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CLASSICAL SOLUTIONS OF QUASILINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS ON THE HAAR PYRAMID

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Abstract. The Cauchy problem for a quasilinear functional differential system is considered. A theorem on the existence of classical solutions defined on the Haar pyramid is proved. The theory of bicharacteristics and the method of successive approximations are used. Differential systems with deviated variables and differential integral systems can be obtained from a general theory by specializing given operators.

1. Introduction

For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let us denote by $M_{k\times n}$ the set of all $k\times n$ matrices with real elements. If $U\in M_{k\times n}$ then U^T denotes the transpose matrix. Let E be the Haar pyramid

$$E=\left\{(t,x)\in\mathbb{R}^{1+n}: t\in[0,a], -b+Mt\leqslant x\leqslant b-Mt\right\},$$

where $a>0$, $M=(M_1,\ldots,M_n)\in\mathbb{R}^n_+$, $\mathbb{R}_+=[0,+\infty)$, $b=(b_1,\ldots,b_n)$ and $b>Ma$. Write for $b_0\in\mathbb{R}_+$

$$E_0=[-b_0,0]\times[-b,b].$$

For $(t,x)\in E$ we define the set $D[t,x]$ as follows

$$D[t,x]=\left\{(\tau,y)\in\mathbb{R}^{1+n}: \tau\geqslant 0, (t+\tau,x+y)\in E_0\cup E\right\}.$$

It is clear that $D[t,x]=D_0[t,x]\cup D_*[t,x]$ where

$$D_0[t,x]=[-b_0-t,-t]\times[-b-x,b-x],$$
$$D_*[t,x]=\{(\tau,y): -t\leqslant\tau\leqslant 0, -b-x+M(\tau+t)\leqslant y\leqslant b-x-M(\tau+t)\}.$$

Write $B=[-b_0-a,0]\times[-2b,2b]$. Then we have $D[t,x]\subset B$ for $(t,x)\in E$. For a function $z\colon E_0\cup E\to\mathbb{R}^k$ and for a point $(t,x)\in E$ we define a function $z_{(t,x)}: D[t,x]\to$

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$\mathbb{R}^k$ as follows: $z_{(t,x)}(\tau,y) = z(t+\tau, x+y)$, $(\tau,y) \in D[t,x]$. Then $z_{(t,x)}$ is the restriction of z to set $(E_0 \cup E) \cap ([-b_0,t] \times \mathbb{R}^n)$ and this restriction is shifted to the set $D[t,x]$. Put $\Omega = E \times C(B, \mathbb{R}^k)$ and suppose that

$$f\colon \Omega \to M_{k\times n}, \quad f = [f_{ij}]_{i=1,\dots,k,\, j=1,\dots,n},$$
$$G\colon \Omega \to \mathbb{R}^k, \quad G = (G_1,\dots,G_k),$$
$$\varphi\colon E_0 \to \mathbb{R}^k, \quad \varphi = (\varphi_1,\dots,\varphi_k),$$
$$\psi_0\colon [0,a] \to \mathbb{R}, \quad \psi'\colon E \to \mathbb{R}^n, \quad \psi' = (\psi_1,\dots,\psi_n)$$

are given function. We assume that $0 \leqslant \psi_0(t) \leqslant t$ for $t \in [0,a]$ and that $(\psi_0(t), \psi'(t,x)) \in E$ for $(t,x) \in E$. Write $\psi(t,x) = (\psi_0(t), \psi'(t,x))$ on E. We will say that the functions f and G satisfy the condition (V) if for each $(t,x) \in E$ and for $w, \tilde{w} \in C(B, \mathbb{R}^k)$ such that $w(\tau,y) = \tilde{w}(\tau,y)$ for $(\tau,y) \in D[\psi(t,x)]$ we have $f(t,x,w) = f(t,x,\tilde{w})$ and $G(t,x,w) = G(t,x,\tilde{w})$. Then the condition (V) means that the values of f and G at the point $(t,x,w) \in \Omega$ depend on (t,x) and on the restriction of w to the set $D[\psi(t,x)]$ only.

Let $z = (z_1,\dots,z_k)$ be an unknown function of the variables $(t,x) = (t,x_1,\dots,x_n)$. We consider the system of differential functional equations

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} f_{ij}(t,x,z_{\psi(t,x)}) \partial_{x_j} z_i(t,x) = G_i(t,x,z_{\psi(t,x)}), \quad i = 1,\dots,k, \tag{1}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0. \tag{2}$$

We assume that f and G satisfy the condition (V) and we consider classical solutions of (1), (2).

If $D[t,x] = \{(t,x)\}$ and $\psi(t,x) = (t,x)$ for $(t,x) \in E$ then (1), (2) reduces to a classical Cauchy problem for a weakly coupled differential system. Such systems and more general quasilinear hyperbolic systems in the Schauder canonic form have been studied in a large number of papers by various authors. Sufficient conditions for the existence and uniqueness of Carathéodory solutions of initial or boundary value problems can be found in [2], [5].

The papers [1], [13], [14] initiated investigations of first order partial differential functional equations. Mixed problems for almost linear or quasilinear systems in two independent variables were considered and existence results on continuous generalized solutions were obtained. A continuous function is a solution of mixed problem if it satisfies integral functional differential system arising from a original system by integrating along bicharacteristics. Results on the existence and uniqueness of Carathéodory solutions for a general class of quasilinear hyperbolic systems with initial conditions which are global with respect to spatial variables can be found in [7]. The papers [8], [9] concern mixed problems and nonlocal boundary value problems for weakly coupled quasilinear systems.

Until now there have not been any results on the existence of classical solutions of initial problems on the Haar pyramid. The aim of this paper is to prove a theorem on the local existence of classical solutions to (1), (2).

Note that different models of the functional dependence in partial equations are used in literature. The first group of results is connected with initial problems for system

$$\partial_t z_i(t,x) = G_i(t,x,z,\partial_x z_i(t,x)), \quad i = 1,\dots,k, \tag{3}$$

where the variable z represents the functional variable. Existence results for (3) can be characterized as follows: theorems have simple assumptions and their proofs are very natural [15], [16]. Unfortunately, a small class of differential functional equations is covered by this theory. It is easy to see that results given in the above papers are not applicable to differential integral systems of the Volterra type and to systems with deviated variables.

There are papers concerning initial value problems for systems

$$\partial_t z_i(t,x) = H_i(t,x,(Vz)(t,x),\partial_x z_i(t,x)), \quad i = 1,\dots,k, \tag{4}$$

where V is an operator of the Volterra type and the function $(H_1,\dots,H_k)$ is defined on finite-dimentional Euclidean space. The main assumptions in existence theorems for (4) concern the operator V. They are formulated in a form of inequalities for norms in some functional spaces [3], [6], [10]. These inequalities are linear and it is the main shortcoming of this theory.

A new model of a functional dependence in partial differential equations is proposed in [11]. It is based on the following idea. Let $B = [-r_0,0]\times[-r,r] \subset \mathbb{R}^{1+n}$. For a function $z\colon [-r_0,a]\times\mathbb{R}^k$, $a>0$ and for a point $(t,x)\in[0,a]\times\mathbb{R}^n$ we define a function $z_{(t,x)}: B\to\mathbb{R}^k$ in the following way:

$$z_{(t,x)}(\tau,y) = z(t+\tau,x+y), \quad (\tau,y)\in B. \tag{5}$$

Then $z_{(t,x)}$ is the restriction of z to the set $[t-r_0,t]\times[x-r,x+r]$ and this restriction is shifted to the set B. If $r\neq(0,\dots,0)\in\mathbb{R}^n$ then there is $(t,x)\in E$ such that the formulation (5) is not suitable for the local Cauchy problem and consequently, the results given in [4] are not applicable to local Cauchy problems. For bibliography on hyperbolic functional differential equations and their applications see the monograph [12].

In the paper we propose a new method of the functional dependence in quasilinear systems considered on the Haar pyramid.

We prove that under natural assumptions on given functions there exists exactly one solution of (1), (2). The method used in this paper is based on the bicharacteristics theory and on the method of successive approximations.

We give examples of quasilinear systems which can be obtained from (1) by specializing f and G.

EXAMPLE 1.1. Suppose that

$$\tilde G = (\tilde G_1,\dots,\tilde G_k)\colon E\times\mathbb{R}^k\to\mathbb{R}^k,\ \tilde f = [f_{ij}]_{i=1,\dots,k,j=1,\dots,n}\colon E\times\mathbb{R}^k\to M_{k\times n}$$

is a given function. Set $f(t,x,w)=\tilde f(t,x,w(0,\theta))$ and $G(t,x,w)=\tilde G(t,x,w(0,\theta))$ where $\theta=(0,\dots,0)\in\mathbb{R}^n$. Then (1) becomes the system of equations with deviated variables

$$\partial_t z_i(t,x) + \sum_{j=1}^n \tilde f_{ij}(t,x,z(\psi(t,x)))\partial_{x_j} z_i(t,x) = \tilde G_i(t,x,z(\psi(t,x))), \quad i=1,\dots,k.$$

EXAMPLE 1.2. Suppose that $\psi(t,x) = (t,x)$. For the above $\tilde{G}$, $\tilde{f}$ we put

$$f(t,x,w) = \tilde{f}(t,x,\int_{D[t,x]} w(\tau,y)d\tau dy), \quad G(t,x,w) = \tilde{G}(t,x,\int_{D[t,x]} w(\tau,y)d\tau dy).$$

Then (1) is equivalent to the system of differential integral equations

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \tilde{f}_{ij}(t,x,\int_{D[t,x]} z_{(t,x)}(\tau,y)d\tau dy)\partial_{x_j} z_i(t,x)$$
$$= \tilde{G}_i(t,x,\int_{D[t,x]} z_{(t,x)}(\tau,y)d\tau dy), \quad i = 1,\dots,k.$$

It is clear that more complicated examples of differential systems with deviated variables and differential integral systems can be obtained from (1) by suitable definitions of f and G.

2. Integral functional equations

For $x \in \mathbb{R}^n$, $p \in \mathbb{R}^k$, $X \in M_{k\times n}$ where $x = (x_1,\dots,x_n)$, $p = (p_1,\dots,p_k)$, $X = [x_{ij}]_{i=1,\dots,k,\, j=1,\dots,n}$ we define the norms

$$||x|| = |x_1| + \dots + |x_n|, \quad ||p||_\infty = \max\{|p_i| : 1 \leqslant i \leqslant k\},$$

$$||X|| = \max\{\sum_{j=1}^{n} |x_{ij}| : 1 \leqslant i \leqslant k\}.$$

The scalar product in $\mathbb{R}^n$ will be denoted by " $\circ$ ". Write $E_t = (E_0 \cup E) \cap ([-b_0,t] \times \mathbb{R}^n)$ where $0 \leqslant t \leqslant a$. For functions $z \in C(E_0 \cup E, \mathbb{R}^k)$, $z = (z_1,\dots,z_k)$, $v \in C(E_0 \cup E, \mathbb{R}^n)$, $v = (v_1,\dots,v_n)$, we define the seminorms:

$$||z_i||_t = \max\{|z_i(\tau,y)| : (\tau,y) \in E_t\}, \quad i = 1,\dots,k,$$
$$||z||_{(t,\mathbb{R}^k)} = \max\{||z(\tau,y)||_\infty : (\tau,y) \in E_t\},$$
$$||v||_{(t,\mathbb{R}^n)} = \max\{||v(\tau,y)|| : (\tau,y) \in E_t\},$$

where $0 \leqslant t \leqslant a$. The norm in the space $C(B,\mathbb{R}^k)$ is given by

$$||w||_B = \max\{||w(\tau,y)||_\infty : (\tau,y) \in B\}.$$

We denote by $LC(B,\mathbb{R})$ the set of all linear and continuous real functions defined by $C(B,\mathbb{R})$. Let $||\cdot||_*$ be the norm in $LC(B,\mathbb{R})$ generated by the maximum norm in the space $C(B,\mathbb{R})$. For $W = [w_{ij}]_{i,j=1,\dots,k,}$ and $w_{ij} \in LC(B,\mathbb{R})$ we write

$$||W||_{*,\infty} = \max\{\sum_{j=1}^{k} ||w_{ij}||_* : 1 \leqslant i \leqslant k\}.$$

Given $\tilde{c} = (c_0,c_1,c_2) \in \mathbb{R}^3_+$. We denote by Φ the class of all $\varphi \in C(E_0,\mathbb{R}^k)$, $\varphi = (\varphi_1,\dots,\varphi_k)$, such that

(i) the derivatives $\partial_x\varphi_i = (\partial_{x_1}\varphi_i,\dots,\partial_{x_n}\varphi_i)$ exist on E_0 and $\partial_x\varphi_i \in C(E_0,\mathbb{R}^n)$ for $i = 1,\dots,k$,

(ii) the estimates

$$||\partial_x\varphi_i(t,x)|| \leqslant c_0,$$
$$||\partial_x\varphi_i(t,x)-\partial_x\varphi_i(\bar{t},\bar{x})|| \leqslant c_1|t-\bar{t}|+c_2||x-\bar{x}||$$

are satisfied on E_0 for $1 \leqslant i \leqslant k$.

Let $\varphi \in \Phi$ be given and $0 < c \leqslant a$. Suppose that $d \in \mathbb{R}_+$ and $d \geqslant c_0$. We denote by $C_{\varphi.c}[d]$ the class of all $z \in C(E_c,\mathbb{R}^k)$ such that $z(t,x) = \varphi(t,x)$ on E_0 and

$$||z(t,x)-z(t,\bar{x})||_\infty \leqslant d||x-\bar{x}|| \quad \text{on} \quad E_c.$$

Assume that $s = (s_0,s_1,s_2) \in \mathbb{R}^3_+$ and $s \geqslant \tilde{c}$. Let $C_{\partial\varphi_i.c}[s]$ be the class of all $v \in C(E_c,\mathbb{R}^n)$ such that $v(t,x) = \partial_x\varphi_i$ on E_0 and

$$||v(t,x)|| \leqslant s_0,$$
$$||v(t,x)-v(\bar{t},\bar{x})|| \leqslant s_1|t-\bar{t}|+s_2||x-\bar{x}|| \quad \text{on} \quad E_c.$$

We put $i = 1,\dots,k$ in the above definitions.

Now we formulate assumptions on ψ and f, G.

Assumption $H[\psi]$ The functions $\psi_0\colon [0,a] \to \mathbb{R}$, $\psi'\colon E \to \mathbb{R}^n$ are continuous and

1) $0 \leqslant \psi_0(t) \leqslant t$ for $t \in [0,a]$ and $\psi(t,x) = (\psi_0(t),\psi'(t,x)) \in E$ for $(t,x) \in E$,

2) there exist the partial derivatives $\partial_x\psi'(t,x) = \left[\partial_{x_j}\psi_i(t,x)\right]_{i,j=1,\dots,n}$ and $\partial_x\psi' \in C(E,M_{n\times n})$,

3) there is $\tilde{Q} \in \mathbb{R}_+$ such that

$$||\partial_x\psi'(t,x)-\partial_x\psi'(t,\bar{x})|| \leqslant \tilde{Q}||x-\bar{x}|| \quad \text{on} \quad E.$$

For $f\colon \Omega \to M_{k\times n}$, $f = [f_{ij}]_{i=1,\dots,k,\,j=1,\dots,n}$ we write $f_{[i]} = (f_{i1},\dots,f_{in})$, $1 \leqslant i \leqslant k$.

Assumption $H[f]$ The function $f\colon \Omega \to M_{k\times n}$ of the variables (t,x,w), $w = (w_1,\dots,w_k)$, is continuous and

1) f satisfies the condition (V) and for $(t,x,w) \in \Omega$ we have

$$(|f_{i1}(t,x,w)|,\dots,|f_{in}(t,x,w)|) \leqslant (M_1,\dots,M_n), \quad i = 1,\dots,k,$$

2) the derivatives
$$\partial_x f_{[i]} = \left[\partial_{x_\nu} f_{i\mu}\right]_{\mu,\nu=1,\dots,n}$$
exist on Ω and $\partial_x f_{[i]} \in C(\Omega,M_{n\times n})$ for $i = 1,\dots,k$,

3) there exist the Fréchet derivatives

$$\partial_w f_{[i]}(P) = \left[\, \partial_{w_\nu} f_{i\mu}(P) \,\right]_{\mu=1,\ldots,n,\nu=1,\ldots,k}, \quad P=(t,x,w)\in\Omega,$$

and $\partial_{w_\nu} f_{i\mu}(P) \in LC(B,\mathbb{R})$ where $P\in\Omega$, $\mu=1,\ldots,n$, $\nu=1,\ldots,k$,

4) there are $A,C\in\mathbb{R}_+$ such that

$$||\partial_x f_{[i]}(t,x,w)|| \leqslant A, \quad ||\partial_w f_{[i]}(t,x,w)||_{*,\infty} \leqslant A,$$
$$||\partial_x f_{[i]}(t,x,w) - \partial_x f_{[i]}(t,\bar{x},\bar{w})|| \leqslant C\left[||x-\bar{x}|| + ||w-\bar{w}||_B\right],$$
$$||\partial_w f_{[i]}(t,x,w) - \partial_w f_{[i]}(t,\bar{x},\bar{w})||_{*,\infty} \leqslant C\left[||x-\bar{x}|| + ||w-\bar{w}||_B\right],$$

where (t,x,w), $(t,\bar{x},\bar{w})\in\Omega$, $i=1,\ldots,k$.

Assumption $H[G]$ The function $G\colon \Omega\to\mathbb{R}^k$ of the variables (t,x,w) is continuous, it satisfies the condition (V) and

1) the derivatives

$$\partial_x G = \left[\, \partial_{x_j} G_i \,\right]_{i=1,\ldots,k,\, j=1,\ldots,n}$$

exist on Ω and $\partial_x G\in C(\Omega, M_{k\times n})$,

2) the Fréchet derivatives

$$\partial_w G(P) = \left[\, \partial_{w_\nu} G_\mu(P) \,\right]_{\mu,\nu=1,\ldots,k}$$

exist for $P=(t,x,w)\in\Omega$ and $\partial_{w_\nu} G_\mu(P)\in LC(B,\mathbb{R})$ for $P\in\Omega$, $\mu,\nu=1,\ldots,k$,

3) the estimates

$$||\partial_x G(t,x,w)|| \leqslant A, \quad ||\partial_w G(t,x,w)||_{*,\infty} \leqslant A,$$
$$||\partial_x G(t,x,w) - \partial_x G(t,\bar{x},\bar{w})|| \leqslant C\left[||x-\bar{x}|| + ||w-\bar{w}||_B\right],$$
$$||\partial_w G(t,x,w) - \partial_w G(t,\bar{x},\bar{w})||_{*,\infty} \leqslant C\left[||x-\bar{x}|| + ||w-\bar{w}||_B\right],$$

are satisfied on Ω.

LEMMA 2.1. *Suppose that* $\Lambda\colon \Omega\to\mathbb{R}^n$ *is continuous and*

1) $\psi=(\psi_0,\psi')$ *satisfies conditions* 1), 2) *of Assumption* $H[\psi]$ *and* $Q_0=\max\{||\partial_x\psi'(t,x)|| : (t,x)\in E\}$,

2) *there is* $\lambda\in\mathbb{R}_+$ *such that*

$$||\Lambda(t,x,w) - \Lambda(t,\bar{x},\bar{w})|| \leqslant \lambda\, [||x-\bar{x}|| + ||w-\bar{w}||_B] \quad on \quad \Omega$$

and Λ *satisfies the condition* (V),

3) $\varphi\in\Phi$ *and* $z\in C_{\varphi.c}[d]$.

Then

$$||\Lambda(t,x,z_{\psi(t,x)}) - \Lambda(t,\bar{x},z_{\psi(t,\bar{x})})|| \leqslant \lambda(1+dQ_0)||x-\bar{x}||.$$

Proof. Note that the functions $z_{\psi(t,x)}$ and $z_{\psi(t,\bar{x})}$ have different domains. Then we need the following construction. There is $\tilde{z}\colon \mathbb{R}^{1+n} \to \mathbb{R}^k$ such that

(i) $\tilde{z}(t,x) = z(t,x)$ on $E_0 \cup E$ and $\tilde{z} \in C(\mathbb{R}^{1+n},\mathbb{R}^k)$,

(ii) $\max\{||z(t,x)||_\infty : (t,x) \in E_0 \cup E\} = \sup\{||\tilde{z}(t,x)||_\infty : (t,x) \in \mathbb{R}^{1+n}\}$ and

$$||\tilde{z}(t,x) - \tilde{z}(t,\bar{x})||_\infty \leqslant d||x-\bar{x}|| \quad \text{on} \quad \mathbb{R}^{1+n}.$$

Let $w,\bar{w}\colon B \to \mathbb{R}^k$ be defined by

$$w(\tau,y) = \tilde{z}_{\psi(t,x)}(\tau,y) = \tilde{z}(\psi(t,x)+(\tau,y)), \quad (\tau,y) \in B,$$
$$\bar{w}(\tau,y) = \tilde{z}_{\psi(t,\bar{x})}(\tau,y) = \tilde{z}(\psi(t,\bar{x})+(\tau,y)), \quad (\tau,y) \in B.$$

Then we have

$$\begin{aligned}||\Lambda(t,x,z_{\psi(t,x)}) - \Lambda(t,\bar{x},z_{\psi(t,\bar{x})})|| &= ||\Lambda(t,x,w) - \Lambda(t,\bar{x},\bar{w})||\\ &\leqslant \lambda\,[||x-\bar{x}|| + ||w-\bar{w}||_B] = \lambda\,\left[||x-\bar{x}|| + ||\tilde{z}_{\psi(t,x)} - \tilde{z}_{\psi(t,\bar{x})}||_B\right]\\ &\leqslant \lambda(1+dQ_0)||x-\bar{x}||.\end{aligned}$$

This completes the proof.

Suppose that Assumptions $H[f]$, $H[\psi]$ are satisfied and $\varphi \in \Phi$, $z \in C_{\varphi.c}[d]$, $1 \leqslant i \leqslant k$. Let us denote by $g_i[z](\cdot,t,x)$ the solution of the Cauchy problem

$$\eta'(\tau) = f_{[i]}(\tau,\eta(\tau),z_{\psi(\tau,\eta(\tau))}), \quad \eta(t) = x, \tag{6}$$

where $(t,x) \in E_c$. The function $g_i[z](\cdot,t,x)$ is the i-th bicharacteristic of (1) corresponding to z. The main properties of the bicharacteristic are presented in the following lemma.

LEMMA 2.2. *Suppose that Assumption $H[f]$ is satisfied and $\varphi,\tilde{\varphi} \in \Phi$, $z \in C_{\varphi.c}[d]$, $\tilde{z} \in C_{\tilde{\varphi}.c}[d]$ where $0 < c \leqslant a$. Then the bicharacteristics $g_i[z](\cdot,t,x)$ and $g_i[\tilde{z}](\cdot,t,x)$, $1 \leqslant i \leqslant k$, exist on intervals $[0,\kappa_i(t,x)]$ and $[0,\tilde{\kappa}_i(t,x)]$ such that*

$$(\kappa_i(t,x), g_i[z](\kappa_i(t,x),t,x)) \in \partial E_c \text{ and } (\tilde{\kappa}_i(t,x), g_i[z](\tilde{\kappa}_i(t,x),t,x)) \in \partial E_c,$$

where ∂E_c is the boundary of E_c. Solutions of (6) are unique and we have the estimates

$$||g_i[z](\tau,t,x) - g_i[z](\tau,\bar{t},\bar{x})|| \leqslant L\left[|t-\bar{t}| + ||x-\bar{x}||\right] \tag{7}$$

where $\tau \in [0,\min\{\kappa_i(t,x),\,\kappa_i(\bar{t},\bar{x})\}]$, $L = \max\{1,||M||,C\}e^{Ca(1+dQ_0)}$, and

$$||g_i[z](\tau,t,x) - g_i[\tilde{z}](\tau,t,x)|| \leqslant L\Big|\int_t^\tau ||z-\tilde{z}||_{(\xi,\mathbb{R}^k)}d\xi\Big| \tag{8}$$

where $\tau \in [0,\min\{\kappa(t,x),\,\tilde{\kappa}(t,x)\}]$.

Proof. The existence and uniqueness of the solution of (6) follows from classical theorems. We prove that the bicharacteristic $g_i[z](\cdot,t,x)$ exists on $[0,\kappa(t,x)]$. Suppose that $[t_0,t]$ is the interval on which the bicharacteristic $g_i[z](\cdot,t,x)$ is defined. Then we have

$$-M \leqslant \frac{d}{d\tau} g_i[z](\tau,t,x) \leqslant M \quad \text{for} \quad \tau \in [t_0,t],$$

and consequently

$$-b + M\tau \leqslant g_i[z](\tau,t,x) \leqslant b - M\tau \quad \text{for} \quad \tau \in [t_0,t].$$

This gives that $(\tau, g_i[z](\tau,t,x)) \in E$ for $\tau \in [t_0,t]$ and the assertion follows.

We conclude from (6) and from Lemma 2.1 that

$$||g_i[z](\tau,t,x) - g_i[z](\tau,\bar{t},\bar{x})|| \leqslant ||x-\bar{x}|| + ||M|| \, |t-\bar{t}| \\ + C(1+dQ_0) \Big| \int_\tau^{\min\{t,\bar{t}\}} ||g_i[z](\xi,t,x) - g_i[z](\xi,\bar{t},\bar{x})|| d\xi \Big|.$$

From the Gronwall inequality we obtain (7). For $z \in C_{\varphi.c}[d]$, $\tilde{z} \in C_{\tilde{\varphi}.c}[d]$ we have the integral inequality

$$||g_i[z](\tau,t,x) - g_i[\tilde{z}](\tau,t,x)|| \\ \leqslant C(1+dQ_0) \Big| \int_0^t ||g_i[z](\tau,t,x) - g_i[\tilde{z}](\tau,t,x)|| d\xi \Big| + C \Big| \int_\tau^t ||z-\tilde{z}||_{(\xi,\mathbb{R}^k)} d\xi \Big|,$$

where $\tau \in [0, \min\{\kappa(t,x), \tilde{\kappa}(t,x)\}]$. From the Gronwall inequality we obtain (8). This completes the proof.

We formulate integral equations corresponding to (1), (2). Let us denote by z and u unknown functions of the variables (t,x) where

$$z = (z_1, \ldots, z_k), \quad u = [u_{ij}]_{i=1,\ldots,k,\, j=1,\ldots,n}$$

and

$$u_{[i]} = (u_{i1}, \ldots, u_{in}), \quad (u_{[i]})_{(t,x)} = \big((u_{i1})_{(t,x)}, \ldots, (u_{in})_{(t,x)}\big), \quad 1 \leqslant i \leqslant k.$$

Let

$$\big(u_{[j]}\big)_{\psi(t,x)} \, \partial_x \psi'(t,x) \colon D[\psi(t,x)] \to \mathbb{R}^n$$

be the function defined by

$$\big(u_{[j]}\big)_{\psi(t,x)} \, \partial_x \psi'(t,x) = \Big(\sum_{\nu=1}^n \partial_{x_1} \psi_\nu(t,x) (u_{j\nu})_{\psi(t,x)}, \ldots, \sum_{\nu=1}^n \partial_{x_n} \psi_\nu(t,x) (u_{j\nu})_{\psi(t,x)} \Big).$$

For $\omega = (\omega_1, \ldots, \omega_n) \in C(B, \mathbb{R}^n)$ we write

$$\partial_{w_j} f_{i\mu}(P)\omega = \big(\partial_{w_j} f_{i\mu}(P)\omega_1, \ldots, \partial_{w_j} f_{i\mu}(P)\omega_n\big), \\ \partial_{w_j} G_i(P)\omega = \big(\partial_{w_j} G_i(P)\omega_1, \ldots, \partial_{w_j} G_i(P)\omega_n\big),$$

and

$$\partial_w G_i(P) * \omega = \sum_{\nu=1}^{k} \partial_{w_\nu} G_i(P)\omega_\nu,$$

where $P = (t,x,w) \in \Omega$, $i,j = 1,\ldots,k$, $\mu = 1,\ldots,n$. Let us denote by $V_{[i\mu]}[z,u]$, $W_{[i]}[z,u]$, $i = 1,\ldots,k$, $\mu = 1,\ldots,n$ the functions defined by

$$V_{[i\mu]}[z,u](t,x) = \partial_x f_{i\mu}(t,x,z_{\psi(t,x)}) + \sum_{j=1}^{k} \partial_{w_j} f_{i\mu}(t,x,z_{\psi(t,x)})(u_{[j]})_{\psi(t,x)} \partial_x \psi'(t,x),$$

and

$$W_{[i]}[z,u](t,x) = \partial_x G_i(t,x,z_{\psi(t,x)}) + \sum_{j=1}^{k} \partial_{w_j} G_i(t,x,z_{\psi(t,x)})(u_{[j]})_{\psi(t,x)} \partial_x \psi'(t,x).$$

We consider the following system of functional integral equations

$$z_i(t,x) = \varphi_i(0,g_i[z](0,t,x)) + \int_0^t G_i(\tau,g_i[z](\tau,t,x),z_{\psi(\tau,g_i[z](\tau,t,x))})d\tau, \tag{9}$$

$$\begin{aligned} u_{[i]}(t,x) = \partial_x \varphi_i(0,g_i[z](0,t,x)) \\ - \sum_{\mu=1}^{n} \int_0^t V_{[i\mu]}[z,u](\tau,g_i[z](\tau,t,x))u_{i\mu}(\tau,g_i[z](\tau,t,x))d\tau \\ + \int_0^t W_{[i]}[z,u](\tau,g_i[z](\tau,t,x))d\tau, \end{aligned} \tag{10}$$

where $i = 1,\ldots,k$ and

$$z_i(t,x) = \varphi(t,x), \quad u_{[i]}(t,x) = \partial_x \varphi_i(t,x) \quad \text{on} \quad E_0 \quad \text{for} \quad i = 1,\ldots,k. \tag{11}$$

We prove that for sufficiently small $c \in (0,a]$ there exist a solution $(\tilde{z}_i, \tilde{u}_{[i]}) \colon E_c \to \mathbb{R} \times \mathbb{R}^n$, $i = 1,\ldots,k$, of above system of integral functional equations and $(\tilde{z}_1,\ldots,\tilde{z}_k)$ is a solution of (1), (2) and $\partial_x \tilde{z}_i = \tilde{u}_i$ for $i = 1,\ldots,k$.

3. Successive approximations for functional integral equations

The proof of the existence of the solutions of (9) - (11) is based on the following method of successive approximations. Suppose that $\varphi \in \Phi$ and that Assumptions $H[f]$, $H[G]$, $H[\psi]$ are satisfied. We consider sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$ where:

$$z^{(m)} = (z_1^{(m)},\ldots,z_k^{(m)}), \quad u^{(m)} = [u_{ij}^{(m)}]_{i=1,\ldots k,\, j=1,\ldots,n},$$
$$u_{[i]}^{(m)} = (u_{i1}^{(m)},\ldots,u_{in}^{(m)}), \quad i = 1,\ldots,k,$$

defined in the following way. Write

$$z_i^{(0)}(t,x) = \varphi_i(t,x) \quad \text{on} \quad E_0, \quad z_i^{(0)}(t,x) = \varphi_i(0,x) \quad \text{for} \quad (t,x) \in E_c \backslash E_0 \tag{12}$$

and

$$u_{[i]}^{(0)}(t,x)=\partial\varphi_i(t,x) \quad \text{on} \quad E_0, \quad u_{[i]}^{(0)}(t,x)=\partial_x\varphi_i(0,x) \quad \text{for} \quad (t,x)\in E_c\backslash E_0. \quad (13)$$

Suppose that $z^{(m)}\colon E_c\to\mathbb{R}^k$, $u^{(m)}\colon E_c\to M_{k\times n}$ are known functions. Then $u_{[i]}^{(m+1)}$ is a solution of the equation

$$u_{[i]}(t,x)=T_{[i]}^{(m)}[u_{[i]}](t,x) \quad (14)$$

where for $(t,x)\in E_c\backslash E_0$,

$$\begin{aligned} T_{[i]}^{(m)}[u_{[i]}](t,x) &= \partial_x\varphi_i(0,g_i[z^{(m)}](0,t,x)) \\ &+\int_0^t W_{[i]}[z^{(m)},u^{(m)}](\tau,g_i[z^{(m)}](\tau,t,x))d\tau \\ &-\sum_{\mu=1}^n\int_0^t V_{[i\mu]}[z^{(m)},u^{(m)}](\tau,g_i[z^{(m)}](\tau,t,x))u_{i\mu}(\tau,g_i[z^{(m)}](\tau,t,x))d\tau, \end{aligned} \quad (15)$$

and for $(t,x)\in E_0$,

$$T_{[i]}^{(m)}[u_{[i]}](t,x)=\partial_x\varphi_i(t,x). \quad (16)$$

The function $z^{(m+1)}$ is given on $E_c\backslash E_0$ by

$$z_i^{(m+1)}(t,x)=\varphi_i(0,g_i[z^{(m)}](0,t,x))+\int_0^t G_i(\tau,g_i[z^{(m)}](\tau,t,x),z^{(m)}_{\psi(\xi,g_i[z^{(m)}](\tau,t,x))})d\tau \quad (17)$$

and on E_0 by

$$z_i^{(m+1)}(t,x)=\varphi_i(t,x). \quad (18)$$

REMARK 3.1. The sequences $\left\{z^{(m)}\right\}$, $\left\{u^{(m)}\right\}$ are obtained in the following way. Suppose that $z^{(m)}\colon E_c\to\mathbb{R}^k$ and $u^{(m)}\colon E_c\to M_{k\times n}$ are known functions. We consider classical solutions of the Cauchy problems

$$\partial_t z_i(t,x)+f_{[i]}(t,x,(z^{(m)})_{\psi(t,x)})\circ\partial_x z_i(t,x)=G_i(t,x,(z^{(m)})_{\psi(t,x)}), \quad (19)$$

$$z_i(0,x)=\varphi_i(0,x) \quad \text{for} \quad x\in[-b,b], \quad (20)$$

where $i=1,\dots,k$. Note that we have obtained separate initial problems for linear equations. We introduce an additional unknown function $u_{[i]}=\partial_x z_i$ in (19). Then we obtain the following differential equations for $u_{[i]}$:

$$\begin{aligned} &\partial_t u_{[i]}(t,x)+f_{[i]}(t,x,(z^{(m)})_{\psi(t,x)})\circ[\partial_x u_{[i]}(t,x)]^T \\ &\qquad = W_{[i]}[z^{(m)},u^{(m)}](t,x)-\sum_{\mu=1}^n V_{[i\mu]}[z^{(m)},u^{(m)}](t,x)u_{i\mu}(t,x). \end{aligned} \quad (21)$$

It is natural to consider the following initial condition for (21):

$$u_{[i]}(0,x) = \partial_x \varphi_i(0,x) \quad \text{for} \quad x \in [-b,b]. \tag{22}$$

We put $i = 1,\ldots,k$ in (21), (22).

Note that differential equations of i-th bicharacteristics for (19) and (21) are the same and they have the form

$$\eta'(\tau) = f_{[i]}(\tau,\eta(\tau),(z^{(m)})_{\psi(\tau,\eta(\tau))}).$$

Then we integrate (19) and (21) along the bicharacteristics $g_i[z^{(m)}](\cdot,t,x)$ and we obtain the system of integral equations (14) for $u_{[i]}^{(m+1)}$ and $z_i^{(m+1)}$ is given by (17), (18). We put $i = 1,\ldots,k$ in these considerations.

Assumption $H[c]$ The constants $d \in \mathbb{R}_+$, $s = (s_0,s_1,s_2) \in \mathbb{R}_+^3$, $c \in (0,a]$ satisfy the conditions

$$\begin{aligned}
s_0 &\geqslant c_0 + cA(1+s_0Q)(1+s_0),\\
s_1 &\geqslant L(c_2 + c\tilde{L}),\\
s_2 &\geqslant L(c_2 + c\tilde{L}) + A(1+s_0)(1+s_0Q),\\
d &\geqslant L(c_0 + cA(1+dQ)),
\end{aligned}$$

where

$$\begin{aligned}
\bar{L} &= C(1+dQ)(1+s_0Q) + AQ(s_0+Qs_2),\\
\tilde{L} &= \bar{L}(1+s_0) + As_2(1+s_0Q).
\end{aligned}$$

REMARK 3.2. If we assume that $s_0 > c_0$, $s_1 > Lc_2$, $s_2 > Lc_2 + A(1+c_0)(1+c_0Q)$ and $d > Lc_0$ then there is $c \in (0,a]$ such that Assumption $H[c]$ is satisfied.

THEOREM 3.3. *If $\varphi \in \Phi$ and Assumptions $H[f], H[G]$, $H[\psi]$, $H[c]$ are satisfied then for any $m \geqslant 0$ we have*

(I_m) *$z^{(m)}$ and $u^{(m)}$ are defined on E_c and $z^{(m)} \in C_{\varphi.c}[d]$, $u_{[i]}^{(m)} \in C_{\partial\varphi_i.c}[s]$ for $m \in \mathbb{N}$, $i = 1,\ldots,k$,*

(II_m) *for $i = 1,\ldots,k$ we have $\partial_x z_i^{(m)} = u_{[i]}^{(m)}$ on E_c.*

Proof. We prove (I_m) and (II_m) by induction. It follows from (13), (18) that conditions (I_0) and (II_0) are satisfied. Suppose that conditions (I_m) and (II_m) hold for a given $m \geqslant 0$. We first prove that there are $u_{[i]}^{(m+1)}\colon E_c \to \mathbb{R}^n$ where $1 \leqslant i \leqslant k$. Suppose that $1 \leqslant i \leqslant k$ is fixed. We claim that

$$T_{[i]}^{(m)}\colon C_{\partial\varphi_i.c}[s] \to C_{\partial\varphi_i.c}[s]. \tag{23}$$

It follows from Assumptions $H[f]$,$H[G]$, $H[\psi]$ that

$$||W_{[i]}[z^{(m)},u^{(m)}](t,x)|| \leqslant A(1+s_0Q),$$
$$||V_{[i\mu]}[z^{(m)},u^{(m)}](t,x)|| \leqslant A(1+s_0Q), \quad \mu=1,\dots,n$$

and consequently

$$||T^{(m)}_{[i]}[u_{[i]}](t,x)|| \leqslant c_0 + cA(1+s_0Q)(1+s_0). \tag{24}$$

It follows from Assumptions $H[f]$,$H[G]$, $H[\psi]$ that

$$||W_{[i]}[z^{(m)},u^{(m)}](t,x) - W_{[i]}[z^{(m)},u^{(m)}](t,\bar{x})|| \leqslant \bar{L}||x-\bar{x}||$$

and

$$||V_{[i\mu]}[z^{(m)},u^{(m)}](t,x) - V_{[i\mu]}[z^{(m)},u^{(m)}](t,\bar{x})|| \leqslant \bar{L}||x-\bar{x}||, \quad \mu=1,\dots,n.$$

We thus get

$$\begin{aligned}||T^{(m)}_{[i]}[u_{[i]}](t,x) &- T^{(m)}_{[i]}[u_{[i]}](\bar{t},\bar{x})||\\ &\leqslant L(c_2+c\tilde{L})(|t-\bar{t}|+||x-\bar{x}||) + A(1+s_0Q)(1+s_0)|t-\bar{t}|,\end{aligned} \tag{25}$$

where $u_{[i]} \in C_{\partial\varphi_i.c}[s]$. We conclude from (24), (25) and from Assumption $H[c]$ that condition (23) is satisfied. Write

$$\big[|u_{[i]}-\bar{u}_{[i]}|\big] = \max\{||(u_{[i]}-\bar{u}_{[i]})(t,x)||e^{-2A(1+s_0Q)t} \,:\, (t,x)\in E_c\}, \quad i=1,\dots,k.$$

Then

$$\begin{aligned}||T^{(m)}_{[i]}&[u^{(m+1)}_{[i]}](t,x) - T^{(m)}_{[i]}[\bar{u}^{(m+1)}_{[i]}](t,x)||\\ &\leqslant A(1+s_0Q)\left|\int_0^t ||(u^{(m+1)}_{[i]}-\bar{u}^{(m+1)}_{[i]})(\tau,x)||e^{-2A(1+s_0Q)\tau}e^{2A(1+s_0Q)\tau}d\tau\right|\\ &\leqslant \left[\left|u^{(m+1)}_{[i]}-\bar{u}^{(m+1)}_{[i]}\right|\right]\int_0^t A(1+s_0Q)e^{2A(1+s_0Q)\tau}d\tau\\ &\leqslant \left[\left|u^{(m+1)}_{[i]}-\bar{u}^{(m+1)}_{[i]}\right|\right]\frac{e^{2A(1+s_0Q)t}}{2}.\end{aligned}$$

Finally we have that

$$\left[\left|T^{(m)}_{[i]}[u^{(m+1)}_{[i]}] - T^{(m)}_{[i]}[\bar{u}^{(m+1)}_{[i]}]\right|\right] \leqslant \frac{1}{2}\left[\left|u^{(m+1)}_{[i]}-\bar{u}^{(m+1)}_{[i]}\right|\right].$$

From Banach fixed point theorem we have that $u^{(m+1)}_{[i]}$ exists on $E_0\cup E_c$ and is unique. The existence of $z^{(m+1)}$ on $E_0\cup E_c$ goes from the definition. We have for $z^{(m)}\in C_{\varphi.c}[d]$ the estimation

$$||z^{(m+1)}_i(t,x) - z^{(m+1)}_i(t,\bar{x})|| \leqslant L_0(c_0+cC(1+dQ))||x-\bar{x}||$$

so $z^{(m+1)} \in C_{\varphi.c}[d]$. From the principle of induction we have that (I_m) is fulfilled for every $m \in \mathbb{N}$. Now we will show the (II_{m+1}). That means that we will prove that for every $m \in \mathbb{N}$ we have $\partial_x z_i^{(m+1)}(t,x) = u_{[i]}^{(m+1)}(t,x)$ for $(t,x) \in E_0 \cup E_c$ and $i = 1,\ldots,k$. Write

$$U(t,x,\bar{x}) = z_i^{(m+1)}(t,\bar{x}) - z_i^{(m+1)}(t,x) - u_{[i]}^{(m+1)}(t,x) \circ (\bar{x} - x)$$

where $1 \leqslant i \leqslant k$ is fixed. We will prove that there is $K > 0$ such that

$$|U(t,x,\bar{x})| \leqslant K||x - \bar{x}||^2 \quad \text{for} \quad (t,x), (t,\bar{x}) \in E_c. \tag{26}$$

For $m \in \mathbb{N}$ denote

$$g_i^{(m)}(\tau,t,x) = g_i[z^{(m)}](\tau,t,x),$$
$$P^{(m)}(\tau,t,x) = \big(\tau, g_i^{(m)}(\tau,t,x), z^{(m)}_{\psi(\tau,g_i^{(m)}(\tau,t,x))}\big).$$

Then we have that

$$\begin{aligned} U(t,x,\bar{x}) &= \varphi_i(0,g_i^{(m)}(0,t,\bar{x})) - \varphi_i(0,g_i^{(m)}(0,t,x)) - \partial_x\varphi_i(0,g_i^{(m)}(0,t,x)) \circ (\bar{x} - x) \\ &+ \int_0^t \Big[G_i(P^{(m)}(\tau,t,\bar{x})) - G_i(P^{(m)}(\tau,t,x))\Big] d\tau \\ &- \int_0^t W_{[i]}[z^{(m)},u^{(m)}](\tau,g_i^{(m)}(\tau,t,x)) d\tau \circ (\bar{x} - x) \\ &+ \sum_{j=1}^n \int_0^t V_{[ij]}[z^{(m)},u^{(m)}](\tau,g_i^{(m)}(\tau,t,x)) u_{ij}^{(m+1)}(\tau,g_i^{(m)}(\tau,t,x)) d\tau \circ (\bar{x} - x). \end{aligned}$$

Write

$$Q^{(m)}(\tau,t,x,\bar{x},\xi) = \xi P^{(m)}(\tau,t,\bar{x}) + (1-\xi)P^{(m)}(\tau,t,x), \quad 0 \leqslant \xi \leqslant 1.$$

Note that $z^{(m)}_{\psi(t,x)}$ and $z^{(m)}_{\psi(t,\bar{x})}$ have different domains. We will need the following construction. There are

$$Z^{(m)}: \mathbb{R}^{1+n} \to \mathbb{R}^k, \quad U_{[i]}^{(m)}: \mathbb{R}^{1+n} \to \mathbb{R}^n, \quad i = 1,\ldots,k$$

such that

(i) $Z^{(m)} \in C(\mathbb{R}^{1+n},\mathbb{R}^k)$, $Z^{(m)} = (Z_1^{(m)},\ldots,Z_k^{(m)})$, $Z^{(m)}(t,x) = z^{(m)}(t,x)$ on $E_0 \cup E_c$ and

$$\sup\{||Z^{(m)}(t,x)||_\infty : (t,x) \in \mathbb{R}^{1+n}\} = \max\{||z^{(m)}(t,x)||_\infty : (t,x) \in E_0 \cup E_c\},$$

(ii) for $i = 1,\ldots,k$ we have:

$$U_{[i]}^{(m)} \in C(\mathbb{R}^{1+n},\mathbb{R}^n), \quad U_{[i]}^{(m)} = (U_{i1}^{(m)},\ldots,U_{in}^{(m)}),$$
$$U_{[i]}^{(m)}(t,x) = u_{[i]}^{(m)}(t,x) \quad \text{on} \quad E_0 \cup E_c,$$

and

$$\sup\{||U_{[i]}^{(m)}(t,x)|| : (t,x) \in \mathbb{R}^{1+n}\} = \max\{||u_{[i]}^{(m)}(t,x)|| : (t,x) \in E_0 \cup E_c\},$$

(*iii*) $\partial_x Z^{(m)} = U^{(m)}_{[i]}$ on $\mathbb{R}^{1+n}$ for $i = 1, \dots, k$.

Then the functions $(Z^{(m)})_{(t,x)}$, $(U^{(m)}_{[i]})_{(t,x)}$, $(t,x) \in E_c$ are defined on B in the following way

$$(Z^{(m)})_{(t,x)}(\tau,y) = (Z^{(m)})(t+\tau, x+y),$$
$$(U^{(m)}_{[i]})_{(t,x)}(\tau,y) = (U^{(m)}_{[i]})(t+\tau, x+y), \quad (\tau,y) \in B, i = 1, \dots, k.$$

From Hadamard mean value theorem we have

$$\begin{aligned}
&U(t,x,\bar{x}) = \varphi_i(0, g_i^{(m)}(0,t,\bar{x})) - \varphi_i(0, g_i^{(m)}(0,t,x)) - \partial_x\varphi_i(0, g_i^{(m)}(0,t,x))(\bar{x}-x) \\
&+ \int_0^t \int_0^1 \partial_x G_i(Q^{(m)}(\tau,t,x,\bar{x},\xi))d\xi \circ \left[g_i^{(m)}(\tau,t,\bar{x}) - g_i^{(m)}(\tau,t,x)\right] d\tau \\
&+ \int_0^t \int_0^1 \partial_w G_i(Q^{(m)}(\tau,t,x,\bar{x},\xi))d\xi * \left[\left(Z^{(m)}\right)_{\psi(\tau, g_i^{(m)}(\tau,t,\bar{x}))} - \left(Z^{(m)}\right)_{\psi(\tau, g_i^{(m)}(\tau,t,x))}\right] d\tau \\
&\quad - \int_0^t W_{[i]}[z^{(m)}, u^{(m)}](\tau, g_i^{(m)}(\tau,t,x))d\tau \circ (\bar{x}-x) \\
&\quad + \sum_{j=1}^n \int_0^t V_{[ij]}[z^{(m)}, u^{(m)}](\tau, g_i^{(m)}(\tau,t,x)) u_{ij}^{(m+1)}(\tau, g_i^{(m)}(\tau,t,x))d\tau \circ (\bar{x}-x).
\end{aligned}$$

For simplicity of formulation of the next properties of the function U we write

$$\begin{aligned}
U_\varphi(t,x,\bar{x}) &= \varphi_i(0, g_i^{(m)}(0,t,\bar{x})) - \varphi_i(0, g_i^{(m)}(0,t,x)) \\
&\qquad - \partial_x\varphi_i(0, g_i^{(m)}(0,t,x)) \circ \left[g_i^{(m)}(0,t,\bar{x}) - g_i^{(m)}(0,t,x)\right],
\end{aligned}$$

and

$$\begin{aligned}
\bar{U}(t,x,\bar{x}) = &\int_0^t \int_0^1 \left[\partial_x G_i(Q^{(m)}(\tau,t,x,\bar{x},\xi)) - \partial_x G_i(P^{(m)}(\tau,t,x))\right] d\xi \\
&\circ \left[g_i^{(m)}(\tau,t,\bar{x}) - g_i^{(m)}(\tau,t,x)\right] d\tau \\
&+ \int_0^t \int_0^1 \left[\partial_w G_i(Q^{(m)}(\tau,t,x,\bar{x},\xi)) - \partial_w G_i(P^{(m)}(\tau,t,x))\right] d\xi \\
&* \left[(Z^{(m)})_{\psi(\tau, g_i^{(m)}(\tau,t,\bar{x}))} - (Z^{(m)})_{\psi(\tau, g_i^{(m)}(\tau,t,x))}\right] d\tau.
\end{aligned}$$

Moreover we put

$$\begin{aligned}
U^*(t,x,\bar{x}) = &\int_0^t \partial_x G_i(P^{(m)}(\tau,t,x)) \circ \left[g_i^{(m)}(\tau,t,\bar{x}) - g_i^{(m)}(\tau,t,x)\right] d\tau \\
&+ \int_0^t \partial_w G_i(P^{(m)}(\tau,t,x)) * \left[(Z^{(m)})_{\psi(\tau, g_i^{(m)}(\tau,t,\bar{x}))} - (Z^{(m)})_{\psi(\tau, g_i^{(m)}(\tau,t,\bar{x}))}\right] d\tau \\
&- \int_0^t W_{[i]}[z^{(m)}, u^{(m)}](\tau, g_i^{(m)}(\tau,t,x)) \circ \left[g_i^{(m)}(\tau,t,\bar{x}) - g_i^{(m)}(\tau,t,x)\right] d\tau,
\end{aligned}$$

$$\Gamma(t,x,\bar{x}) = -\partial_x\varphi_i(0,g_i^{(m)}(0,t,x)) \circ \int_0^t \Big[f_{[i]}(P^{(m)}(\xi,t,\bar{x})) - f_{[i]}(P^{(m)}(\xi,t,x))\Big]\,d\xi$$
$$-\int_0^t W_{[i]}[z^{(m)},u^{(m)}](\tau,g_i^{(m)}(\tau,t,x)) \circ \int_\tau^t \Big[f_{[i]}(P^{(m)}(\xi,t,\bar{x})) - f_{[i]}(P^{(m)}(\xi,t,x))\Big]\,d\xi d\tau$$
$$+\sum_{j=1}^n \int_0^t u_{ij}^{(m+1)}(\tau,g_i^{(m)}(\tau,t,x)) V_{[ij]}[z^{(m)},u^{(m)}](\tau,g_i^{(m)}(\tau,t,x))$$
$$\circ \int_\tau^t \Big[f_{[i]}(P^{(m)}(\xi,t,\bar{x})) - f_{[i]}(P^{(m)}(\xi,t,x))\Big]\,d\xi d\tau.$$

It follows from (6) that the bicharacteristics satisfy the condition

$$g_i^{(m)}(\tau,t,\bar{x}) - g_i^{(m)}(\tau,t,x) = \bar{x} - x + \int_t^\tau \Big[f_{[i]}(P^{(m)}(\xi,t,\bar{x})) - f_{[i]}(P^{(m)}(\xi,t,x))\Big]\,d\xi.$$

Then we have

$$U(t,x,\bar{x}) = U_\varphi(t,x,\bar{x}) + \bar{U}(t,x,\bar{x}) + U_*(t,x,\bar{x}) + \Gamma(t,x,\bar{x})$$
$$+\sum_{j=1}^n \int_0^t u_{ij}^{(m+1)}(\tau,g_i^{(m)}(\tau,t,x)) V_{[ij]}[z^{(m)},u^{(m)}](\tau,g_i^{(m)}(\tau,t,x))$$
$$\circ \Big[g_i^{(m)}(\tau,t,\bar{x}) - g_i^{(m)}(\tau,t,x)\Big]\,d\tau.$$

We conclude from Assumptions $H[G]$, $H[\psi]$ and from Lemma 2.2 that there is $\tilde{C} > 0$ such that

$$|U_\varphi(t,x,\bar{x})| + |\bar{U}(t,x,\bar{x})| \leqslant \tilde{C}||x-\bar{x}||^2, \quad (t,x),\,(t,\bar{x}) \in E_c. \tag{27}$$

It follows from Lemma 2.2 that the bicharacteristics satisfy the condition

$$g_i^{(m)}(\tau,\xi,g_i^{(m)}(\xi,t,x)) = g_i^{(m)}(\tau,t,x),\ (t,x) \in E_c,\ \tau,\xi \in [0,\kappa_i(t,x)].$$

The above relations and (14), (15) imply

$$u_{[i]}^{(m+1)}(\xi,g_i^{(m)}(\xi,t,x)) = \partial_x\varphi_i(0,g_i^{(m)}(0,t,x)) + \int_0^\xi W_{[i]}[z^{(m)},u^{(m)}](\tau,g_i^{(m)}(\tau,t,x))d\tau$$
$$-\sum_{j=1}^n \int_0^\xi V_{[ij]}[z^{(m)},u^{(m)}](\tau,g_i^{(m)}(\tau,t,x)) u_{ij}^{(m+1)}(\tau,g_i^{(m)}(\tau,t,x))d\tau.$$

Then we have that

$$\Gamma(t,x,\bar{x}) = -\int_0^t \Big[f_{[i]}(P^{(m)}(\xi,t,\bar{x})) - f_{[i]}(P^{(m)}(\xi,t,x))\Big] \circ u_{[i]}^{(m+1)}(\xi,g_i^{(m)}(\xi,t,x))d\xi.$$

Write

$$U_*(t,x,\bar{x}) = \sum_{j=1}^n \int_0^t u_{ij}^{(m+1)}(\tau,g_i^{(m)}(\tau,t,x)) V_{[ij]}[z^{(m)},u^{(m)}](\tau,g_i^{(m)}(\tau,t,x))$$
$$\circ \Big[g_i^{(m)}(\tau,t,\bar{x}) - g_i^{(m)}(\tau,t,x)\Big]\,d\tau$$
$$-\int_0^t \Big[f_{[i]}(P^{(m)}(\xi,t,\bar{x})) - f_{[i]}(P^{(m)}(\xi,t,x))\Big] \circ u_{[i]}^{(m+1)}(\xi,g_i^{(m)}(\xi,t,x))d\xi.$$

Then we have

$$U(t,x,\bar{x}) = U_\varphi(t,x,\bar{x}) + \bar{U}(t,x,\bar{x}) + U^*(t,x,\bar{x}) + U_*(t,x,\bar{x}).$$

It follows that

$$U_*(t,x,\bar{x}) = -\sum_{j=1}^{n}\int_0^t u_{ij}^{(m+1)}(\tau, g_i^m(\tau,t,x))\{f_{ij}(P^{(m)}(\tau,t,\bar{x}))$$

$$-f_{ij}(P^{(m)}(\tau,t,x)) - V_{[ij]}[z^{(m)},u^{(m)}](\tau,g_i^{(m)}(\tau,t,x))\circ[g_i^{(m)}(\tau,t,\bar{x}) - g_i^{(m)}(\tau,t,x)]\}d\tau$$

and

$$U^*(t,x,\bar{x}) = \int_0^t \sum_{j=1}^{k} \partial_{w_j} G_i(P^{(m)}(\tau,t,x))(\Delta Z_j^{(m)})(\tau,t,x)d\tau,$$

where

$$(\Delta Z_j^{(m)})(\tau,t,x) = (Z_j^{(m)})_{\psi(\tau,g_i^{(m)}(\tau,t,\bar{x}))} - (Z_j^{(m)})_{\psi(\tau,g_i^{(m)}(\tau,t,x))}$$

$$-\sum_{\nu=1}^{n}\sum_{\mu=1}^{n} \partial_{x_\mu}\psi_\nu(\tau,g_i^{(m)}(\tau,t,x))[g_{i\mu}^{(m)}(\tau,t,\bar{x}) - g_{i\mu}^{(m)}(\tau,t,x)](U_{j\nu}^{(m)})_{\psi(\tau,g_i^{(m)}(\tau,t,x))}.$$

It follows from the definitions of $V_{[ij]}$ and from (II_m) that there is $\bar{K} > 0$ such that

$$|U_*(t,x,\bar{x})| + |U^*(t,x,\bar{x})| \leqslant \bar{K}||x-\bar{x}||^2 \quad \text{for} \quad (t,x),(t,\bar{x}) \in E_c.$$

The above inequality and (27) imply (26). Then $\partial_x z_i^{(m+1)} = u_{[i]}^{(m+1)}$ on E_c for $i = 1,\ldots,k$. This completes the proof.

Now we formulate a theorem on the existence of classical solutions of (1), (2).

THEOREM 3.4. *If Assumptions $H[f]$, $H[G]$, $H[\psi]$, $H[c]$ are satisfied, and $\varphi \in \Phi$ then there is a classical solutions $\bar{z}\colon E_0 \cup E_c \to \mathbb{R}^k$ of (1), (2). If $\tilde{\varphi}\colon E_0 \to \mathbb{R}$ is such that $\tilde{\varphi} \in \Phi$ and $\tilde{z}\colon E_0 \cup E_c \to \mathbb{R}$ is a classical solution of (1) with the initial condition $\tilde{z}(t,x) = \tilde{\varphi}(t,x)$ on E_0 then there is $0 \leqslant A^*$ such that for $t \in [0,c]$ and $i = 1,\ldots,k$,*

$$\begin{aligned} &||\bar{z}_i - \tilde{z}_i||_t + ||\partial_x \bar{z}_i - \partial_x \tilde{z}_i||_{(t,\mathbb{R}^n)} \\ &\qquad \leqslant e^{A^* t}\left[\max_{1\leqslant j\leqslant k}||\varphi_j - \tilde{\varphi}_j||_0 + \max_{1\leqslant j\leqslant k}||\partial_x\varphi_j - \partial_x\tilde{\varphi}_j||_{(0,\mathbb{R}^n)}\right]. \end{aligned} \tag{28}$$

Proof. We first prove that the sequences $\{z_i^{(m)}\}$, $\{u_{[i]}^{(m)}\}$ are uniformly convergent on $E_0 \cup E_c$ for $i = 1,\ldots,k$. Write

$$\Lambda_i^{(m)}(t) = ||z_i^{(m)} - z_i^{(m-1)}||_t, \quad \tilde{\Lambda}_i^{(m)}(t) = ||u_{[i]}^{(m)} - u_{[i]}^{(m-1)}||_{(t,\mathbb{R}^n)}$$

where $i = 1,\ldots,k$ and

$$\Lambda^{(m)}(t) = \max\{\Lambda_i^{(m)}(t) : 1 \leqslant i \leqslant k\}, \quad \tilde{\Lambda}^{(m)}(t) = \max\{\tilde{\Lambda}_i^{(m)}(t) : 1 \leqslant i \leqslant k\}.$$

It follows from Assumptions $H[f]$, $H[G]$ and from (14) - (18) that there is $\tilde{K} > 0$ such that

$$\tilde{\Lambda}_i^{(m+1)}(t) \leqslant \tilde{K}\big[\Lambda^{(m)}(t) + \tilde{\Lambda}^{(m)}(t) + \int_0^t \tilde{\Lambda}^{(m+1)}(s)ds\big], \quad i = 1,\dots,k.$$

We conclude from the above inequalities and from the Gronwall inequality that

$$\tilde{\Lambda}_i^{(m+1)}(t) \leqslant \tilde{K}e^{\tilde{K}c} \int_0^t \big[\Lambda^{(m)}(\tau) + \tilde{\Lambda}^{(m)}(\tau)\big]d\tau, \quad i = 1,\dots,k, t \in [0,c].$$

There is $\bar{K} > 0$ such that

$$\Lambda^{(m+1)}(t) \leqslant \bar{K} \int_0^t \Lambda^{(m)}(\tau)d\tau, \quad t \in [0,c].$$

There is $K > 0$ such that

$$\Lambda^{(m+1)}(t) + \tilde{\Lambda}^{(m+1)}(t) \leqslant K \int_0^t \big[\Lambda^{(m)}(s) + \tilde{\Lambda}^{(m)}(s)\big]ds, \quad t \in [0,c]. \tag{29}$$

Write

$$Q^{(m)}(t) = \max\{[\Lambda^{(m)}(\tau) + \tilde{\Lambda}^{(m)}(\tau)]e^{-2K\tau} \,:\, \tau \in [0,c]\}.$$

We conclude from (29) that

$$\Lambda^{(m+1)}(t) + \tilde{\Lambda}^{(m+1)}(t) \leqslant \frac{1}{2}Q^{(m)}(t)e^{2Kt}, \quad t \in [0,c]$$

and consequently

$$Q^{(m+1)}(t) \leqslant \frac{1}{2}Q^{(m)}(t), \quad t \in [0,c].$$

There is $C_0 \in \mathbb{R}_+$ such that $Q^{(1)}(t) \leqslant C_0$ for $t \in [0,c]$. We thus get

$$\lim_{m\to\infty} Q^{(m)}(t) = 0 \quad \text{uniformly on} \quad [0,c]$$

and there are $\bar{z} \in C_{\varphi_i.c}[d]$ and $\bar{u}_{[i]} = (\bar{u}_{i1},\dots,\bar{u}_{in}) \in C_{\partial\varphi_i.c}[s]$ such that

$$\bar{z}_i(t,x) = \lim_{m\to\infty} z_i^{(m)}(t,x), \quad \bar{u}_{[i]}(t,x) = \lim_{m\to\infty} u_{[i]}^{(m)}(t,x) \quad \text{uniformly on} \quad E_c.$$

It follows from Theorem 3.3 that $\partial_t \bar{z}_i$ and $\partial_x \bar{z}_i$ exist on E_c and $\partial_x \bar{z}_i = \bar{u}_{[i]}$. Furthermore, we have that $\bar{u}_{[i]} = T_{[i]}[\bar{u}_{[i]}](t,x)$ and

$$\bar{z}_i(t,x) = \varphi_i(0, g_i[\bar{z}](0,t,x)) + \int_0^t G_i(\tau, g_i[\bar{z}](\tau,t,x), z^{(m)}_{\psi(\xi, g_i[\bar{z}](\tau,t,x))})d\tau, \tag{30}$$

where $(t,x) \in E_c$. For a given $(t,x) \in E_c$ let us put $y = g_i[\bar{z}](0,t,x)$. It follows that $g_i[\bar{z}](\tau,t,x) = g_i[\bar{z}](\tau,0,y)$ for $\tau \in [0,\kappa_i(t,x)]$ where $[0,\kappa_i(t,x)]$ is a domain of $g_i[\bar{z}](\cdot,t,x)$. Then the relation (30) imply

$$\bar{z}_i(t, g_i[\bar{z}](t,0,y)) = \varphi_i(0,y) + \int_0^t G_i(\tau, g_i[\bar{z}](\tau,0,y), \bar{z}_{\psi(\tau, g_i[\bar{z}](\tau,0,y))})d\tau. \tag{31}$$

The relations $y = g_i[\bar{z}](0,t,x)$ and $x = g_i[\bar{z}](t,0,y)$ are equivalent. By differentiating (31) with respect to t and by putting again $x = g_i[\bar{z}](t,0,y)$ we obtain that $\bar{z}_i$ satisfies (1) on E_c. Now we prove (28). There is $A^* \geqslant 0$ such that

$$\begin{aligned}
||\bar{z}_i - \tilde{z}_i||_t + ||\partial_x \bar{z}_i - \partial_x \tilde{z}_i||_{(t,\mathbb{R}^n)} &\leqslant ||\varphi_i - \tilde{\varphi}_i||_0 + ||\partial_x \varphi_i - \partial_x \tilde{\varphi}_i||_{(0,\mathbb{R}^n)} \\
&\quad + A^* \int_0^t \left[\max_{1 \leqslant j \leqslant k} ||\bar{z}_j - \tilde{z}_j||_\tau + \max_{1 \leqslant j \leqslant k} ||\partial_x \bar{z}_j - \partial_x \tilde{z}_j||_{(\tau,\mathbb{R}^n)} \right] d\tau \\
&\leqslant \max_{1 \leqslant j \leqslant k} ||\varphi_j - \tilde{\varphi}_j||_0 + \max_{1 \leqslant j \leqslant k} ||\partial_x \varphi_j - \partial_x \tilde{\varphi}_j||_{(0,\mathbb{R}^n)} \\
&\quad + A^* \int_0^t \left[\max_{1 \leqslant j \leqslant k} ||\bar{z}_j - \tilde{z}_j||_\tau + \max_{1 \leqslant j \leqslant k} ||\partial_x \bar{z}_j - \partial_x \tilde{z}_j||_{(\tau,\mathbb{R}^n)} \right] d\tau, \quad t \in (0,c].
\end{aligned}$$

Then we obtain (28) from Gronwall inequality and from the properities of the function maximum. This completes proof.

REMARK 3.5. Let z and $\bar{z}$ be the solutions of the Cauchy problem (1) with the initial condition (2). Then from the Theorem 3 we have that the solutions z and $\bar{z}$ are the same on the whole domain.

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