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GENERALIZED SOLUTIONS TO A NON–LIPSCHITZ GOURSAT PROBLEM

VICTOR DÉVOUÉ

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Abstract. We study the semilinear wave equation in canonical form with non-Lipschitz nonlinearity by using the recent theories of generalized functions. We investigate solutions to the Goursat problem. We turn this non-Lipschitz Goursat problem with irregular data into a biparameter family of problems. The first parameter replaces the problem by a family of Lipschitz problems and the second one regularizes the data. Finally the family of problems is solved in an appropriate biparametric $(\mathscr{C},\mathscr{E},\mathscr{P})$ algebra.

1. Introduction

The distribution theory has some limitations when nonlinear problems are considered. The theories of algebras of generalized functions [1], [11], which form at least presheaves of differential algebras, seem to be an efficient tool to overcome these limitations. They have already been used to solve many nonlinear and irregular problems. For example, in the case of singular data and Lipschitz nonlinearity, a method consists in replacing the given problem with a one-parameter family of smooth problems and has been successfully used in [5], [15], [16], [18] among other references. With similar techniques, various type of nonlinearities are considered in [17], [19].

The main purpose of this paper is to establish the existence of a global solution for the non-Lipschitz Goursat problem formally written as $\left(P_{form}\right)$: $\dfrac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u)$ (for example $F(\cdot,\cdot,u)) = u^2$), in the case of irregular data given along the characteristic curve $C = (Ox)$ and along a monotonic curve γ of equation $x = g(y)$. We want to investigate solutions to this nonlinear problem with distributions or other generalized functions as data. This justifies to search for solutions in algebras which are invariant under nonlinear functions and contain the space of distributions. To do this, we use some regularization processes and cutoff techniques described in the framework of $(\mathscr{C},\mathscr{E},\mathscr{P})$-algebras of Marti [12], [13], [14], [15], [16] which are an improvement and generalization of the algebras of Colombeau [1], [11]. These algebras are designed to admit multiparametric families of smooth functions as representatives of generalized functions.

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The mentioned ill-posed problem remains in general unsolvable in classical functional spaces. To overcome this difficulty, we associate to problem $\left(P_{form}\right)$ a generalized one (P_{gen}) well formulated in a convenient algebra $\mathscr{A}(\mathbb{R}^2)$.

We begin by giving general conditions for such an association: a stability condition of $\mathscr{A}(\mathbb{R}^2)$ in relationship with the nonlinearity and a compatibility condition leading to a generalization of the usual restriction.

In the case we are studying, the problem (P_{gen}) is constructed by means of a family $(P_\lambda)_{\lambda\in\Lambda}$ of regularized problems, where $\lambda = (\varepsilon,\rho)$ lies in the set $\Lambda = (0,1]\times(0,1]$. Our techniques use a family of cutoff functions $(f_\varepsilon)_\varepsilon$ and a family mollifiers $\left(\theta_\rho\right)_\rho$ regularizing the data in singular cases. Therefore, like in [10], the parameter ε is used to render the problem Lipschitz, and the parameter ρ to make it regular.

We treat in details the case of irregular data given along the characteristic curve $C = (Ox)$ and along a monotonic curve γ of equation $x = g(y)$. We add some remarks for the case of regular data.

The classical successive approximations technique used in [8], [9] permits to obtain, for each λ, a global solution u_λ to (P_λ). Using the precise estimates given subsection 3.5, we show that the class of $(u_\lambda)_\lambda$ in $\mathscr{A}(\mathbb{R}^2)$ is the expected solution u of the generalized problem (P_{gen}). Thus, we obtain a global generalized solution, when the classical smooth solutions often break down in finite time as it is pointed out in [20]. We show that this solution is unique in the constructed algebra. However, the generalized problem (P_{gen}), and obviously its solution, a priori depend on the choice of the cutoff functions and, in the case of irregular data, on the family of mollifiers. With regard to the regularization, we show that this solution depends solely on the class of the cutoff functions as a generalized function, not on the particular representative. In the case of irregular data, the solution of the problem (P_{gen}) depends on the family of mollifiers but not on a class of that family.

Moreover, if the initial problem $\left(P_{form}\right)$ admits a smooth solution u satisfying appropriate growth estimates on some open subset Ω of $\mathbb{R}^2$, then this solution and the generalized one are equal in a meaning given in Theorem 13. So the theory of generalized functions appears as the natural continuation of the classical theory of functions and distributions. In the example we take advantage of our results to give a new approach of a blow-up problem. The local classical solution extends to a global generalized solution which absorbs the blow-up.

2. Algebras of generalized functions

2.1. The presheaves of $(\mathscr{C},\mathscr{E},\mathscr{P})$-algebras

2.1.1. Definitions

We refer the reader to [7], [12], [13], [14], [15] for more details. Take

- Λ a set of indices;
- A a solid subring of the ring $\mathbb{K}^\Lambda$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), that is A has the following stability property: whenever $(|s_\lambda|)_\lambda \leqslant (r_\lambda)_\lambda$ (i.e. for any λ, $|s_\lambda| \leqslant r_\lambda$) for any

pair $((s_\lambda)_\lambda,(r_\lambda)_\lambda) \in \mathbb{K}^\Lambda \times |A|$, it follows that $(s_\lambda)_\lambda \in A$, with $|A| = \{(|r_\lambda|)_\lambda : (r_\lambda)_\lambda \in A\}$;

- I_A an solid ideal of A with the same property;
- $\mathscr{E}$ a sheaf of $\mathbb{K}$-topological algebras on a topological space X, such that for any open set Ω in X, the algebra $\mathscr{E}(\Omega)$ is endowed with a family $\mathscr{P}(\Omega) = (p_i)_{i\in I(\Omega)}$ of seminorms satisfying

$$\forall i \in I(\Omega),\ \exists (j,k,C) \in I(\Omega)\times I(\Omega)\times \mathbb{R}_+^*,\ \forall f,g \in \mathscr{E}(\Omega) : p_i(fg) \leqslant C p_j(f) p_k(g).$$

Assume that

- For any two open subsets Ω_1, Ω_2 of X such that $\Omega_1 \subset \Omega_2$, we have $I(\Omega_1) \subset I(\Omega_2)$ and if ρ_1^2 is the restriction operator $\mathscr{E}(\Omega_2) \to \mathscr{E}(\Omega_1)$, then, for each $p_i \in \mathscr{P}(\Omega_1)$, the seminorm $\widetilde{p}_i = p_i \circ \rho_1^2$ extends p_i to $\mathscr{P}(\Omega_2)$;
- For any family $\mathscr{F} = (\Omega_h)_{h\in H}$ of open subsets of X if $\Omega = \cup_{h\in H}\Omega_h$, then, for each $p_i \in \mathscr{P}(\Omega)$, $i \in I(\Omega)$, there exists a finite subfamily $\Omega_1,...,\Omega_{n(i)}$ of $\mathscr{F}$ and corresponding seminorms $p_1 \in \mathscr{P}(\Omega_1),...,p_{n(i)} \in \mathscr{P}(\Omega_{n(i)})$, such that, for each $u \in \mathscr{E}(\Omega)$,

$$p_i(u) \leqslant p_1\left(u_{|\Omega_1}\right) + ... + p_{n(i)}(u_{|\Omega_{n(i)}}).$$

Set

$$\mathscr{X}_{(A,\mathscr{E},\mathscr{P})}(\Omega) = \{(u_\lambda)_\lambda \in [\mathscr{E}(\Omega)]^\Lambda : \forall i \in I(\Omega),\ ((p_i(u_\lambda))_\lambda \in |A|\},$$
$$\mathscr{N}_{(I_A,\mathscr{E},\mathscr{P})}(\Omega) = \{(u_\lambda)_\lambda \in [\mathscr{E}(\Omega)]^\Lambda : \forall i \in I(\Omega),\ (p_i(u_\lambda))_\lambda \in |I_A|\},$$
$$\mathscr{C} = A/I_A.$$

One can prove that $\mathscr{X}_{(A,\mathscr{E},\mathscr{P})}$ is a sheaf of subalgebras of the sheaf $\mathscr{E}^\Lambda$ and $\mathscr{N}_{(I_A,\mathscr{E},\mathscr{P})}$ is a sheaf of ideals of $\mathscr{X}_{(A,\mathscr{E},\mathscr{P})}$ [13]. Moreover, the constant sheaf $\mathscr{X}_{(A,\mathbb{K},|.|)}/\mathscr{N}_{(I_A,\mathbb{K},|.|)}$ is exactly the sheaf $\mathscr{C} = A/I_A$.

DEFINITION 1. We call presheaf of $(\mathscr{C},\mathscr{E},\mathscr{P})$-algebra the factor presheaf of algebras over the ring $\mathscr{C} = A/I_A$

$$\mathscr{A} = \mathscr{X}_{(A,\mathscr{E},\mathscr{P})}/\mathscr{N}_{(I_A,\mathscr{E},\mathscr{P})}.$$

We denote by $[u_\lambda]$ the class in $\mathscr{A}(\Omega)$ defined by the representative

$$(u_\lambda)_{\lambda\in\Lambda} \in \mathscr{X}_{(A,\mathscr{E},\mathscr{P})}(\Omega).$$

2.1.2. Overgenerated rings

Let $B_p = \left\{\left(r_{n,\lambda}\right)_\lambda \in (\mathbb{R}_+^*)^\Lambda : n = 1,...,p\right\}$ and B be the subset of $(\mathbb{R}_+^*)^\Lambda$ obtained as rational functions with coefficients in $\mathbb{R}_+^*$ of elements in B_p as variables. Define

$$A = \left\{(a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \exists (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leqslant b_\lambda\right\}.$$

DEFINITION 2. In the above situation, we say that A is *overgenerated* by B_p (and it is easy to see that A is a solid subring of $\mathbb{K}^\Lambda$). If I_A is some solid ideal of A, we also say that $\mathscr{C} = A/I_A$ is *overgenerated* by B_p.

EXAMPLE 1. For example, as a "canonical" ideal of A, we can take

$$I_A = \left\{ (a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \forall (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leqslant b_\lambda \right\}.$$

REMARK 1. We can see that with this definition B is stable by inverse.

2.1.3. Relationship with distribution theory

Let Ω an open subset of $\mathbb{R}^n$. The space of distributions $\mathscr{D}'(\Omega)$ can be embedded into $\mathscr{A}(\Omega)$. If $\left(\theta_\rho\right)_{\rho \in (0,1]}$ is a family of mollifiers $\theta_\rho(x) = \frac{1}{\rho^n}\theta\left(\frac{x}{\rho}\right)$, $x \in \mathbb{R}^n$, $\int \theta(x)\,dx = 1$ and if $T \in \mathscr{D}'(\mathbb{R}^n)$, the convolution product family $\left(T * \theta_\rho\right)_\rho$ is a family of smooth functions slowly increasing in $1/\rho$. Then, taking ρ as a component of the multi-index $\lambda \in \Lambda$, we shall choose the subring A overgenerated by some B_p of $(\mathbb{R}_+^*)^\Lambda$ containing the family $(\rho)_\lambda$ [3], [18].

2.1.4. The association process

We assume that Λ is left-filtering for a given partial order relation $\prec$. We denote by Ω an open subset of X, E a given sheaf of topological $\mathbb{K}$-vector spaces containing $\mathscr{E}$ as a subsheaf, a a given map from Λ to $\mathbb{K}$ such that $(a(\lambda))_\lambda = (a_\lambda)_\lambda$ is an element of A. We also assume that

$$\mathscr{N}_{(I_A,\mathscr{E},\mathscr{P})}(\Omega) \subset \left\{ (u_\lambda)_\lambda \in \mathscr{X}_{(A,\mathscr{E},\mathscr{P})}(\Omega) : \lim_{E(\Omega),\Lambda} u_\lambda = 0 \right\}.$$

DEFINITION 3. We say that $u = [u_\lambda]$ and $v = [v_\lambda] \in \mathscr{E}(\Omega)$ are a-E associated if

$$\lim_{E(\Omega),\Lambda} a_\lambda (u_\lambda - v_\lambda) = 0.$$

That is to say, for each neighborhood V of 0 for the E -topology, there exists $\lambda_0 \in \Lambda$ such that $\lambda \prec \lambda_0 \implies a_\lambda (u_\lambda - v_\lambda) \in V$. We write

$$u \underset{E(\Omega)}{\overset{a}{\sim}} v.$$

REMARK 2. We can also define an association process between $u = [u_\lambda]$ and $T \in \mathscr{E}(\Omega)$ by writing simply

$$u \sim T \iff \lim_{E(\Omega),\Lambda} u_\lambda = T.$$

Taking $E = \mathscr{D}'$, $\mathscr{E} = \mathrm{C}^\infty$, $\Lambda = (0,1]$, we recover the association process defined in the literature (J.-F. Colombeau [1]).

2.2. $\mathscr{D}'$ -singular support

Assume that

$$\mathscr{N}^{\mathscr{A}}_{\mathscr{D}'}(\Omega)=\left\{(u_\lambda)_\lambda\in\mathscr{X}(\Omega):\lim_{\lambda\to 0}u_\lambda=0\text{ in }\mathscr{D}'(\Omega)\right\}\supset\mathscr{N}(\Omega).$$

Set

$$\mathscr{D}'_{\mathscr{A}}(\Omega)=\left\{[u_\lambda]\in\mathscr{A}(\Omega):\exists T\in\mathscr{D}'(\Omega),\ \lim_{\lambda\to 0}(u_\lambda)=T\text{ in }\mathscr{D}'(\Omega)\right\}.$$

$\mathscr{D}'_{\mathscr{A}}(\Omega)$ is clearly well defined because the limit is independent of the chosen representative; indeed, if $(i_\lambda)_\lambda\in\mathscr{N}(\Omega)$ we have $\lim\limits_{\substack{\lambda\to 0\\ \mathscr{D}'(\mathbb{R})}} i_\lambda=0$.

$\mathscr{D}'_{\mathscr{A}}(\Omega)$ is an $\mathbb{R}$-vector subspace of $\mathscr{A}(\Omega)$. Therefore we can consider the set $\mathscr{O}_{D'_{\mathscr{A}}}$ of all x having a neighborhood V on which u is associated to a distribution:

$$\mathscr{O}_{D'_{\mathscr{A}}}(u)=\left\{x\in\Omega:\exists V\in\mathscr{V}(x),\ u|_V\in\mathscr{D}'_{\mathscr{A}}(V)\right\},$$

$\mathscr{V}(x)$ being the set of all neighborhoods of x.

DEFINITION 4. The $\mathscr{D}'$ -singular support of $u\in\mathscr{A}(\Omega)$, denoted $\operatorname{singsupp}_{\mathscr{D}'}(u)=S^{\mathscr{A}}_{\mathscr{D}'_{\mathscr{A}}}(u)$, is the set

$$S^{\mathscr{A}}_{\mathscr{D}'_{\mathscr{A}}}(u)=\Omega\backslash\mathscr{O}_{D'_{\mathscr{A}}}(u).$$

2.3. Algebraic framework for our problem

Set $\mathscr{E}=\mathrm{C}^\infty$, $X=\mathbb{R}^d$ for $d=1,2$, $E=\mathscr{D}'$ and Λ a set of indices, $\lambda\in\Lambda$. For any open set Ω, in $\mathbb{R}^d$, $\mathscr{E}(\Omega)$ is endowed with the $\mathscr{P}(\Omega)$ topology of uniform convergence of all derivatives on compact subsets of Ω. This topology may be defined by the family of the seminorms

$$P_{K,l}(u_\lambda)=\sup_{|\alpha|\leqslant l}P_{K,\alpha}(u_\lambda)\ \text{ with } P_{K,\alpha}(u_\lambda)=\sup_{x\in K}|D^\alpha u_\lambda(x)|,\ K\Subset\Omega$$

and $D^\alpha=\dfrac{\partial^{\alpha_1+\ldots+\alpha_d}}{\partial z_1^{\alpha_1}\ldots\partial z_d^{\alpha_d}}$ for $z=(z_1,\ldots,z_d)\in\Omega$, $l\in\mathbb{N}$, $\alpha=(\alpha_1,\ldots,\alpha_d)\in\mathbb{N}^d$. Let A be a subring of the ring $\mathbb{R}^\Lambda$ of family of reals with the usual laws. We consider a solid ideal I_A of A. Then we have

$$\begin{aligned}
&\mathscr{X}(\Omega)=\{(u_\lambda)_\lambda\in[\mathrm{C}^\infty(\Omega)]^\Lambda:\forall K\Subset\Omega,\forall l\in\mathbb{N},\ \left(P_{K,l}(u_\lambda)\right)_\lambda\in|A|\},\\
&\mathscr{N}(\Omega)=\{(u_\lambda)_\lambda\in[\mathrm{C}^\infty(\Omega)]^\Lambda:\forall K\Subset\Omega,\forall l\in\mathbb{N},\ \left(P_{K,l}(u_\lambda)\right)_\lambda\in|I_A|\},\\
&\mathscr{A}(\Omega)=\mathscr{X}(\Omega)/\mathscr{N}(\Omega).
\end{aligned}$$

The generalized derivation $D^\alpha:u(=[u_\lambda])\mapsto D^\alpha u=[D^\alpha u_\lambda]$ provides $\mathscr{A}(\Omega)$ with a differential algebraic structure.

EXAMPLE 2. Set $\Lambda = (0,1]$. Consider

$$A = \mathbb{R}_M^\Lambda = \left\{ (m_\lambda)_\lambda \in \mathbb{R}^\Lambda : \exists p \in \mathbb{R}_+^*,\ \exists C \in \mathbb{R}_+^*,\ \exists \mu \in\ (0,1],\ \forall \lambda \in\ (0,\mu],\ |m_\lambda| \leqslant C\lambda^{-p} \right\}$$

and the ideal

$$I_A = \left\{ (m_\lambda)_\lambda \in \mathbb{R}^\Lambda : \forall q \in \mathbb{R}_+^*,\ \exists D \in \mathbb{R}_+^*,\ \exists \mu \in\ (0,1],\ \forall \lambda \in\ (0,\mu],\ |m_\lambda| \leqslant D\lambda^{q} \right\}.$$

In this case we denote $\mathscr{X}^s(\Omega) = \mathscr{X}(\Omega)$ and $\mathscr{N}^s(\Omega) = \mathscr{N}(\Omega)$. The sheaf of factor algebras $\mathscr{G}(\cdot) = \mathscr{X}^s(\cdot)/\mathscr{N}^s(\cdot)$ is called the sheaf of *simplified Colombeau algebras* [1].

We have the analogue of theorem 1.2.3. of [11] for $(\mathscr{C},\mathscr{E},\mathscr{P})$-algebras. We suppose here that Λ is left filtering and give this proposition for $\mathscr{A}(\mathbb{R}^2)$, although it is valid in more general situations.

PROPOSITION 1. *Assume that there exists* $(a_\lambda)_\lambda \in B$ *with* $\lim_\Lambda a_\lambda = 0$. *Consider* $(u_\lambda)_\lambda \in \mathscr{X}(\mathbb{R}^2)$ *such that*

$$\forall K \Subset \mathbb{R}^2,\ \ (P_{K,0}(u_\lambda))_\lambda \in |I_A|.$$

Then $(u_\lambda)_\lambda \in \mathscr{N}(\mathbb{R}^2)$.

We refer the reader to [4], [7] for a detailed proof.

2.4. Some regularizing conditions

2.4.1. Generalized operator associated to a stability property

When $\Lambda = \Lambda_1 \times \Lambda_2$, we denote by $\lambda = (\varepsilon,\rho)$ an element of Λ and we shall set $\varepsilon = \mu(\lambda)$, $\rho = \nu(\lambda)$. When $\Lambda = \Lambda_1$, we denote by $\lambda = \varepsilon$ an element of Λ and we shall set $\varepsilon = \mu(\lambda) = \lambda$.

If we use the notation λ, we also use $\mu(\lambda)$ and $\nu(\lambda)$ in the same expression, else we use ε and ρ.

DEFINITION 5. Let Ω be an open subset of $\mathbb{R}^2$, $\Omega' = \Omega \times \mathbb{R} \subset \mathbb{R}^3$. Let $F_{\mu(\lambda)} \in \mathrm{C}^\infty(\Omega',\mathbb{R})$. We say that the algebra $\mathscr{A}(\Omega)$ is *stable under the family* $\left(F_{\mu(\lambda)}\right)_\lambda$ if the following two conditions are satisfied:

- For each $K \Subset \mathbb{R}^2$ $l \in \mathbb{N}$ and $(u_\lambda)_\lambda \in \mathscr{X}(\Omega)$, there is a positive finite sequence C_0 ,..., C_l such that

$$P_{K,l}(F_{\mu(\lambda)}(\cdot,\cdot,u_\lambda)) \leqslant \sum_{i=0}^{l} C_i P_{K,l}^i(u_\lambda).$$

- For each $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$, $(v_\lambda)_\lambda$ and $(u_\lambda)_\lambda \in \mathscr{X}(\Omega)$, there is a positive finite sequence D_1,..., D_l such that

$$P_{K,l}(F_{\mu(\lambda)}(\cdot,\cdot,v_\lambda) - F_{\mu(\lambda)}(\cdot,\cdot,u_\lambda)) \leqslant \sum_{j=1}^{l} D_j P_{K,l}^{j}(v_\lambda - u_\lambda).$$

REMARK 3. If $\mathscr{A}(\Omega)$ is stable under $\left(F_{\mu(\lambda)}\right)_\lambda$, then for all $(u_\lambda)_\lambda \in \mathscr{X}(\Omega)$ and $(i_\lambda)_\lambda \in \mathscr{N}(\Omega)$, we have

$$\left(F_{\mu(\lambda)}(\cdot,\cdot,u_\lambda)\right)_\lambda \in \mathscr{X}(\Omega);\ \left(F_{\mu(\lambda)}(\cdot,\cdot,u_\lambda + i_\lambda) - F_{\mu(\lambda)}(\cdot,\cdot,u_\lambda)\right)_\lambda \in \mathscr{N}(\Omega).$$

Set $f \in \mathrm{C}^\infty\left(\mathbb{R}^2\right)$, we define

$$\begin{aligned}
&\mathrm{C}^\infty\left(\mathbb{R}^2\right) \to \mathrm{C}^\infty\left(\mathbb{R}^2\right),\\
&f \mapsto H_\lambda\left(f\right) = F_{\mu(\lambda)}\left(\cdot,\cdot,f\right),\\
&H_\lambda\left(f\right) = F_{\mu(\lambda)}\left(\cdot,\cdot,f\right) : (x,y) \mapsto F_{\mu(\lambda)}\left(x,y,f\left(x,y\right)\right).
\end{aligned}$$

Clearly, the family $(H_\lambda)_\lambda$ maps $\left(\mathrm{C}^\infty\left(\mathbb{R}^2\right)\right)^\Lambda$ into $\left(\mathrm{C}^\infty\left(\mathbb{R}^2\right)\right)^\Lambda$ and allows to define a map from $\mathscr{A}(\mathbb{R}^2)$ into $\mathscr{A}(\mathbb{R}^2)$. For $u = [u_\lambda] \in \mathscr{A}(\mathbb{R}^2)$, $\left(\left[F_{\mu(\lambda)}(.,.,u_\lambda)\right]\right)$ is a well defined element of $\mathscr{A}(\mathbb{R}^2)$ (i.e. not depending on the representative $(u_\lambda)_\lambda$ of u). This leads to the following definition [7]:

DEFINITION 6. If $\mathscr{A}(\mathbb{R}^2)$ if stable under $\left(F_{\mu(\lambda)}\right)_\lambda$, the operator

$$\mathscr{F} : \mathscr{A}(\mathbb{R}^2) \to \mathscr{A}(\mathbb{R}^2),\ \ u = [u_\lambda] \mapsto \left[F_{\mu(\lambda)}(.,.,u_\lambda)\right] = \left[H_\lambda\left(u_\lambda\right)\right]$$

is called the *generalized operator associated to the family* $\left(F_{\mu(\lambda)}\right)_\lambda$.

DEFINITION 7. Let $F \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R})$ and $\left(f_{\mu(\lambda)}\right)_{\mu(\lambda)} \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_1}$, we define

$$F_{\mu(\lambda)}(x,y,z) = F(x,y,zf_{\mu(\lambda)}(z)).$$

The family $\left(F_{\mu(\lambda)}\right)_\lambda$ is called the *family associated to* F *via the family* $\left(f_{\mu(\lambda)}\right)_{\mu(\lambda)}$. If $\mathscr{A}(\mathbb{R}^2)$ is stable under $\left(F_{\mu(\lambda)}\right)_\lambda$, the operator

$$\mathscr{F} : \mathscr{A}(\mathbb{R}^2) \to \mathscr{A}(\mathbb{R}^2),\ \ u = [u_\lambda] \mapsto \left[F_{\mu(\lambda)}(.,.,u_\lambda)\right] = \left[H_\lambda\left(u_\lambda\right)\right]$$

is called the *generalized operator associated to* F *via the family* $\left(f_{\mu(\lambda)}\right)_{\mu(\lambda)}$.

2.4.2. Generalized restriction mappings

Set $g \in \mathrm{C}^{\infty}(\mathbb{R})$. We define L_g by

$$\begin{aligned} &\mathrm{C}^{\infty}\left(\mathbb{R}^2\right) \mapsto \mathrm{C}^{\infty}(\mathbb{R}) \\ &f \mapsto (y \mapsto f(g(y),y)) \end{aligned}$$

and R_g by

$$\begin{aligned} &\mathrm{C}^{\infty}\left(\mathbb{R}^2\right) \mapsto \mathrm{C}^{\infty}(\mathbb{R}) \\ &f \mapsto (x \mapsto f(x,g(x))). \end{aligned}$$

DEFINITION 8. The smooth function g is *compatible with first side restriction* (resp. *second side restriction*) if

$$\forall (u_\lambda)_\lambda \in \mathscr{X}(\mathbb{R}^2),\ (u_\lambda(g(\cdot),\cdot))_\lambda \in \mathscr{X}(\mathbb{R});\ \forall (i_\lambda)_\lambda \in \mathscr{N}(\mathbb{R}^2),\ (i_\lambda(g(\cdot),\cdot))_\lambda \in \mathscr{N}(\mathbb{R}),$$

(resp.

$$\forall (u_\lambda)_\lambda \in \mathscr{X}(\mathbb{R}^2),\ (u_\lambda(\cdot,g(\cdot)))_\lambda \in \mathscr{X}(\mathbb{R}); \forall (i_\lambda)_\lambda \in \mathscr{N}(\mathbb{R}^2),\ (i_\lambda(\cdot,g(\cdot)))_\lambda \in \mathscr{N}(\mathbb{R})).$$

Clearly, if $u = [u_\lambda] \in \mathscr{A}(\mathbb{R}^2)$ then $[u_\lambda(g(\cdot),\cdot)]$ (resp. $[u_\lambda(\cdot,g(\cdot))]$) is a well defined element of $\mathscr{A}(\mathbb{R})$ (i.e. not depending on the representative of u). This leads to the following:

DEFINITION 9. If the smooth function g is compatible with first side restriction (resp. second side restriction), the mapping

$$\mathscr{L}_g : \mathscr{A}(\mathbb{R}^2) \to \mathscr{A}(\mathbb{R}), \quad u = [u_\lambda] \mapsto [u_\lambda(g(\cdot),\cdot)] = [L_g(u_\lambda)]$$

$$(\text{resp. } \mathscr{R}_g : \mathscr{A}\left(\mathbb{R}^2\right) \to \mathscr{A}(\mathbb{R}), \quad u = [u_\lambda] \mapsto [u_\lambda(\cdot,g(\cdot))] = [R_g(u_\lambda)])$$

is called the *generalized first side restriction* (resp. *second side restriction*) *mapping* associated to the function g.

PROPOSITION 2. If function g is c-bounded for each $K \Subset \mathbb{R}$ it exists $K' \Subset \mathbb{R}$ such that $g(K) \subset K'$) then the function g is compatible with first side restriction (resp. second side restriction).

Proof. Take $(u_\lambda)_\lambda$ (resp. $(i_\lambda)_\lambda$) in $\mathscr{X}(\mathbb{R}^2)$ (resp. $\mathscr{N}(\mathbb{R}^2)$) and set $v_\lambda(y) = u_\lambda(g(y),y)$. We have

$$\begin{aligned} &p_{K,0}(v_\lambda) \leqslant p_{K'\times K,0}(u_\lambda) \\ &P_{K,1}(v_\lambda) \leqslant p_{K'\times K,(1,0)}(u_\lambda)\, p_{K,1}(g) + p_{K'\times K,(0,1)}(u_\lambda). \end{aligned}$$

By induction we can see that for each $K \Subset \mathbb{R}$, and each $l \in \mathbb{N}$, $p_{K,l}(v_\lambda)$ is estimated by sums or products of terms like $p_{K'\times K,(n,m)}(u_\lambda)$ for $n+m \leqslant l$, or $p_{K,k}(g)$ for $k \leqslant l$, then $p_{K,l}(v_\lambda)$ is in $|A|$. Similarly, setting $j_\lambda(t) = i_\lambda(g(y),y)$ leads to $p_{K,l}(j_\lambda) \in |I_A|$. Then $(u_\lambda(g(\cdot),\cdot))_\lambda$ (resp. $i_\lambda(g(\cdot),\cdot)$) belongs to $\mathscr{X}(\mathbb{R})$ (resp. $\mathscr{N}(\mathbb{R})$).

3. A non Lipschitz Goursat problem

We study the differential Goursat problem formally written as

$$(P_{form})\begin{cases}\dfrac{\partial^2}{\partial x\partial y}u=F(\cdot,\cdot,u),\\ u|_{(Ox)}=s,\\ u|_\gamma=t\end{cases}$$

where F, a nonlinear function of its arguments, may be non Lipschitz (in u), γ is the monotonic curve of equation $x=g(y)$, the data s, t may be as irregular as distributions. We don't have a classical surrounding in which we can pose (and a fortiori solve) the problem.

We treat in details the case of irregular data given along the characteristic curve $C=(Ox)$ and along the curve γ, we add some remarks for the case of regular data.

3.1. Cut off procedure

Let $(r_\varepsilon)_\varepsilon$ be in $\mathbb{R}_*^{(0,1]}$ such that $r_\varepsilon>0$ and $\lim\limits_{\varepsilon\to 0}r_\varepsilon=+\infty$. Set $L_\varepsilon=[-r_\varepsilon,r_\varepsilon]$. Consider a family of smooth one-variable functions $(f_\varepsilon)_\varepsilon$ such that

$$\sup_{z\in L_\varepsilon}|f_\varepsilon(z)|=1,\ f_\varepsilon(z)=\begin{cases}0 & \text{if } |z|\geqslant r_\varepsilon,\\ 1 & \text{if } -r_\varepsilon+1\leqslant z\leqslant r_\varepsilon-1,\end{cases}\qquad (H1)$$

and $\dfrac{\partial^n f_\varepsilon}{\partial z^n}$ is bounded on L_ε for any integer n, $n>0$. Set

$$\sup_{z\in L_\varepsilon}\left|\frac{\partial^n f_\varepsilon}{\partial z^n}(z)\right|=M_n.$$

Let $\phi_\varepsilon(z)=zf_\varepsilon(z)$. We approximate the function F by the family of functions $(F_\varepsilon)_\varepsilon=\left(F_{\mu(\lambda)}\right)_{\mu(\lambda)}$ defined by

$$F_\varepsilon(x,y,z)=F(x,y,\phi_\varepsilon(z)).$$

3.2. Construction of $\mathscr{A}\left(\mathbb{R}^2\right)$

We recall that $\lambda=(\mu(\lambda),\nu(\lambda))=(\varepsilon,\rho)\in\Lambda_1\times\Lambda_2=\Lambda$, $\Lambda_1=\Lambda_2=(0,1]$ where the parameter ρ is used to regularize the distributions s and t, the more general case. Consider the previous family $(r_\varepsilon)_\varepsilon$. We take

$$\begin{cases}\mathscr{C}=A/I_A \text{ the ring overgenerated by the following elements of } \mathbb{R}_*^{(0,1]\times(0,1]}\\ \qquad (\mu(\lambda))_\lambda,(\nu(\lambda))_\lambda,\left(r_{\mu(\lambda)}\right)_\lambda,(e^{r_{\mu(\lambda)}})_\lambda.\end{cases}\qquad (H2)$$

Then $\mathscr{A}(\mathbb{R}^2)=\mathscr{X}(\mathbb{R}^2)/\mathscr{N}(\mathbb{R}^2)$ is built on the ring $\mathscr{C}$ of generalized constants with $(\mathscr{E},\mathscr{P})=\left(\mathrm{C}^\infty(\mathbb{R}^2),\left(P_{K,l}\right)_{K\Subset\mathbb{R}^2,l\in\mathbb{N}}\right)$. In the same way $\mathscr{A}(\mathbb{R})=\mathscr{X}(\mathbb{R})/\mathscr{N}(\mathbb{R})$ is built on $\mathscr{C}$ with $(\mathscr{E},\mathscr{P})=\left(\mathrm{C}^\infty(\mathbb{R}),\left(P_{K,l}\right)_{K\Subset\mathbb{R},l\in\mathbb{N}}\right)$.

3.3. Stability of $\mathscr{A}(\mathbb{R}^2)$

PROPOSITION 3. *Set* $S_n = \left\{\alpha \in \mathbb{N}^3 : |\alpha| = n\right\}$ *when* $n \in \mathbb{N}^*$. *Let* $F \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R})$, F_ε *defined as above in Section* 3.1. *Assume that*

$$\forall \varepsilon \in (0,1], \forall (x,y) \in \mathbb{R}^2, F_\varepsilon(x,y,0) = 0, \tag{$H3$}$$

$$\exists p > 0, \forall n \in \mathbb{N}, \exists\, c_n > 0, \forall \varepsilon \in (0,1], \forall K \Subset \mathbb{R}^2,$$
$$\sup_{(x,y)\in K;\, z\in\mathbb{R};\alpha\in Sn} |D^\alpha F_\varepsilon(x,y,z)| \leqslant c_n r_\varepsilon^p, \tag{$H4$}$$

then $\mathscr{A}\left(\mathbb{R}^2\right)$ *is stable under the family* $\left(F_{\mu(\lambda)}\right)_\lambda$.

We refer the reader to [10] for a detailed proof.

COROLLARY 4. *Set* $F(x,y,z) = G(z) = z^p$, $G_\varepsilon(z) = F_\varepsilon(x,y,z)$, *then* $\mathscr{A}(\mathbb{R}^2)$ *is stable under* $\left(G_{\mu(\lambda)}\right)_\lambda$.

Proof. We have $|G_\varepsilon(z)| = \left|z^p f_\varepsilon^p(z)\right| \leqslant r_\varepsilon^p$, then $\sup_{(x,y)\in\mathbb{R};z\in\mathbb{R}} |G_\varepsilon(z)| \leqslant r_\varepsilon^p$. As $\phi_\varepsilon(z) = z f_\varepsilon(z)$, we obtain

$$\frac{\partial^n \phi_\varepsilon}{\partial z^n}(z) = z\frac{\partial^n f_\varepsilon}{\partial z^n}(z) + n\frac{\partial^{n-1} f_\varepsilon}{\partial z^{n-1}}(z).$$

Thus

$$\left|\frac{\partial^n \phi_\varepsilon}{\partial z^n}(z)\right| \leqslant r_\varepsilon M_n + nM_{n-1} \leqslant \alpha_n r_\varepsilon,$$

where $\alpha_n = 2\max(M_n; nM_{n-1})$. Set $w(z) = z^p$, then $\dfrac{\partial^m w}{\partial z^m}(z) = \left(\prod_{i=0}^{i=m-1}(p-i)\right) z^{p-m}$ for $1 \leqslant m \leqslant p$. According Francesco Faà di Bruno's formula, the nth order derivative of $G_\varepsilon = w \circ \phi_\varepsilon$ can be written

$$\frac{\partial^n G_\varepsilon}{\partial z^n} = \sum_{m=1}^{n} \sum_{\substack{i_1 \geqslant \ldots \geqslant i_m \\ i_1+\ldots+i_m=n}} t_{i_1,\ldots,i_m} w^{(m)} \circ \phi_\varepsilon \prod_{k=1}^{m} \phi_\varepsilon^{(i_k)},$$

where the coefficients $t_{i_1,\ldots,i_m}$ are integers. Then we get

$$\left|\frac{\partial^n G_\varepsilon}{\partial z^n}(z)\right| \leqslant \sum_{m=1}^{p} \sum_{\substack{i_1 \geqslant \ldots \geqslant i_m \\ i_1+\ldots+i_m=n}} t_{i_1,\ldots,i_m} \left(\prod_{i=0}^{i=m-1}(p-i)\right) r_\varepsilon^{p-m} \prod_{k=1}^{m} \alpha_{i_k} r_\varepsilon \leqslant c_n r_\varepsilon^p$$

where c_n is independent of ε . Then assumptions *(*$H3$*), (*$H4$*)* are fulfilled.

3.4. A generalized differential problem associated to the formal one

Our goal is to give a meaning to the differential Goursat problem formally written as $\left(P_{form}\right)$.

As the data s and t are as irregular as distributions, we set

$$\varphi_\rho = s * \theta_\rho \text{ and } \varphi = \left[\varphi_\rho\right], \tag{$d1$}$$

$$\psi_\rho = t * \theta_\rho \text{ and } \psi = \left[\psi_\rho\right], \tag{$d2$}$$

where $\left(\theta_\rho\right)_\rho$ is a chosen family of mollifiers. Then the data φ, ψ belong to $\mathscr{A}(\mathbb{R})$ and u is searched in the algebra $\mathscr{A}\left(\mathbb{R}^2\right)$.

Let $(f_\varepsilon)_\varepsilon \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_1}$ and $\mathscr{F}$ the generalized operator associated to F *via* the family $(f_\varepsilon)_\varepsilon$ in Definition 7. Let $\mathscr{R}_\theta$ and $\mathscr{L}_g$ given by Definition 9 with $\theta(x) = 0$.

The problem associated to $\left(P_{form}\right)$ can be written as the well formulated one:

$$(P_{gen}) \begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = \mathscr{F}(u), \\ \mathscr{R}_\theta(u) = \varphi, \\ \mathscr{L}_g(u) = \psi. \end{cases}$$

In terms of representatives, and thanks to the stability and restriction hypothesis, solving (P_{gen}) leads to find a family $(u_\lambda)_\lambda \in \mathscr{X}(\mathbb{R}^2)$ such that

$$\begin{cases} \dfrac{\partial^2 u_\lambda}{\partial x \partial y}(x,y) - F_{\mu(\lambda)}(x,y,u_\lambda(x,y)) = i_\lambda(x,y), \\ u_\lambda(x,0) - \varphi_{\nu(\lambda)}(x) = j_{\nu(\lambda)}(x), \\ u_\lambda(g(y),y) - \psi_{\nu(\lambda)}(y) = l_{\nu(\lambda)}(y), \end{cases}$$

where $(i_\lambda)_\lambda \in \mathscr{N}\left(\mathbb{R}^2\right)$, $\left(j_{\nu(\lambda)}\right)_\lambda$, $\left(l_{\nu(\lambda)}\right)_\lambda \in \mathscr{N}(\mathbb{R})$. Suppose we can find $u_\lambda \in \mathrm{C}^\infty\left(\mathbb{R}^2\right)$ verifying

$$(P_\lambda) \begin{cases} \dfrac{\partial^2 u_\lambda}{\partial x \partial y}(x,y) = F_{\mu(\lambda)}(x,y,u_\lambda(x,y)), \\ u_\lambda(x,0) = \varphi_{\nu(\lambda)}(x), \\ u_\lambda((g(y),y)) = \psi_{\nu(\lambda)}(y). \end{cases} \tag{1}$$

Then, if we can prove that $(u_\lambda)_\lambda \in \mathscr{X}(\mathbb{R}^2)$, $u = [u_\lambda]$ is a solution of (P_{gen}).

REMARK 4. Uniqueness in the algebra $\mathscr{A}(\mathbb{R}^2)$. Let $v = [v_\lambda]$ another solution to (P_{gen}). There are $(i_\lambda)_\lambda \in \mathscr{N}\left(\mathbb{R}^2\right)$, $\left(\alpha_{\nu(\lambda)}\right)_\lambda$, $\left(\beta_{\nu(\lambda)}\right)_\lambda \in \mathscr{N}(\mathbb{R})$, such that

$$\begin{cases} \dfrac{\partial^2 v_\lambda}{\partial x \partial y}(x,y) = F_{\mu(\lambda)}(x,y,v_\lambda(x,y)) + i_\lambda(x,y), \\ v_\lambda(x,0) = \varphi_{\nu(\lambda)}(x) + \alpha_{\nu(\lambda)}(x), \\ v_\lambda(g(y),y) = \psi_{\nu(\lambda)}(y) + \beta_{\nu(\lambda)}(y). \end{cases}$$

The uniqueness of the solution to (P_{gen}) will be the consequence of $(w_\lambda)_\lambda = (v_\lambda - u_\lambda)_\lambda \in \mathscr{N}(\mathbb{R}^2)$.

REMARK 5. Dependence on some regularizing family. The problem (P_{gen}) itself, so a solution of it, a priori depends on the family of cutoff functions and, in the case of irregular data, on the family of mollifiers. If $\left(\theta_\rho\right)_{\rho\in\Lambda_2}$ and $\left(\tau_\rho\right)_{\rho\in\Lambda_2}$ are families of mollifiers in $\mathscr{D}(\mathbb{R})$ and $T \in \mathscr{D}'(\mathbb{R})$, it is well known that generally $\left[T*\theta_\rho\right] \neq \left[T*\tau_\rho\right]$ in the Colombeau simplified algebra even if $\left[\theta_\rho\right] = \left[\tau_\rho\right]$ in these algebra. Therefore, in the case of irregular data the solution of problem (P_{gen}) in some Colombeau algebra depends on the family of mollifiers $\left(\theta_\rho\right)_\rho$ but not on a class of that family. We have associated the generalized operator $\mathscr{F}$ to F *via* the family $(f_\varepsilon)_\varepsilon$. Let $(h_\varepsilon)_\varepsilon \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_1}$ another family representative of the class $[f_\varepsilon] = f$ in a meaning specified in Section 5 and leading to another generalized operator $\mathscr{H}$ associated to F. We can prove that in fact $\mathscr{H} = \mathscr{F}$, that is to say problem (P_{gen}) only depends on the class f of cutoff functions.

3.5. Estimates for a parametrized regular problem

To solve the problem (P_{gen}) associated to $\left(P_{form}\right)$ we can consider (as it is done in Subsection 3.4) the family of problems $(P_\lambda)_\lambda$. First we are going to prove that (P_λ) has a unique smooth solution under the following assumption

$$\begin{cases} \text{a) } g \in \mathrm{C}^\infty(\mathbb{R}),\, g' \geqslant 0,\, g(\mathbb{R}) = \mathbb{R} \\ \text{b) } F_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R}),\, \forall K \Subset \mathbb{R}^2,\ \sup\limits_{(x,y)\in K; z\in\mathbb{R}} \left|\partial_z F_\varepsilon(x,y,z)\right| = m_{K,\varepsilon} < +\infty \\ \text{c) } \varphi_\rho \text{ and } \psi_\rho \in \mathrm{C}^\infty(\mathbb{R}),\, \psi_\rho(0) = \varphi_\rho(g(0)). \end{cases} \qquad (H)$$

Following [8], one can prove that (P_λ) is equivalent to the integral formulation

$$\left(P'_\lambda\right) : u_\lambda(x,y) = u_{0,\lambda}(x,y) + \iint_{D(x,y,g_\eta)} F_{\mu(\lambda)}(\xi,\zeta,u_\lambda(\xi,\zeta))d\xi d\zeta \qquad (2)$$

where $u_{0,\lambda}(x,y) = \psi_{\nu(\lambda)}(y) + \varphi_{\nu(\lambda)}(x) - \varphi_{\nu(\lambda)}(g(y))$, with

$$D(x,y,g) = \begin{cases} \{(\xi,\eta) :\ g(y) \leqslant \xi \leqslant x,\ 0 \leqslant \eta \leqslant y\} \text{ if } g(y) \leqslant x \text{ and } 0 \leqslant y, \\ \{(\xi,\eta) : x \leqslant \xi \leqslant g(y),\ 0 \leqslant \eta \leqslant y\} \text{ if } g(y) \geqslant x \text{ and } 0 \leqslant y, \\ \{(\xi,\eta) : x \leqslant \xi \leqslant g(y),\ y \leqslant \eta \leqslant 0\} \text{ if } g(y) \geqslant x \text{ and } y \leqslant 0, \\ \{(\xi,\eta) :\ g(y) \leqslant \xi \leqslant x,\ y \leqslant \eta \leqslant 0\} \text{ if } g(y) \leqslant x \text{ and } y \leqslant 0. \end{cases}$$

THEOREM 5. *Under Assumption (H), problem (P_λ) has a unique solution in $C^\infty(\mathbb{R}^2)$.*

We refer the reader to [8], [10] for a detailed proof. The main idea consists in a Picard's procedure to define a sequence of successive approximations.

$$u_{n,\lambda}(x,y) = u_{0,\lambda}(x,y) + \iint_{D(x,y,g)} F_{\mu(\lambda)}(\xi,\zeta,u_{n-1,\lambda}(\xi,\zeta))d\xi d\zeta.$$

From the assumptions, putting $v_{n,\lambda} = u_{n,\lambda} - u_{n-1,\lambda}$, we can prove that

$$\left\|v_{n,\lambda}\right\|_{\infty,K_{a,\eta}} \leqslant \frac{\Phi_{a,\lambda}}{m_{a,\mu(\lambda)}} \frac{\left[m_{a,\mu(\lambda)}\left(g(a)-g(-a)\right)a\right)]^n}{n!}$$

when $K_a = [g(-a), g(a)] \times [-a, a]$, with

$$m_{K_a,\mu(\lambda)} = m_{a,\mu(\lambda)} = \sup_{(x,y)\in K_a;\, t\in\mathbb{R}} \left| \frac{\partial F_{\mu(\lambda)}}{\partial z}(x,y,t) \right|$$

and

$$\Phi_{a,\lambda} = \left\| F_{\mu(\lambda)}(\cdot,\cdot,0) \right\|_{\infty,K_a} + m_{a,\mu(\lambda)} \left\| u_{0,\lambda} \right\|_{\infty,K_a}.$$

Finally the sequence $u_{n,\lambda}$ converges uniformly on any compact set to

$$u_\lambda = u_{0,\lambda} + \sum_{n\geqslant 1} v_{n,\lambda}$$

which verifies $\left(P'_\lambda\right)$. Gronwall's lemma gives the uniqueness of u_λ. Moreover, we have the estimate

$$\|u_\lambda\|_{\infty,K} \leqslant \|u_\lambda\|_{\infty,K_a} \leqslant \left\|u_{0,\lambda}\right\|_{\infty,K_a,} + \frac{\Phi_{a,\lambda}}{m_{a,\mu(\lambda)}} \exp[m_{a,\mu(\lambda)}(g(a)-g(-a))a]. \tag{3}$$

PROPOSITION 6. *If $F(x,y,z) = G(z) = z^p$ then problem (P_λ) has a unique solution in $C^\infty(\mathbb{R}^2)$.*

Proof. We have

$$F_{\mu(\lambda)}(x,y,u_\lambda(x,y)) = G_{\mu(\lambda)}(u_\lambda(x,y)) = \left(\phi_{\mu(\lambda)}(u_\lambda(x,y))\right)^p.$$

We compute $\dfrac{\partial G_{\mu(\lambda)}}{\partial z}(z) = p\phi^{p-1}_{\mu(\lambda)}(z)\phi'_{\mu(\lambda)}(z)$. Thus

$$\left|\frac{\partial G_{\mu(\lambda)}}{\partial z}(z)\right| \leqslant pr^{p-1}_{\mu(\lambda)} \left|f_{\mu(\lambda)}(z) + zf'_{\mu(\lambda)}(z)\right| \leqslant pr^{p-1}_{\mu(\lambda)} \left|1 + r_{\mu(\lambda)}M_1\right| \leqslant c_1 r^p_{\mu(\lambda)}$$

and $c_1 = 2p\max(M_1, 1)$ is independent of $\mu(\lambda)$. Then assumption *(H)* is verified and problem (P_λ) has a unique solution in $\mathrm{C}^\infty(\mathbb{R}^2)$.

4. Solution to (P_{gen})

THEOREM 7. *With the previous Assumptions (H), (H3), (H4), if u_λ is the solution to problem (P_λ) then problem (P_{gen}) admits $u = [u_\lambda]_{\mathscr{A}(\mathbb{R}^2)}$ as solution.*

Proof. According to [8], $u = [u_\lambda]$ is solution to (P_{gen}) if $(u_\lambda)_\lambda \in \mathscr{X}(\mathbb{R}^2)$. Then we shall prove that

$$\forall K \Subset \mathbb{R}^2, \forall l \in \mathbb{N}, \left(P_{K,l}(u_\lambda)\right)_\lambda \in A.$$

We proceed by induction. We have: $\forall K \Subset \mathbb{R}^2\ \exists\ K_a \Subset \mathbb{R}^2\ K \subset K_a$,

$$\|u_\lambda\|_{\infty,K} \leqslant \|u_\lambda\|_{\infty,K_a} \leqslant \left\|u_{0,\lambda}\right\|_{\infty,K_a} + \frac{\Phi_{a,\lambda}}{m_{a,\lambda}} \exp[m_{a,\mu(\lambda)}(g(a)-g(-a))a]$$
$$\leqslant \left\|u_{0,\lambda}\right\|_{\infty,K_a} \left(1 + \exp[ac_1 r^p_{\mu(\lambda)}(g(a)-g(-a))]\right).$$

As $\left(\left\|u_{0,\lambda}\right\|_{\infty,K_a}\right)_\lambda \in A$ we have

$$\left(\left\|u_{0,\lambda}\right\|_{\infty,K_a}(1+\exp[ac_1 r^p_{\mu(\lambda)}(g(a)-g(-a))])\right)_\lambda \in A.$$

As A is stable, we deduce $(P_{K,0}(u_\lambda))_\lambda \in A$, then the 0th order estimate is verified.

We have

$$\frac{\partial u_\lambda}{\partial x}(x,y) = \frac{\partial u_{0,\lambda}}{\partial x}(x,y) + \int_0^y F_{\mu(\lambda)}(x,\zeta,u_\lambda(x,\zeta))d\zeta,$$

hence

$$P_{K,(1,0)}(u_\lambda) \leqslant \sup_K \left|\frac{\partial u_{0,\lambda}}{\partial x}(x,y)\right| + a\Big(\sup_{K_a}\left|F_{\mu(\lambda)}(x,\zeta,u_\lambda(x,\zeta))\right|\Big).$$

Moreover

$$P_{K_a,(0,0)}(F_{\mu(\lambda)}(\cdot,\cdot,u_\lambda)) \leqslant P_{K_a,0}(F_{\mu(\lambda)}(\cdot,\cdot,u_\lambda)) \leqslant \mu_0 r^p_{\mu(\lambda)},$$

then

$$P_{K,(1,0)}(u_\lambda) \leqslant \left\|\frac{\partial u_{0,\lambda}}{\partial x}\right\|_{\infty,K} + c_0 r^p_{\mu(\lambda)} a.$$

As A is stable, we get $\left(P_{K,(1,0)}(u_\lambda)\right)_\lambda \in A$. We have

$$\begin{aligned}\frac{\partial u}{\partial y}(x,y) &= \frac{\partial u_{0,\lambda}}{\partial y}(x,y)\\ &\quad + \int_{g(y)}^x F_{\mu(\lambda)}(\xi,y,u_\lambda(\xi,y))d\xi - g'(y)\int_0^y F_{\mu(\lambda)}(g(y),\zeta,u_\lambda(g(y),\zeta))d\zeta,\end{aligned}$$

thus

$$P_{K,(0,1)}(u_\lambda) \leqslant \sup_K \left|\frac{\partial u_{0,\lambda}}{\partial y}(x,y)\right| + \big((g(a)-g(-a))+ag'(y)\big)\sup_{K_a}\left|F_{\mu(\lambda)}(x,y,u_\lambda(x,y))\right|.$$

Hence

$$P_{K,(0,1)}(u_\lambda) \leqslant \left\|\frac{\partial u_{0,\lambda}}{\partial y}\right\|_{\infty,K} + c_0 r^p_{\mu(\lambda)}\big(g(a)-g(-a)+ag'(y)\big)$$

and, as previously, $\left(P_{K,(0,1)}(u_\lambda)\right)_\lambda \in A$. Finally

$$(P_{K,1}(u_\lambda))_\lambda \in A.$$

Assume that $\left(P_{K,l}(u_\lambda)\right)_\lambda \in A$ for any $l \leqslant n$. In fact we have

$$P_{K,n+1} = \max(P_{K,n}, P_{1,n}, P_{2,n}, P_{3,n}, P_{4,n})$$

with

$$\begin{aligned}&P_{1,n} = P_{K,(n+1,0)}\ ;\ P_{2,n} = P_{K,(0,n+1)K,(0,n+1)}\\ &P_{3,n} = \sup_{\alpha+\beta=n;\beta\geqslant 1} P_{K,(\alpha+1,\beta)}\ ;\ P_{4,n} = \sup_{\alpha+\beta=n;\alpha\geqslant 1} P_{K,(\alpha,\beta+1)}.\end{aligned}$$

For $n \geqslant 1$, we have by successive derivations

$$\frac{\partial^{n+1} u_\lambda}{\partial x^{n+1}}(x,y) = \frac{\partial^{n+1} u_{0,\lambda}}{\partial x^{n+1}}(x,y) + \int_0^y \frac{\partial^n}{\partial x^n} F_{\mu(\lambda)}(x,\zeta,u_\lambda(x,\zeta))d\zeta.$$

As $K \subset K_a$, we can write

$$\sup_{(x,y)\in K}\left|\frac{\partial^{n+1} u_\lambda}{\partial x^{n+1}}(x,y)\right| \leqslant \left\|\frac{\partial^{n+1} u_{0,\lambda}}{\partial x^{n+1}}\right\|_{\infty,K} + a\left(\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial x^n} F_{\mu(\lambda)}(x,y,u_\lambda(x,y))\right|\right).$$

We have

$$\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial x^n} F_{\mu(\lambda)}(x,y,u_\lambda(x,y))\right| \leqslant P_{K,n}(F_{\mu(\lambda)}(\cdot,\cdot,u_\lambda)).$$

Moreover

$$\left(\left\|\frac{\partial^{n+1} u_{0,\lambda}}{\partial x^{n+1}}\right\|_{\infty,K}\right)_\lambda \in A.$$

According to the stability hypothesis, a simple calculation shows that for any $K \Subset \mathbb{R}^2$, $\left(P_{K,(n+1,0)}(u_\lambda)\right)_\lambda \in A$, then $(P_{1,n}(u_\lambda))_\lambda \in A$. Let us show that $(P_{2,n}(u_\lambda))_\lambda \in A$ for every $n \in \mathbb{N}$. We have by successive derivations, for $n \geqslant 1$,

$$\begin{aligned}
&\frac{\partial^{n+1} u_\lambda}{\partial y^{n+1}}(x,y) = \frac{\partial^{n+1} u_{0,\lambda}}{\partial y^{n+1}}(x,y)\\
&-\sum_{j=0}^{n-1} C_n^j g^{(n-j)}(y)\frac{\partial^j}{\partial y^j} F_{\mu(\lambda)}(g(y),y,\psi_{\nu(\lambda)}(y)) - \int_x^{g(y)} \frac{\partial^n}{\partial y^n} F_{\mu(\lambda)}(\xi,y,u_\lambda(\xi,y))d\xi\\
&-\sum_{j=0}^{n-1} C_n^{j+1} g^{(n-j)}(y)\frac{\partial^j}{\partial y^j} F_{\mu(\lambda)}(g(y),y,\psi_{\nu(\lambda)}(y))\\
&-g^{(n+1)}(y)\int_0^y F_{\mu(\lambda)}(g(y),\zeta,u_\lambda(g(y),\zeta))d\zeta.
\end{aligned}$$

As $K \subset K_a$, we can write

$$\begin{aligned}
\sup_{(x,y)\in K}\left|\frac{\partial^{n+1} u_\lambda}{\partial y^{n+1}}(x,y)\right| \leqslant{}& \sup_{y\in[-a,a]}\sum_{j=0}^{n-1} C_{n+1}^{j+1}\left|g^{(n-j)}(y)\right|\left|\frac{\partial^j}{\partial y^j} F_{\mu(\lambda)}(g(y),y,\psi_{\nu(\lambda)}(y))\right|\\
&+(g(a)-g(a))\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial y^n} F_{\mu(\lambda)}(x,y,u_\lambda(x,y))\right|\\
&+ag^{(n+1)}(y)\sup_{(x,y)\in K}\left|F_{\mu(\lambda)}(x,y,u_\lambda(x,y))\right| + P_{K,(0,n+1)}\left(u_{0,\lambda}\right).
\end{aligned}$$

For any $K \Subset \mathbb{R}^2$, we have

$$\sup_{(x,y)\in K}\left|\frac{\partial^i}{\partial y^i} F(x,y,u_\lambda(x,y))\right| \leqslant P_{K,n}(F(\cdot,\cdot,u_\lambda).$$

Then, for any $n \in \mathbb{N}$, $\left(P_{K,(0,n+1)}(u_\lambda)\right)_\lambda \in A$. We deduce that $(P_{2,n}(u_\lambda))_\lambda \in A$.

For $\alpha + \beta = n$ and $\beta \geqslant 1$, we have now

$$\begin{aligned} P_{K,(\alpha+1,\beta)}(u_\lambda) &= \sup_{(x,y)\in K} \left| D^{(\alpha,\beta-1)} D^{(1,1)} u_\lambda(x,y) \right| \\ &= \sup_{(x,y)\in K} \left| D^{(\alpha,\beta-1)} F_{\mu(\lambda)}(x,y,u_\lambda(x,y)) \right| \leqslant P_{K,n}(F_{\mu(\lambda)}(\cdot,\cdot,u_\lambda)). \end{aligned}$$

We finally obtain

$$P_{3,n}(u_\lambda) = \sup_{\alpha+\beta=n;\beta\geqslant 1} P_{K,(\alpha+1,\beta)}(u_\lambda) \leqslant P_{K,n}(F_{\mu(\lambda)}(\cdot,\cdot,u_\lambda))$$

and the stability hypothesis implies $(P_{3,n}(u_\lambda))_\lambda \in A$. In the same way for $\alpha + \beta = n$ and $\alpha \geqslant 1$, we have

$$P_{K,(\alpha,\beta+1)}(u_\lambda) = \sup_{(x,y)\in K} \left| D^{(\alpha-1,\beta)} F_{\mu(\lambda)}(x,y,u_\lambda(x,y)) \right| \leqslant P_{K,n}(F_{\mu(\lambda)}(\cdot,\cdot,u_\lambda)).$$

We deduce

$$P_{4,n}(u_\lambda) = \sup_{\alpha+\beta=n;\alpha\geqslant 1} P_{K,(\alpha,\beta+1)}(u_\lambda) \leqslant P_{K,n}(F_{\mu(\lambda)}(\cdot,\cdot,u_\lambda))$$

and the stability hypothesis implies $(P_{4,n}(u_\lambda))_\lambda \in A$. Finally, we have $(P_{K,n+1}(u_\lambda))_\lambda \in A$.

THEOREM 8. *problem (P_{gen}) has a unique solution in the algebra $\mathscr{A}\left(\mathbb{R}^2\right)$.*

Proof. Let $u = [u_\lambda]_{\mathscr{A}\left(\mathbb{R}^2\right)}$ the solution to (P_{gen}) obtain in Theorem 7. Let $v = [v_\lambda]$ another solution to (P_{gen}). There are $(i_\lambda)_\lambda \in \mathscr{N}\left(\mathbb{R}^2\right)$, $\left(\alpha_{\nu(\lambda)}\right)_\lambda \in \mathscr{N}(\mathbb{R})$, $\left(\beta_{\nu(\lambda)}\right)_\lambda \in \mathscr{N}(\mathbb{R})$, such that

$$\begin{cases} \dfrac{\partial^2 v_\lambda}{\partial x \partial y}(x,y) = F_{\mu(\lambda)}(x,y,v_\lambda(x,y)) + i_\lambda(x,y), \\ v_\lambda(x,0) = \varphi_{\nu(\lambda)}(x) + \alpha_{\nu(\lambda)}(x), \\ v_\lambda(g(y),y) = \psi_{\nu(\lambda)}(y) + \beta_{\nu(\lambda)}(y). \end{cases}$$

It is easy to see that

$$\left((x,y) \mapsto \iint_{D(x,y,f)} i_\lambda(\xi,\zeta)) d\xi d\zeta \right)_\lambda \in \mathscr{N}\left(\mathbb{R}^2\right).$$

Then there is $(j_\lambda)_\lambda \in \mathscr{N}\left(\mathbb{R}^2\right)$ such that

$$v_\lambda(x,y) = v_{0,\lambda}(x,y) + \iint_{D(x,y,g)} F_{\mu(\lambda)}(\xi,\zeta,v_\lambda(\xi,\zeta)) d\xi d\zeta + j_\lambda(x,y),$$

with $v_{0,\lambda}(x,y) = u_{0,\lambda}(x,y) + \theta_\lambda(x,y)$, where

$$u_{0,\lambda}(x,y) = \psi_{\nu(\lambda)}(y) + \varphi_{\nu(\lambda)}(x) - \varphi_{\nu(\lambda)}(g(y)),$$
$$\theta_\lambda(x,y) = \beta_{\nu(\lambda)}(y) + \alpha_{\nu(\lambda)}(x) - \beta_{\nu(\lambda)}(g(y)).$$

Then $(\theta_\lambda)_\lambda$ belongs to $\mathscr{N}(\mathbb{R}^2)$. So there is $(\sigma_\lambda)_\lambda \in \mathscr{N}\left(\mathbb{R}^2\right)$ such that

$$v_\lambda(x,y) = u_{0,\lambda}(x,y) + \iint_{D(x,y,g)} F_{\mu(\lambda)}(\xi,\zeta,u_\lambda(\xi,\zeta))d\xi d\zeta + \sigma_\lambda(x,y). \tag{4}$$

The uniqueness of the solution to (P_{gen}) will be the consequence of

$$(w_\lambda)_\lambda = (v_\lambda - u_\lambda)_\lambda \in \mathscr{N}(\mathbb{R}^2).$$

First, we will prove that $\forall K \Subset \mathbb{R}^2, (P_{K,0}(w_\lambda))_\lambda \in |I_A|$. We have

$$w_\lambda(x,y) = \sigma_\lambda(x,y) + \iint_{D(x,y,g)} \left(F_{\mu(\lambda)}(\xi,\zeta,v_\lambda(\xi,\zeta)) - F_{\mu(\lambda)}(\xi,\zeta,u_\lambda(\xi,\zeta))\right) d\xi d\zeta.$$

However

$$\begin{aligned} &F_{\mu(\lambda)}(\xi,\zeta,v_\lambda(\xi,\zeta)) - F_{\mu(\lambda)}(\xi,\zeta,u_\lambda(\xi,\zeta)) \\ &\qquad = w_\lambda(\xi,\zeta)\int_0^1 \frac{\partial F_{\mu(\lambda)}}{\partial z}(\xi,\zeta,u_\lambda(\xi,\zeta) + \eta(v_\lambda(\xi,\zeta) - u_\lambda(\xi,\zeta)))d\eta, \end{aligned}$$

then we get

$$\begin{aligned} &w_\lambda(x,y) = \sigma_\lambda(x,y) \\ &+ \iint_{D(x,y,f)} w_\lambda(\xi,\zeta)\left(\int_0^1 \frac{\partial F_{\mu(\lambda)}}{\partial z}(\xi,\zeta,u_\lambda(\xi,\zeta) + \eta(w_\lambda(\xi,\zeta)))d\eta\right) d\xi d\zeta. \end{aligned}$$

Let $(x,y) \in K$, we have $D(x,y,g) \subset K_a$. If $g(y) \leqslant x$, then

$$\begin{aligned} |w_\lambda(x,y)| &\leqslant m_{a,\mu(\lambda)} \int_{g(y)}^x \int_0^y |w_\lambda(\xi,\zeta)|\, d\xi d\zeta + \|\sigma_\lambda\|_{\infty,K_a} \\ &\leqslant m_{a,\mu(\lambda)} \int_{-g(a)}^{+g(a)} \int_0^y |w_\lambda(\xi,\zeta)|\, d\xi d\zeta + \|\sigma_\lambda\|_{\infty,K_a}. \end{aligned}$$

Take $e_\lambda(y) = \sup\limits_{\xi\in[g(-a);g(a)]} |w_\lambda(\xi,y)|$, $2a' = (g(a) - g(-a))$, then

$$|w_\lambda(x,y)| \leqslant m_{a,\mu(\lambda)} 2a' \int_0^y e_\lambda(\zeta)d\zeta + \|\sigma_\lambda\|_{\infty,K_a}.$$

We deduce that $e_\lambda(y) \leqslant m_{a,\mu(\lambda)} 2a' \int_0^y e_\lambda(\zeta)d\zeta + \|\sigma_\lambda\|_{\infty,K_a}$ for any $y \in [0,a]$. Thus according to the Gronwall's lemma

$$e_\lambda(y) \leqslant \|\sigma_\lambda\|_{\infty,K_a} \exp(\int_0^y m_{a,\mu(\lambda)} 2a' d\zeta).$$

Then

$$e_\lambda(y) \leqslant \|\sigma_\lambda\|_{\infty,K_a} \exp(m_{a,\mu(\lambda)} 2a'y) \leqslant \|\sigma_\lambda\|_{\infty,K_a} \exp(m_{a,\mu(\lambda)} 2a'a).$$

We obtain the same result in the other cases, hence

$$\forall y \in [-a,a],\, e_\lambda(y) \leqslant \|\sigma_\lambda\|_{\infty,K_a} \exp(m_{a,\mu(\lambda)} 2a'a),$$

consequently

$$\|w_\varepsilon\|_{\infty,K_a} \leqslant \|\sigma_\lambda\|_{\infty,K_a} \exp(m_{a,\mu(\lambda)} 2a'a).$$

As $(\sigma_\lambda)_\lambda \in \mathscr{N}(\mathbb{R}^2)$ then $\left(\|\sigma_\lambda\|_{\infty,K_a}\right)_\lambda \in |I_A|$. Moreover $\|\sigma_\lambda\|_{\infty,K_a} \exp(m_{a,\mu(\lambda)} 2a'a)$ is a constant, consequently $\left(\|w_\lambda\|_{\infty,K_a}\right)_\lambda \in |I_A|$. Which implies the 0th order estimate.

According to Proposition 1, we deduce $(w_\lambda)_\lambda \in \mathscr{N}(\mathbb{R}^2)$; then u is the unique solution to (P_{gen}) for the family $\left(F_{\mu(\lambda)}\right)_\lambda$.

EXAMPLE 3. A degenerate Goursat problem in $(\mathscr{C},\mathscr{E},\mathscr{P})$-algebras. We study the Goursat problem in the case where φ and ψ are one-variable generalized functions, $\gamma = (Oy)$. (We take $g = 0$). To solve the problem (P_{gen}) associated to $\left(P_{form}\right)$ we can consider, as previously, the family of problems

$$(P_\lambda) \begin{cases} \dfrac{\partial^2 u_\lambda}{\partial x \partial y}(x,y) = F_{\mu(\lambda)}(x,y,u_\lambda(x,y)), \\ u_\lambda(x,0) = \varphi_{\nu(\lambda)}(x), \\ u_\lambda(0,y)) = \psi_{\nu(\lambda)}(y), \end{cases}$$

where $\left(\varphi_{\nu(\lambda)}\right)_{\nu(\lambda)}$ and $\left(\psi_{\nu(\lambda)}\right)_{\nu(\lambda)}$ are representatives of φ and ψ in $\mathscr{A}(\mathbb{R})$. If u_λ is the solution to problem $\left(P_{(\lambda)}\right)$ then problem (P_{gen}) admits $u = [u_\lambda]_{\mathscr{A}(\mathbb{R}^2)}$ as solution. Moreover

$$\begin{aligned} u_\lambda(x,y) &= u_{0,\lambda}(x,y) + \iint_{D(x,y,0)} F_{\mu(\lambda)}(\xi,\eta,u_\lambda(\xi,\eta))\,d\xi\, d\eta \\ &= u_{0,\lambda}(x,y) + \int_0^x \left(\int_0^y F_{\mu(\lambda)}(\xi,\eta,u_\lambda(\xi,\eta))\,d\eta \right) d\xi \end{aligned}$$

with

$$u_{0,\lambda}(x,y) = \psi_{\nu(\lambda)}(y) + \varphi_{\nu(\lambda)}(x) - \varphi_{\nu(\lambda)}(0).$$

The case of the degenerate Goursat problem solved in [16] is a particular case of our study. If $\Lambda = \Lambda_1 \times \Lambda_2 = (0,1] \times (0,1]$, if $F = 0$ and $\varphi = \psi = \delta$ then $u = \left[u_{0,\lambda}\right] = u_1 + u_2$ where $u_1 \sim 1_x \otimes \delta_y + \delta_x \otimes 1_y$ but u_2 cannot be associated with a distribution. Thus, even in the linear case, the Goursat problem with distribution data has a generalized solution which is not (associated to) a distribution.

REMARK 6. Construction of $\mathscr{A}(\mathbb{R}^2)$ in the case of regular data. If the data s and t are smooth, we take $\Lambda = \Lambda_1 = (0,1]$, and $\mu(\lambda) = \lambda = \varepsilon$. Let $(r_\varepsilon)_\varepsilon$ be in $(\mathbb{R}_*^+)^{(0,1]}$ such that $\lim_{\varepsilon\to 0} r_\varepsilon = +\infty$. We take $\mathscr{C} = A/I_A$ the ring overgenerated by $(\lambda)_\lambda$, $(r_\lambda)_\lambda$, $(e^{r_\lambda})_\lambda$, elements of $(\mathbb{R}_*^+)^{(0,1]}$. Then $\mathscr{A}\left(\mathbb{R}^2\right) = \mathscr{X}(\mathbb{R}^2)/\mathscr{N}(\mathbb{R}^2)$ is built on the ring $\mathscr{C}$ of generalized constants with $(\mathscr{E},\mathscr{P}) = \left(\mathrm{C}^\infty(\mathbb{R}^2), \left(P_{K,l}\right)_{K\Subset\mathbb{R}^2, l\in\mathbb{N}}\right)$ and, in the same way, $\mathscr{A}(\mathbb{R}) = \mathscr{X}(\mathbb{R})/\mathscr{N}(\mathbb{R})$ is built on $\mathscr{C}$ with

$$(\mathscr{E},\mathscr{P}) = \left(\mathrm{C}^\infty(\mathbb{R}), \left(P_{K,l}\right)_{K\Subset\mathbb{R}, l\in\mathbb{N}}\right).$$

Nonetheless, the algebra $\mathscr{A}(\mathbb{R}^2)$ is not the same in the two cases, regular data and irregular data. We set $\varphi = s$ and $\psi = t$, elements of $\mathrm{C}^\infty(\mathbb{R})$ canonically embedded in $\mathscr{A}(\mathbb{R})$. If $\alpha \in \mathscr{A}(\mathbb{R})$ we take $\alpha_{\nu(\lambda)} = \alpha$, if $\alpha \in \mathscr{N}(\mathbb{R})$ we take $\alpha_{\nu(\lambda)} = 0$. Then we can rewrite this section and we get similar results. We have the same definitions as previously and we obtain the same theorems, the same proofs replacing $\varphi_{\nu(\lambda)}$ by φ and $\psi_{\nu(\lambda)}$ by ψ. As previously (Theorem 7 and Theorem 8), we can prove that problem (P_{gen}) has a unique generalized solution $u = [u_\lambda]$ in the algebra $\mathscr{A}(\mathbb{R}^2)$.

5. Independence of the generalized solution from the class of cut off functions

Recall that $\Lambda_1 = (0,1]$, set

$$\mathscr{X}_1(\mathbb{R}) = \{(f_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\mathbb{R})]^{\Lambda_1} : \forall K \Subset \mathbb{R}, \forall l \in \mathbb{N}, \left(P_{K,l}(f_\varepsilon)\right)_\varepsilon \in |A|\},$$
$$\mathscr{N}_1(\mathbb{R}) = \{(f_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\mathbb{R})]^{\Lambda_1} : \forall K \Subset \mathbb{R}, \forall l \in \mathbb{N}, \left(P_{K,l}(f_\varepsilon)\right)_\varepsilon \in |I_A|\},$$
$$\mathscr{A}_1(\mathbb{R}) = \mathscr{X}_1(\mathbb{R})/\mathscr{N}_1(\mathbb{R}).$$

Consider $\mathscr{T}(\mathbb{R})$ the set of families of smooth one-variable functions $(h_\varepsilon)_{\varepsilon\in\Lambda_1} \in \mathscr{X}_1(\mathbb{R})$, verifying the following assumptions

$$\exists (s_\varepsilon)_\varepsilon \in \mathbb{R}_*^{(0,1]} : \sup_{z\in[-s_\varepsilon, s_\varepsilon]} |h_\varepsilon(z)| = 1, \tag{5}$$

$$h_\varepsilon(z) = \begin{cases} 0 \text{ if } & |z| \geqslant s_\varepsilon, \\ 1 \text{ if } & -s_\varepsilon + 1 \leqslant z \leqslant s_\varepsilon - 1, \end{cases}$$

$$\exists\, q \in \mathbb{N}^*, \forall (h_\varepsilon)_\varepsilon \in \mathscr{T}(\mathbb{R}), \forall \varepsilon, s_\varepsilon \leqslant r_\varepsilon^q. \tag{6}$$

Moreover assume that $\dfrac{\partial^n h_\varepsilon}{\partial z^n}$ is bounded on $J_\varepsilon = [-s_\varepsilon, s_\varepsilon]$ for any integer n, $n > 0$.

We have $(f_\varepsilon)_{\varepsilon\in\Lambda_1} \in \mathscr{T}(\mathbb{R})$. Recall that $\phi_\varepsilon(z) = z f_\varepsilon(z)$ for $z \in \mathbb{R}$, $F_\varepsilon(x,y,z) = F(x,y,\phi_\varepsilon(z))$ for $(x,y,z) \in \mathbb{R}^3$ and

$$\sup_{z\in[-r_\varepsilon, r_\varepsilon]} \left| \frac{\partial^n f_\varepsilon}{\partial z^n}(z) \right| = M_n.$$

Let $f \in \mathscr{T}(\mathbb{R})/\mathscr{N}_1(\mathbb{R})$ be the class of $(f_\varepsilon)_\varepsilon$. Take $(h_\varepsilon)_\varepsilon$ another representative of f, that is to say $(h_\varepsilon)_\varepsilon \in \mathscr{T}(\mathbb{R})$ and

$$(f_\varepsilon - h_\varepsilon)_\varepsilon \in \mathscr{N}_1(\mathbb{R}). \tag{7}$$

Set $\sigma_\varepsilon(z) = zh_\varepsilon(z)$ for $z \in \mathbb{R}$, $H_\varepsilon(x,y,z) = F(x,y,\sigma_\varepsilon(z))$ for $(x,y,z) \in \mathbb{R}^3$ and

$$\sup_{z\in[-s_\varepsilon,s_\varepsilon]} \left| \frac{\partial^n h_\varepsilon}{\partial z}(z) \right| = M'_n.$$

Our choice is made such that $(\operatorname{supp}(h_\varepsilon))_\varepsilon$ have the same growth as $(\operatorname{supp}(f_\varepsilon))_\varepsilon$ with respect to the scale $\left(r_\varepsilon^q\right)_\varepsilon$, in this way the corresponding solutions are lying in the same algebra $\mathscr{A}(\mathbb{R}^2)$.

PROPOSITION 9. *Set $F \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R})$, $\phi \in \mathrm{C}^\infty(\mathbb{R},\mathbb{R})$, $\mathfrak{F}(x,y,z) = F(x,y,\phi(z))$. For any $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$, $\alpha_3 \geqslant 0$ with $|\alpha| = \alpha_1+\alpha_2+\alpha_3 = n \neq 0$, we have*

$$\frac{\partial^n \mathfrak{F}}{\partial x^{\alpha_1}\partial y^{\alpha_2}\partial z^{\alpha_3}}(x,y) = \sum_{1\leqslant|\beta|\leqslant n} \left(D^\beta F\right)(x,y,\phi(z)) \sum_{i=1}^{n} \sum_{p_i(\alpha,\beta)} d_{i,\alpha,\beta} \prod_{j=1}^{i} \left(\frac{\partial^{l_j}}{\partial z^{l_j}} \phi(z) \right)^{k_j}$$

where $\beta \in \mathbb{N}^3$. The set $p_i(\alpha,\beta)$ mentioned in the inner sum consists of all nonzero multi-indices $(k_1,\ldots,k_i,l_1,\ldots,l_i) \in (\mathbb{N})^{2i}$, such that

$$0 < l_1 < \ldots < l_i, \quad \sum_{j=1}^{i} k_j = \beta_3, \quad \sum_{j=1}^{i} k_j l_j = \alpha_3.$$

The proof uses the Multivariate Faà di Bruno's formula [2].

COROLLARY 10. *Set $F \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R})$, $\sigma_\varepsilon(z) = zh_\varepsilon(z)$ with $(h_\varepsilon)_\varepsilon \in \mathscr{T}(\mathbb{R})$,*

$$H_\varepsilon(x,y,z) = F(x,y,\sigma_\varepsilon(z)),$$

$\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$, $\alpha_3 \geqslant 0$ with $|\alpha| = \alpha_1+\alpha_2+\alpha_3 = n \neq 0$. Then, for $\beta \in \mathbb{N}^3$, $1 \leqslant |\beta| \leqslant n$, there exist constants $C_{|\beta|}$ which no depend of F and ϕ_ε, such that $\forall K \Subset \mathbb{R}^2$, $\forall (x,y) \in K$, $\forall z \in [-s_\varepsilon,s_\varepsilon]$,

$$\left| \frac{\partial^n H_\varepsilon}{\partial x^{\alpha_1}\partial y^{\alpha_2}\partial z^{\alpha_3}}(x,y,z) \right| \leqslant \sum_{1\leqslant|\beta|\leqslant n} P_{K,|\beta|}(F)\, C_{|\beta|} s_\varepsilon^{\alpha_3}.$$

Proof. We have

$$\frac{\partial^n \sigma_\varepsilon}{\partial z^n}(z) = z\frac{\partial^n h_\varepsilon}{\partial z^n}(z) + n\frac{\partial^{n-1} h_\varepsilon}{\partial z^{n-1}}(z).$$

Thus

$$\left| \frac{\partial^n \sigma_\varepsilon}{\partial z^n}(z) \right| \leqslant s_\varepsilon M'_n + nM'_{n-1} \leqslant \alpha_n s_\varepsilon \leqslant \alpha_n r_\varepsilon^q,$$

where $\alpha_n = 2\max(M'_n; nM'_{n-1})$. Thus we deduce the formula. Moreover, according to (5), we have $s_\varepsilon \leqslant r_\varepsilon^q$, then

$$\left| \frac{\partial^n H_\varepsilon}{\partial x^{\alpha_1}\partial y^{\alpha_2}\partial z^{\alpha_3}}(x,y,z) \right| \leqslant \sum_{1\leqslant|\beta|\leqslant n} P_{K,|\beta|}(F)\, C_{|\beta|} r_\varepsilon^{q\alpha_3}.$$

COROLLARY 11. *Set* $S_n = \left\{\alpha \in \mathbb{N}^3 : |\alpha| = n\right\}$ *when* $n \in \mathbb{N}^*$. *Let* $F \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R})$, H_ε *defined by*

$$H_\varepsilon(x,y,z) = F(x,y,\sigma_\varepsilon(z)).$$

Assume that

$$\forall (x,y) \in \mathbb{R}^2, F(x,y,0) = 0\ ,$$

$$\exists p_0 > 0, \forall \alpha \in \mathbb{N}^3, |\alpha| = n > p_0, D^\alpha F(x,y,z) = 0,$$

$$\forall n \in \mathbb{N}, n \leqslant p_0, \exists d_n > 0, \forall \varepsilon \in (0,1], \forall K \Subset \mathbb{R}^2, \sup_{(x,y)\in K;\, z\in J_\varepsilon; \alpha \in Sn} |D^\alpha F(x,y,z)| \leqslant d_n r_\varepsilon^{p_0}, \tag{8}$$

then $\mathscr{A}(\mathbb{R}^2)$ *is stable under the family* $\left(H_{\mu(\lambda)}\right)_\lambda$.

Proof. Indeed, we have $\forall K \Subset \mathbb{R}^2$, $\forall (x,y) \in K$, $\forall z \in J_{\mu(\lambda)}$, $\forall \alpha \in \mathbb{N}^3$,

$$\left|\frac{\partial^n H_\varepsilon}{\partial x^{\alpha_1} \partial^{\alpha_2} y \partial z^{\alpha_3}}(x,y,z)\right| \leqslant \sum_{1\leqslant|\beta|\leqslant n} P_{K,|\beta|}(F)\, C_{|\beta|} r_\varepsilon^{q\alpha_3} \leqslant \sum_{1\leqslant|\beta|\leqslant p} d_{|\beta|} r_\varepsilon^{p_0} C_{|\beta|} r_\varepsilon^{qp_0}$$
$$\leqslant c_n r_\varepsilon^{p_0(1+q)}$$

where c_n no depends to ε and r_ε. As $\sigma_\varepsilon(z) = 0$ if $z \notin J_\varepsilon$, we have

$$\sup_{(x,y)\in K;\, z\in\mathbb{R}; \alpha\in Sn} |D^\alpha H_\varepsilon(x,y,z)| \leqslant c_n r_\varepsilon^{p_0(1+q)},$$

and, according to Proposition 3, $\mathscr{A}\left(\mathbb{R}^2\right)$ is stable under the family $\left(H_{\mu(\lambda)}\right)_\lambda$.

THEOREM 12. *Assume that* $p = p_0(1+q)$. *Under the same hypotheses as Corollary* 11, *problem* (P_{gen}), *a fortiori its solution, does not depend of the choice of the representative* $(f_\varepsilon)_{\varepsilon\in\Lambda_1}$ *of the class* $f \in \mathscr{T}(\mathbb{R})/\mathscr{N}_1(\mathbb{R})$.

Proof. We have associated the generalized operator $\mathscr{F}$ to F *via* the family $(f_\varepsilon)_\varepsilon$. Let $(h_\varepsilon)_\varepsilon \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_1}$ another family representative of the class $[f_\varepsilon] = f$ and leading to another generalized operator $\mathscr{H}$ associated to F. We have to prove that $\mathscr{H} = \mathscr{F}$, that is to say $\mathscr{H}(u) = \mathscr{F}(u)$ for any $u \in \mathscr{A}(\mathbb{R}^2)$. Then, in terms of representatives, we have to prove that, if $(u_\lambda)_\lambda$, $(v_\lambda)_\lambda \in \mathscr{X}(\mathbb{R}^2)$ and $(w_\lambda)_\lambda = (v_\lambda - u_\lambda)_\lambda \in \mathscr{N}(\mathbb{R}^2)$, then

$$\left(F(\cdot,\cdot,\sigma_{\mu(\lambda)}(v_\lambda)) - F(\cdot,\cdot,\phi_{\mu(\lambda)}(u_\lambda))\right)_\lambda \in \mathscr{N}(\mathbb{R}^2).$$

Let

$$\Delta_\lambda(x,y) = \sigma_{\mu(\lambda)}(v_\lambda(x,y)) - \phi_{\mu(\lambda)}(u_\lambda(x,y)).$$

We have $\forall K \Subset \mathbb{R}^2$, $\forall (x,y) \in K$,

$$\Delta_\lambda(x,y) = v_\lambda(x,y) h_{\mu(\lambda)}(v_\lambda(x,y)) - u_\lambda(x,y) f_{\mu(\lambda)}(u_\lambda(x,y)),$$

thus

$$\Delta_\lambda(x,y) = w_\lambda(x,y) h_{\mu(\lambda)}(v_\lambda(x,y)) + u_\lambda(x,y)\left(h_{\mu(\lambda)}(v_\lambda(x,y)) - f_{\mu(\lambda)}(u_\lambda(x,y))\right), \tag{9}$$

moreover

$$h_{\mu(\lambda)} \circ v_\lambda - f_{\mu(\lambda)} \circ u_\lambda = \left(h_{\mu(\lambda)} \circ v_\lambda - h_{\mu(\lambda)} \circ u_\lambda\right) + \left(h_{\mu(\lambda)} \circ u_\lambda - f_{\mu(\lambda)} \circ u_\lambda\right).$$

As

$$\begin{aligned} & h_{\mu(\lambda)}\left(v_\lambda(x,y)\right) - h_{\mu(\lambda)}\left(u_\lambda(x,y)\right) \\ & = \left(v_\lambda(x,y) - u_\lambda(x,y)\right) \int_0^1 \frac{\partial h_{\mu(\lambda)}}{\partial z}\left(u_\lambda(x,y) + \eta\left(v_\lambda(x,y) - u_\lambda(x,y)\right)\right) d\eta, \end{aligned} \tag{10}$$

we have

$$\begin{aligned} & h_{\mu(\lambda)}\left(v_\lambda(x,y)\right) - f_{\mu(\lambda)}\left(u_\lambda(x,y)\right) \\ & = w_\lambda(x,y) \int_0^1 \frac{\partial h_{\mu(\lambda)}}{\partial z}\left(u_\lambda(x,y) + \eta w_\lambda(x,y)\right) d\eta + \left(h_{\mu(\lambda)} - f_{\mu(\lambda)}\right)\left(u_\lambda(x,y)\right). \end{aligned} \tag{11}$$

We deduce that $\forall (x,y) \in K$,

$$\begin{aligned} \left|h_{\mu(\lambda)}\left(v_\lambda(x,y)\right) - f_{\mu(\lambda)}\left(u_\lambda(x,y)\right)\right| & \leqslant \left|w_\lambda(x,y)\right| \int_0^1 M_1' d\eta + \left|\left(h_{\mu(\lambda)} - f_{\mu(\lambda)}\right)\left(u_\lambda(x,y)\right)\right| \\ & \leqslant \left|w_\lambda(x,y)\right| M_1' + p_{J_{\mu(\lambda)},1}\left(h_{\mu(\lambda)} - f_{\mu(\lambda)}\right). \end{aligned}$$

where $J_{\mu(\lambda)} = \left[-s_{\mu(\lambda)}, s_{\mu(\lambda)}\right]$. Then

$$\begin{aligned} \left|\Delta_\lambda(x,y)\right| & \leqslant \left|w_\lambda(x,y)\right| + \left|u_\lambda(x,y)\right| \left(\left|w_\lambda(x,y)\right| M_1' + p_{J_{\mu(\lambda)},1}\left(h_{\mu(\lambda)} - f_{\mu(\lambda)}\right)\right) \\ & \leqslant \left|w_\lambda(x,y)\right| \left(1 + \left|u_\lambda(x,y)\right| M_1'\right) + \left|u_\lambda(x,y)\right| p_{J_{\mu(\lambda)},1}\left(h_{\mu(\lambda)} - f_{\mu(\lambda)}\right). \end{aligned}$$

Consequently,

$$\left|\Delta_\lambda(x,y)\right| \leqslant \left\|w_\lambda\right\|_{\infty,K} \left(1 + \left\|u_\lambda\right\|_{\infty,K} M_1'\right) + \left\|u_\lambda\right\|_{\infty,K} p_{J_{\mu(\lambda)},1}\left(h_{\mu(\lambda)} - f_{\mu(\lambda)}\right).$$

Let

$$d_\lambda = P_{K,0}\left(w_\lambda\right)\left(1 + P_{K,0}\left(u_\lambda\right) M_1'\right) + P_{K,0}\left(u_\lambda\right) p_{J_{\mu(\lambda)},1}\left(h_{\mu(\lambda)} - f_{\mu(\lambda)}\right).$$

According to (7), we have $\left(p_{J_{\mu(\lambda)},1}\left(h_{\mu(\lambda)} - f_{\mu(\lambda)}\right)\right)_{\mu(\lambda)} \in |I_A|$, moreover $(w_\lambda)_\lambda \in \mathcal{N}(\mathbb{R}^2)$, then we have $\left(P_{K,0}(w_\lambda)\right)_\lambda \in |I_A|$ and $(d_\lambda)_\lambda \in |I_A|$. As

$$\begin{aligned} & F\left(x,y,\sigma_{\mu(\lambda)}\left(v_\lambda(x,y)\right)\right) - F\left(x,y,\phi_{\mu(\lambda)}\left(u_\lambda(x,y)\right)\right) \\ & = \Delta_\lambda(x,y) \left(\int_0^1 \frac{\partial F}{\partial z}\left(x,y,\phi_{\mu(\lambda)}\left(u_\lambda(x,y)\right)\right) + \eta\left(\sigma_{\mu(\lambda)}\left(v_\lambda(x,y)\right) - \phi_{\mu(\lambda)}\left(u_\lambda(x,y)\right)\right) d\eta\right) \\ & = \Delta_\lambda(x,y) \left(\int_0^1 \frac{\partial F}{\partial z}\left(x,y,\phi_{\mu(\lambda)}\left(u(x,y)\right)\right) + \eta \Delta_\lambda(x,y)\right) d\eta\right), \end{aligned} \tag{12}$$

we get

$$\begin{aligned}&\left|F(x,y,\sigma_{\mu(\lambda)}(v_\lambda(x,y)))-F(x,y,\phi_{\mu(\lambda)}(u_\lambda(x,y)))\right|\\&\qquad\leqslant d_1 r_{\mu(\lambda)}^{p_0}\left|\Delta_\lambda(x,y)\right|\leqslant d_1 r_{\mu(\lambda)}^{p_0} d_\lambda .\end{aligned}$$

Then we have

$$\left(P_{K,0}\left(F(\cdot,\cdot,\sigma_{\mu(\lambda)}(v_\lambda))-F(\cdot,\cdot,\phi_{\mu(\lambda)}(u_\lambda))\right)\right)_\lambda\in|I_A|.$$

It implies the 0th order estimate. According to Proposition 1, we deduce

$$\left(F(\cdot,\cdot,\sigma_{\mu(\lambda)}(v_\lambda))-F(\cdot,\cdot,\phi_{\mu(\lambda)}(u_\lambda))\right)_\lambda\in\mathcal{N}(\mathbb{R}^2).$$

REMARK 7. In the case of regular data, we can show analogously that the solution to problem (P_{gen}) does not depend of the choice of the representative $(f_\varepsilon)_\varepsilon$ of the class $f\in\mathcal{T}(\mathbb{R})/\mathcal{N}_1(\mathbb{R})$, as in Theorem 12.

6. Comparison with classical solutions

Even if the data are as irregular as distributions, it may happen that the initial formal ill-posed problem $\left(P_{form}\right)$ has nonetheless a local smooth solution as it will be seen in the example 4. We are going to prove that this solution is exactly the restriction (according to the sheaf theory sense) of the generalized one.

The generalized solution to problem (P_{gen}) is defined from the integral representation (2). Thus, we are going to study the relationship between this generalized function and the classical solutions to $\left(P_{form}\right)$ (when they exist) on a domain Ω such that $\forall(x,y)\in\Omega$, $D(x,y,g)\subset\Omega$. This justified to choose $\Omega=(g(\mu),g(\nu))\times(\mu,\nu)$ when $(\mu,\nu)\in\mathbb{R}^2$ with $\mu<0<\nu$.

REMARK 8. If the non regularized problem $\left(P_{form}\right)$ has a smooth solution v on Ω then, necessarily we have $\Omega\subset\mathbb{R}^2\backslash\operatorname{singsupp}(u)$.

Recall that there exists a canonical sheaf embedding of $\mathrm{C}^\infty(\cdot)$ into $\mathcal{A}(\cdot)$, through the morphism of algebra

$$\sigma_O:\mathrm{C}^\infty(O)\to\mathcal{A}(O),\ \ f\mapsto[f_\lambda]\ \ \text{(where } O \text{ is any open subset of } \mathbb{R}^2 \text{and } f_\lambda=f).$$

The presheaf $\mathcal{A}$ allows to restriction and as usually we denote by $u|_O$ the restriction on O of $u\in\mathcal{A}\left(\mathbb{R}^2\right)$.

THEOREM 13. *Let $u=[u_\lambda]$ be the solution to problem (P_{gen}) given in Theorem 7. Let Ω be an open subset of $\mathbb{R}^2$ such that $\Omega\subset\mathbb{R}^2\backslash\operatorname{singsupp}(u)$. Assume that $\Omega=\bigcup_{\varepsilon\in\Lambda_1}\Omega_\varepsilon$ with $(\Omega_\varepsilon)_\varepsilon$ is an increasing family of open subsets of $\mathbb{R}^2$ such that $\Omega_\varepsilon=(g(a_\varepsilon),g(b_\varepsilon))\times(a_\varepsilon,b_\varepsilon)$ when $(a_\varepsilon,b_\varepsilon)\in\mathbb{R}^2$ with $a_\varepsilon<0<b_\varepsilon$. Assume that problem $\left(P_{form}\right)$ has a smooth solution v on Ω such that $\sup_{(x,y)\in\Omega_\varepsilon}|v(x,y)|<r_\varepsilon-1$ for any ε. Then v (element of $\mathrm{C}^\infty(\Omega)$ canonically embedded in $\mathcal{A}(\Omega)$) is the restriction (according to the sheaf theory sense) of u to Ω, $v=u|_\Omega$.*

Proof. We clearly have $\forall (x,y) \in \Omega, \exists \varepsilon_0, \forall \varepsilon \leqslant \varepsilon_0, (x,y) \in \Omega_\varepsilon$. Then $D(x,y,g) \subset \Omega_\varepsilon \subset \Omega$ and following [8], [9],

$$v(x,y) = v_0(x,y) + \iint_{D(x,y,g)} F(\xi,\zeta,v(\xi,\zeta))d\xi d\zeta.$$

We take as representative of u the family $(u_\lambda)_\lambda$ given by Theorem 7, we have

$$\forall (x,y) \in \Omega,\ u_\lambda(x,y) = u_{0,\lambda}(x,y) + \iint_{D(x,y,g)} F_{\mu(\lambda)}(\xi,\zeta,u_\lambda(\xi,\zeta))d\xi d\zeta$$

and $v_0(x,y) = u_{0,\lambda}(x,y)$. Set $(w_\lambda)_\lambda = (u_\lambda|_\Omega - v)_\lambda$ and take $K \Subset \Omega$. There exists ε_1 such that, for all $\varepsilon < \varepsilon_1$, $K \Subset \Omega_\varepsilon$. According the definition of Ω_ε, there exists a, $0 < a < (b_\varepsilon - a_\varepsilon)/2$, such that $K \subset Q_a \subset \Omega$ with $Q_a = [g(a_\varepsilon + a), g(b_\varepsilon - a)] \times [a_\varepsilon + a, b_\varepsilon - a]$. Take $(x,y) \in K$, then $D(x,y,g) \subset Q_a$. Note that, for $(\xi,\varsigma,z) \in \Omega_\varepsilon \times (-r_\varepsilon + 1, r_\varepsilon - 1)$, we have $F(\xi,\varsigma,z) = F_\varepsilon(\xi,\varsigma,z)$ by construction of F_ε. Thus v, which values are in $(-r_\varepsilon + 1, r_\varepsilon - 1)$, and u_λ are solutions of the same integral equation, which admits a unique solution since F_ε is a smooth function of its arguments. Thus, for all $\varepsilon = \mu(\lambda) \leqslant \varepsilon_1$, v and u_λ are equal on Ω_ε. Then $(P_{K,n}(v))_\lambda \in A$ for any $K \Subset \Omega$ and $n \in \mathbb{N}$. Then v (identified with $[(v)_\lambda]$) belongs to $\mathscr{A}(\Omega)$. Moreover, for all $\varepsilon = \mu(\lambda) \leqslant \varepsilon_1$, $\sup_{(x,y)\in Q_a} |w_\lambda(x,y)| = 0$, hence $(P_{K,l}(w_\lambda))_\lambda \in |I_A|$ for any $l \in \mathbb{N}$ as w_λ vanishes on K. Thus $(w_\lambda)_\lambda \in \mathscr{N}(\Omega)$ and $v = u|_\Omega$ as claimed.

EXAMPLE 4. Assume that $\lambda = (\mu(\lambda), \nu(\lambda)) = (\varepsilon, \rho) \in \Lambda_1 \times \Lambda_2 = \Lambda$, $\Lambda_1 = \Lambda_2 = (0,1]$. Consider the problem

$$(P_{form}) \begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = u^2, \\ u|_{(Ox)} = vp(\frac{1}{1-x}), \\ u|_{(Oy)} = vp(\frac{1}{1-y}). \end{cases}$$

This problem is classically highly ill-posed. Let be (P_{gen}) the generalized associated problem as it is done in Subsection 3.4.

$$(P_{gen}) \begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = \mathscr{F}(u), \\ \mathscr{R}_\theta(u) = \varphi, \\ \mathscr{L}_g(u) = \psi, \end{cases}$$

where $\mathscr{F}$ is associated to $F = u^2$ via the family $(f_\varepsilon)_\varepsilon$ given in Subsection 3.1, by Definition 7. The generalized functions $\varphi = [\varphi_\rho] \in \mathscr{A}(\mathbb{R})$ and $\psi = [\psi_\rho] \in \mathscr{A}(\mathbb{R})$ are constructed from

$$\varphi_\rho(x) = \left(\theta_\rho * vp(\frac{1}{1-\cdot})\right)(x) = \langle vp(\frac{1}{1-z}), z \mapsto \theta_\rho(x-z)\rangle = \lim_{\varepsilon\to 0} \int_{|1-z|>\varepsilon} \frac{\theta_\rho(x-z)}{1-z} dz$$

$$\psi_\rho(y) = \left(\theta_\rho * vp(\frac{1}{1-\cdot})\right)(y) = \langle vp(\frac{1}{1-z}), z \mapsto \theta_\rho(y-z)\rangle = \lim_{\varepsilon\to 0} \int_{|1-z|>\varepsilon} \frac{\theta_\rho(y-z)}{1-z} dz$$

where $(\theta_\rho)_\rho$ is a chosen family of mollifiers. Then φ_ρ (resp. ψ_ρ) regularize $vp(\frac{1}{1-x})$ (resp. $vp(\frac{1}{1-y})$). To solve the problem (P_{gen}) associate to (P_{form}) we can consider (as it is done in Subsection 3.4) the family of problems

$$(P_\lambda)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u_\lambda(x,y) = \left(u_\lambda(x,y)\right) f_{\mu(\lambda)}(u_\lambda(x,y)))^2, \\ u_\lambda(x,0) = \varphi_{\nu(\lambda)}(x), \\ u_\lambda(0,y) = \psi_{\nu(\lambda)}(y). \end{cases}$$

If u_λ is a solution to (P_λ) then $u = [u_\lambda]$ is solution to (P_{gen}). We have the restrictions

$$vp(\frac{1}{1-x})\Big|_{(Ox)} = \left(x \mapsto \frac{1}{1-x}\right); vp(\frac{1}{1-y})\Big|_{(Oy)} = \left(y \mapsto \frac{1}{1-y}\right).$$

Then (P_{form}) has the classical solution v in $\mathrm{C}^\infty(\Omega)$ where $\Omega = (-\infty, 1) \times (-\infty, 1)$,

$$v(x,y) = \frac{1}{(1-x)(1-y)},$$

and Theorem 13 shows that the restriction of $u \in \mathscr{A}(\mathbb{R}^2)$ to Ω is precisely v. The local classical solution v which blows-up for $x = 1$, $y = 1$, extends to a global generalized solution u which absorbs this blow-up.

REMARK 9. In the case of regular data, whenever problem (P_{gen}) admits a classical smooth function v on some open subset Ω, we can show that v (element of $\mathrm{C}^\infty(\Omega)$ canonically embedded in $\mathscr{A}(\Omega)$) is the restriction (according to the sheaf theory sense) of u to Ω as it is described in Theorem 13.

REFERENCES

[1] J.-F. COLOMBEAU, *Elementary introduction to new generalized functions*, North Holland Math. Studies 113. North-Holland, Amsterdam, 1984.

[2] G. M. CONSTANTINE, H. SAVITS, *A multivariate Faà di Bruno formula with applications,* Trans. Amer. Math. Soc., **348**, 2 (1996), 503–520.

[3] A. DELCROIX, *Remarks on the embedding of spaces of distributions into spaces of Colombeau generalized functions*, Novisad J. math., **35**, 2 (2005), 27–40.

[4] A. DELCROIX, *Some properties of $(\mathscr{C},\mathscr{E},\mathscr{P})$ -algebras : Overgeneration and 0^{th} -order estimates.* Preprint.

[5] A. DELCROIX, D. SCARPALÉZOS, *Topology on Asymptotic Algebras of Generalized Functions and Applications*, Monatsh. Math., **129** (2000), 1–14.

[6] A. DELCROIX, V. DÉVOUÉ, J.-A. MARTI, *Generalized solutions of singular differential problems. Relationship with classical solutions.* J. Math. Anal. Appl., **353**, 1 (2009), 386–402.

[7] A. DELCROIX, V. DÉVOUÉ, J.-A. MARTI, *Wellposed differential problems in Algebras of Generalized Functions*, Preprint.

[8] V. DÉVOUÉ, *Sur les singularités de certains problèmes différentiels*, Thèse de doctorat, Université des Antilles et de la Guyane (2005).

[9] V. DÉVOUÉ, *On generalized solutions to the wave equation in canonical form*, Dissertationes Math., **443**, (2007), 1–69.

[10] V. DÉVOUÉ, *Generalized solutions to a non Lipschitz Cauchy problem*, J. Appl. Anal. Vol., **15**, 1 (2009), 1–32.

[11] M. GROSSER, M. KUNZINGER, M. OBERGUGGENBERGER, R. STEINBAUER, *Geometric theory of generalized functions with applications to general relativity*, Mathematics and its Applications 537. Kluwer Academic Publ., Dordrecht, 2001.

[12] J.-A. MARTI, *Fundamental structures and asymptotic microlocalization in sheaves of generalized functions*, Integral Transforms Spec. Funct., **6**, 1-4 (1998), 223–228.

[13] J.-A. MARTI, $(\mathscr{C},\mathscr{E},\mathscr{P})$ *-Sheaf Structures and Applications*, In: Nonlinear Theory of Generalized Functions, M. Grosser and alii, Eds. Research notes in Mathematics. Chapman & Hall/CRC (1999), 175–186.

[14] J.-A. MARTI, *Multiparametric Algebras and Characteristic Cauchy Problem*, Non-linear algebraic analysis and applications, Proceeding of the International Conference on Generalized functions (ICGF 2000). CSP (2004), 181–192.

[15] J.-A. MARTI, *Non linear Algebraic Analysis of Delta Shock wave solutions to Burger's Equation*, Pacific J. Math., **210**, 1 (2003), 165–187.

[16] J.-A. MARTI, S. P. NUIRO, V. S. VALMORIN, *A non linear Goursat problem with irregular data*, Integral Transforms Spec. Funct., **6**, 1-4 (1998), 229–246.

[17] M. NEDELJKOV, M. OBERGUGGENBERGER, S. PILIPOVIĆ, *Generalized solution to a semilinear wave equation*, Nonlinear Anal., **61** (2005), 461–475.

[18] M. OBERGUGGENBERGER, *Multiplication of Distributions and Applications to Partial Differential Equations*, Pitman Research Notes in Mathematics 259. Longman, Harlow, 1992.

[19] J. SHATAH, M. STRUWE, *Geometric Wave Equations*, CIMS Lecture Notes, vol. 2, Courant Inst. Math. Sci., New York, 1998.

[20] M. E. TAYLOR, *Partial Differential equations III. Nonlinear Equations*, Applied Mathematical Sciences 117. Springer-Verlag, New York, 1996.

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Victor DÉVOUÉ
Equipe Analyse Algébrique Non Linéaire
Département Scientifique Interfacultaire
Université des Antilles et de la Guyane
Campus de Schoelcher BP 7209
97275 Schoelcher Cedex Martinique F.W.I.
France
e-mail: victor.devoue@orange.fr