x)y(x) = 0$ is 2-point oscillatory if $\phi \in L^1[0,\infty)$ which is equivalent to the Hartman-Wintner condition $f^{-\frac{1}{4}}(f^{-\frac{1}{4}})'' \in L^1(I)$. This provides an alternative proof of Theorem 3.3. $\square$

4. Two-point rectifiable oscillations

The problem of oscillations of any real continuous function $y(x)$ mostly considered on an infinite interval (x_0,∞), and so the graph $G(y)$ as a curve in $\mathbf{R}^2$ posses the infinite length. However, in some recent papers [8], [12], [14], [13], and [25], the oscillations of all solutions $y(x)$ of the second order differential equations $y'' + f(x)y = 0$ is studied on a finite interval, where the problem of finiteness or infiniteness of the graph $G(y)$ was naturally arises. The length of the graph $G(y)$ is determined as usually by,

$$\text{length}(G(y)) = \sup \sum_{i=1}^{m} ||(t_i, y(t_i)) - (t_{i-1}, y(t_{i-1}))||_2,$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \ldots < t_m = 1$ of the interval $\bar{I}$ and $||\ ||_2$ denotes the norm in $\mathbf{R}^2$. It is known that the graph $G(y)$ is said to be *rectifiable* curve in $\mathbf{R}^2$ provided $length(G(y)) < \infty$. Otherwise, $G(y)$ is said to be *unrectifiable* curve in $\mathbf{R}^2$. See for instance [6, Chapter 5.2].

DEFINITION 4.1. A 2-point oscillatory function $y(x)$ on I is said to be 2*-point rectifiable oscillatory* on I, if its graph $G(y)$ is a rectifiable curve in $\mathbf{R}^2$. The equation $y'' + f(x)y = 0$ is said to be 2*-point rectifiable oscillatory* on I, if all its non-trivial solutions are 2-point rectifiable oscillatory on I.

In this section, we study 2-point rectifiable oscillations of equation (9) and some other model-equations on I. Under the assumption of Hartman-Wintner condition (16), we prove an integral criterion for 2-point rectifiable oscillations of equation $y'' + f(x)y = 0$ on I.

THEOREM 4.2. *Let* $q(x)$ *satisfy* (5), (6), *and* (7). *If*

$$\left(|q'|^{-\frac{3}{2}}|q''|+|q'|^{\frac{1}{2}}\right) \in L^1(I), \tag{32}$$

then equation (9) *is* 2*-point rectifiable oscillatory on* I.

Proof. The proof is based on the following elementary geometric fact, see for instance [5, Theorem 1, p.217].

LEMMA 4.3. *The graph* $G(y)$ *is a rectifiable curve in* $\mathbf{R}^2$, *if and only if,* $y' \in L^1(I)$.

Now, let $y(x)$ be a solution of equation (9). By (8), we have that

$$|y'(x)| \leqslant |c_1 y_1'(x)| + |c_1 y_2'(x)| \leqslant c(|q'(x)|^{-\frac{3}{2}}|q''(x)| + |q'(x)|^{\frac{1}{2}}).$$

By (32), we have $y' \in L^1(I)$, which together with Lemma 4.3 proves Theorem 4.2. □

EXAMPLE 4.4. The equation (1) is 2-point rectifiable oscillatory on I. From Example 2.2 we know that equation (1) is 2-point oscillatory on I and that the function $q(x) = \rho \ln\ln(1/x)$, $\rho = (\lambda - 1/4)^{1/2}$, satisfies the conditions (5), (6), and (7). Also,

$$|q'(x)|^{-\frac{3}{2}}|q''(x)| + |q'(x)|^{\frac{1}{2}} = \frac{1}{[x\ln(1/x)]^{1/2}}(\ln(1/x) - 1 - \rho) \in L^1(I).$$

Now, by Theorem 4.2 we observe that (1) is 2-point rectifiable oscillatory on I. □

Next, we give a simple example for the function $q(x)$ which satisfies the conditions (5), (6), and (7), but does not satisfy the condition (32).

EXAMPLE 4.5. Let $q(x) = (1-2x)/(x-x^2)$. By Example 2.4 we know that such defined $q(x)$ satisfies (5), (6), and (7). However, the function $q(x)$ does not satisfy the condition (32) since

$$|q'(x)|^{-\frac{3}{2}}|q''(x)| + |q'(x)|^{\frac{1}{2}} = \frac{2|1-2x|(x^2-x+1)}{(2x^2-2x+1)^{3/2}} + \frac{(2x^2-2x+1)^{3/2}}{x-x^2} \notin L^1(I).$$

This gives a sub-class of equation (9) which are not 2-point rectifiable oscillatory on I. In Section 5, we shall give a condition to $q(x)$ which ensures that equation (9) is 2-point unrectifiable oscillatory on I (see Theorem 5.2). □

The most simple case for the function $q(x)$ satisfying the conditions (5), (6), (7), and (32) is the following.

EXAMPLE 4.6. Let $0 < \beta < 1$ and let $q(x) = (1-2x)/(x-x^2)^\beta$, $x \in I$. By Example 2.4 we know that such a class of functions $q(x)$ satisfies (5), (6), and (7). Moreover, since $0 < \beta < 1$ we have

$$|q'(x)|^{-\frac{3}{2}}|q''(x)| + |q'(x)|^{\frac{1}{2}} \leqslant c(x-x^2)^{\frac{\beta-1}{2}} + c(x-x^2)^{-\frac{\beta+1}{2}} \in L^1(I),$$

which shows that $q(x)$ satisfies the condition (32). □

THEOREM 4.7. *Let $f \in C^2(I)$, $f(x) > 0$ on I, $f(0+) = f(1-) = \infty$, and let $f(x)$ satisfy the Hartman-Wintner condition* (16). *If*

$$f^{1/4} \in L^1(I), \tag{33}$$

then equation $y'' + f(x)y = 0$ is 2-point rectifiable oscillatory on I.

Proof. Let $y(x)$ be a solution of equation $y'' + f(x)y = 0$. According to Theorem 3.3 we know that $y(x)$ is 2-point oscillatory on I. From (24) and (33) we conclude that $y' \in L^1(I)$ which by Lemma 4.3 proves that $G(y)$ is a rectifiable curve in $\mathbf{R}^2$. Hence, $y(x)$ is 2-point rectifiable oscillatory on I. □

As the main consequence of Theorem 4.7, we establish 2-point rectifiable oscillations on I of our principal model-equation (4).

COROLLARY 4.8. *If $\sigma \in (2,4)$, then equation* (4) *is 2-point rectifiable oscillatory on I.*

Proof. From $f(x) = c(x)/(x-x^2)^\sigma$, $x \in I$, follows $f^{1/4}(x) = c^{1/4}/(x-x^2)^{\sigma/4} \in L^1(I)$ provided $\sigma \in (2,4)$, which together by Example 3.4 implies that $f(x)$ satisfies all assumptions of Theorem 4.7. Hence, equation (4) is 2-point rectifiable oscillatory on I. □

5. Two-point unrectifiable oscillations

In this section, we study 2-point unrectifiable oscillations of second-order linear differential equations on the interval I. In Example 4.5, we show that such a kind of 2-point oscillations is very possible.

DEFINITION 5.1. A 2-point oscillatory function $y(x)$ on I is said to be *2-point unrectifiable oscillatory* on I, if its graph $G(y)$ is an unrectifiable curve in $\mathbf{R}^2$. The equation $y'' + f(x)y = 0$ is said to be *2-point unrectifiable oscillatory* on I, if all its non-trivial solutions are 2-point unrectifiable oscillatory on I.

At first we give a sufficient condition on the function $q(x)$ such that equation (9) is 2-point unrectifiable oscillatory on I. It completes preceding Theorem 2.1 and Theorem 4.2 about 2-point oscillations of equation (9) on the interval I.

THEOREM 5.2. *Let $q(x)$ satisfy the conditions* (5), (6), *and* (7). *We suppose that $|q'(x)|^{-1}$ is increasing near $x = 0$ and decreasing near $x = 1$. If the series,*

$$\sum_k |q'(q^{-1}(k\pi))|^{-\frac{1}{2}} \ \text{or} \ \sum_k |q'(q^{-1}(-k\pi))|^{-\frac{1}{2}} \tag{34}$$

are divergent, then the equation (9) *is 2-point unrectifiable oscillatory on I.*

In order to prove Theorem 5.2, we need the following elementary fact.

LEMMA 5.3. *Let $s_k \in I$ and $t_k \in I$ be two sequences of consecutive zeros of $y'(x)$ such that $s_k \searrow 0$ and $t_k \nearrow 1$. If $\sum_k |y(s_k)|$ or $\sum_k |y(t_k)|$ is divergent, then the graph $G(y)$ is an unrectifiable curve in $\mathbf{R}^2$.*

For the proofs of this lemma, we refer reader to [12, Proposition 4.2].

Proof of Theorem 5.2. Let $y(x)$ be a solution of equation (9). According to (5), (6), (7), and Theorem 2.1, we know that $y(x)$ is 2-point oscillatory on I. By means of general solution (8), we derive two cases: either $y(x) = c_2 y_2(x)$ or $y(x)$ and $y_2(x)$ are linearly independent, where $y_2(x) = |q'(x)|^{-1/2} \sin q(x)$. Suppose that $y(x) = c_2 y_2(x)$, $a_k = q^{-1}(k\pi)$, $b_k = q^{-1}(-k\pi)$, $s_k = q^{-1}(\pi/2 + k\pi)$, and $t_k = q^{-1}(-\pi/2 - k\pi)$. Since $q(x)$ is decreasing, we have that $y(a_k) = y(b_k) = 0$, $s_k \in (a_{k+1}, a_k)$, $t_k \in (b_k, b_{k+1})$, and

$$\begin{aligned}\sum_k |y(q^{-1}(\pm(k+1/2)\pi))| &= |c_2| \sum_k |y_2(q^{-1}(\pm(k+1/2)\pi))| \\ &= |c_2| \sum_k |q'(q^{-1}(\pm(k+1/2)\pi))|^{-\frac{1}{2}} \\ &\geqslant |c_2| \sum_k |q'(q^{-1}(\pm(k+1)\pi))|^{-\frac{1}{2}}.\end{aligned}$$

This together with (34) imply that $\sum_k |y(s_k)|$ or $\sum_k |y(t_k)|$ is divergent. Hence by Lemma 5.3, $y(x) = c_2 y_2(x)$ is 2-point unrectifiable oscillatory on I.

In the second case when $y(x)$ and $y_2(x)$ are two linearly independent solutions of equation (9), by Theorem 2.1, there are a_k and $b_k \in I$, two sequences of consecutive zeros of $y(x)$ such that a_k is decreasing and $a_k \searrow 0$, and b_k is increasing and $b_k \nearrow 1$. Let $s_k = q^{-1}(k\pi)$ and $t_k = q^{-1}(-k\pi)$ be two sequences of consecutive zeros of $y_2(x)$. By Sturm comparison principle and since $y(x)$ and $y_2(x)$ are linearly independent, we know that there is $k_0 \in \mathbf{N}$ such that $s_k \in (a_{k-k_0+1}, a_{k-k_0})$ and $t_k \in (b_{k-k_0}, b_{k-k_0+1})$ for all $k \geqslant k_0$. Obviously, $y_2(q^{-1}(\pm k\pi)) = 0$ and $|y_2'(q^{-1}(\pm k\pi))| = |q'(q^{-1}(\pm k\pi))|^{\frac{1}{2}}$. Since the Wronskian $|W(y_2, y)(x)| = c > 0$ for each $x \in I$, it implies that

$$\begin{aligned}0 < c = |W(y_2, y)(q^{-1}(\pm k\pi))| &= |y_2'(q^{-1}(\pm k\pi)) y(q^{-1}(\pm k\pi))| \\ &= |q'(q^{-1}(\pm k\pi))|^{\frac{1}{2}} |y(q^{-1}(\pm k\pi))|, \text{ for } k > k_0.\end{aligned} \tag{35}$$

From (35) follows

$$\sum_k |y(q^{-1}(\pm k\pi))| = c \sum_k |q'(q^{-1}(\pm k\pi))|^{-\frac{1}{2}}.$$

Now, the assumption (34) implies that either one of the sequences $\sum_k |y(s_k)|$ and $\sum_k |y(t_k)|$ is divergent. Hence from Lemma 5.3 follows that $y(x)$ is 2-point unrectifiable oscillatory on I. $\square$

EXAMPLE 5.4. Let $q(x) = (1-2x)/(x-x^2)^\beta$, $x \in I$, where $\beta \geqslant 1$. By Example 2.4, we know that $q(x)$ satisfies (5), (6), and (7). Also, from (12) we have that

$$|q'(x)|^{-\frac{1}{2}} \geqslant c(x-x^2)^{\frac{\beta+1}{2}} \geqslant c\left(\frac{|1-2x|}{|q(x)|}\right)^{\frac{\beta+1}{2\beta}} \quad \text{for all } x \in I. \tag{36}$$

Since $s_k = q^{-1}(\pi/2+k\pi)$ tends to 0 and $t_k = q^{-1}(-\pi/2-k\pi)$ tends to 1, from equalities $q(s_k)=\pi/2\pm k\pi$, $q(t_k)=\pm k\pi$, and (36) with $x=d_k$, where $d_k=s_k$ or $d_k=t_k$ and k is sufficiently large, together yield:

$$|q'(d_k)|^{-\frac{1}{2}} \geqslant c\left(\frac{|1-2d_k|}{|q(d_k)|}\right)^{\frac{\beta+1}{2\beta}} \geqslant c_1\left(\frac{1}{k}\right)^{\frac{\beta+1}{2\beta}}.$$

Since $\beta\geqslant 1$, it implies that series $\sum|q'(d_k)|^{-\frac{1}{2}}$ is divergent, so by (34) and Theorem 5.2, equation (9) is 2-point unrectifiable oscillatory on I. $\square$

THEOREM 5.5. *Let $f\in C^2(I)$, $f(x)>0$ on I, $f(0+)=f(1-)=\infty$, and let $f(x)$ satisfy the Hartman-Wintner condition* (16). *If*

$$f^{\frac{1}{4}} \notin L^1(I), \tag{37}$$

then equation $y''+f(x)y=0$ is 2-point unrectifiable oscillatory on I.

Proof. The main idea is taken from a proof of unrectifiable oscillations of p-Laplacian equation, see [15, Theorem 3.2]. Let $y(x)$ be a solution of equation $y''+f(x)y=0$. Let $s_k,t_k\in I$ be two sequences of consecutive zeros of $y'(x)$ such that $s_k\searrow 0$ and $t_k\nearrow 1$ when k goes to infinity. Let $\varphi(x)$ and $\psi(x)$ be two functions defined on $(0,\frac{1}{2}]$ and $[\frac{1}{2},1)$ respectively by

$$\varphi(x)=\int_x^{1/2} f^{\frac{1}{2}}(\xi)d\xi,\ x\in(0,\frac{1}{2}],\quad \text{and}\quad \psi(x)=\int_{1/2}^{x} f^{\frac{1}{2}}(\xi)d\xi,\ x\in[\frac{1}{2},1).$$

It is clear that $\varphi(x)$ is decreasing on $(0,\frac{1}{2}]$ and $\psi(x)$ is increasing on $[\frac{1}{2},1)$. Also, from (17) follows that $\lim_{x\to 0+}\varphi(x)=\lim_{x\to 1-}\psi(x)=\infty$. Therefore, there exist the inverse functions $\varphi^{-1}(t)$ and $\psi^{-1}(t)$, and two sequences S_k and T_k such that $s_k=\varphi^{-1}(S_k)$ and $t_k=\psi^{-1}(T_k)$ respectively. Let k_0 be a sufficiently large natural number. In order to show that $G(y)$ is an unrectifiable curve in $\mathbf{R}^2$, by Lemma 5.3 it is enough to show that at least one of $\sum|y(\varphi^{-1}(S_k))|$ and $\sum|y(\psi^{-1}(T_k))|$ is divergent. Let $F(x)=f^{-\frac{1}{4}}(x)$ and let $\sigma_k\in[S_k,S_{k+1}]$ and $\tau_k\in[T_k,T_{k+1}]$ be two sequences of real numbers defined by

$$F(\varphi^{-1}(\sigma_k))=\max_{t\in[S_k,S_{k+1}]}F(\varphi^{-1}(t)) \text{ and } F(\psi^{-1}(\tau_k))=\max_{t\in[T_k,T_{k+1}]}F(\psi^{-1}(t)).$$

We claim that

$$F(\varphi^{-1}(S_k))\geqslant \tfrac{3}{4}F(\varphi^{-1}(\sigma_k)) \text{ and } F(\psi^{-1}(T_k))\geqslant \tfrac{3}{4}F(\psi^{-1}(\tau_k)),\ k\geqslant k_0. \tag{38}$$

Indeed, for $k\geqslant k_0$, by Lagrange mean-value theorem we obviously have:

$$\begin{cases} F(\varphi^{-1}(S_k))\geqslant F(\varphi^{-1}(\sigma_k))-\max_{t\in[S_k,S_{k+1}]}|\frac{d}{dt}[F(\varphi^{-1}(t))]||S_k-\sigma_k|,\\ F(\psi^{-1}(T_k))\geqslant F(\psi^{-1}(\tau_k))-\max_{t\in[T_k,T_{k+1}]}|\frac{d}{dt}[F(\psi^{-1}(t))]||T_k-\tau_k|.\end{cases}$$

Hence, with the help of (26) we have $|S_k - S_{k+1}| \leqslant c_1\pi$ and $|T_k - T_{k+1}| \leqslant c_1\pi$, and so,

$$\begin{cases} F(\varphi^{-1}(S_k)) \geqslant F(\varphi^{-1}(\sigma_k)) - c_1\pi \max_{t\in[S_k,S_{k+1}]} |\frac{d}{dt}[F(\varphi^{-1}(t))]|, \\ F(\psi^{-1}(T_k)) \geqslant F(\psi^{-1}(\tau_k)) - c_1\pi \max_{t\in[T_k,T_{k+1}]} |\frac{d}{dt}[F(\psi^{-1}(t))]|. \end{cases} \tag{39}$$

From (18) we also have $|f^{-\frac{3}{2}}(x)f'(x)| < 1/(c_1\pi)$ near $x = 0$ and $x = 1$. Since $|\varphi'(x)| = |\psi'(x)| = f^{\frac{1}{2}}(x)$, it implies that

$$\begin{aligned} |\frac{d}{dt}[F(\varphi^{-1}(t))]| &= \frac{1}{4}\frac{f^{-\frac{5}{4}}(x)|f'(x)|}{|\varphi'(x)|} = \frac{1}{4}\frac{f^{-\frac{5}{4}}(x)|f'(x)|}{f^{\frac{1}{2}}(x)} \\ &= \frac{1}{4}f^{-\frac{3}{2}}(x)|f'(x)|F(\varphi^{-1}(t)) \leqslant \frac{1}{4c_1\pi}F(\varphi^{-1}(t)). \end{aligned}$$

Putting the above inequality into (39) we obtain the desired statement (38). Next, since $F(x) = f^{-1/4}(x)$, the statement (25) can be rewritten in the form:

$$|y(\varphi^{-1}(S_k))| \geqslant c_0 F(\varphi^{-1}(S_k)) \ \text{ and } \ |y(\psi^{-1}(T_k))| \geqslant c_0 F(\psi^{-1}(T_k)). \tag{40}$$

Now, from (38) and (40), we observe that:

$$\begin{aligned} \sum_k |y(s_k)| &= \sum_k |y(\varphi^{-1}(S_k))| \geqslant c_0 \sum_{k\geqslant k_0} F(\varphi^{-1}(S_k)) \\ &\geqslant \frac{3c_0}{4}\sum_{k\geqslant k_0} F(\varphi^{-1}(\sigma_k)) = \frac{3c_0}{4}\sum_{k\geqslant k_0} f^{-\frac{1}{4}}(\varphi^{-1}(\sigma_k)) \\ &= \frac{3c_0}{4}\sum_{k\geqslant k_0}\frac{1}{|S_k - S_{k+1}|}\int_{S_k}^{S_{k+1}} f^{-\frac{1}{4}}(\varphi^{-1}(\sigma_k))dt \\ &\geqslant \frac{3c_0}{4c_1\pi}\sum_{k\geqslant k_0}\int_{S_k}^{S_{k+1}} f^{-\frac{1}{4}}(\varphi^{-1}(t))dt = c_3\lim_{\varepsilon\to 0}\int_{S_{k_0}}^{\varphi(\varepsilon)} f^{-\frac{1}{4}}(\varphi^{-1}(t))dt \\ &= -c_3\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\varphi^{-1}(S_{k_0})} f^{-\frac{1}{4}}(x)\varphi'(x)dx = c_3\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\varphi^{-1}(S_{k_0})} f^{-\frac{1}{4}}(x)f^{\frac{1}{2}}(x)dx \\ &= c_3\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\varphi^{-1}(S_{k_0})} f^{\frac{1}{4}}(x)dx, \end{aligned} \tag{41}$$

where $c_3 = 3c_0/(4c_1\pi)$. Analogously, we obtain for the sequence t_k that

$$\sum_k |y(t_k)| \geqslant c_3\lim_{\varepsilon\to 0}\int_{\psi^{-1}(T_{k_0})}^{1-\varepsilon} f^{\frac{1}{4}}(x)dx. \tag{42}$$

Since by (37) we have that $f^{1/4} \notin L^1(I)$ and so, from (41) and (42), we have that at least one of $\sum_k |y(s_k)|$ and $\sum_k |y(t_k)|$ is divergent. Hence by Lemma 5.3 the graph $G(y)$ is an unrectifiable curve in $\mathbf{R}^2$, which together by Theorem 3.3 implies that equation $y'' + f(x)y = 0$ is 2-point unrectifiable oscillatory on I. □

As a consequence of Theorem 5.5 we establish 2-point unrectifiable oscillation on I of equation (4) as follows.

COROLLARY 5.6. *If* $\sigma \geqslant 4$, *then equation* (4) *is* 2*-point unrectifiable oscillatory on* I.

Proof. Let $f(x) = c(x)/(x-x^2)^{\sigma}$, $x \in I$. Since $\sigma \geqslant 4$, we have

$$f^{1/4}(x) = \frac{c^{1/4}(x)}{(x-x^2)^{\sigma/4}} \notin L^1(I),$$

which together by Example 3.4 implies that the function $f(x)$ satisfies all assumptions of Theorem 5.5. Hence, equation (4) is 2-point unrectifiable oscillatory on I. $\square$

6. Two-point fractal oscillations of equation (4)

In the next two sections, we discuss 2-point oscillations of a function $y(x)$ on I, where the graph $G(y)$ is a fractal curve in $\mathbf{R}^2$. The fractality of a graph $G(y)$ will be expressed in terms of its upper Minkowski-Bouligand dimension, also known as box-counting dimension,

$$\dim_M G(y) = \limsup_{\varepsilon \to 0} \big(2 - \frac{\log |G_\varepsilon(y)|}{\log \varepsilon}\big),$$

and corresponding d-dimensional upper Minkowski content

$$M^d(G(y)) = \limsup_{\varepsilon \to 0} (2\varepsilon)^{d-2} |G_\varepsilon(y)|,\ d \in [1,2).$$

Here, the $\varepsilon-$ neighbourhood $G_\varepsilon(y)$ of the graph $G(y)$ is given by $G_\varepsilon(y) = \{(t_1,t_2) \in \mathbf{R}^2 : d((t_1,t_2),G(y)) \leqslant \varepsilon\}$, where $\varepsilon > 0$ and $d((t_1,t_2),G(y))$ denotes the distance from (t_1,t_2) to $G(y)$, and $|G_\varepsilon(y)|$ denotes the Lebesgue measure of $G_\varepsilon(y)$.

DEFINITION 6.1. Let $y(x)$ be 2-point oscillatory function on I. If there is an $d \in (1,2)$ such that $\dim_M G(y) = d$ and $0 < M^d(G(y)) < \infty$, then $y(x)$ is said to be 2*-point fractal oscillatory* on I.

In order to find a number $d \in (1,2)$ such that $\dim_M G(y) = d$ and $0 < M^d(G(y)) < \infty$, we will use two geometric lemmas: the first one deals with $\dim_M G(y) \geqslant d$ and $M^d(G(y)) > 0$ and the second one with $\dim_M G(y) \leqslant d$ and $M^d(G(y)) < \infty$. Since the statements $\dim_M G(y) = d$ and $0 < M^d(G(y)) < \infty$ imply that $G(y)$ is an unrectifiable curve in $\mathbf{R}^2$, 2-point fractal oscillations on I present a refinement of 2-point unrectifiable oscillations on I.

DEFINITION 6.2. The equation $y'' + f(x)y = 0$ is said to be 2*-point fractal oscillatory* on I, if all its non-trivial solutions are 2-point fractal oscillatory on I.

In this section, we study 2-point fractal oscillations of second-order linear differental equation $y'' + f(x)y = 0$ where $f(x)$ satisfies the Hartman-Wintner asymptotic condition (16) and prove results concerning 2-point fractal oscillations of equation (4).

THEOREM 6.3. *Let $f \in C^2(I)$, $f(x) > 0$ on I, $f(0+) = f(1-) = \infty$, and let $f(x)$ satisfy the Hartman-Wintner condition* (16). *Let $\sigma > 4$ and let $f(x) \sim (x-x^2)^{-\sigma}$ near $x = 0$ and $x = 1$, that is, there exist constants $\lambda_0 > 0$, $\lambda_1 > 0$, and $\delta \in I$ such that*

$$\frac{\lambda_0}{(x-x^2)^\sigma} \leqslant f(x) \leqslant \frac{\lambda_1}{(x-x^2)^\sigma} \quad \text{for all} \quad x \in (0,\delta) \cup (1-\delta,1). \tag{43}$$

Then equation $y'' + f(x)y = 0$ is 2-point fractal oscillatory on I with the dimensional number $d = 3/2 - 2/\sigma$.

Proof. We first need the following result concerning the zeros of solution $y(x)$ of the second-order linear differential equation $y'' + f(x)y = 0$, where $f(x)$ satisfies (43). It will be proved in Appendix of the paper.

LEMMA 6.4. *Let $f \in C^2(I)$, $f(x) > 0$ on I, and $f(0+) = f(1-) = \infty$. Let $f(x)$ satisfy the conditions* (16) *and* (43), *where $\sigma > 4$. Let $y(x)$ be a nontrivial solution of equation $y'' + f(x)y = 0$. Let a_k and b_k be two sequences of consecutive zeros of $y(x)$ such that $a_k \searrow 0$ and $b_k \nearrow 1$, when $k \to \infty$. Then there is $k_0 \in \mathbf{N}$ such that for all $k \geqslant k_0$:*

$$2^{-\sigma/2}\lambda_1^{-1/2} a_{k+1}^{\sigma/2} \leqslant (a_k - a_{k+1}) \leqslant \pi\lambda_0^{-1/2} a_k^{\sigma/2}, \tag{44}$$

and

$$2^{-\sigma/2}\lambda_1^{-1/2}(1-b_{k+1})^{\sigma/2} \leqslant (b_{k+1} - b_k) \leqslant \pi\lambda_0^{-1/2}(1-b_k)^{\sigma/2}. \tag{45}$$

Next, for $m > 0$ small enough and $M > 0$ large enough there is a $k_0 \in \mathbf{N}$ such that:

$$\left(\frac{m}{2\pi}\right)^{\frac{2}{\sigma-2}} \left(\frac{1}{k+k_0}\right)^{\frac{2}{\sigma-2}} \leqslant a_k \leqslant 2\left(\frac{M}{\pi}\right)^{\frac{2}{\sigma-2}} \left(\frac{1}{k-k_0}\right)^{\frac{2}{\sigma-2}}, \quad k \geqslant k_0, \tag{46}$$

and

$$1 - 2\left(\frac{M}{\pi}\right)^{\frac{2}{\sigma-2}} \left(\frac{1}{k-k_0}\right)^{\frac{2}{\sigma-2}} \leqslant b_k \leqslant 1 - \left(\frac{m}{2\pi}\right)^{\frac{2}{\sigma-2}} \left(\frac{1}{k+k_0}\right)^{\frac{2}{\sigma-2}}, \quad k \geqslant k_0. \tag{47}$$

Furthermore, $y(x)$ is convex-concave function on (a_{k+1}, a_k) and (b_k, b_{k+1}), and there are a constant $c > 0$ and two sequences $s_k \in (a_k, a_{k+1})$ and $t_k \in (b_k, b_{k+1})$ such that for all $k \geqslant k_0$:

$$|y(s_k)| \geqslant c\, s_k^{\sigma/4} \text{ and } |y(t_k)| \geqslant c(1-t_k)^{\sigma/4}. \tag{48}$$

Next, we need the following 2-point version of a result in [13, Lemma 4.1], which is valid for any arbitrarily given continuous function $y(x)$ on $\bar{I}$.

LEMMA 6.5. *Let $y = y(x)$ be a real function, $y \in C(\bar{I})$, and $y(0) = y(1) = 0$. Let $a_k \in I$ and $b_k \in I$ be two sequences of consecutive zeros of $y(x)$ such that a_k is decreasing, $a_k \searrow 0$, and b_k is increasing, $b_k \nearrow 1$. For any $\varepsilon \in (0, \min\{\varepsilon_0, \varepsilon_1\})$, where ε_0 and ε_1 are two positive constants, we suppose that there are two natural numbers $k_0(\varepsilon), k_1(\varepsilon) \in \mathbf{N}$ such that*

$$\max\{|a_k - a_{k+1}|, |b_{k+1} - b_k|\} \leqslant \varepsilon/2, \text{ for each } k \geqslant \max\{k_0(\varepsilon), k_1(\varepsilon)\}. \tag{49}$$

If there are four sequences of real numbers, $\delta_k, \gamma_k > 0$, and $s_k \in (a_{k+1}, a_k)$, $t_k \in (b_k, b_{k+1})$ such that

$$|y(s_k)| \geqslant \delta_k \text{ and } |y(t_k)| \geqslant \gamma_k, \text{ for each } k \geqslant \max\{k_0(\varepsilon), k_1(\varepsilon)\}, \tag{50}$$

then for all $\varepsilon \in (0, \min\{\varepsilon_0, \varepsilon_1\})$, the function $y(x)$ satisfies:

$$|G_\varepsilon(y)| \geqslant \max\{\sum_{k=k_0(\varepsilon)}^{\infty} \delta_k(a_k - a_{k+1}), \sum_{k=k_1(\varepsilon)}^{\infty} \gamma_k(b_{k+1} - b_k)\}. \tag{51}$$

In order to prove Lemma 6.5, it is enough to follow the same argument from the proof of [11, Lemma 2.1], see also [8, Appendix].

Now, we proceed with the proof of Theorem 6.3. We will show that any solution $y(x)$ of equation $y'' + f(x)y = 0$ satisfies assumptions of Lemma 6.5. Let $k(\varepsilon)$ be a natural number determined for all $\varepsilon \in (0, \varepsilon_0)$ by

$$c_0 \varepsilon^{-\frac{\sigma-2}{\sigma}} + k_0 < k_0(\varepsilon) = k_1(\varepsilon) < 2c_0 \varepsilon^{-\frac{\sigma-2}{\sigma}} - k_0 - 1, \tag{52}$$

where the constants c_0 and ε_0 satisfy

$$c_0 = M\left(\frac{2^{\sigma^2-4}}{\pi^4 \lambda_0^{\sigma-2}}\right)^{1/(2\sigma)} \quad \text{and} \quad \varepsilon_0 = \left(\frac{c_0}{2k_0+2}\right)^{\sigma/(\sigma-2)}.$$

Here the constants M, λ_0, and k_0 are from $(44)-(47)$. From (44), (45), (46), and (47), for all $k \geqslant k_0$ we derive:

$$\begin{cases} |a_k - a_{k+1}| \leqslant \pi \lambda_0^{-1/2} a_k^{\sigma/2} \leqslant \frac{1}{2} c_0^{\frac{\sigma}{\sigma-2}} (\frac{1}{k-k_0})^{\frac{\sigma}{\sigma-2}}, \\ |b_{k+1} - b_k| \leqslant \pi \lambda_0^{-1/2} (1-b_k)^{\sigma/2} \leqslant \frac{1}{2} c_0^{\frac{\sigma}{\sigma-2}} (\frac{1}{k-k_0})^{\frac{\sigma}{\sigma-2}}. \end{cases} \tag{53}$$

Putting (52) into (53), for $d_k = a_k$, $d_k = b_k$, and $k \geqslant k_i(\varepsilon)$, we obtain that $|d_k - d_{k+1}| \leqslant \frac{1}{2} c_0^{\frac{\sigma}{\sigma-2}} (1/(k-k_i(\varepsilon)))^{\frac{\sigma}{\sigma-2}} \leqslant \varepsilon/2$, which together with (48) implies that $y(x)$ satisfies all assumptions of Lemma 6.5. Hence by (51) we derive that

$$|G_\varepsilon(y)| \geqslant c \max\{\sum_{k=k_0(\varepsilon)}^{\infty} s_k^{\sigma/4}(a_k - a_{k+1}), \sum_{k=k_1(\varepsilon)}^{\infty} (1-t_k)^{\sigma/4}(b_{k+1} - b_k)\}. \tag{54}$$

Since $s_k \geqslant a_{k+1}$ and $t_k \leqslant b_{k+1}$, from (44), (45), and (54) follows

$$|G_\varepsilon(y)| \geqslant c_1 \max\{\sum_{k=k_0(\varepsilon)}^{\infty} a_{k+1}^{3\sigma/4}, \sum_{k=k_1(\varepsilon)}^{\infty} (1-b_{k+1})^{3\sigma/4}\}.$$

Therefore, from (46) and (47) follows

$$\begin{aligned} |G_\varepsilon(y)| &\geqslant c_2 \sum_{k=k_i(\varepsilon)}^{\infty} \left(\frac{1}{k+1+k_0}\right)^{\frac{3\sigma}{2\sigma-4}} \geqslant c_3 \left(\frac{1}{k_i(\varepsilon)+1+k_0}\right)^{\frac{3\sigma}{2\sigma-4}-1} \\ &\geqslant c_4 \varepsilon^{\frac{\sigma-2}{\sigma}\frac{\sigma+4}{2\sigma-4}} = c_4 \varepsilon^{\frac{\sigma+4}{2\sigma}}. \end{aligned}$$

Applying the definitions of $\dim_M G(y)$ and $M^d(G(y))$ to the previous inequality, we observe that

$$\dim_M G(y) \geqslant d = 3/2 - 2/\sigma \text{ and } M^d(G(y)) > 0. \tag{55}$$

In addition to Lemma 6.4 and Lemma 6.5, we need the following geometric lemma, which is the 2-point version of [13, Lemma 5.1].

LEMMA 6.6. *Let $y = y(x)$ be a real function, $y \in C^2(I) \cup C(\bar{I})$. Let $a_k, b_k \in I$, $s_k \in (a_{k+1}, a_k)$, and $t_k \in (b_k, b_{k+1})$ be four sequences of consecutive zeros of $y(x)$ and $y'(x)$ respectively such that:*

$$\begin{cases} y(a_k) = y(b_k) = 0,\ a_k \searrow 0, \text{ and } b_k \nearrow 1, \\ y(x) \text{ is convex-concave on } (a_{k+1}, a_k) \text{ and } (b_k, b_{k+1}), \\ y'(s_k) = y'(t_k) = 0. \end{cases} \tag{56}$$

Let $k_0, k_1 \in \mathbf{N}$ be large enough, and let $k_0(\varepsilon)$ and $k_1(\varepsilon)$ be two natural numbers depending on ε such that $k_i(\varepsilon) \geqslant k_i + 1$. There is a positive constant $c > 0$ such that

$$\begin{aligned} |G_\varepsilon(y|_I)| \leqslant c\varepsilon &+ 2a_{k_0(\varepsilon)}|y(s_{k_0(\varepsilon)})| + 2(1 - b_{k_1(\varepsilon)})|y(t_{k_1(\varepsilon)})| \\ &+ \varepsilon \sum_{k=k_0+1}^{k_0(\varepsilon)} [2|y(s_k)| + a_k - a_{k+1}] + \varepsilon \sum_{k=k_1+1}^{k_1(\varepsilon)} [2|y(t_k)| + b_{k+1} - b_k]. \end{aligned} \tag{57}$$

In order to prove this lemma we suggest to use the partitions of I in the following way: $I_1 = [0, a_{k_0(\varepsilon)}]$, $I_2 = [a_{k_0(\varepsilon)}, a_{k_0}]$, $I_3 = [a_{k_0}, b_{k_1}]$, $I_4 = [b_{k_1}, b_{k_1(\varepsilon)}]$, and $I_5 = [b_{k_1(\varepsilon)}, 1]$. If $y|_J$ denotes the function-restriction of $y(x)$ on a closed interval J, then

$$G_\varepsilon(y|_I) = \cup_{i=1}^5 G_\varepsilon(y|_{I_i}) \ \text{ and } \ |G_\varepsilon(y|_I)| \leqslant \cup_{i=1}^5 |G_\varepsilon(y|_{I_i})|. \tag{58}$$

Obviously, there is a constant $c_3 > 0$ such that $|G_\varepsilon(y|_{I_3})| \leqslant c_3\varepsilon$. By an easy geometric argument (for an exact calculation see [11, Lemma 2.2]), the desired inequality (57) follows from (56) and (58).

Next, we now return to the proof of Theorem 6.3 and let $s_k \in (a_{k+1}, a_k)$ and $t_k \in (b_k, b_{k+1})$ such that $y'(s_k) = y'(t_k) = 0$. From (24) and (43) follows that $|y(x)| \leqslant \lambda_1^{-1/4}(x - x^2)^{\sigma/4}$ near $x = 0$ and $x = 1$. Since $s_k \leqslant a_k$ and $t_k \geqslant b_k$, from previous inequality we conclude:

$$|y(s_k)| \leqslant c_0 a_k^{\sigma/4} \text{ and } |y(t_k)| \leqslant c_0(1 - b_k)^{\sigma/4}, \text{ for all } k > k_0, \tag{59}$$

where k_0 is sufficiently large. Furthermore, since $f(x) > 0$ on I, we have that $y(x)$ is convex-concave function on (a_{k+1}, a_k) and (b_k, b_{k+1}), and so, the statement (56) is satisfied. Now, with the help of (44), (46), (52), and (59) we obtain:

$$a_{k_0(\varepsilon)}|y(s_{k_0(\varepsilon)})| \leqslant c_0(a_{k_0(\varepsilon)})^{1+\frac{\sigma}{4}} \leqslant c_1\left(\frac{1}{k - k_0(\varepsilon)}\right)^{\frac{\sigma+4}{2\sigma-4}} \leqslant c_2(\varepsilon^{\frac{\sigma-2}{\sigma}})^{\frac{\sigma+4}{2\sigma-4}} = c_2\varepsilon^{\frac{\sigma+4}{2\sigma}},$$

and

$$\varepsilon \sum_{k=k_0+1}^{k_0(\varepsilon)} [2|y(s_k)|+a_k-a_{k+1}] \leqslant c_3\varepsilon \sum_{k=k_0+1}^{k_0(\varepsilon)} [a_k^{\sigma/4}+a_k^{\sigma/2}] \leqslant c_4\varepsilon \sum_{k=k_0+1}^{k_0(\varepsilon)} a_k^{\sigma/4}$$
$$\leqslant c_5\varepsilon \sum_{k=k_0+1}^{k_0(\varepsilon)} (\frac{1}{k-k_0})^{\frac{\sigma}{2\sigma-4}} \leqslant c_6\varepsilon(\frac{1}{k-k_0(\varepsilon)})^{\frac{\sigma}{2\sigma-4}-1}$$
$$\leqslant c_7\varepsilon(\varepsilon^{\frac{\sigma-2}{\sigma}})^{\frac{4-\sigma}{2\sigma-4}} = c_7\varepsilon^{\frac{\sigma+4}{2\sigma}}.$$

Also, from (45), (47), (52), (59), and by similar reasoning as in the previous inequalities, we derive:

$$(1-b_{k_1(\varepsilon)})|y(t_{k_1(\varepsilon)})| \leqslant c_0(1-b_{k_1(\varepsilon)})^{1+\frac{\sigma}{4}} \leqslant c_8(\frac{1}{k-k_1(\varepsilon)})^{\frac{\sigma+4}{2\sigma-4}} \leqslant c_9\varepsilon^{\frac{\sigma+4}{2\sigma}},$$

and

$$\varepsilon \sum_{k=k_1+1}^{k_1(\varepsilon)} [2|y(t_k)|+b_{k+1}-b_k] \leqslant c_3\varepsilon \sum_{k=k_1+1}^{k_1(\varepsilon)} [(1-b_k)^{\sigma/4}+(1-b_k)^{\sigma/2}]$$
$$\leqslant c_{10}\varepsilon \sum_{k=k_1+1}^{k_1(\varepsilon)} (1-b_k)^{\sigma/4} \leqslant c_{11}\varepsilon \sum_{k=k_1+1}^{k_1(\varepsilon)} \left(\frac{1}{k-k_1}\right)^{\frac{\sigma}{2\sigma-4}} \leqslant c_{12}\varepsilon^{\frac{\sigma+4}{2\sigma}}.$$

Using all the above inequalities into (57) we observe that

$$|G_\varepsilon(y)| \leqslant c\varepsilon + 2c_2\varepsilon^{\frac{\sigma+4}{2\sigma}} + c_7\varepsilon^{\frac{\sigma+4}{2\sigma}} + 2c_9\varepsilon^{\frac{\sigma+4}{2\sigma}} + c_{12}\varepsilon^{\frac{\sigma+4}{2\sigma}} \leqslant c_{13}\varepsilon^{\frac{\sigma+4}{2\sigma}}.$$

Hence, by the definitions of $\dim_M G(y)$ and $M^d(G(y))$, from previous inequality follows $\dim_M G(y) \leqslant d = 3/2-2/\sigma$ and $M^d(G(y)) < \infty$. This together with (55) proves that the equation $y''+f(x)y=0$ is 2-point fractal oscillatory on I, where the dimensional number $d = 3/2-2/\sigma$. □

As a consequence of Theorem 6.3, we complete results on 2-point rectifiable and unrectifiable oscillations concerning equation (4) given in Corollary 4.8 and Corollary 5.6.

COROLLARY 6.7. *If $\sigma > 4$, then equation (4) is 2-point fractal oscillatory on I with the dimensional number $d = 3/2-2/\sigma$.*

7. Two-point fractal oscillations of equation (9)

In this section, we will show that the function

$$y(x) = (x-x^2)^\alpha \sin\frac{1-2x}{(x-x^2)^\beta},$$

where $x \in I$ and $0 < \alpha < \beta$, are 2-point fractal oscillatory on I, where the dimensional number $d = 2-(\alpha+1)/(\beta+1)$. Here, the function $\sin(t)$ can be repleaced by $\cos(t)$

or by any T-periodic and smooth function $W(t)$ such that $W(t_*) = 0$ for some $t_* \in \mathbf{R}$. We then enlarge our discussion on 2-point fractal oscillations to include equation (9) where its fundamental solution (8) is of the form just described. Therefore, we firstly derive the lower bounds for $\dim_M G(y)$ and $M^d(G(y))$, where the function $y(x)$ is given in the form $y(x) = p(x)\sin q(x)$. Here $p(x)$ is the amplitude and $q(x)$ is the frequence of $y(x)$. The crucial role in our procedure to bound $\dim_M G(y)$ and $M^d(G(y))$ from below will be played by the inverse function $q^{-1}(t)$ of $q(x)$ satisfying conditions (5), (6), and (7). In many examples of the function $q(x)$, where the corresponding $y(x)$ is 2-point oscillatory on I, it is not easy to determine explicitly $q^{-1}(t)$, for instance $q(x) = (1-2x)/(x-x^2)^\beta$. Since the calculation of the lower bounds of $\dim_M G(y)$ and $M^d(G(y))$ involves explicitly determination of $q^{-1}(t)$, the function $q(x)$ need to be repleaced by its asymptotic approximations $q_0(x)$ and $q_1(x)$ near $x=0$ and $x=1$ respectively, where $q_0^{-1}(t)$ and $q_1^{-1}(t)$ exist. It is also useful to consider asymptotic approximations $p_0(x)$ and $p_1(x)$ of the function $p(x)$ near $x=0$ and $x=1$ respectively. We use the notation $f(x) \sim g(x)$ near $x = x_0$ to represent $\lim_{x\to x_0} f(x)/g(x) = 1$.

LEMMA 7.1. *Let $y(x) = p(x)\sin q(x)$, $x \in I$, where $p \in C(I)$, $p(0) = p(1) = 0$, $q \in C(I)$, $q(0+) = -q(1-) = \infty$, and $q(x)$ is decreasing on I. Let $h_0(x)$ and $h_1(x)$ be two positive functions, where $h_0(x)$ is increasing near $x=0$ and $h_1(x)$ is decreasing near $x=1$, such that for all $s<t$,*

$$\begin{cases} \mu\, h_0(q^{-1}(t))(t-s) \leqslant |q^{-1}(s)-q^{-1}(t)| \leqslant \nu\, h_0(q^{-1}(s))(t-s), & (s,t) \subseteq J_0, \\ \mu\, h_1(q^{-1}(s))(t-s) \leqslant |q^{-1}(s)-q^{-1}(t)| \leqslant \nu\, h_1(q^{-1}(t))(t-s), & (s,t) \subseteq J_1, \end{cases} \tag{60}$$

where μ_1 and ν are two positive constants, and $J_0 = (t_0,\infty)$ and $J_1 = (-\infty,-t_0)$, $t_0 > 0$. Let $\varepsilon_0, \varepsilon_1 > 0$ and let $k_0(\varepsilon)$ and $k_1(\varepsilon)$ be two natural numbers such that

$$k_i(\varepsilon) \geqslant \pm\frac{1}{\pi} q(h_i^{-1}(\frac{\varepsilon}{2\pi\nu})), \varepsilon \in (0,\varepsilon_i), \tag{61}$$

where $\pm = +$ for $i=0$ and $\pm = -$ for $i=1$. We suppose that

$$\begin{cases} |p(x)| \sim p_0(x) \text{ and } |q(x)| \sim q_0(x) \text{ near } x=0, \\ |p(x)| \sim p_1(x) \text{ and } |q(x)| \sim q_1(x) \text{ near } x=1, \end{cases} \tag{62}$$

where the functions $p_i \in C(\bar{I})$ and $q_i \in C(I)$ satisfy:

$$\begin{cases} p_0(x) \text{ is increasing and } q_0(x) \text{ is decreasing near } x=0, \\ p_1(x) \text{ is decreasing and } q_1(x) \text{ is increasing near } x=1. \end{cases} \tag{63}$$

If there are $\sigma_0, \sigma_1 \in (0,1)$ and $c_0, c_1 > 0$ such that

$$\sum_{k=k_i(\varepsilon)+1}^{\infty} |p_i(q_i^{-1}(2k\pi))| h_i(q_i^{-1}(2k\pi)) \geqslant c_i \varepsilon^{\sigma_i},\ \varepsilon \in (0,\varepsilon_i), \tag{64}$$

then

$$\dim_M G(y) \geqslant d_* = 2 - \min\{\sigma_0,\sigma_1\} > 1 \text{ and } M^{d_*}(G(y)) > 0. \tag{65}$$

Proof. Let $y(x) = p(x)\sin q(x)$, $x \in I$. We will show that such defined $y(x)$ satisfies all assumptions of Lemma 6.5. Since $p \in C(\bar{I})$, $p(0) = p(1) = 0$, and $q \in C(I)$, we have that $y \in C(\bar{I})$, and $y(0) = y(1) = 0$. Next, since $q(x)$ is supposed to be decreasing on I, its inverse function $q^{-1}(t)$ exists. Therefore, the zeros and stationary points of $y(x)$ can be given by $a_k = q^{-1}(k\pi)$, $b_k = q^{-1}(-k\pi)$, $s_k = q^{-1}(\frac{\pi}{2} + k\pi)$, and $t_k = q^{-1}(-\frac{\pi}{2} - k\pi)$. Since $|q(0+)| = |q(1-)| = \infty$ and since $q^{-1}(t)$ is decreasing in t, it is clear that $a_k \searrow 0$, $b_k \nearrow 1$, $s_k \in (a_{k+1}, a_k)$, and $t_k \in (b_k, b_{k+1})$. From (60) and (61), since $h_0(x)$ is increasing near $x = 0$ and $h_1(x)$ is decreasing near $x = 1$, for each $k \geqslant \max\{k_0(\varepsilon), k_1(\varepsilon)\}$ we obtain:

$$|a_k - a_{k+1}| = |q^{-1}(k\pi) - q^{-1}((k+1)\pi)| \leqslant \pi\nu\, h_0(q^{-1}(k\pi))$$
$$\leqslant \pi\nu\, h_0(q^{-1}(k_0(\varepsilon)\pi)) \leqslant \varepsilon/2,$$

and

$$|b_{k+1} - b_k| = |q^{-1}(-(k+1)\pi) - q^{-1}(-k\pi)| \leqslant \pi\nu\, h_1(q^{-1}(-k\pi))$$
$$\leqslant \pi\nu\, h_1(q^{-1}(-k_1(\varepsilon)\pi)) \leqslant \varepsilon/2.$$

Hence, the statement (49) is satisfied for such choice of a_k, b_k, and $k_i(\varepsilon)$. Next, since $f(x) \sim g(x)$ near $x = x_0$ means that $\lim_{x\to x_0} f(x)/g(x) = 1$ we have: if $f(x) > 0$ and $g(x) > 0$ on I, then

$$f(x) \sim g(x) \text{ near } x = x_0 \text{ implies } \tfrac{1}{2}g(x) \leqslant f(x) \leqslant 2g(x) \text{ near } x = x_0. \qquad (66)$$

Therefore, from (62), (63), and (66) follows:

$$|y(s_k)| = |p(s_k)||\sin q(s_k)| = |p(s_k)| \geqslant \frac{1}{2}|p_0(s_k)| \geqslant \frac{1}{2}|p_0(a_{k+1})|,$$

$$|y(t_k)| = |p(t_k)||\sin q(t_k)| = |p(t_k)| \geqslant \frac{1}{2}|p_1(t_k)| \geqslant \frac{1}{2}|p_1(b_{k+1})|.$$

Hence, the statement (50) is satisfied for $\delta_k = \frac{1}{2}|p_0(a_{k+1})|$ and $\gamma_k = \frac{1}{2}|p_1(b_{k+1})|$. Thus, all hypotheses of Lemma 6.5 are fulfiled, and so we may use the statement (51) to observe that

$$|G_\varepsilon(y)| \geqslant \frac{1}{2}\max\{\sum_{k=k_0(\varepsilon)}^{\infty} |p_0(a_{k+1})|(a_k - a_{k+1}), \sum_{k=k_1(\varepsilon)}^{\infty} |p_1(b_{k+1})|(b_{k+1} - b_k)\}. \quad (67)$$

Next, with the help of the left inequality from (60), we get:

$$|a_k - a_{k+1}| = |q^{-1}(k\pi) - q^{-1}((k+1)\pi)| \geqslant \pi\mu\, h_0(q^{-1}((k+1)\pi)) = \pi\mu\, h_0(a_{k+1}),$$

$$|b_{k+1} - b_k| = |q^{-1}(-(k+1)\pi) - q^{-1}(-k\pi)| \geqslant \pi\mu\, h_1(q^{-1}(-(k+1)\pi)) = \pi\mu\, h_1(b_{k+1}).$$

Putting these inequalities into (67), we obtain:

$$|G_\varepsilon(y)| \geqslant \frac{\pi\mu}{2}\max\{\sum_{k=k_0(\varepsilon)}^{\infty} |p_0(a_{k+1})|h_0(a_{k+1}), \sum_{k=k_1(\varepsilon)}^{\infty} |p_1(b_{k+1})|h_1(b_{k+1})\}. \qquad (68)$$

Next, from (62) and (66) follows that $\frac{1}{2}q_i(x) \leqslant |q(x)| \leqslant 2q_i(x)$ near $x=0$ when $i=0$ and near $x=1$ when $i=1$. In particular for $x=a_k$ and $x=b_k$, since $|q(a_k)| = |q(b_k)| = k\pi$, we get $\frac{1}{2}q_0(a_k) \leqslant k\pi \leqslant 2q_0(a_k)$ and $\frac{1}{2}q_1(b_k) \leqslant k\pi \leqslant 2q_1(b_k)$. Using these estimates, we can bound a_k's and b_k's in terms of $q_0^{-1}(t)$ and $q_1^{-1}(t)$ as follows:

$$\begin{cases} q_0^{-1}(2k\pi) \leqslant a_k = q^{-1}(k\pi) \leqslant q_0^{-1}(k\pi/2), \\ q_1^{-1}(k\pi/2) \leqslant b_k = q^{-1}(-k\pi) \leqslant q_1^{-1}(2k\pi). \end{cases} \tag{69}$$

Since $p_0(x)$ and $h_0(x)$ are increasing near $x=0$, and $p_1(x)$ and $h_1(x)$ are decreasing near $x=1$, by putting (69) into (68), we obtain that for $i=1,2$,

$$\begin{aligned} |G_\varepsilon(y)| &\geqslant c \sum_{k=k_i(\varepsilon)}^{\infty} |p_i(q_i^{-1}((2k+2)\pi))| h_i(q_i^{-1}((2k+2)\pi))| \\ &= c \sum_{k=k_i(\varepsilon)+1}^{\infty} |p_i(q_i^{-1}(2k\pi))| h_i(q_i^{-1}(2k\pi))|. \end{aligned} \tag{70}$$

Now, by combining the hypothesis (64) with the conclusion (70), we finally observe that $|G_\varepsilon(y)| \geqslant \max\{c_0\varepsilon^{\sigma_0}, c_1\varepsilon^{\sigma_1}\}$, which by using the corresponding definitions for $d = \dim_M G(y)$ and $M^d(G(y))$ proves the desired statement (65). $\square$

In many examples of the function $q(x)$, the inverse function $q^{-1}(t)$ is not explicitly determined and hence, the verification of (60) is not a simple procedure. However, if $q \in C^1(I)$ and $q'(x) \neq 0$ on I, then (60) can be easy verified by using Lagrange mean-value theorem. In that case, the functions $h_i(x)$ which appear in (60) can be explicitly given in the dependence of $q'(x)$ as in the following variant of Lemma 7.1.

COROLLARY 7.2. *Let* $y(x) = p(x)\sin q(x)$, $x \in I$, *where* $p \in C(\bar{I})$, $p(0) = p(1) = 0$, $q \in C^1(I)$, $q(0+) = -q(1-) = \infty$, $q'(x) < 0$ *on* I, *and*

$$|q'(x)|^{-1} \textit{ is increasing near } x=0 \textit{ and decreasing near } x=1. \tag{71}$$

Let $p(x)$ *and* $q(x)$ *satisfy the asymptotic conditions* (62) *and* (63). *Let the natural numbers* $k_0(\varepsilon)$ *and* $k_1(\varepsilon)$ *be determined by* (61), *where* $h_0^{-1}(t)$ *and* $h_1^{-1}(t)$ *are the inverse functions respectively of* $h_0(x) = |q'(x)|^{-1}$ *near* $x=0$ *and* $h_1(x) = |q'(x)|^{-1}$ *near* $x=1$. *If there are* $\sigma_0, \sigma_1 \in (0,1)$ *and* $c_0, c_1 > 0$ *such that*

$$\sum_{k=k_i(\varepsilon)+1}^{\infty} |p_i(q_i^{-1}(2k\pi))| q'(q_i^{-1}(2k\pi))|^{-1} \geqslant c_i\varepsilon^{\sigma_i},\ \varepsilon \in (0,\varepsilon_i), \tag{72}$$

then

$$\dim_M G(y) \geqslant d_* = 2 - \min\{\sigma_0, \sigma_1\} > 1 \textit{ and } M^{d_*}(G(y)) > 0. \tag{73}$$

Proof. Since $q'(x) \neq 0$ on I, by the use of Lagrange mean-value theorem, for any $s < t$ there is $\xi \in (s,t)$ such that $q^{-1}(t) - q^{-1}(s) = [q'(q^{-1}(\xi))]^{-1}(t-s)$. Since $|q'(x)|^{-1}$ is increasing near $x=0$ and decreasing near $x=1$, and $q^{-1}(t)$ is decreasing in all t, for all $s<t$ we have that $|q'(q^{-1}(t))|^{-1} \leqslant |q'(q^{-1}(\xi))|^{-1} \leqslant |q'(q^{-1}(s))|^{-1}$, $(s,t) \subseteq J_0$, and $|q'(q^{-1}(s))|^{-1} \leqslant |q'(q^{-1}(\xi))|^{-1} \leqslant |q'(q^{-1}(t))|^{-1}$, $(s,t) \subseteq J_1$, where $J_0=(t_0,\infty)$ and $J_1=(-\infty,-t_0)$, $t_0>0$. Thus, the condition (60) is satisfied, where $h_0(x)=|q'(x)|^{-1}$ near $x=0$ and $h_1(x)=|q'(x)|^{-1}$ near $x=1$. Also, for such choice of $h_i(x)$, the hypotheses (64) and (72) are equivalent. Hence, the function $q(x)$ satisfies all assumptions of Lemma 7.1 and the desired statement (73) immediately follows from (65). $\square$

The most interesting example for the function $y(x)$ which satisfies all hypotheses of previous Corollary 7.2 is the following.

EXAMPLE 7.3. Let $1/2<\alpha<\beta$ and let $y(x)=(x-x^2)^{\alpha}\sin[(1-2x)/(x-x^2)^{\beta}]$, $x \in I$. With the help of Example 2.4, one can check that the functions $p(x)=(x-x^2)^{\alpha}$ and $q(x)=(1-2x)/(x-x^2)^{\beta}$ satisfy all assumptions of Corollary 7.2. For instance, from (12) follows (71). Also, $|p(x)| \sim p_0(x)=x^{\alpha}$ and $|q(x)| \sim q_0(x)=x^{-\beta}$ near $x=0$, and $|p(x)| \sim p_1(x)=(1-x)^{\alpha}$ and $|q(x)| \sim q_1(x)=(1-x)^{-\beta}$ near $x=1$. It is clear that such defined functions $p_i(x)$ and $q_i(x)$ satisfy the conditions (62) and (63). Hence, all assumptions of Corollary 7.2 are fulfiled. Next, with the help of (61) we derive the natural numbers $k_0(\varepsilon)$ and $k_1(\varepsilon)$ determined by

$$c_0\varepsilon^{-\frac{\beta}{\beta+1}} \leqslant k_0(\varepsilon)=k_1(\varepsilon) \leqslant 2c_0\varepsilon^{-\frac{\beta}{\beta+1}} \text{ for all } \varepsilon \in (0,\varepsilon_0=\varepsilon_1). \tag{74}$$

We claim that $y(x)$ satisfies the condition (72) in particular for $\sigma_0=\sigma_1=\frac{\alpha+1}{\beta+1}$. Indeed, one can easily check that $q_0^{-1}(t)=t^{-1/\beta}$, $q_1^{-1}(t)=1-t^{-1/\beta}$, $p_i(q_i^{-1}(t))=t^{-\alpha/\beta}$, and with the help of Example 2.4 follows:

$$|q'(q_i^{-1}(t))|^{-1}=\frac{[q_i^{-1}(t)-(q_i^{-1}(t))^2]^{\beta+1}}{Q(q_i^{-1}(t))} \geqslant c_1[q_i^{-1}(t)-(q_i^{-1}(t))^2]^{\beta+1} \geqslant c_2 t^{-\frac{\beta+1}{\beta}},$$

which together with (74) implies:

$$\begin{aligned}\sum_{k=k_i(\varepsilon)}^{\infty} |p_i(q_i^{-1}((2k+2)\pi))||q'(q_i^{-1}((2k+2)\pi))|^{-1} &\geqslant c_3 \sum_{k=k_i(\varepsilon)}^{\infty} k^{-\frac{\alpha}{\beta}} k^{-\frac{\beta+1}{\beta}} \\ &= c_4 \sum_{k=k_i(\varepsilon)}^{\infty} k^{-\frac{\alpha+\beta+1}{\beta}} \geqslant c_5\big(k_i(\varepsilon)\big)^{-\frac{\alpha+\beta+1}{\beta}+1} \geqslant c_6\varepsilon^{\frac{\beta}{\beta+1}\frac{\alpha+1}{\beta}} = c_6\varepsilon^{\frac{\alpha+1}{\beta+1}}.\end{aligned}$$

Hence, the statement (72) is fulfiled, where $\sigma_0=\sigma_1=\frac{\alpha+1}{\beta+1}$. Now by Corollary 7.2 follows (73). Thus, we derive that $\dim_M G(y) \geqslant d_*=2-\frac{\alpha+1}{\beta+1}$ and $M^{d_*}(G(y))>0$. $\square$

Using the same argument as in Example 7.3, one can show that the following class of functions:

$$y(x)=x^{\alpha_0}(1-x)^{\alpha_1}\sin[(1-2x)/(x^{\beta_0}(1-x)^{\beta_1})],\ x \in I,\ 1/2<\alpha_i<\beta_i, \tag{75}$$

also satisfies the statements $\dim_M G(y) \geqslant d_*$ and $M^{d_*}(G(y))>0$, like the function $y(x)$ from Example 7.3, but with different dimensional number d_*.

In the sequel, we study the upper bounds for $\dim_M G(y)$ and $M^d(G(y))$ of the functions $y(x) = p(x)\sin q(x)$. We begin with the following geometric lemma.

LEMMA 7.4. *Let $y(x) = p(x)\sin q(x)$, $x \in I$, where $p \in C(\bar{I})$, $p(0) = p(1) = 0$, $q \in C^2(I)$, $q(0+) = -q(1-) = \infty$, $q(x)$ is decreasing on I, and satisfies (60). We suppose that*

$$y(x) \text{ is either convex or concave function between its zeros.} \tag{76}$$

Let $p(x)$ and $q(x)$ satisfy the conditions (62) and (63). Let $\varepsilon_0, \varepsilon_1 > 0$, let $k_0, k_1 \in \mathbf{N}$, and let $k_0(\varepsilon)$ and $k_1(\varepsilon)$ be two natural numbers which satisfy

$$k_i + 1 < k_i(\varepsilon), \text{ for all } \varepsilon \in (0, \varepsilon_i). \tag{77}$$

Let $q_{i,\varepsilon} = q_i^{-1}(k_i(\varepsilon)\pi/2)$ and let $h_i(x)$ be from (60). If there are constants $\sigma_0, \sigma_1 \in (0,1)$ and $c_0, c_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_i)$,

$$q_{0,\varepsilon}\, p_0(q_{0,\varepsilon}) + (1 - q_{1,\varepsilon})p_1(q_{1,\varepsilon}) + \varepsilon \sum_{k=k_i+1}^{k_i(\varepsilon)} \left[|p_i(q_i^{-1}(k\pi/2))| + h_i(q_i^{-1}(k\pi/2))\right] \leqslant c_i \varepsilon^{\sigma_i}, \tag{78}$$

then

$$\dim_M G(y) \leqslant d^* = 2 - \max\{\sigma_0, \sigma_1\} \text{ and } M^{d^*}(G(y)) < \infty. \tag{79}$$

Proof. Let $y(x) = p(x)\sin q(x)$, $x \in I$, and let $a_k = q^{-1}(k\pi)$, $b_k = q^{-1}(-k\pi)$, $s_k = q^{-1}(\frac{\pi}{2} + k\pi)$, and $t_k = q^{-1}(-\frac{\pi}{2} - k\pi)$. With the help of (76), we get that $y(x)$ satisfies all assumptions of Lemma 6.6. Next, from (62), (63), and (66) follows: $|y(s_k)| = |p(s_k)||\sin q(s_k)| = |p(s_k)| \leqslant 2p_0(s_k) \leqslant 2p_0(a_k)$ and $|y(t_k)| = |p(t_k)||\sin q(t_k)| = |p(t_k)| \leqslant 2p_1(t_k) \leqslant 2p_1(b_k)$. Since $p_0(x)$ is increasing near $x = 0$ and $p_1(x)$ is decreasing near $x = 1$, from previous inequalities and (69) we observe that:

$$|y(s_k)| \leqslant 2p_0(q_0^{-1}(k\pi/2)) \text{ and } |y(t_k)| \leqslant 2p_1(q_1^{-1}(k\pi/2)). \tag{80}$$

Also, from (60) follows

$$|a_k - a_{k+1}| = |q^{-1}(k\pi) - q^{-1}((k+1)\pi)| \leqslant \pi\nu\, h_0(q^{-1}(k\pi)) = \pi\nu\, h_0(a_k),$$

$$|b_{k+1} - b_k| = |q^{-1}(-(k+1)\pi) - q^{-1}(-k\pi)| \leqslant \pi\nu\, h_1(q^{-1}(-k\pi)) = \pi\nu\, h_1(b_k),$$

which together with (69) imply

$$|a_k - a_{k+1}| \leqslant \pi\nu\, h_0(q_0^{-1}(k\pi/2)) \text{ and } |b_{k+1} - b_k| \leqslant h_1(q_1^{-1}(k\pi/2)). \tag{81}$$

Since $q_{i,\varepsilon} = q_i^{-1}(k_i(\varepsilon)\pi/2)$, from (69) and (80) we obtain:

$$a_{k_0(\varepsilon)}|y(s_{k_0(\varepsilon)})| \leqslant 2q_0^{-1}(k_0(\varepsilon)\pi/2)p_0(q_0^{-1}(k_0(\varepsilon)\pi/2)) = 2q_{0,\varepsilon}\, p_0(q_{0,\varepsilon}), \tag{82}$$

$$
\begin{aligned}
(1-b_{k_1(\varepsilon)})|y(t_{k_1(\varepsilon)})| &\leqslant 2[1-q_1^{-1}(k_1(\varepsilon)\pi/2)]p_1(q_1^{-1}(k_1(\varepsilon)\pi/2)) \\
&= 2(1-q_{1,\varepsilon})p_1(q_{1,\varepsilon}). \qquad (83)
\end{aligned}
$$

Also, from (69), (80), and (81) we obtain:

$$\sum_{k=k_0+1}^{k_0(\varepsilon)}[2|y(s_k)|+a_k-a_{k+1}] \leqslant \sum_{k=k_0+1}^{k_0(\varepsilon)}[4p_0(q_0^{-1}(k\pi/2))+\pi\nu h_0(q_0^{-1}(k\pi/2))], \quad (84)$$

$$\sum_{k=k_1+1}^{k_1(\varepsilon)}[2|y(t_k)|+b_{k+1}-b_k] \leqslant \sum_{k=k_1+1}^{k_1(\varepsilon)}[4p_1(q_1^{-1}(k\pi/2))+\pi\nu h_1(q_1^{-1}(k\pi/2))]. \quad (85)$$

Involving (82), (83), (84), and (85) into (57), and using the hypothesis (78) we observe that $|G_\varepsilon(y|_I)| \leqslant c_i\varepsilon^{\sigma_i}$. By the definitions of $\dim_M G(y)$ and $M^{d^*}(G(y))$, it gives the desired statement (79). □

In the case when $y(x)$ is a solution of (9), the hypothesis (76) is satisfied if we assume that $S(q')(x) > 0$ on I. Also, in the case when $y(x) = p(x)\sin q(x)$, we can ensure that the hypothesis (76) holds if d^jp/dx^j and d^jq/dx^j do not change the sign near $x=0$ and $x=1$, where $j=0,1,2$. Next, we present an example for the functions $q(x)$ which satisfies all assumptions of Lemma 7.4.

EXAMPLE 7.5. Let $0<\alpha<\beta$ and let $y(x) = (x-x^2)^\alpha \sin[(1-2x)/(x-x^2)^\beta]$, $x \in I$. Let $p(x) = (x-x^2)^\alpha$ and $q(x) = (1-2x)/(x-x^2)^\beta$. Let us remark that we do not explicitly expresed the inverse function $q^{-1}(t)$ of $q(x)$. By (12) and by Lagrange mean-value theorem, we obtain

$$c_1[q^{-1}(t)-(q^{-1}(t))^2]^{\beta+1} \leqslant |q^{-1}(s)-q^{-1}(t)| \leqslant c_2[q^{-1}(s)-(q^{-1}(s))^2]^{\beta+1}.$$

Therefore, the statement (60) is satisfied for such defined $q(x)$, where $h_0(x) = x^{\beta+1}$ near $x=0$ and $h_1(x) = (1-x)^{\beta+1}$ near $x=1$, and so, the functions $p(x)$ and $q(x)$ satisfy all assumptions of Lemma 7.4. Next, as in Example 7.3, we have that $p(x) \sim p_0(x) = x^\alpha$ and $q(x) \sim q_0(x) = x^{-\beta}$ near $x=0$, and $p(x) \sim p_1(x) = (1-x)^\alpha$ and $q(x) \sim q_1(x) = (1-x)^{-\beta}$ near $x=1$, and that these functions satisfy the conditions (62) and (63). Let $k_0(\varepsilon)$ and $k_1(\varepsilon)$ be two natural numbers defined in (74). Since $q_0^{-1}(t) = t^{-1/\beta}$, $q_1^{-1}(t) = 1-t^{-1/\beta}$, $p_i(q_i^{-1}(t)) = t^{-\alpha/\beta}$ and $h_i(q_i^{-1}(t)) = t^{-(\beta+1)/\beta}$, we observe that:

$$q_{0,\varepsilon}\,p_0(q_{0,\varepsilon}) = [q_0^{-1}(k_0(\varepsilon)\pi/2)]^{\alpha+1} \leqslant c_1(k_0(\varepsilon))^{-\frac{\alpha+1}{\beta}} \leqslant c_2\varepsilon^{\frac{\beta}{\beta+1}\frac{\alpha+1}{\beta}} = c_2\varepsilon^{\frac{\alpha+1}{\beta+1}},$$

$$(1-q_{1,\varepsilon})p_1(q_{1,\varepsilon}) = [1-q_i^{-1}(k_1(\varepsilon)\pi/2)]^{\alpha+1} \leqslant c_1(k_1(\varepsilon))^{-\frac{\alpha+1}{\beta}} \leqslant c_2\varepsilon^{\frac{\beta}{\beta+1}\frac{\alpha+1}{\beta}} = c_2\varepsilon^{\frac{\alpha+1}{\beta+1}},$$

$$\varepsilon \sum_{k=k_i+1}^{k_i(\varepsilon)} |p_i(q_i^{-1}(k\pi/2))| \leqslant c_3\varepsilon \sum_{k=k_i+1}^{k_i(\varepsilon)} k^{-\frac{\alpha}{\beta}} \leqslant c_4\varepsilon[k_i(\varepsilon)]^{-\frac{\alpha}{\beta}+1}$$

$$\leqslant c_5\varepsilon\,\varepsilon^{\frac{\beta}{\beta+1}\frac{\alpha-\beta}{\beta}} = c_5\varepsilon^{\frac{\alpha+1}{\beta+1}},$$

$$\varepsilon \sum_{k=k_i+1}^{k_i(\varepsilon)} h_i(q_i^{-1}(k\pi/2)) \leqslant c_6\varepsilon \sum_{k=k_i+1}^{k_i(\varepsilon)} k^{-\frac{\beta+1}{\beta}} \leqslant c_7\,\varepsilon(k_i(\varepsilon))^{-\frac{\beta+1}{\beta}+1} \leqslant c_8\,\varepsilon\varepsilon^{\frac{\beta}{\beta+1}\frac{1}{\beta}} \leqslant c_9\varepsilon,$$

which all together imply that (78) is fulfiled in particular for $\sigma_1 = \sigma_2 = \frac{\alpha+1}{\beta+1}$. Now, by means of Lemma 7.4 follows that $\dim_M G(y) \leqslant d^* = 2 - \frac{\alpha+1}{\beta+1}$ and $M^{d^*}(G(y)) < \infty$. $\square$

It can be generalized to the function defined in (75) and hence, we obtain

$$\dim_M G(y) \leqslant d^* = 2 - \max\{\tfrac{\alpha_0+1}{\beta_0+1}, \tfrac{\alpha_1+1}{\beta_1+1}\} \text{ and } M^{d^*}(G(y)) < \infty.$$

As a consequence of Lemma 6.5, Lemma 6.6, Corollary 7.2, and Lemma 7.4, the main result of this section gives sufficient conditions on $q(x)$ such that equation (9) is 2-point fractal oscillatory on the interval I.

THEOREM 7.6. *Let $q(x)$ satisfy the conditions* (5), (6), (7), (71), *and* $S(q')(x) > 0$ *on I. Let $|q(x)| \sim q_i(x)$ as in* (63). *Let the natural numbers $k_0(\varepsilon)$ and $k_1(\varepsilon)$ satisfy*

$$k_i(\varepsilon) \geqslant 1 \pm \frac{1}{\pi} q(h_i^{-1}(\frac{\varepsilon}{4\pi})),\ \varepsilon \in (0,\varepsilon_i), \tag{86}$$

where $\pm = +$ for $i = 0$ and $\pm = -$ for $i = 1$, and where $h_0^{-1}(t)$ and $h_1^{-1}(t)$ are the inverse functions respectively of $h_0(x) = |q'(x)|^{-1}$ near $x = 0$ and $h_1(x) = |q'(x)|^{-1}$ near $x = 1$. Let k_1 be a natural number determined as in (77) *and let $q_{i,\varepsilon} = q_i^{-1}(k_i(\varepsilon)\pi)$. If there are $\sigma_0, \sigma_1 \in (0,1)$ and $c_0, c_1 > 0$ such that*

$$\sum_{k=k_i(\varepsilon)+1}^{\infty} |q'(q_i^{-1}(4k\pi))|^{-\frac{3}{2}} \geqslant c_i\varepsilon^{\sigma_i},\ \varepsilon \in (0,\varepsilon_i), \tag{87}$$

$$\begin{aligned} &q_{0,\varepsilon}\,|q'(q_{0,\varepsilon})|^{-\frac{1}{2}} + (1-q_{1,\varepsilon})|q'(q_{1,\varepsilon})|^{-\frac{1}{2}} \\ &\qquad + \varepsilon \sum_{k=k_i}^{k_i(\varepsilon)} \left[|q'(q_i^{-1}(k\pi/2))|^{-\frac{1}{2}} + |q'(q_i^{-1}(k\pi/2))|^{-1}\right] \leqslant c_i\varepsilon^{\sigma_i}, \end{aligned} \tag{88}$$

then for all nontrivial solutions $y(x)$ of equation (9),

$$\begin{cases} 2 - \min\{\sigma_0, \sigma_1\} = d_* \leqslant \dim_M G(y) \leqslant d^* = 2 - \max\{\sigma_0, \sigma_1\}, \\ 0 < M^{d_*}(G(y)) \text{ and } M^{d^*}(G(y)) < \infty. \end{cases} \tag{89}$$

Moreover, if $\sigma_0 = \sigma_1$ then equation (9) *is 2-point fractal oscillatory on I.*

Proof. Let $y(x)$ be a solution of equation (9). Obviously, the following conclusion there holds true:

$$S(q')(x) > 0 \text{ on } I \text{ implies that } y(x) \text{ is either convex or concave on } I. \tag{90}$$

As in the proof of Theorem 5.2, we consider two cases of $y(x)$: the first one when $y(x) = c_2 y_2(x)$ and the second one when $y(x)$ and $y_2(x)$ are two linearly independent solutions of (9), where $y_2(x) = |q'(x)|^{-\frac{1}{2}} \sin q(x)$ defined in (8). At the first, let $y(x) = c_2 y_2(x)$ and let $p_i(x)$ be two functions such that $p_0(x) = |q'(x)|^{-\frac{1}{2}}$ near $x = 0$ and $p_1(x) = |q'(x)|^{-\frac{1}{2}}$ near $x = 1$. Since (71), such defined $p_i(x)$ satisfy all assumptions of Corollary 7.2. Also, from (71) and by using the assumption (87) we get:

$$\begin{aligned} c_i \varepsilon^{\sigma_i} &\leqslant \sum_{k=k_i(\varepsilon)+1}^{\infty} |q'(q_i^{-1}(4k\pi))|^{-\frac{3}{2}} \leqslant \sum_{k=k_i(\varepsilon)+1}^{\infty} |q'(q_i^{-1}(2k\pi))|^{-\frac{3}{2}} \\ &= \sum_{k=k_i(\varepsilon)+1}^{\infty} |q'(q_i^{-1}(2k\pi))|^{-\frac{1}{2}} |q'(q_i^{-1}(2k\pi))|^{-1} \\ &= \sum_{k=k_i(\varepsilon)+1}^{\infty} p_i(q_i^{-1}(2k\pi)) |q'(q_i^{-1}(2k\pi))|^{-1}. \end{aligned}$$

Hence, the condition (72) is fulfiled and by Corollary 7.2 follows that $\dim_M G(y) \geqslant d_* = 2 - \min\{\sigma_0, \sigma_1\} > 1$ and $M^{d_*}(G(y)) > 0$. Moreover, since it is supposed that $S(q')(x) > 0$ on I, from (88) and (90) follow that all assumptions of Lemma 7.4 are fulfiled and therefore by (79), the desired statement (89) is shown in the case when $y(x) = c_2 y_2(x)$.

In the second case, when $y(x)$ and $y_2(x)$ are two linearly independent solutions of equation (9), let $a_k, b_k \in I$ be two sequences of consecutive zeros of $y(x)$ such that $a_k \searrow 0$ and $b_k \nearrow 1$. Such defined sequences exist by Theorem 2.1. Let us remark that in the proof of Theorem 5.2, we have derived two sequences $s_k \in (a_{k+1}, a_k)$, $t_k \in (b_k, b_{k+1})$, $s_k = q^{-1}(k\pi)$, and $t_k = q^{-1}(-k\pi)$, which satisfy the equality (35). It together with (69) implies that

$$\begin{cases} |y(s_k)| = |y(q^{-1}(k\pi))| = c|q'(q^{-1}(k\pi))|^{-\frac{1}{2}} \geqslant c|q'(q_0^{-1}(2k\pi))|^{-\frac{1}{2}}, \\ |y(t_k)| = |y(q^{-1}(-k\pi))| = c|q'(q^{-1}(-k\pi))|^{-\frac{1}{2}} \geqslant c|q'(q_1^{-1}(2k\pi))|^{-\frac{1}{2}}. \end{cases}$$

Next, let $d_k = a_k$ for $i = 0$ and $d_k = b_k$ for $i = 1$. By means of Lagrange mean-value theorem, we have:

$$\begin{aligned} |d_k - d_{k+1}| &\leqslant |q^{-1}(\pm(k-1)\pi) - q^{-1}(\pm(k+1)\pi)| \\ &\leqslant 2\pi |q'(q^{-1}(\pm(k-1)\pi))|^{-1} = 2\pi h_i(q^{-1}(\pm(k-1)\pi)). \end{aligned} \tag{91}$$

Putting (86) into (91), we obtain that $|d_k - d_{k+1}| \leqslant 2\pi h_i(q^{-1}(\pm(k_i(\varepsilon)-1)\pi)) \leqslant \varepsilon/2$. Hence, the function $y(x)$ satisfies all assumptions of Lemma 6.5, and we observe that:

$$\begin{aligned}
|G_\varepsilon(y)| &\geqslant c \sum_{k=k_i(\varepsilon)}^{\infty} |q'(q_i^{-1}(2k\pi))|^{-\frac{1}{2}} |d_k - d_{k+1}| \\
&\geqslant c \sum_{k=k_i(\varepsilon)}^{\infty} \min\{|q'(q_i^{-1}(2k\pi))|^{-\frac{1}{2}}, |q'(q_i^{-1}((2k+2)\pi))|^{-\frac{1}{2}}\} |d_k - d_{k+2}| \\
&= c \sum_{k=k_i(\varepsilon)}^{\infty} |q'(q_i^{-1}((2k+2)\pi))|^{-\frac{1}{2}} |d_k - d_{k+2}|,
\end{aligned}$$

that is,

$$|G_\varepsilon(y)| \geqslant c \sum_{j=1}^{\infty} |q'(q_i^{-1}((2k_i(\varepsilon)+2j)\pi))|^{-\frac{1}{2}} |d_{k_i(\varepsilon)+2j-2} - d_{k_i(\varepsilon)+2j}|. \tag{92}$$

Since

$$\begin{aligned}
|d_{k_i(\varepsilon)+2j-2} - d_{k_i(\varepsilon)+2j}| &\geqslant |q^{-1}(\pm(k_i(\varepsilon)+2j-2)\pi) - q^{-1}(\pm(k_i(\varepsilon)+2j-1)\pi)| \\
&\geqslant \pi |q'(q^{-1}(\pm(k_i(\varepsilon)+2j-1)\pi))|^{-1} \geqslant \pi |q'(q_i^{-1}((2k_i(\varepsilon)+4j-2)\pi))|^{-1},
\end{aligned}$$

from (92) follows:

$$\begin{aligned}
|G_\varepsilon(y)| &\geqslant c \sum_{j=1}^{\infty} |q'(q_i^{-1}((2k_i(\varepsilon)+2j)\pi))|^{-\frac{1}{2}} |q'(q_i^{-1}((2k_i(\varepsilon)+4j-2)\pi))|^{-1} \\
&\geqslant c \sum_{j=1}^{\infty} |q'(q_i^{-1}((4k_i(\varepsilon)+4j)\pi))|^{-\frac{1}{2}} |q'(q_i^{-1}((4k_i(\varepsilon)+4j)\pi))|^{-1} \\
&\geqslant c \sum_{j=1}^{\infty} |q'(q_i^{-1}(4(k_i(\varepsilon)+j)\pi))|^{-\frac{3}{2}} \geqslant \sum_{k=k_i(\varepsilon)+1}^{\infty} |q'(q_i^{-1}(4k\pi))|^{-\frac{3}{2}}.
\end{aligned}$$

Hence from assumption (87) follows that $|G_\varepsilon(y)| \geqslant \max\{c_0\varepsilon^{\sigma_0}, c_1\varepsilon^{\sigma_1}\}$. Now using the definitions for $d = \dim_M G(y)$ and $M^d(G(y))$, we conclude:

$$\dim_M G(y) \geqslant d_* = 2 - \min\{\sigma_0, \sigma_1\} > 1 \text{ and } M^{d_*}(G(y)) > 0. \tag{93}$$

Next, let $s_k \in (a_{k+1}, a_k)$ and $t_k \in (b_k, b_{k+1})$ such that $y'(s_k) = y'(t_k) = 0$. Since $y(x)$ and $y_2(x)$ are two linearly independent solutions of equation (9), we have that $q^{-1}(k\pi) \in (a_{k+1}, a_k)$ and $q^{-1}(-k\pi) \in (b_k, b_{k+1})$ or

$$s_k \leqslant a_k \leqslant q^{-1}((k-1)\pi) \text{ and } t_k \geqslant b_k \geqslant q^{-1}(-(k-1)\pi). \tag{94}$$

Since $|y(x)| \leqslant (|c_1|+|c_2|)|q'(x)|^{-1/2}$, with the help of (69) and (94), we obtain:

$$\begin{cases}
|y(s_k)| \leqslant c|q'(s_k)|^{-\frac{1}{2}} \leqslant c|q'(q_0^{-1}((k-1)\pi))|^{-\frac{1}{2}} \leqslant c|q'(q_0^{-1}(\frac{(k-1)\pi}{2}))|^{-\frac{1}{2}}, \\
|y(t_k)| \leqslant c|q'(t_k)|^{-\frac{1}{2}} \leqslant c|q'(q_1^{-1}(-(k-1)\pi))|^{-\frac{1}{2}} \leqslant c|q'(q_1^{-1}(\frac{-(k-1)\pi}{2}))|^{-\frac{1}{2}}.
\end{cases} \tag{95}$$

Also, from (69), (91), (94), and by definitions of $h_i(x)$, it follows:

$$
\begin{aligned}
|d_k - d_{k+1}| &\leqslant 2\pi h_i(q^{-1}(\pm(k-1)\pi)) = 2\pi |q'(q^{-1}(\pm(k-1)\pi))|^{-1} \\
&\leqslant 2\pi |q'(q_i^{-1}(\frac{(k-1)\pi}{2}))|^{-1}.
\end{aligned} \tag{96}
$$

Since $S(q')(x) > 0$ on I, from (90) follows (56) and hence (57) holds. Very similar to the proof of (79), we now put (95) and (96) into (57) to obtain:

$$
\begin{aligned}
|G_\varepsilon(y)| &\leqslant q_{0,\varepsilon}\, |q'(q_{0,\varepsilon})|^{-\frac{1}{2}} + (1-q_{1,\varepsilon})|q'(q_{1,\varepsilon})|^{-\frac{1}{2}} \\
&\quad + \varepsilon \sum_{k=k_i+1}^{k_i(\varepsilon)} \big[|q'(q_i^{-1}(\frac{(k-1)\pi}{2}))|^{-\frac{1}{2}} + |q'(q_i^{-1}(\frac{(k-1)\pi}{2}))|^{-1}\big].
\end{aligned}
$$

Shifting the index k into $k+1$ on right hand side in the previous inequality and by using the assumption (88), we get $|G_\varepsilon(y|_I)| \leqslant c_i \varepsilon^{\sigma_i}$. Applying the definitions of $\dim_M G(y)$ and $M^{d^*}(G(y))$, we get $\dim_M G(y) \leqslant d^* = 2 - \max\{\sigma_0, \sigma_1\}$ and $M^{d_*}(G(y)) < \infty$, which by (93) proves the desired statement (89) in the second case of $y(x)$ too. $\square$

Now, using the same argument as in Example 7.3 and Example 7.5, where $p(x) = |q(x)|^{-1/2}$, we can show that the function $q(x) = (1-2x)/(x-x^2)^\beta$, $x \in I$, $\beta > 1$, satisfies all assumptions of Theorem 7.6. Hence, the equation (9) is 2-point fractal oscillatory on I, where the dimensional number $d_* = d^* = (3\beta+1)/(2\beta+2)$.

8. Two-sided oscillations

In this section, we consider equation (P): $y'' + f(x)y = 0$, where the coefficient function $f(x)$ has an interior singularity $x_0 \in I$. A solution $y(x)$ of (P) is said to be 2-sided oscillatory at x_0 if there exist two sequences of zeros of $y(x)$ such that:

$$\{s_i: \ s_i < s_{i+1} < x_0,\ y(s_i) = 0,\ \lim_{i\to\infty} s_i = x_0\},$$

$$\{t_j: \ x_0 < t_{j+1} < t_j,\ y(t_j) = 0,\ \lim_{j\to\infty} t_j = x_0\}.$$

Equation (P) is said to be 2-sided oscillatory at x_0 if all solutions of (P) are 2-sided oscillatory at x_0. The concept of 2-point rectifiable (unrectifiable) oscillations can be extended to study 2-sided oscillations of (P). Likewise, equation (P) is said to be 2-sided rectifiable (unrectifiable) oscillatory at x_0 if all its solution curves are 2-sided oscillatory at x_0 and have finite (infinite) arclength on $\bar{I}$. Furthermore, equation (P) is said to be 2-sided fractal oscillatory at x_0 if all its solutions $y(x)$ are 2-sided oscillatory at x_0 and there is $d \in (1,2)$ such that $\dim_M G(y) = d$ and $0 < M^d(G(y)) < \infty$.

EXAMPLE 8.1. Consider equation (E_1): $y'' + (2x-1)^{-4}y = 0$. Here the function $f(x) = (2x-1)^{-4}$ satisfies the Hartman-Wintner condition (16) so solutions of

(E_1) satisfy the asymptotic formula given by Hartman-Wintner theorem near $x=1/2$, namely,

$$y(x)=|1-2x|\left[c_1\cos\left(\frac{1+2x}{2|1-2x|}\right)+c_2\sin\left(\frac{1+2x}{2|1-2x|}\right)+o(1)\right]. \tag{97}$$

Clearly $y(x)$ given in (97) is oscillatory on $[0,\frac{1}{2})$ and $(\frac{1}{2},1]$ and has two sequences of zeros converging from the left and the right to $x=1/2$. $\square$

Similar to equation (4), we consider the model equation for 2-sided oscillations:

$$y''+\frac{c(x)}{|2x-1|^{\sigma}}y=0,\ x\in I, \tag{98}$$

where $c(x)$ is a positive and continuous function on $\bar{I}$. Following Theorem 3.3, Theorem 4.7, and Theorem 5.5, we can also prove similar results for 2-sided oscillations of $y''+f(x)y=0$.

THEOREM 8.2. *Let $f\in C(\bar{I}-x_0)\cap C^2(I-x_0)$, $f(x)>0$ on I, where x_0 is an interior point of $\bar{I}$. Suppose that $f(x)$ satisfies the Hartman-Wintner condition* (16). *Then equation (P): $y''+f(x)y=0$ is 2-sided oscillatory at x_0. Furthermore:*

(a) *(P) is 2-sided rectifiable oscillatory at x_0 if $f\in L^1(I)$;*

(b) *(P) is 2-sided unrectifiable oscillatory at x_0 if $f\notin L^1(I)$.*

In the case when $f(x)\sim(x-x_0)^{\sigma}$ near $x=x_0$, $\sigma>4$, we can also prove a similar result as Theorem 6.3.

THEOREM 8.3. *Let $f(x)$ satisfy all assumptions in Theorem* 8.2. *In addition $f(x)$ satisfies for $\lambda_0,\lambda_1>0$,*

$$\lambda_0|2x-1|^{-\sigma}\leqslant f(x)\leqslant\lambda_1|2x-1|^{-\sigma}\ \textit{near}\ x=x_0,\ \sigma>4.$$

Then equation (98) *is 2-sided fractal oscillatory at $x=1/2$ with the box dimension $d=3/2-2/\sigma$.*

The proofs of Theorem 8.2 and Theorem 8.3 are similar to the proofs of Theorem 3.3, Theorem 4.7, Theorem 5.5, and Theorem 6.3 and we leave the details to the reader.

Using the concept of 2-sided oscillations which deals with the case when $f(x)$ has interior singularities, we consider the equation

$$y''+\frac{R(x)}{|P(x)|}y=0,\ x\in I, \tag{99}$$

where $P(x)=x^n+a_{n-1}x^{n-1}+....+a_1x+a_0$ is a n^{th} degree polynomial and $R(x)$ is a positive rational function. Assume that $P(x)$ has real roots $x_1,x_2,...,x_k$ all lie in I (note that complex roots can be grouped into the rational function $R(x)$), with multiplicity

$m_1, m_2, ..., m_k$ such that $\sum_{j=1}^{k} m_j = n$ and $m_j \geqslant 2$. We can then claim that equation (99) is 2-sided oscillatory at all multiple zeros of $P(x)$ and when the multiplicity $\geqslant 4$ the solutions are unrectifiable 2-sided oscillatory with fractal dimension $3/2 - 2/m_j$, $m_j \geqslant 4$.

EXAMPLE 8.4. Consider the equation

$$y'' + (1-x^2)(x+1)^{-1}(2x-1)^{-5}y = 0,\ x \in I. \tag{100}$$

Here the denominator has a multiple root at $x = 1/2$ with multiplicity 5 and $(1-x^2)(x+1)^{-1}$ is a positive rational function on I. So by Theorem 8.3, we conclude that equation (100) is fractal oscillatory at $x = 1/2$ with fractal dimension $11/10$. □

A result due to Laguerre (see [1] and also [17]) states that if all roots of the polynomial $P(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ are real then they must lie between an interval $[A,B]$, where A,B are given by

$$A,B = -\frac{a_{n-1}}{n} \pm \frac{n-1}{n}\Big(a_{n-1}^2 - \frac{2n}{n-1}a_{n-2}\Big)^{1/2}.$$

We can rescale the independent variable x in $[A,B]$ by the linear translation $\xi = (x-A)(B-A)$ to an equation $y''(\xi) + \hat{f}(\xi)y(\xi) = 0$, $\xi \in \bar{I}$, where $\hat{f}(\xi)$ satisfies the Hartman-Wintner condition whenever $f(x)$ does.

EXAMPLE 8.5. Consider the equation

$$y'' + (x^2 - x + 1)(x-1)^{-3}(x-2)^{-4}y = 0,\ x \geqslant 0. \tag{101}$$

We can use Laguerre formula to determine an optimal interval, i. e. $[\frac{11}{7} - \frac{6\sqrt{2}}{7}, \frac{11}{7} + \frac{6\sqrt{2}}{7}]$, in which the singular points $x = 1,2$ lie. However, it is simpler to make a scale change $\xi = x/3$ to transforms (101) to

$$\frac{d^2y}{d\xi^2} + 9(9\xi^2 - 3\xi + 1)(3\xi - 1)^{-3}(3\xi - 2)^{-4}y = 0, \tag{102}$$

where $y(\xi) = y(x)$ and $\xi \in \bar{I} = [0,1]$. We can now appply Theorem 8.2 and Theorem 8.3 to conclude that equation (101) is:

(a) 2-sided oscillatory at $x = 1$ and $x = 2$;

(b) 2-sided rectifiable oscillatory at $x = 1$;

(c) 2-sided unrectifiable oscillatory at $x = 2$.

9. Appendix

In this section we give the proof of Proposition 3.2 and Lemma 6.4.

Proof of Proposition 3.2. The main idea is taken from the proof of [15, Proposition 2.9]. Since $F^{A-1}F'' \in L^1(I)$, there are two constants $K_0, K_1 > 0$ such that

$$K_0 = \int_0^{1/2} F^{A-1}(x)|F''(x)|dx \quad \text{and} \quad K_1 = \int_{1/2}^1 F^{A-1}(x)|F''(x)|dx.$$

Hence,

$$F^{A-1}(1/2)F'(1/2) - F^{A-1}(s)F'(s) - (A-1)\int_s^{1/2} F^{A-2}(x)(F'(x))^2dx$$
$$= \int_s^{1/2} F^{A-1}(x)F''(x)dx \geqslant -K_0,\ s \in (0,1/2), \quad (103)$$

and

$$F^{A-1}(t)F'(t) - F^{A-1}(1/2)F'(1/2) - (A-1)\int_{1/2}^{t} F^{A-2}(x)(F'(x))^2dx$$
$$= \int_{1/2}^{t} F^{A-1}(x)F''(x)dx \geqslant -K_1,\ t \in (1/2,1). \quad (104)$$

Now, from (103) and (104) follows

$$\begin{cases} F^{A-1}(s)F'(s) \leqslant K_0 + F^{A-1}(1/2)|F'(1/2)|, s \in (0,1/2), \\ -F^{A-1}(t)F'(t) \leqslant K_1 + F^{A-1}(1/2)|F'(1/2)|, t \in (1/2,1). \end{cases}$$

Integrating these two inequalities respectively over $(0,x)$ where $x \in (0,1/2)$ and $(x,1)$ where $x \in (1/2,1)$, and using that $F(0) = F(1) = 0$ we obtain

$$F^{-A}(x) \geqslant \tfrac{c_0}{x},\ x \in (0,\tfrac{1}{2}) \ \text{ and } \ F^{-A}(x) \geqslant \tfrac{c_1}{1-x},\ x \in (\tfrac{1}{2},1),$$

which show that $F^{-A} \notin L^1(0,\frac{1}{2})$ and $F^{-A} \notin L^1(\frac{1}{2},1)$.

Next, from (103) we also obtain,

$$(A-1)\int_s^{1/2} F^{A-2}(x)(F'(x))^2dx \leqslant K_0 + F^{A-1}(1/2)|F'(1/2)| - F^{A-1}(s)F'(s).$$

Since $F(0) = 0$ and $F(x) > 0$ for all $x \in I$, from the mean value theorem we get a sequence $s_n \in I$ such that $s_n \to 0$ and $F'(s_n) > 0$. Putting for $s = s_n$ in the previous inequality and passing to the limit, we obtain that

$$\int_0^{1/2} F^{A-2}(x)(F'(x))^2dx = \lim_{s_n \to 0}\int_{s_n}^{1/2} F^{A-2}(x)(F'(x))^2dx \leqslant \frac{K_0 + F^{A-1}(\frac{1}{2})|F'(\frac{1}{2})|}{A-1},$$

which proves that $F^{A-2}(F')^2 \in L^1(0,1/2)$ which together with (103) implies that there exists constant c such that $\lim_{s\to 0} F^{A-1}(s)F'(s) = c$. If $c \neq 0$, then

$$\int_0^{1/2} F^{-A}(x)dx = \int_0^{1/2} \frac{F^{A-2}(x)(F'(x))^2}{(F^{A-1}(x)F'(x))^2}dx < \infty,$$

which is not possible since $F^{-A} \notin L^1(0,1/2)$. Hence, $c = 0$ and so $\lim_{s\to 0} F^{A-1}(s)F'(s) = 0$. Analogously to the above observation, since $F(1) = 0$ and $F(x) > 0$ for all $x \in I$, from (104) follows $\lim_{s\to 1} F^{A-1}(s)F'(s) = 0$ which completes the proof of Proposition 3.2. □

Proof of (44) *and* (45). The proofs of (44) and (45) are very similar to the proofs of corresponding inequalities for the Euler equation presented in the proofs of [13, Lemma 3.3]. □

Proof of (46) *and* (47). Let λ_0 and λ_1 be from (43). Let $\sigma > 2$ and $\beta = (\sigma-2)/2 > 0$. Let m, M be two real numbers such that

$$0 < m < \frac{\sqrt{\lambda_0}}{\max\{\beta, 1/4\}} \quad \text{and} \quad M > \frac{\sqrt{\lambda_1}}{\min\{\beta, 1/4\}}. \tag{105}$$

For the function $q(x) = (1-2x)/(x-x^2)^\beta$, $x \in I$, since $S(\lambda q) = S(q)$, $\lambda \neq 0$, by (10) and (11) we derive a $\gamma_0 \in I$ only depending on λ_0, m, σ and a $\gamma_1 \in I$ only depending on λ_1, M, σ such that

$$(mq')^2(x) + \frac{1}{2}S(mq)(x) \leqslant \frac{\lambda_0}{(x-x^2)^\sigma}, \quad x \in (0,\gamma_0) \cup (1-\gamma_0, 1),$$

and

$$\frac{\lambda_1}{(x-x^2)^\sigma} \leqslant (Mq')^2(x) + \frac{1}{2}S(Mq)(x),\ x \in (0,\gamma_1) \cup (1-\gamma_1, 1),$$

which together with (43) yield:

$$(mq')^2(x) + \frac{1}{2}S(mq)(x) \leqslant f(x) \leqslant (Mq')^2(x) + \frac{1}{2}S(Mq)(x), \tag{106}$$

for all $x \in (0,\varepsilon) \cup (1-\varepsilon, 1)$ and $\varepsilon = \min\{\delta, \gamma_0, \gamma_1\}$, where δ is from (43). Next, by (8) and (9), the function $y_\lambda(x) = \lambda^{-1/2}|q'(x)|^{-1/2}\sin(\lambda q(x))$ satisfies the equation

$$y_\lambda'' + \big[(\lambda q')^2(x) + \frac{1}{2}S(\lambda q)(x)\big]y_\lambda = 0,\ x \in I. \tag{107}$$

Since $q(x)$ is decreasing on I with $q(0+) = \infty$ and $q(1-) = -\infty$, there is a $k_1 \in \mathbf{N}$ and there are two sequences $u_{\lambda,k} \searrow 0$ and $v_{\lambda,k} \nearrow 1$ of consecutive zeros of $y_\lambda(x)$ which satisfy $\lambda q(u_{\lambda,k}) = k\pi$ and $\lambda q(v_{\lambda,k}) = -k\pi$ for all $k \geqslant k_1$. We claim that there is a $k_2 \geqslant k_1$ such that:

$$\Big(\frac{m}{2\pi}\Big)^{\frac{1}{\beta}}\Big(\frac{1}{k}\Big)^{\frac{1}{\beta}} \leqslant u_{m,k} \leqslant 2\Big(\frac{m}{\pi}\Big)^{\frac{1}{\beta}}\Big(\frac{1}{k}\Big)^{\frac{1}{\beta}} \text{ for all } k \geqslant k_2, \tag{108}$$

and

$$1-2\Big(\frac{M}{\pi}\Big)^{\frac{1}{\beta}}\Big(\frac{1}{k}\Big)^{\frac{1}{\beta}} \leqslant v_{M,k} \leqslant 1-\Big(\frac{M}{2\pi}\Big)^{\frac{1}{\beta}}\Big(\frac{1}{k}\Big)^{\frac{1}{\beta}} \text{ for all } k \geqslant k_2, \tag{109}$$

where m, M are from (105). Indeed, the starting identity $\lambda q(x) = \pm k\pi$ can be written in the form:

$$\frac{\lambda(1-2x)}{x^{\beta}(1-x)^{\beta}} = \pm k\pi. \tag{110}$$

We know that:

$$\frac{\lambda}{2x^{\beta}} \leqslant \frac{\lambda(1-2x)}{x^{\beta}(1-x)^{\beta}} \leqslant \frac{\lambda 2^{\beta}}{x^{\beta}} \text{ for all } x \in (0, \tfrac{1}{4}), \tag{111}$$

and

$$-\frac{\lambda 2^{\beta}}{(1-x)^{\beta}} \leqslant \frac{\lambda(1-2x)}{x^{\beta}(1-x)^{\beta}} \leqslant -\frac{\lambda}{2(1-x)^{\beta}} \text{ for all } x \in (\tfrac{3}{4}, 1). \tag{112}$$

Putting $x = u_{m,k}$ into (111) and $x = v_{M,k}$ into (112) and using (110), we derive:

$$\frac{\lambda}{2u_{m,k}^{\beta}} \leqslant k\pi \leqslant \frac{\lambda 2^{\beta}}{u_{m,k}^{\beta}},\ k \geqslant k_2, \tag{113}$$

and

$$-\frac{\lambda 2^{\beta}}{(1-v_{M,k})^{\beta}} \leqslant -k\pi \leqslant -\frac{\lambda}{2(1-v_{M,k})^{\beta}},\ k \geqslant k_2, \tag{114}$$

where $k_2 \geqslant k_1$ such that $u_{m,k} \in (0, \frac{1}{4})$ and $v_{M,k} \in (\frac{3}{4}, 1)$ for all $k \geqslant k_2$. Now, from (113) and (114) immediately follows the desired inequalities (108) and (109).

Next, let $y(x)$ be a solution of equation $y'' + f(x)y = 0$ on I and let a_k, b_k be two sequences of consecutive zeros of $y(x)$ such that $a_k \searrow 0$ and $b_k \nearrow 1$, when $k \to \infty$. It gives the existence of a $k_0 \in \mathbf{N}$ such that $k_0 \geqslant k_2$ and $a_k \in (0, \varepsilon)$ and $b_k \in (1-\varepsilon, 1)$ for all $k > k_0$. Now, by means of the left inequality from (106) and Sturm comparison theorem applied to equations (107) with $\lambda = m$ and $y'' + f(x)y = 0$, we conclude that between two consecutive zeros u_{m,k_0+1} and u_{m,k_0} of $y_m(x)$ there is at least one zero a_{i_0} of $y(x)$ such that $u_{m,k_0+1} < a_{i_0} < u_{m,k_0}$. Repeating this procedure to all u_{m,k_0+k} and u_{m,k_0+k-1}, we observe that $u_{m,k_0+k} < a_{i_0+k-1}$ for all $k \geqslant 1$, which together with left inequality from (108) imply:

$$\Big(\frac{m}{2\pi}\Big)^{\frac{2}{\sigma-2}}\Big(\frac{1}{k+k_0}\Big)^{\frac{2}{\sigma-2}} = \Big(\frac{m}{2\pi}\Big)^{\frac{1}{\beta}}\Big(\frac{1}{k_0+k}\Big)^{\frac{1}{\beta}} \leqslant u_{m,k_0+k} \leqslant a_{i_0+k-1} \leqslant a_k,\ k \geqslant 1.$$

It proves the left inequality in (46).

From the right inequality in (106) and Sturm comparison theorem applied to equations $y'' + f(x)y = 0$ and (107) with $\lambda = M$, we conclude that between two consecutive zeros a_{k_0+1} and a_{k_0} of $y(x)$ there is at least one zero u_{M,i_0} of $y_M(x)$ such that $a_{k_0+1} < u_{M,i_0} < a_{k_0}$. Applying this procedure to all pairs a_{k_0+j} and a_{k_0+j-1}, we get $a_{k_0+j} < u_{M,i_0+j-1}$ for each $j \geqslant 1$. Hence, for $k = k_0 + j$ we derive $a_k < u_{M,k-k_0+i_0-1} \leqslant u_{M,k-k_0}$ for each $k > k_0$, which together with the right inequality from (108) imply:

$$a_k \leqslant u_{M,k-k_0} \leqslant 2\Big(\frac{M}{\pi}\Big)^{\frac{1}{\beta}}\Big(\frac{1}{k}\Big)^{\frac{1}{\beta}} = \Big(\frac{M}{\pi}\Big)^{\frac{2}{\sigma-2}}\Big(\frac{1}{k-k_0}\Big)^{\frac{2}{\sigma-2}},\ k > k_0.$$

It proves the right inequality in (46) and thus, the inequality (46) is proved. Inequality (47) concerning the sequence b_k can be proved similarly. □

Proof of (48). According to assumptions of Lemma 6.4 we may use Corollary 3.5. Hence, by (25) and (43), we derive two sequences $s_k, t_k \in I$ of consecutive zeros of $y'(x)$, $s_k \searrow 0$, $t_k \nearrow 1$, such that for sufficiently large k,

$$|y(s_k)| \geqslant c_0 f^{-\frac{1}{4}}(s_k) \geqslant \frac{c_0}{\lambda_1^{1/4}}(s_k - s_k^2)^{\sigma/4} = c_1 s_k^{\sigma/4}(1-s_k)^{\sigma/4} \geqslant c_2 s_k^{\sigma/4},$$

and

$$|y(t_k)| \geqslant c_0 f^{-\frac{1}{4}}(t_k) \geqslant \frac{c_0}{\lambda_1^{1/4}}(t_k - t_k^2)^{\sigma/4} = c_1 t_k^{\sigma/4}(1-t_k)^{\sigma/4} \geqslant c_3 (1-t_k)^{\sigma/4}.$$

It proves (48). □

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Mervan Pašić
Department of Mathematics
FER
Unska 3
University of Zagreb, 10000 Zagreb
Croatia
e-mail: pasic@net.hr

James S. W. Wong
Department of Mathematics
University of Hong Kong
2308, Wing On Centre
Central
Hong Kong
China
e-mail: jsww@chinneyhonkwok.com