Differential
Equations
& Applications
Volume 1, Number 1 (2009), 123–138

A PARABOLIC REGULARIZATION PROPERTY OF p–LOGARITHMIC SOBOLEV GENERATORS

GABRIELE GRILLO

(*communicated by C. Trombetti*)

Abstract. Let N be a Riemannian manifold, $M \subset N$ be a domain with smooth boundary, μ a positive measure on M such that M has unit μ–volume. Consider the evolution driven by the p–Laplace–type operator ($p > 2$) associated to the natural p–energy functional $\mathscr{E}^{(p)}$ constructed from μ, homogeneous Dirichlet boundary conditions on ∂M being assumed. Assume that a single suitable logarithmic inequality holds for $\mathscr{E}^{(p)}$.Then we show that the evolution brings any data belonging to the natural domain of the evolution instantaneously into L^q for any $q > 2$, with quantitative bounds on the L^q norms.

1. Introduction, basic properties of the evolution, and statement of the results

The aim of this paper is to prove an instantaneous L^2–L^q regularizing property ($q > 2$ arbitrary) for the evolution equation associated to (possibly degenerate or singular) p–Laplacian–like operators on *finite volume* domains of Riemannian manifolds, Dirichlet boundary conditions being assumed, provided the associated energy functional satisfy a *single* logarithmic Sobolev inequality.

This parallels, in the present case, the results discovered by L. Gross in his celebrated paper [11] for the linear case (see also [12] and, without any claim of completeness, the fundamental papers of Federbush, Nelson, Simon and Høegh–Krohn [9], [13], [15]), but shows a substantial and unexpected difference with that situation, in which it is well known that no more than a L^2–$\mathrm{L}^{p(t)}$ regularization holds, with $p(t)$ smooth and increasing, $p(0) = 2$, $p(t) \to +\infty$ as $t \to +\infty$. The Ornstein–Uhlenbeck semigroup shows the sharpness of that result in the linear case, this being particularly evident in the fact that the eigenfunctions of such operator are *unbounded*.

To start with we shall introduce our setting and the corresponding notation. We consider a connected, smooth Riemannian manifold (N, g) of dimension n endowed with the associated Riemannian measure m. Let $M \subset N$ be an open domain with smooth boundary and consider a measurable function V on M. It will assumed hereafter, and will be crucial in what follows, that e^V is a probability measure on M, so that the μ–volume of M is one, where we set $\mathrm{d}\mu := e^V\,\mathrm{d}m$. All L^p spaces and norms will

Mathematics subject classification (2000): 47H20, 35K55, 58J35.
Keywords and phrases: p-Laplacian, nonlinear evolutions on manifolds, logarithmic Sobolev inequality, contractivity properties.

Paper DEA-01-06

be considered w.r.t. μ. Notice that e^V can be *degenerate* or *singular* at the boundary as well.

Consider a (m–)measurable, locally integrable metric a on the tangent bundle TM. The central object of this paper will be the following *energy functional*:

$$\mathscr{E}^{(p)}(u) := \int_M a_x(\nabla u(x), \nabla u(x))|\nabla u(x)|_x^{p-2}\,\mathrm{d}\mu. \tag{1.1}$$

here ∇ is the Riemannian gradient and $|\cdot|_x$ is the Riemannian length in the tangent space T_xM.

The functional above is first considered on the space $C_c^\infty(M)$. Then we define the space $\mathscr{A}_0^p(M)$ to be the completion of $C_c^\infty(M)$ under the norm $\|u\|_p + \mathscr{E}^{(p)}(u)^{1/p}$, and consider the functional at hand as being finite only on the Sobolev space $\mathscr{A}_0^p(M)$. We shall say that homogeneous Dirichlet boundary conditions on the boundary hold. Clearly $\mathscr{A}_0^p(M)$ depends on the choices of a and μ as well. It will be required hereafter, without further comment, that the constants do not belong to $\mathscr{A}_0^p(M)$. Qualitatively this is related to the fact that, if e^V tends to zero at the boundary, such convergence must not occur too fast.

The subgradient of this functional (cf. [4],[14]) is a version of the (a,p,μ)–Laplacian $A_{a,p,\mu}$ acting on $\mathrm{L}^2(M,\mu)$, an operator which *in the flat case* reads formally as

$$A_{a,p,\mu}u := e^{-V}\left[2\partial_i(e^V a_{i,j}|\nabla u|^{p-2}\partial_j u) + (p-2)\partial_k\left(e^V \frac{a_{i,j}\partial_i u \partial_j u}{|\nabla u|^2}|\nabla u|^{p-2}\partial_k u\right)\right],$$

where the summation convention is used and $a_{i,j}$ are the coefficient of the (symmetric) matrix a at x. Notice that, formally, $(u, A_{a,p,\mu}u)_{\mathrm{L}^2(\mu)} = -p\mathscr{E}^{(p)}(u)$. If a belongs to the same conformal class of g, i.e. if $a = \sigma g$ for some smooth nonnegative scalar σ, then the generator has a simpler form which, in the flat case, reads

$$A_{\sigma,p,\mu} := pe^{-V}\partial_i(e^V\sigma|\nabla u|^{p-2}\partial_i u).$$

In the non–flat case one should replace, in local coordinates, the gradient operator by $g^{ij}\partial_j$ and the divergence operators by the operator acting on vector fields ξ as $\mathrm{g}^{-1/2}\partial_i(\mathrm{g}^{1/2}\xi_i)$, where g is the determinant of the metric tensor. For an excellent discussion of the evolution equation associated to p–Laplacian–like operators in the Euclidean case see [8].

Several different assumption can be made in order that $\mathscr{E}^{(p)}$ is lower semicontinuous in the L^2 topology. We shall in the sequel use the following one.

ASSUMPTION 1.1. *The metric a is locally strictly elliptic w.r.t. the metric g in the sense that, for all compact $K \subset\subset M$, there exists $\lambda_K > 0$ such that $a_x(\xi(x),\xi(x)) \geqslant \lambda_K g_x(\xi(x)\xi(x))$ for all smooth vector fields ξ. Moreover, e^V belongs to the Sobolev space $W^{1,2}(M,\mu)$.*

LEMMA 1.2. *The functional (1.1), considered as finite on $\mathscr{A}_0^p(M)$ and infinite otherwise, is convex and lower semicontinuous in the strong topology of $\mathrm{L}^2(\mu)$, so that* *(see* [4]*) its subgradient defines a nonexpansive semigroup* $\{T_t\}_{t\geqslant 0}$ *in* $\mathrm{L}^2(\mu)$. *Such semigroup enjoys the Markov property, in the sense that it preserves order, namely* $u \leqslant v$ *implies* $T_t u \leqslant T_t v$ *for all* $t > 0$, *and that it is nonexpansive (in particular contractive) on each* $\mathrm{L}^p(\mu)$ *space for any* $p \in [1,+\infty]$, *namely* $\|T_t u - T_t v\|_p \leqslant \|u-v\|_p$ *for all* $t > 0$, *for all* $u,v \in L^p(\mu)$.

Proof. The only point is to prove the lower semicontinuity, in the $\mathrm{L}^2(\mu)$ topology, of $\mathscr{E}^{(p)}$. This can be shown by minor modifications of the methods of [7], Sections 4.1, 4.2. We outline the only point in which there is a difference and in which the assumption on V has a role. As shown in [7], Sections 4.1, 4.2, it suffices to deal with the case $a = g$. Take u_n converging in L^2 to a function u and consider the sequence $a_n = \mathscr{E}^{(p)}(u_n)$. Suppose that $a := \liminf_{n\to+\infty} a_n$ is finite, otherwise there is nothing to prove. Take any subsequence, still denoted by u_n, such that $\mathscr{E}^{(p)}(u_n) \to a$. Then the set $\{\nabla u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^p(TM,\mu)$ so that, since this latter space is reflexive, it is relatively weakly compact. We can then extract a subsequence, still indicated by u_n, such that ∇u_n is weakly convergent to an element $X \in \mathrm{L}^p(TM,\mu)$. To prove that $X = \nabla u$ notice that, denoting by $\langle\cdot,\cdot\rangle$ the scalar product on $T_x M$:

$$\begin{aligned}
&\lim_{n\to+\infty}\int_M \langle \nabla u_n, \chi\rangle \,\mathrm{d}\mu = -\lim_{n\to+\infty}\int_M u_n \mathrm{div}\left[e^V \chi\right]\,\mathrm{d}m \\
&= -\lim_{n\to+\infty}\int_M u_n\left(e^{-V}\mathrm{div}\left[e^V\chi\right]\right)\mathrm{d}\mu = -\lim_{n\to+\infty}\int_M u_n\left(\mathrm{div}\chi + \chi\nabla(e^V)\right)\mathrm{d}\mu \\
&= -\int_M u\left(e^{-V}\mathrm{div}\left[e^V\chi\right]\right)\mathrm{d}\mu = -\int_M u\,\mathrm{div}\left[e^V\chi\right]\,\mathrm{d}m = \int_M \langle\nabla u,\chi\rangle\,\mathrm{d}\mu
\end{aligned}$$

for any $\chi \in C_c^\infty(TM)$ so that $X = \nabla u \in \mathrm{L}^p(TM,\mu)$. Here the assumption on V has been used to be allowed to pass to the limit in the fourth step, since $\nabla(e^V)]$ is (μ–)square integrable by assumption.

By the weak lower semicontinuity of the L^p norm we thus have:

$$\mathscr{E}_p(u) \leqslant \liminf_{n\to+\infty}\mathscr{E}_p(u_n)$$

with the present choice of u_n. This holds for all such $\mathrm{L}^p(TM,\mu)$ weakly convergent subsequence of ∇u_n, which thus converges weakly to the same limit ∇u. Lower semicontinuity then holds.

As for the Markov property we refer to Section 4.3 of [7], where such property is proved in much greater generality, once the lower semicontinuity of the energy functional is given, for a class of evolution including the present one. □

REMARK 1.3. One could deal with p–homogeneous functionals similar to $\mathscr{E}^{(p)}$ but constructed starting from a L^2–closable vector valued *derivation*. We content ourselves instead with the present assumption to avoid cumbersome notation and to focus on the core of the argument, but stress that the results can be shown to hold in somewhat greater generality. A relevant situation in which our discussion can be carried up is the case in which $\mathscr{E}^p$ is replaced by

$$\mathscr{E}_X^{(p)}(u) := \int_M |Xu|^p\,\mathrm{d}\mu$$

and $|Xu|^2 = \sum_{i=1}^m (X_i u)^2$, $\{X_i\}_{i=1}^m$ being a collection of vector fields for which the relevant lower semicontinuity property and the requested logarithmic Sobolev inequality (see below) holds.

Our next, crucial assumption on the functional under consideration is the following.

ASSUMPTION 1.4. *[logarithmic Sobolev inequality] Consider a fixed number $p > 2$. The following logarithmic Sobolev inequality:*

$$\int_M u^p \log\left(\frac{u}{\|u\|_p}\right) \mathrm{d}\mu \leqslant c_1 \mathscr{E}^{(p)}(u) \tag{1.2}$$

is required to hold for any $\mathscr{A}_0^p(M)$ and a suitable real constant c_1.

REMARK 1.5. An inequality like (1.2) *cannot hold* in the whole maximal domain of $\mathscr{E}^{(p)}$, call it $\mathscr{A}^p(M)$, if M has finite μ–measure. In fact, even when M is compact and $\mathscr{E}^{(p)}(f) = \int_M |\nabla f|^p \, \mathrm{d}m$, notice that the constant functions belong to $\mathscr{A}^p$. One may then gauge the validity of the claimed inequality on functions of the type $f = 1 + sg$ with g a smooth function and s a small parameter. It is clear that the r.h.s. of (1.2) behaves like s^p, while the l.h.s. behaves like s^2 for small s. Such inequality is then false on $\mathscr{A}^p$ since $p > 2$ by assumption.

Logarithmic Sobolev inequalities involving the p–energy functional for general $p \neq 2$ appeared first, to our knowledge, in [6] as a consequence of Sobolev inequalities, and were used there to prove some regularizing properties of the p–heat equations in the euclidean and uniformly elliptic case. The essential difference between the paper [6] and the present one lies in the fact that the result of the former were proved starting from the much stronger *ordinary* Sobolev inequalities so that the L^p–L^q regularizing properties proved there had, in that case, a direct correspondence with what happens in the linear case. The validity of an instantaneous L^2–L^q smoothing under the validity of a single logarithmic Sobolev inequality has on the contrary no linear analogue.

Notice that the evolution considered has a natural maximal domain, i.e. the $\mathrm{L}^2(M)$ closure of the domain of its generator. This set coincide with the $\mathrm{L}^2(M)$ closure of the domain of the generating functional (cf. [4], Prop. 2.11), i.e. with $\overline{\mathscr{A}_0^p(M)}$ in this case (the running assumptions need not imply that $\mathscr{A}_0^p(M)$ is dense in $\mathrm{L}^2(M)$).

We are now ready to state our results, which take a different form in the case in which the datum belongs to $\mathscr{A}_0^p(M)$ or to the maximal domain of the evolution.

THEOREM 1.6. *Consider a solution $u(t) = T_t u_0$ to the evolution equation associated to the subgradient of the energy functional $\mathscr{E}^p$ with domain $\mathscr{A}_0^p(M)$, where assumptions 1.1 and 1.4 are required to hold. Fix $q > 2$ and define, for a suitable numerical constant C (independent of q):*

$$t_q(u_0) := \frac{\|u_0\|_2^2}{\mathscr{E}^{(p)}(u_0)} \left\{ \exp\left[C \frac{\mathscr{E}^{(p)}(u_0)}{\|u_0\|_2^p} \left(q^{p-2} - 2^{p-2}\right)\right] - 1 \right\}. \tag{1.3}$$

Then the bound

$$\|u(t)\|_q \leqslant \|u_0\|_2 \left(\frac{t_q(u_0)}{t}\right)^{1/(p-2)} \tag{1.4}$$

for all $t \leqslant t_q(u_0)$, *for all* $u_0 \in \mathscr{A}_0^p$ *and for a suitable* $B > 0$ *independent of* q.

The above bound is scale invariant, i.e. it invariant when applied to the scaled solution $Ru(R^{p-2}s)$ *corresponding to the initial datum* Ru_0.

For general data in the L^2*–closure of* $\mathscr{A}_0^p(M)$ *one still has that* $u(t)$ *belongs to* $\mathrm{L}^q(M)$ *for all positive times and for all* $q > 2$, *and the bound*

$$\|u(t)\|_q \leqslant Ae^{B/t} \tag{1.5}$$

holds for any u_0 *in the* L^2*–closure of* $\mathscr{A}_0^p(M)$ *for suitable positive* A, B *depending on* q *and* $\|u_0\|_2$, *but not on* $\mathscr{E}^{(p)}(u_0)$, *and for any* t *sufficiently small.*

REMARK 1.7. Clearly the bound (1.5) is weaker then the bound (1.4) as concerns the rate of decay as $t \to 0$, but besides the fact that it holds for the maximal class of data, one notices that the constants involved there do not depend on the energy $\mathscr{E}^{(p)}(u_0)$ of the initial datum, then making such bound closer to the kind of supercontractive–type bounds valid in the linear case.

REMARK 1.8. The above result, and the whole present paper, deals with the L^2–L^q regularization properties of the evolution considered and therefore concerns short times only. Clearly t above can be replaced, by the L^q contraction property of the evolution, by $t \wedge t_q(u_0)$ with no upper bound on t, but power–type decay bounds on the long time behaviour can be obtained if an L^p Poincarè inequality holds. In turn, the L^p Poincarè inequality holds when an L^2 logarithmic Sobolev inequality holds, because the latter implies an L^2 Poincarè inequality which in turns implies the required L^p Poincarè inequality since $p > 2$. In fact, there is a different strategy of proof to get bounds of the above form starting from the L^p Poincarè inequality, but we are not aware of any existing proof that our L^p logarithmic Sobolev inequality implies the L^p Poincaré inequality, so that this topic requires a different treatment and will be dealt with elsewhere.

The validity of the assumed L^p logarithmic Sobolev inequality can often be proved when a similar L^2 logarithmic Sobolev inequality holds: we shall recall at the end of the paper some cases in which this happens. The following corollary makes the above statement precise, and has the goal of showing that the validity of the widely studied L^2 logarithmic Sobolev inequality has consequences on the instantaneous L^2–L^q regularization of the *nonlinear* evolution associated to the p–analogue of the energy functional involved, in the case of homogeneous Dirichlet boundary conditions on the boundary of a suitable $M \subset N$.

COROLLARY 1.9. *Suppose that $a = g$, so that $\mathscr{E}^{(p)}(u) = \int_M |\nabla u|^p \,\mathrm{d}\mu$. Assume that the* L^2*–logarithmic Sobolev inequality*

$$\int_M u^2 \log\left(\frac{u}{\|u\|_2}\right) \mathrm{d}\mu \leqslant c_1 \mathscr{E}^{(2)}(u) \tag{1.6}$$

holds for any $\mathscr{A}_0^2(M)$ and a suitable real constant c_1. Consider, given $p > 2$, the evolution driven by the subgradient of the functional $\mathscr{E}_0^{(p)}$. Then the evolution brings each L^2 *data instantaneously into* L^q *for all $q > 2$. The same bounds of Theorem* 1.6 *hold.*

To prove the Corollary we notice that logarithmic Sobolev inequalities in L^2 imply suitable logarithmic Sobolev inequalities in L^p when $p > 2$, a familiar fact when dealing with ordinary Sobolev inequalities. The L^2–logarithmic Sobolev inequalities is widely studied in the literature, so that significative examples in the case at hand can be deduced from the L^2–theory. The include for example the Gaussian case, but several more general examples are known: for example a complete characterization in the one–dimensional case of the measures for which a L^2 logarithmic Sobolev inequality holds is known, and several variants of the Bakry–Emery criterion can be used in the multi–dimensional case. Since we are not concerned here with such topic we confine ourself to addressing the interested reader e.g. to [1] and references quoted.

The proof of the Theorem will consist in several steps. In the second section we shall prove some properties of a suitable entropic functional and use them to estimate the time dependence of $\|u(s)\|_{r(s)}$ for $r(s)$ increasing and bounded data. In section 3 we prove a lower bound on $\|u(s)\|_2$. The latter, combined with the former, allows us to prove in section 4 that one can choose a specific $r(s)$, *depending on the initial datum*, such that $\|u(s)\|_{r(s)}$ is *decreasing*. The specific form of r, a scaling argument and the use of the order preserving property for the evolution at hand, will make it possible to prove the main claim. The last claim is then proved by making use of a result of Brezis [4].

2. On the behaviour of some $\mathrm{L}^{r(s)}$–norms

From now on we shall consider a nonnegative essentially bounded initial datum u and denote its time evolved by $u(t)$. Essential boundedness and nonnegativity are, by Lemma 1.2, conserved by the evolution: notice indeed that $T_t 0 = 0$ in the present case. Essential boundedness is a technical requirement which will be removed later, while nonnegativity is not necessary in what follows but simplifies notations. We also comment here that the order property of the evolution will however be crucial later on.

In the sequel we shall also use the notation

$$J(q,u) := \int_M \frac{u^q}{\|u\|_q^q} \log\left(\frac{u}{\|u\|_q}\right) \mathrm{d}\mu.$$

Minor modifications of the results of [6] then show the following:

LEMMA 2.1. *Let r is a smooth function of time s and $u = u(s)$ a solution to the evolution equation at hand corresponding to a nonnegative, essentially bounded datum. Then:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|u\|_r = \frac{\dot r}{r}J(r,u) - \frac{(r-1)p^p}{(r+p-2)^p}\frac{\mathscr{E}^{(p)}\left(u^{\frac{r+p-2}{p}}\right)}{\|u\|_r^r}. \tag{2.1}$$

We shall always consider r in the sequel, without further comment, such that $r(0) = 2$ and r is increasing. Notice that we have not indicated explicitly the time dependence of u and r in the above formula and we shall keep this notational convention throughout the paper.

LEMMA 2.2. *The inequality*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\|u\|_r \leqslant \|u\|_r^{-r}\Bigg[\frac{\dot r}{r}\int_M u^r\log\left(\frac{u}{\|u\|_r}\right)\mathrm{d}\mu \\ - \frac{p^{p-1}(r-1)}{c_1(r+p-2)^{p-1}}\int_M u^{r+p-2}\log\left(\frac{u}{\|u\|_{r+p-2}}\right)\mathrm{d}\mu\Bigg]\end{aligned} \tag{2.2}$$

holds true for any solution corresponding to a nonnegative essentially bounded datum.

Proof. We may rewrite the logarithmic Sobolev inequality (1.2) as:

$$\mathscr{E}^{(p)}(u) \geqslant \frac{1}{c_1}\|u\|_p^p J(p,u)$$

so that we may deduce from (2.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|u\|_r \leqslant \frac{\dot r}{r}J(r,u) - \frac{(r-1)p^p}{c_1(r+p-2)^p}\frac{\|u\|_{r+p-2}^{r+p-2}}{\|u\|_r^r}\left(J(p,u^{(r+p-2)/p})\right)$$

where we have used the elementary identity $\|u^{(r+p-2)/p}\|_p^p = \|u\|_{r+p-2}^{r+p-2}$. Equivalently:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\|u\|_r &\leqslant \|u\|_r^{-r}\Bigg[\frac{\dot r}{r}\int_M u^r\log\left(\frac{u}{\|u\|_r}\right)\mathrm{d}\mu - \frac{p^p(r-1)}{c_1(r+p-2)^p}\|u\|_{r+p-2}^{r+p-2}\times \\ &\qquad \times\left(p^{-1}\int_M \frac{u^{r+p-2}}{\|u^{(r+p-2)/p}\|_p^p}\log\left(\frac{u^{r+p-2}}{\|u^{(r+p-2)/p}\|_p^p}\right)\mathrm{d}\mu\right)\Bigg] \\ &= \|u\|_r^{-r}\Bigg[\frac{\dot r}{r}\int_M u^r\log\left(\frac{u}{\|u\|_r}\right)\mathrm{d}\mu - \frac{p^p(r-1)}{c_1(r+p-2)^p}\times \\ &\qquad \times\left(\frac{r+p-2}{p}\int_M u^{r+p-2}\log\left(\frac{u}{\|u\|_{r+p-2}}\right)\mathrm{d}\mu\right)\Bigg] \\ &= \|u\|_r^{-r}\Bigg[\frac{\dot r}{r}\int_M u^r\log\left(\frac{u}{\|u\|_r}\right)\mathrm{d}\mu \\ &\qquad - \frac{p^{p-1}(r-1)}{c_1(r+p-2)^{p-1}}\int_M u^{r+p-2}\log\left(\frac{u}{\|u\|_{r+p-2}}\right)\mathrm{d}\mu\Bigg] \quad \square\end{aligned}$$

To prove a hypercontractive–like property for the nonlinear semigroup at hand it is hence necessary to deal with the ϑ–dependence of the functional $\int_M u^\vartheta \log\left(\frac{u}{\|u\|_\vartheta}\right) \mathrm{d}\mu$. It will be more convenient to deal with the functional:

$$H(\vartheta,u) = \vartheta \int_M u^\vartheta \log\left(\frac{u}{\|u\|_\vartheta}\right) \mathrm{d}\mu,$$

defined on $\mathrm{L}^\infty_+(M,\mu)$.

LEMMA 2.3. *Let $u \in \mathrm{L}^\infty_+(M,\mu)$. Then the map*

$$\vartheta \mapsto \vartheta e^{-\int_2^\vartheta \log \|u\|_s \mathrm{d}s} \int_M u^\vartheta \log\left(\frac{u}{\|u\|_\vartheta}\right) \mathrm{d}\mu, \quad \vartheta \geqslant 1$$

is non–decreasing.

Proof. To prove the claim we first notice that the derivative of the functional $\vartheta \mapsto \|u\|_\vartheta^\vartheta$ is given by $\int_M u^\vartheta \log u \,\mathrm{d}\mu$, that the the derivative of the functional $r \mapsto \int_M u^\vartheta \log u \,\mathrm{d}\mu$ is given by $\int_M u^\vartheta \log^2 u \,\mathrm{d}\mu$, and then compute:

$$\begin{aligned}
H'(\vartheta,u) &= \frac{\mathrm{d}}{\mathrm{d}\vartheta}\left[\vartheta\left(\int_M u^\vartheta \log u \,\mathrm{d}\mu - \|u\|_\vartheta^\vartheta \log \|u\|_\vartheta\right)\right] \\
&= \int_M u^\vartheta \log u \,\mathrm{d}\mu + \vartheta \int_M u^\vartheta \log^2 u \,\mathrm{d}\mu - \frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\|u\|_\vartheta^\vartheta \log \|u\|_\vartheta^\vartheta\right) \\
&= \int_M u^\vartheta \log u \,\mathrm{d}\mu + \vartheta \int_M u^\vartheta \log^2 u \,\mathrm{d}\mu \\
&\qquad - \left(\int_M u^\vartheta \log u \,\mathrm{d}\mu\right) \log \|u\|_\vartheta^\vartheta - \int_M u^\vartheta \log u \,\mathrm{d}\mu \\
&= \vartheta\left[\int_M u^\vartheta \log^2 u \,\mathrm{d}\mu - \left(\int_M u^\vartheta \log u \,\mathrm{d}\mu\right) \log \|u\|_\vartheta\right] \\
&= \vartheta \int_M u^\vartheta (\log u) \log\left(\frac{u}{\|u\|_\vartheta}\right) \mathrm{d}\mu.
\end{aligned}$$

We may also write

$$\begin{aligned}
&\frac{1}{\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left[\vartheta \int_M u^\vartheta \log\left(\frac{u}{\|u\|_\vartheta}\right) \mathrm{d}\mu\right] \\
&\qquad = \int_M u^\vartheta \log^2\left(\frac{u}{\|u\|_\vartheta}\right) \mathrm{d}\mu + \int_M u^\vartheta \log\left(\frac{u}{\|u\|_\vartheta}\right) \mathrm{d}\mu \log \|u\|_\vartheta.
\end{aligned}$$

Let now A be a differentiable function of ϑ, to be chosen later. We compute using the last formula, a prime denoting derivative with respect to ϑ:

$$\frac{1}{\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left[A(\vartheta)\vartheta\int_M u^\vartheta \log\left(\frac{u}{\|u\|_\vartheta}\right)\mathrm{d}\mu\right]$$
$$=A'\int_M u^\vartheta \log\left(\frac{u}{\|u\|_\vartheta}\right)\mathrm{d}\mu + A\int_M u^\vartheta \log^2\left(\frac{u}{\|u\|_\vartheta}\right)\mathrm{d}\mu$$
$$+A\int_M u^\vartheta \log\left(\frac{u}{\|u\|_\vartheta}\right)\mathrm{d}\mu\,\log\|u\|_\vartheta$$
$$=A\int_M u^\vartheta \log^2\left(\frac{u}{\|u\|_\vartheta}\right)\mathrm{d}\mu + \int_M u^\vartheta \log\left(\frac{u}{\|u\|_\vartheta}\right)\mathrm{d}\mu\,\left(A\log\|u\|_\vartheta + A'\right).$$

We now choose A so that the above derivative is nonnegative. To this end it suffices to require that A is nonnegative and that

$$A' + A\log\|u\|_\vartheta = 0.$$

This equation has the solution, for $\vartheta \geqslant 2$ and a given u:

$$A(\vartheta) = Ce^{-\int_2^\vartheta \log\|u\|_s\,\mathrm{d}s}.$$

We have therefore proven that

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left[\vartheta e^{-\int_2^\vartheta \log\|u\|_s\,\mathrm{d}s}\int_M u^\vartheta \log\left(\frac{u}{\|u\|_\vartheta}\right)\mathrm{d}\mu\right] \geqslant 0$$

as claimed. □

We may rewrite the conclusion of the last lemma as follows: for any positive δ and any $r \geqslant 1$ one has

$$(r+\delta)e^{-\int_2^{r+\delta}\log\|u\|_s\,\mathrm{d}s}\int_M u^{r+\delta}\log\left(\frac{u}{\|u\|_{r+\delta}}\right)\mathrm{d}\mu \geqslant re^{-\int_2^r \log\|u\|_s\,\mathrm{d}s}\int_M u^r\log\left(\frac{u}{\|u\|_r}\right)\mathrm{d}\mu$$

or equivalently

$$\int_M u^{r+\delta}\log\left(\frac{u}{\|u\|_{r+\delta}}\right)\mathrm{d}\mu \geqslant \frac{r}{r+\delta}e^{\int_r^{r+\delta}\log\|u\|_s\,\mathrm{d}s}\int_M u^r\log\left(\frac{u}{\|u\|_r}\right)\mathrm{d}\mu. \tag{2.3}$$

We now use the latter inequality in (2.2) and deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|u\|_r \leqslant \left(\frac{\dot r}{r} - \frac{p^{p-1}r(r-1)}{c_1(r+p-2)^p}e^{\int_r^{r+p-2}\log\|u\|_s\,\mathrm{d}s}\right)\int_M \frac{u^r}{\|u\|_r^r}\log\left(\frac{u}{\|u\|_r}\right)\mathrm{d}\mu. \tag{2.4}$$

It is in the following Lemma that we need to assume that M has unit μ–measure, a property which has not been used so far.

LEMMA 2.4. *The inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|u\|_r \leqslant \left(\frac{\dot r}{r} - \frac{p^{p-1}r(r-1)}{c_1(r+p-2)^p}\|u\|_2^{p-2}\right)\int_M \frac{u^r}{\|u\|_r^r}\log\left(\frac{u}{\|u\|_r}\right)\mathrm{d}\mu \tag{2.5}$$

holds true for any solution corresponding to a positive essentially bounded datum.

Proof. Jensen's inequality applied to the convex function $x \mapsto x\log x$ implies, since the μ–mass of M is one, that

$$\int_M \frac{u^r}{\|u\|_r^r} \log\left(\frac{u}{\|u\|_r}\right) \mathrm{d}\mu \geqslant 0.$$

Notice now that, again because the μ–mass of M is one, one has $\|u\|_r \leqslant \|u\|_s$ whenever $r \leqslant s$ so that

$$e^{\int_r^{r+p-2} \log\|u\|_s \, \mathrm{d}s} \geqslant e^{(p-2)\log\|u\|_r} = \|u\|_r^{p-2} \geqslant \|u\|_2^{p-2}.$$

Putting back this inequalities into (2.4) we then obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \log\|u\|_r \leqslant \left(\frac{\dot r}{r} - \frac{p^{p-1} r(r-1)}{c_1(r+p-2)^p}\|u\|_r^{p-2}\right) \int_M \frac{u^r}{\|u\|_r^r} \log\left(\frac{u}{\|u\|_r}\right) \mathrm{d}\mu$$

and finally, using also that $\|u\|_r \geqslant \|u\|_2$ since r is increasing and $r(0) = 2$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \log\|u\|_r \leqslant \left(\frac{\dot r}{r} - \frac{p^{p-1} r(r-1)}{c_1(r+p-2)^p}\|u\|_2^{p-2}\right) \int_M \frac{u^r}{\|u\|_r^r} \log\left(\frac{u}{\|u\|_r}\right) \mathrm{d}\mu \tag{2.6}$$

as claimed. □

3. A lower bound

To proceed further in discussing the consequences of the bounds proved in the previous Section we need to prove a *lower* bound on the L^2 norm of the solution. This can be done by using a technique due to Alikakos and Rostamian [2] and modifying it slightly in order to take care of the dependence of the bound upon the initial datum.

We comment that the constant C below depends on p only and could be explicitly estimated by the interested reader, as the other numerical constant appearing in the paper, by following carefully the proofs.

LEMMA 3.1. *Let u be a solution to the equation at hand corresponding to the initial datum $u_0 \in \mathscr{A}_0^p(M)$. Then the inequality*

$$\|u(t)\|_2 \geqslant \frac{\|u_0\|_2}{\left(1 + C\frac{\mathscr{E}^{(p)}(u_0)}{\|u_0\|_2^2}t\right)^{1/(p-2)}} \tag{3.1}$$

holds true.

Proof. The proof follows closely [2] and we underline only the points which are more relevant to our discussion. Consider the functional

$$E(u) := \mathscr{E}^{(p)}(u) - \frac{1}{2(p-2)}\|u\|_2^2.$$

Let also $v(t) := u(t)(1+t)^{1/(p-2)}$ and consider the time scaling $\tau = \log(t+1)$.

Consider the easily verified formula $(t+1)\dot{v}(t) = \triangle_p v + v(t)/(p-2)$. This formula, the fact that $A_{a,p,\mu}$ is the generator of the evolution and that such evolution is associated to the subgradient of the functional $\mathscr{E}^{(p)}(u)$ then yield, for any $\tau > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}E(v(\tau)) = -\|\dot{v}(\tau)\|_2^2 \leqslant 0$$

so that $E(v(\tau)) \leqslant E(v(\varepsilon))$ for all $\tau \geqslant \varepsilon > 0$. Recall that for any positive time the time evolved of any L^2 datum belongs to the domain of the energy functional associated to the evolution, i.e. in the present case to $\mathscr{A}_0^p$, see [4]. For an initial datum *belonging to* $\mathscr{A}_0^p$ we may of course replace ε with zero, and then we rewrite the formula as

$$\mathscr{E}^{(p)}(u) \leqslant \frac{1}{2(p-2)}\|u\|_2^2 + E_0 \tag{3.2}$$

where $E_0 := E(v(0)) = E(u_0)$.

It is easily verified that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|v(\tau)\|_2^2 = -p\mathscr{E}^{(p)}(v(\tau)) + \frac{1}{p-2}\|v(\tau)\|_2^2.$$

The latter bound implies, using (3.2), that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\|v(\tau)\|_2^2 \geqslant -\|v(\tau)\|_2^2 - 2pE_0(v(0)). \tag{3.3}$$

Integrating the above inequality gives

$$\|v(\tau)\|_2^2 \geqslant e^{-\tau}\|v(0)\|_2^2 - 2pE_0(v(0))(1-e^{-\tau})$$

and it is then immediate to see (notice that E_0 may be negative) that

$$\|v(\tau)\|_2^2 \geqslant \min(\|v(0)\|_2^2, -2pE_0(v(0))) \quad \forall \tau > 0. \tag{3.4}$$

Since the map $t \mapsto t(\tau)$ is a bijection of $[0,+\infty)$ into itself and τ is arbitrary a similar estimate holds with t replacing τ. Coming back to the original solution u and recalling that $v(0) = u_0$ this shows that, whenever $E_0 \leqslant 0$:

$$\|u(t)\|_2 \geqslant \frac{\sqrt{\min(\|u_0\|_2^2, -2pE_0(u_0))}}{(1+t)^{1/(p-2)}} \quad \forall t > 0. \tag{3.5}$$

Notice now that the homogeneity of the generator at hand allows to conclude that, if $u(t)$ is the solution corresponding to the datum u_0, then $\hat{u}(t) := Ru(R^{p-2}t)$ is the solution corresponding to the datum Ru_0 given any positive R. Let us notice that

$$E_0(\hat{u}(0)) = R^p\mathscr{E}^{(p)}(u_0) - \frac{R^2}{2(p-2)}\|u_0\|_2^2 = R^2\left(R^{p-2}\mathscr{E}^{(p)}(u_0) - \frac{1}{2(p-2)}\|u_0\|_2^2\right).$$

Choose now

$$R = R_{u_0} := \frac{1}{2}\left(\frac{\|u_0\|_2^2}{2(p-2)\mathscr{E}^{(p)}(u_0)}\right)^{1/(p-2)}$$

where we notice that by assumption the constants do not belong to the domain of the energy functional and hence we can suppose that $\mathscr{E}^{(p)}(u_0)^{1/p} \neq 0$. Therefore $E_0(\hat{u}(0)) = -CR_{u_0}^2\|u_0\|_2^2$ for a suitable positive constant C. The bound (3.5) and the previous calculation then gives

$$\|\hat{u}(t)\|_2 \geqslant \frac{\sqrt{\min[\|\hat{u}_0\|_2^2, -2pE_0(\hat{u}_0)]}}{(1+t)^{1/(p-2)}} = \text{const.}\,\frac{R_{u_0}\|u_0\|_2}{(1+t)^{1/(p-2)}}$$

that is

$$\|u(R_{u_0}^{p-2}t)\|_2 \geqslant C\frac{\|u_0\|_2}{(1+t)^{1/(p-2)}}.$$

We can rewrite the above formula as follows, where we set $s = R^{p-2}t$ and inessential numerical constants may hereafter change from line to line:

$$\|u(s)\|_2 \geqslant \frac{C\|u_0\|_2^{p/(p-2)}}{\left(C\|u_0\|_2^2 + s\mathscr{E}^{(p)}(u_0)\right)^{1/(p-2)}} = \frac{\|u_0\|_2}{\left(1 + C\frac{\mathscr{E}^{(p)}(u_0)}{\|u_0\|_2^2}s\right)^{1/(p-2)}} \tag{3.6}$$

where C is a positive constant independent of u_0. □

4. Proof of the main results

Proof of the Theorem. We now insert the bound (3.1) into the formula (2.5) to yield:

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|u\|_r \leqslant \left(\frac{\dot{r}}{r} - \frac{p^{p-1}r(r-1)}{c_1 p(r+p-2)^p}\frac{\|u_0\|_2^{p-2}}{\left(1 + C\frac{\mathscr{E}^{(p)}(u_0)}{\|u_0\|_2^2}t\right)}\right)\int_M \frac{u^r}{\|u\|_r}\log\left(\frac{u}{\|u\|_r^r}\right)\mathrm{d}m.$$

Finally we may also write, C denoting below a constant depending only on p and on c_1:

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|u\|_r \leqslant \left(\frac{\dot{r}}{r} - \frac{C}{r^{p-2}}\frac{\|u_0\|_2^{p-2}}{\left(1 + \frac{\mathscr{E}^{(p)}(u_0)}{\|u_0\|_2^2}t\right)}\right)\int_M \frac{u^r}{\|u\|_r^r}\log\left(\frac{u}{\|u\|_r}\right)\mathrm{d}m. \tag{4.1}$$

We shall need the following proposition.

PROPOSITION 4.1. *[state dependent hypercontractivity] Let, for a suitable numeric constant C, for all positive t and all data in $\mathscr{A}_0^p$,*

$$r_{u_0}(t) := \left[2^{p-2} + C\frac{\|u_0\|_2^p}{\mathscr{E}^{(p)}(u_0)}\log\left(1 + \frac{\mathscr{E}^{(p)}(u_0)}{\|u_0\|_2^2}t\right)\right]^{1/(p-2)}.$$

Then the state dependent hypercontractive bound

$$\|u(t)\|_{r_{u_0}(t)} \leqslant \|u_0\|_2, \ \ \forall t \geqslant 0$$

holds. The latter bound is scale invariant, i.e. it is invariant when applied to the scaled solution $Ru(R^{p-2}t)$ corresponding to the initial datum u_0.

Proof. We choose r, *depending on the initial datum u_0,* so that the right–hand side of (4.1) vanishes. In fact, let r solve the ordinary differential equation

$$\dot{r}r^{p-3} = \frac{C\|u_0\|_2^{p-2}}{1+\frac{\mathscr{E}^{(p)}(u_0)}{\|u_0\|_2^2}t}$$

Then r is strictly increasing in time and, recalling that $r(0)=2$, it has the explicit expression

$$r(t) = r_{u_0}(t) = \left[2^{p-2} + C\frac{\|u_0\|_2^p}{\mathscr{E}^{(p)}(u_0)}\log\left(1+\frac{\mathscr{E}^{(p)}(u_0)}{\|u_0\|_2^2}t\right)\right]^{1/(p-2)}. \tag{4.2}$$

Notice that $r(t) \to +\infty$ as $t \to +\infty$.

We have thus shown that $\|u(t)\|_{r_{u_0}(t)} \leqslant \|u_0\|_2$ for all $t \leqslant t(u_0)$ and for all data which belong to $L^\infty \cap \mathscr{A}_0^p$.

The assumption of essential boundedness of the data can be removed. Take indeed a sequence $\{u_n\}_{n\in\mathbb{N}} \subset L^\infty \cap \mathscr{A}_0^p$ with $u_n \to u$ in $\mathscr{A}_0^p$. Then in particular $u_n \to u$ in L^2 as well since M has finite measure. Our previous bound implies that, for any fixed $t>0$, $\|u_n(t)\|_{r_{u_n}(t)} \leqslant \|u_n\|_2$ for all t. Thus the sequence $\{\|u_n(t)\|_{r_{u_n}(t)}\}$ is bounded and, since $r_{u_n}(t) \to r_{u_0}(t)$, so is the sequence $\{\|u_n(t)\|_{r_{u_0}(t)-\varepsilon}\}$ for any given positive ε. By the Banach–Alaoglu Theorem there exists a function $f \in L^{r_{u_0}(t)-\varepsilon}$ such that, possibly by passing to a subsequence, $u_n(t) \to f$ in the weak-* topology of such space. Weak–* lower semicontinuity of the norm then implies that $\|f\|_{r_{u_0}(t)-\varepsilon} \leqslant \|u_0\|_2$ for all $\varepsilon > 0$ so that $\|f\|_{r_{u_0}(t)} \leqslant \|u_0\|_2$ as well. The identification of f with $u(t)$ follows from the Markov property for the evolution at hand which implies that $\|u_n(t)-u_0(t)\|_q \leqslant \|u_n-u_0\|_q$ for all $q \in [1,+\infty]$.

Therefore, we have shown that

$$\|u(t)\|_{r_{u_0}(t)} \leqslant \|u_0\|_2 \ \ \forall t < t(u_0), \ \forall u_0 \in \mathscr{A}_0^p. \tag{4.3}$$

It is of some importance to notice that the present bound is *scale invariant*. Indeed take a solution u to the equation at hand and consider the rescaled solution $v(t) = Ru(R^{p-2}t)$, corresponding to the initial datum Ru_0. Writing the above bound for v leads to the inequality $\|u(s)\|_{r_{Ru_0}(s/R^{p-2})} \leqslant \|u_0\|_2$ for $s \leqslant t(u_0)$, a formula which is obtained by setting $s = R^{p-2}t$. It is then elementary to check that $r_{Ru_0}(s/R^{p-2}) = r_{u_0}(s)$ as claimed. $\square$

Proof of Theorem 1.6. Consider now a fixed $q>2$ and notice that $r_{u_0}(t)=q$ if and only if, C denoting again an inessential numerical constant (independent of q):

$$t=t_q(u_0):=\frac{\|u_0\|_2^2}{\mathscr{E}^{(p)}(u_0)}\left\{\exp\left[C\frac{\mathscr{E}^{(p)}(u_0)}{\|u_0\|_2^p}\left(q^{p-2}-2^{p-2}\right)\right]-1\right\}.$$

To proceed further, take now a *nonnegative* initial datum u_0.

Then the function $v_\lambda:=\lambda u_0$ is pointwise not smaller then u_0 whenever $\lambda\geqslant 1$. Therefore the Markov property implies $0\leqslant T_t u_0\leqslant T_t v_\lambda$ for all t where for notational clarity we have indicated, in the present steps only, by T_t the nonlinear semigroup which represents the evolution at hand. Therefore, fixing any q larger than 2:

$$\|T_t u_0\|_q\leqslant\|T_t v_\lambda\|_q\leqslant\|v_\lambda\|_2=\lambda\|u_0\|_2$$

provided t and λ are related by the condition $t=t_q(\lambda u_0)=t_q(u_0)/\lambda^{p-2}$. Equivalently one must have, given a fixed $t>0$, $\lambda=[t_q(u_0)/t]^{1/(p-2)}$. Since λ must be not smaller than one one must require that $t\leqslant t_q(u_0)$ as well. With this choice of λ it follows that,

$$\|u(t)\|_q\leqslant\frac{t_q(u_0)^{1/(p-2)}\|u_0\|_2}{t^{1/(p-2)}}.$$

To end with, we finally notice that the case of initial data with changing sign can be considered similarly starting from the fact that $-|u_0|\leqslant u_0\leqslant|u_0|$, using the order preserving property of the evolution and noticing finally that the homogeneity properties of the evolution imply that the solution corresponding to a nonpositive datum $v_0=-u_0$ is obtained by changing sign to the time evolved of u_0.

To prove the latter claim we shall use Theorem 3.2, formula (13) of [4]. Noticing that $A_{a,p,\mu}(0)=0$ in the case at hand and then choosing $v=0$ in the mentioned result gives

$$\|A_{a,p,\mu}(u(t))\|_2\leqslant\frac{\|u_0\|_2}{t}.$$

This fact also implies that

$$\mathscr{E}^{(p)}(u(t))\leqslant C\frac{\|u_0\|_2^2}{t}, \tag{4.4}$$

a bound which, incidentally, is easily shown to be scale invariant as well. Use this fact as follows. Write the conclusion of the main Theorem, using the semigroup property, choosing $t/2$ instead of zero as initial time. Then:

$$\begin{aligned}\|u(t)\|_q&\leqslant\left(\frac{\|u(t/2)\|_2}{\mathscr{E}^{(p)}(u(t/2))^{1/p}}\right)^{p/(p-2)}\left[\exp\left(B_q\frac{\mathscr{E}^{(p)}(u(t/2))}{\|u(t/2)\|_2^p}\right)-1\right]^{1/(p-2)}\frac{C}{t^{1/(p-2)}}\\&\leqslant\left(\frac{\|u_0\|_2}{\mathscr{E}^{(p)}(u(t/2))^{1/p}}\right)^{p/(p-2)}\left[\exp\left(B_q\frac{\mathscr{E}^{(p)}(u(t/2))}{\|u(t/2)\|_2^p}\right)-1\right]^{1/(p-2)}\frac{C}{t^{1/(p-2)}}\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\|u_0\|_2\left[\exp\left(B_q\frac{\mathscr{E}^{(p)}(u(t/2))}{\|u(t/2)\|_2^p}\right)-1\right]^{1/(p-2)}\\
&\leqslant C\|u_0\|_2\left[\exp\left(B_q\frac{\mathscr{E}^{(p)}(u(t/2))}{\|u_0\|_2^p}\right)-1\right]^{1/(p-2)}\\
&\leqslant C\|u_0\|_2\left[\exp\left(\frac{B_q}{t\|u_0\|_2^{p-2}}\right)-1\right]^{1/(p-2)}
\end{aligned}
$$

for t sufficiently small, the constant B_q and C being as usual allowed to change from line to line. We have used the fact that the present evolution is L^2–contractive in the second step (since we deal with a nonexpansive semigroup $\{T_t\}_{t\geqslant 0}$ for which $T_t 0=0$ for all times), the bound (4.4) in the third and in the fifth step, and in the fourth step the fact that, due to the strong L^2–continuity of the evolution, there exists a positive t_1 such that for $t\leqslant t_1$ one has $\|u(t)\|_2\geqslant\|u_0\|_2/2$. From the above bound the assertion follows. □

To prove the corollary, we now consider the case in which $a=g$, so that

$$
\mathscr{E}^{(p)}(u):=\int_M|\nabla u(x)|_x^p\,\mathrm{d}\mu \tag{4.5}
$$

Proof of Corollary 1.9. We notice that the if the logarithmic Sobolev inequality

$$
\int_M u^2\log\left(\frac{u}{\|u\|_2}\right)\mathrm{d}\mu\leqslant c_1\mathscr{E}^{(2)}(u), \tag{LS_2}
$$

holds for any $u\in\mathscr{A}_0^2(M)$, then the logarithmic Sobolev inequality

$$
\int_M u^p\log\left(\frac{u}{\|u\|_p}\right)\mathrm{d}\mu\leqslant c_1\mathscr{E}^{(p)}(u), \tag{LS_p}
$$

holds for any $p>2$ and any function $u\in\mathscr{A}_0^p(M)$.

To prove such claim we first notice that the obvious identity $J(r,u^\gamma)=\gamma J(\gamma r,u)$ holds true for all $\gamma,r>0$. Then we compute, first for positive bounded functions in $\mathscr{A}^p$:

$$
\begin{aligned}
pJ(p,u)&=2\frac{p}{2}J\left(2\frac{p}{2},u\right)=2J\left(2,u^{\frac{p}{2}}\right)\leqslant C\frac{\mathscr{E}_2\left(u^{\frac{p}{2}}\right)}{\left\|u^{\frac{p}{2}}\right\|_2^2}\\
&=C\frac{\int_M u^{2\left(\frac{p}{2}-1\right)}|\nabla u|^2\,\mathrm{d}\mu}{\|u\|_p^p}\leqslant C\frac{\|u^{p-2}\|_{\sigma'}\left\||\nabla u|^2\right\|_\sigma}{\|u\|_p^p}\\
&=C\frac{\|u\|_p^{p-2}\|\nabla u\|_p^2}{\|u\|_p^p}=C\frac{\|\nabla u\|_p^2}{\|u\|_p^2}.
\end{aligned} \tag{4.6}
$$

where we also used Hölder inequality with the choice of the two conjugate exponents $\sigma = \frac{p}{2}$ and $\sigma' = \frac{p}{p-2}$. To conclude the proof it is then clearly sufficient to prove the fact that the Poincaré inequality $\|u\|_p \leqslant C\|\nabla u\|_p$ holds for any $u \in \mathscr{A}_0^p$. To prove such claim notice that, by a variant of a result of Gross (see [5], Theorem 3.1 and Example 3.2) the bottom λ_0 of the L^2–spectrum of the (nonnegative) self–adjoint operator associated to $\mathscr{E}_0^{(2)}$ is an eigenvalue with finite multiplicity. The quantity λ_0 cannot then equal zero because of the Dirichlet boundary condition which would force in such case the corresponding eigenfunction to vanish identically. This amounts to the validity of the L^2–Poincaré inequality $\|u\|_2 \leqslant C\|\nabla u\|_2$. That this latter inequality implies the corresponding one in L^p is standard, see however the much more general arguments of [10], section 3, or [3]. □

REFERENCES

[1] C. ANÉ, S. BLACHÈRE, D. CHAFAÏ, P. FOUGÈRES, I. GENTIL, F. MALRIEU, C. ROBERTO, G. SCHEFFER, *Sur les inégalités de Sobolev logarithmiques*, Panoramas et Synthèses 10. Société Mathématique de France, Paris, 2000.

[2] N. ALIKAKOS, R. ROSTAMIAN, *Lower bound estimates and separable solutions for homogeneous equations of evolution in Banach space*, J. Differential Equations, **43** (1982), 323–344.

[3] M. BIROLI, U. MOSCO, *Sobolev inequalities on homogeneous spaces*, Potential Anal., **4** (1995), 311–324.

[4] H. BREZIS, *Operateurs Maximaux Monotones'*, North-Holland Publishing Company, Amsterdam/London, 1973.

[5] F. CIPRIANI, *Sobolev–Orlicz embeddings, weak compactness, and spectrum*, J. Funct. Anal., **177** (2000), 89–106.

[6] F. CIPRIANI, G. GRILLO, *Uniform bounds for solutions to quasilinear evolution equations*, J. Differential Equations, **177** (2001), 209–234.

[7] F. CIPRIANI, G. GRILLO, *Nonlinear Dirichlet forms and nonlinear Markov semigroups*, J. Reine Angew. Math., **562** (2003), 201–235.

[8] E. DIBENEDETTO, *Degenerate Parabolic Equations*, Springer Verlag, New York/Berlin, 1993.

[9] P. FEDERBUSH, *A partially alternate derivation of a result of Nelson*, J. Math. Phys., **10** (1969), 50–52.

[10] B. FRANCHI, C. PEREZ, R.L. WHEEDEN, *Self–improving properties of John–Nirenberg and Poincaré inequalities on spaces of homogeneous type*, J. Funct. Anal., **153** (1998), 108–146.

[11] L. GROSS, *Logarithmic Sobolev inequalities*, Amer. J. Math., **97** (1976), 1061–1083.

[12] L. GROSS, *Logarithmic Sobolev inequalities and contractivity properties of semigroups*, in: Dirichlet Forms, Lecture Notes in Mathematics 1563, pp. 54–88.

[13] E. NELSON, *The free Markoff field*, J. Funct. Anal., **12** (1973), 211–227.

[14] R.E. SHOWALTER, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations'*, Mathematical Surveys and Monographs, vol. **49**, American Mathematical Society, 1997.

[15] B. SIMON, R. HØEGH–KROHN, *Hypercontractive semigroups and two dimensional self-coupled Bose fields*, J. Funct. Anal., **9** (1972), 121–180.

(Received October 23, 2008)

Gabriele Grillo
Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino
Italy
e-mail: gabriele.grillo@polito.it