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CHEBYSHEV'S DIFFERENTIAL EQUATION AND ITS HYERS–ULAM STABILITY

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Abstract. We solve the inhomogeneous Chebyshev's differential equation and apply this result to obtain a partial solution to the Hyers-Ulam stability problem for the Chebyshev's differential equation.

1. Introduction

Let X be a normed space over a scalar field $\mathbb{K}$ and let $I \subset \mathbb{R}$ be an open interval, where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Assume that $a_0, a_1, \ldots, a_n : I \to \mathbb{K}$ and $g : I \to X$ are given continuous functions, and that $y : I \to X$ is an n times continuously differentiable function satisfying the inequality

$$\|a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) + g(t)\| \leqslant \varepsilon$$

for all $t \in I$ and for a given $\varepsilon > 0$. If there exists an n times continuously differentiable function $y_0 : I \to X$ satisfying

$$a_n(t)y_0^{(n)}(t) + a_{n-1}(t)y_0^{(n-1)}(t) + \cdots + a_1(t)y_0'(t) + a_0(t)y_0(t) + g(t) = 0$$

and $\|y(t) - y_0(t)\| \leqslant K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression of ε with $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$, then we say that the above differential equation has the Hyers-Ulam stability. For more detailed definitions of the Hyers-Ulam stability, we refer the reader to [2, 3, 4, 5, 6, 18, 20].

Alsina and Ger first investigated the Hyers-Ulam stability of differential equations (see [1]): They proved that if a differentiable function $f : I \to \mathbb{R}$ satisfies the inequality $|y'(t) - y(t)| \leqslant \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t) - f_0(t)| \leqslant 3\varepsilon$ for any $t \in I$. Their result was generalized by Takahasi, Miura and Miyajima: Indeed, it was proved in [19] that the Hyers-Ulam stability holds true for the Banach space valued differential equation $y'(t) = \lambda y(t)$ (see also [14, 15]).

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Moreover, Miura, Miyajima and Takahasi [16] investigated the Hyers-Ulam stability of n-th order linear differential equation with complex coefficients. They [17] also proved the Hyers-Ulam stability of linear differential equations of first order, $y'(t)+g(t)y(t)=0$, where $g(t)$ is a continuous function.

Jung also proved the Hyers-Ulam stability of various linear differential equations of first order (ref. [7, 8, 9, 10]). Moreover, he could successfully apply the power series method to the study of the Hyers-Ulam stability of Legendre's differential equation (see [11]). Subsequently, the authors [13] investigated the Hyers-Ulam stability problem for Bessel's differential equation by applying the same method.

In §2 of this paper, by using the ideas from [11, 12, 13], we investigate the general solution of the inhomogeneous Chebyshev's differential equation of the form

$$(1-x^2)y''(x)-xy'(x)+n^2y(x)=\sum_{m=0}^{\infty}a_mx^m, \tag{1}$$

where n is a given positive integer. §3 will be devoted to a partial solution of the Hyers-Ulam stability problem for the Chebyshev's differential equation (2) in a subclass of analytic functions.

2. Inhomogeneous Chebyshev's equation

A function is called a Chebyshev function if it satisfies the Chebyshev's differential equation

$$(1-x^2)y''(x)-xy'(x)+n^2y(x)=0. \tag{2}$$

The Chebyshev's equation plays a great role in physics and engineering. In particular, this equation is most useful for treating the boundary value problems exhibiting certain symmetries.

In this section, we define $c_0=c_1=0$ and for $m\geqslant 2$,

$$c_m=\begin{cases}\displaystyle\sum_{i=0}^{\frac{m-2}{2}}\frac{(2i)!}{m!}a_{2i}\prod_{\substack{j=2i+2\\ j\text{ even}}}^{m-2}(j^2-n^2) & \text{for } m \text{ even},\\ \displaystyle\sum_{i=0}^{\frac{m-3}{2}}\frac{(2i+1)!}{m!}a_{2i+1}\prod_{\substack{j=2i+3\\ j\text{ odd}}}^{m-2}(j^2-n^2) & \text{for } m \text{ odd},\end{cases} \tag{3}$$

where we refer to (1) for the a_m's. Here we have the convention $\prod_{j=m}^{m-2}(j^2-n^2)=1$. We can easily check that c_m's satisfy the following

$$(m+2)(m+1)c_{m+2}-(m^2-n^2)c_m=a_m \tag{4}$$

for any $m\in\{0,1,2,\ldots\}$.

LEMMA 1. (a) *If the power series* $\sum_{m=0}^{\infty} a_m x^m$ *converges for all* $x \in (-\rho, \rho)$ *with* $\rho > 1$, *then the power series* $\sum_{m=2}^{\infty} c_m x^m$ *with* c_m*'s given in* (3) *satisfies the inequality* $|\sum_{m=2}^{\infty} c_m x^m| \leqslant \frac{C_1}{1-|x|}$ *for some positive constant* C_1 *and for any* $x \in (-1, 1)$.

(b) *If the power series* $\sum_{m=0}^{\infty} a_m x^m$ *converges for all* $x \in (-\rho, \rho)$ *with* $\rho \leqslant 1$, *then for any positive* $\rho_0 < \rho$ *the power series* $\sum_{m=2}^{\infty} c_m x^m$ *with* c_m*'s given in* (3) *satisfies the inequality* $|\sum_{m=2}^{\infty} c_m x^m| \leqslant C_2$ *for any* $x \in [-\rho_0, \rho_0]$ *and for some positive constant* C_2 *which depends on* ρ_0. *Since* ρ_0 *is arbitrarily close to* ρ, *this means that* $\sum_{m=2}^{\infty} c_m x^m$ *is convergent for all* $x \in (-\rho, \rho)$.

Proof. (a) Since the power series $\sum_{m=0}^{\infty} a_m x^m$ is absolutely convergent on its interval of convergence, with $x = 1$, $\sum_{m=0}^{\infty} a_m$ converges absolutely, i.e., $\sum_{m=0}^{\infty} |a_m| < M_1$ by some number M_1. Now, if m is an even integer not less than 2, then it follows from (3) that

$$
\begin{aligned}
|c_m| &\leqslant \sum_{i=0}^{\frac{m-2}{2}} \frac{(2i)!}{m!} |a_{2i}| \prod_{\substack{j=2i+2 \\ j \text{ even}}}^{m-2} |j^2 - n^2| \\
&= \sum_{i=0}^{\frac{m-2}{2}} |a_{2i}| \frac{1}{m} \frac{|(m-2)^2 - n^2|}{(m-1)(m-2)} \frac{|(m-4)^2 - n^2|}{(m-3)(m-4)} \cdots \frac{|(2i+2)^2 - n^2|}{(2i+3)(2i+2)} \frac{1}{(2i+1)},
\end{aligned}
$$

where each factor $\frac{|j^2-n^2|}{(j+1)j}$ (with j even) is either less than 1 if $j \geqslant n$ or is less than n^2 if $j \leqslant n$. Therefore, the whole summand is less than $|a_{2i}|(n^2)^{n/2} = |a_{2i}|n^n$ because j can run through even integers at most from 2 to n. Hence

$$
|c_m| \leqslant \sum_{i=0}^{\frac{m-2}{2}} |a_{2i}| n^n \leqslant M_1 n^n \equiv C_1
$$

and this holds similarly for c_m with m odd. Therefore we have

$$
\left| \sum_{m=0}^{\infty} c_m x^m \right| \leqslant \sum_{m=0}^{\infty} |c_m||x^m| \leqslant C_1 \sum_{m=0}^{\infty} |x|^m \leqslant \frac{C_1}{1-|x|}
$$

for every $x \in (-1, 1)$.

(b) The power series $\sum_{m=0}^{\infty} a_m x^m$ is absolutely convergent on its interval of convergence, and therefore for any given $\rho_0 < \rho \leqslant 1$, the series $\sum_{m=0}^{\infty} |a_m x^m|$ is convergent on $[-\rho_0, \rho_0]$ and

$$
\sum_{m=0}^{\infty} |a_m||x|^m \leqslant \sum_{m=0}^{\infty} |a_m| \rho_0^m \equiv M_2 \tag{5}
$$

for any $x \in [-\rho_0, \rho_0]$. It now follows from (3) that

$$\begin{aligned}
\left| \sum_{m=2}^{\infty} c_m x^m \right| &\leqslant \sum_{\substack{m=2 \\ m \text{ even}}}^{\infty} |c_m| \rho_0^m + \sum_{\substack{m=3 \\ m \text{ odd}}}^{\infty} |c_m| \rho_0^m \\
&\leqslant \sum_{\substack{m=2 \\ m \text{ even}}}^{\infty} \rho_0^m \sum_{i=0}^{\frac{m-2}{2}} \frac{(2i)!}{m!} |a_{2i}| \prod_{\substack{j=2i+2 \\ j \text{ even}}}^{m-2} |j^2 - n^2| \\
&\quad + \sum_{\substack{m=3 \\ m \text{ odd}}}^{\infty} \rho_0^m \sum_{i=0}^{\frac{m-3}{2}} \frac{(2i+1)!}{m!} |a_{2i+1}| \prod_{\substack{j=2i+3 \\ j \text{ odd}}}^{m-2} |j^2 - n^2| \qquad (6) \\
&\leqslant \sum_{\substack{m=2 \\ m \text{ even}}}^{\infty} \sum_{i=0}^{\frac{m-2}{2}} |a_{2i}| \rho_0^{2i} \frac{\rho_0^2}{m(m-1)} \prod_{\substack{j=2i+2 \\ j \text{ even}}}^{m-2} \frac{|j^2 - n^2| \rho_0^2}{j(j-1)} \\
&\quad + \sum_{\substack{m=3 \\ m \text{ odd}}}^{\infty} \sum_{i=0}^{\frac{m-3}{2}} |a_{2i+1}| \rho_0^{2i+1} \frac{\rho_0^2}{m(m-1)} \prod_{\substack{j=2i+3 \\ j \text{ odd}}}^{m-2} \frac{|j^2 - n^2| \rho_0^2}{j(j-1)}.
\end{aligned}$$

Choose a positive integer $j_1 > n$ such that $\frac{j_1}{j_1-1} < \frac{1}{\rho_0^2}$. This means that for any $j > j_1$, $\frac{j}{j-1} < \frac{j_1}{j_1-1} < \frac{1}{\rho_0^2}$. If $j > j_1 > n$, then

$$\frac{|j^2 - n^2| \rho_0^2}{j(j-1)} < \frac{j}{j-1} \rho_0^2 < \frac{j_1}{j_1-1} \rho_0^2 < 1$$

and for $j \leqslant j_1$, since $j \geqslant 2$, $\rho_0 < \rho \leqslant 1$, and $\frac{\rho_0^2}{j-1} < \frac{1}{j}$, we get

$$\frac{|j^2 - n^2| \rho_0^2}{j(j-1)} < \max\{j^2, n^2\}.$$

Thus, if m is an even integer not less than 2, then we have

$$\begin{aligned}
\prod_{\substack{j=2i+2 \\ j \text{ even}}}^{m-2} \frac{|j^2 - n^2| \rho_0^2}{j(j-1)} &= \left(\prod_{\substack{j>j_1 \\ j \text{ even}}}^{m-2} \frac{|j^2 - n^2| \rho_0^2}{j(j-1)} \right) \left(\prod_{\substack{j=2i+2 \\ j \text{ even}}}^{j_1} \frac{|j^2 - n^2| \rho_0^2}{j(j-1)} \right) \\
&< \prod_{\substack{j=2i+2 \\ j \text{ even}}}^{j_1} \frac{|j^2 - n^2| \rho_0^2}{j(j-1)} \qquad (7) \\
&= \left(\prod_{\substack{j>n \\ j \text{ even}}}^{j_1} \frac{|j^2 - n^2| \rho_0^2}{j(j-1)} \right) \left(\prod_{\substack{j=2i+2 \\ j \text{ even}}}^{n} \frac{|j^2 - n^2| \rho_0^2}{j(j-1)} \right) \\
&< (j_1!)^2 (n^2)^{n/2}
\end{aligned}$$

and similarly if m is an odd integer not less than 3, then we obtain

$$\prod_{\substack{j=2i+3\\ j \text{ odd}}}^{m-2} \frac{|j^2-n^2|\rho_0^2}{j(j-1)} < (j_1!)^2 n^n. \tag{8}$$

Hence, it follows from (5), (6), (7) and (8) that

$$\begin{aligned}
\left|\sum_{m=2}^{\infty} c_m x^m\right| &\leqslant \sum_{\substack{m=2\\ m \text{ even}}}^{\infty} \sum_{i=0}^{\frac{m-2}{2}} |a_{2i}|\rho_0^{2i} \frac{\rho_0^2}{m(m-1)}(j_1!)^2 n^n \\
&\quad + \sum_{\substack{m=3\\ m \text{ odd}}}^{\infty} \sum_{i=0}^{\frac{m-3}{2}} |a_{2i+1}|\rho_0^{2i+1} \frac{\rho_0^2}{m(m-1)}(j_1!)^2 n^n \\
&\leqslant \sum_{m=2}^{\infty} \frac{\rho_0^2}{m(m-1)} M_2 (j_1!)^2 n^n \\
&\leqslant \sum_{m=2}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m}\right) \rho_0^2 M_2 (j_1!)^2 n^n \\
&\leqslant \rho_0^2 M_2 (j_1!)^2 n^n \equiv C_2
\end{aligned}$$

for any $x \in [-\rho_0, \rho_0]$. □

LEMMA 2. *Suppose that the power series $\sum_{m=0}^{\infty} a_m x^m$ converges for all $x \in (-\rho, \rho)$ with some positive ρ. Let $\rho_1 = \min\{1, \rho\}$. Then the power series $\sum_{m=2}^{\infty} c_m x^m$ with c_m's given in* (3) *is convergent for all $x \in (-\rho_1, \rho_1)$. Further for any positive $\rho_0 < \rho_1$, $|\sum_{m=2}^{\infty} c_m x^m| \leqslant C$ for any $x \in [-\rho_0, \rho_0]$ and for some positive constant C which depends on ρ_0.*

Proof. The first statement follows from the second one. Therefore, let us prove the second statement. If $\rho \leqslant 1$, then $\rho_1 = \rho$. By Lemma 1 (b), for any positive $\rho_0 < \rho = \rho_1$, $|\sum_{m=2}^{\infty} c_m x^m| \leqslant C_2$ for each $x \in [-\rho_0, \rho_0]$ and for some positive constant C_2 which depends on ρ_0.

If $\rho > 1$, then by Lemma 1 (a), for any positive $\rho_0 < 1 = \rho_1$, we get

$$\left|\sum_{m=2}^{\infty} c_m x^m\right| \leqslant \frac{C_1}{1-|x|} \leqslant \frac{C_1}{1-\rho_0} \equiv C$$

for $x \in [-\rho_0, \rho_0]$ and for some positive constant C which depends on ρ_0. □

Using these definitions and the lemmas above, we will show that $\sum_{m=2}^{\infty} c_m x^m$ is a particular solution of the inhomogeneous Chebyshev's equation (1).

THEOREM 3. *Assume that* n *is a given positive integer and the radius of convergence of the power series* $\sum_{m=0}^{\infty} a_m x^m$ *is* $\rho > 0$. *Let* $\rho_1 = \min\{1, \rho\}$. *Then, every solution* $y : (-\rho_1, \rho_1) \to \mathbb{C}$ *of the Chebyshev's differential equation* (1) *can be expressed by*

$$y(x) = y_h(x) + \sum_{m=2}^{\infty} c_m x^m,$$

where $y_h(x)$ *is a Chebyshev function and* c_m*'s are given by* (3).

Proof. We show that $\sum_{m=2}^{\infty} c_m x^m$ satisfies the equation (1). By Lemma 2, the power series $\sum_{m=2}^{\infty} c_m x^m$ is convergent for each $x \in (-\rho_1, \rho_1)$.

Substituting $\sum_{m=2}^{\infty} c_m x^m$ for $y(x)$ in (1) and collecting like powers together, it follows from (3) and (4) that

$$\begin{aligned}
&(1-x^2)y''(x) - xy'(x) + n^2 y(x) \\
&\quad = 2c_2 + 6c_3 x + \sum_{m=2}^{\infty} \left[(m+2)(m+1)c_{m+2} - (m^2 - n^2)c_m \right] x^m \\
&\quad = a_0 + a_1 x + \sum_{m=2}^{\infty} a_m x^m \\
&\quad = \sum_{m=0}^{\infty} a_m x^m
\end{aligned}$$

for all $x \in (-\rho_1, \rho_1)$.

Therefore, every solution $y : (-\rho_1, \rho_1) \to \mathbb{C}$ of the inhomogeneous Chebyshev's differential equation (1) can be expressed by

$$y(x) = y_h(x) + \sum_{m=2}^{\infty} c_m x^m,$$

where $y_h(x)$ is a Chebyshev function. □

3. Partial solution to Hyers-Ulam stability problem

In this section, we will investigate a property of the Chebyshev's differential equation (2) concerning the Hyers-Ulam stability problem. That is, we will try to answer the question, whether there exists a Chebyshev function near any approximate Chebyshev function.

THEOREM 4. *Let* $y : (-\rho, \rho) \to \mathbb{C}$ *be a given analytic function which can be represented by a power series* $\sum_{m=0}^{\infty} b_m x^m$ *whose radius of convergence is at least* $\rho > 0$. *Suppose there exists a constant* $\varepsilon > 0$ *such that*

$$\left| (1-x^2)y''(x) - xy'(x) + n^2 y(x) \right| \leqslant \varepsilon \tag{9}$$

for all $x \in (-\rho, \rho)$ *and for some positive integer* n. *Let* $\rho_1 = \min\{1, \rho\}$. *Let* a_m *be a sequence such that* $(1-x^2)y''(x) - xy'(x) + n^2 y(x) = \sum_{m=0}^{\infty} a_m x^m$ *and*

$$\sum_{m=0}^{\infty} |a_m x^m| \leqslant K \left| \sum_{m=0}^{\infty} a_m x^m \right|$$

for all $x \in (-\rho, \rho)$ *and for some constant* K. *Then there exists a Chebyshev function* $y_h : (-\rho_1, \rho_1) \to \mathbb{C}$ *such that*

$$|y(x) - y_h(x)| \leqslant C\varepsilon$$

for all $x \in [-\rho_0, \rho_0]$, *where* $\rho_0 < \rho_1$ *is any positive number and* C *is some constant which depends on* ρ_0.

Proof. We assumed that $y(x)$ can be represented by a power series and

$$(1-x^2)y''(x) - xy'(x) + n^2 y(x) = \sum_{m=0}^{\infty} a_m x^m$$

also satisfies

$$\sum_{m=0}^{\infty} |a_m x^m| \leqslant K \left| \sum_{m=0}^{\infty} a_m x^m \right| \leqslant K\varepsilon$$

for all $x \in (-\rho, \rho)$ from (9).

According to Theorem 3, $y(x)$ can be written as $y_h(x) + \sum_{m=2}^{\infty} c_m x^m$ for all $x \in (-\rho_1, \rho_1)$, where y_h is some Chebyshev function and c_m's are given by (3). Then by Lemmas 1 and 2 and their proofs (replace M_1 and M_2 with $K\varepsilon$ in Lemma 1), we obtain

$$|y(x) - y_h(x)| = \left| \sum_{m=2}^{\infty} c_m x^m \right| \leqslant C\varepsilon$$

for all $x \in [-\rho_0, \rho_0]$, where $\rho_0 < \rho_1$ is any positive number and C is some constant which depends on ρ_0. This completes the proof of our theorem. □

4. Example

In this section, we show that there certainly exist functions $y(x)$ which satisfy all the conditions given in Theorem 4. We introduce an example related to the Chebyshev's differential equation (1) for $n = 1$.

EXAMPLE. Let $y_h(x)$ be a Chebyshev function and let $y : (-1, 1) \to \mathbb{R}$ be an analytic function given by

$$y(x) = y_h(x) + \frac{450}{509} \varepsilon \sum_{m=0}^{\infty} \frac{x^{2m}}{10^{2m}}, \tag{10}$$

where ε is a positive constant. Then, we have

$$(1-x^2)y''(x) - xy'(x) + y(x) = \sum_{m=0}^{\infty} a_m x^m,$$

where

$$a_m = \begin{cases} \frac{450}{509} \cdot \frac{-396m^2+6m+102}{10^{2m+2}}\varepsilon & \text{for } m \in \{0,2,4,\ldots\}, \\ 0 & \text{for } m \in \{1,3,5,\ldots\}. \end{cases}$$

It is obvious that $a_0 = \frac{450}{509}\frac{102}{100}\varepsilon$, $a_m \leqslant 0$, and $|a_{2m}| < \frac{450}{509}\frac{1}{10^m}\varepsilon$ for any $m \geqslant 1$. Thus, for each $x \in (-1,1)$, we have

$$\begin{aligned} |(1-x^2)y''(x) - xy'(x) + y(x)| &= \left| \sum_{m=0}^{\infty} a_m x^m \right| < \sum_{m=0}^{\infty} |a_m| \\ &< \frac{450}{509}\varepsilon \left(\frac{102}{100} + \frac{1}{9} \right) = \varepsilon. \end{aligned}$$

Moreover, for $x \in (-1,1)$, it follows from the last inequality that

$$\sum_{m=0}^{\infty} |a_m x^m| < \sum_{m=0}^{\infty} |a_m| < \varepsilon.$$

On the other hand, we obtain

$$\begin{aligned} \left| \sum_{m=0}^{\infty} a_m x^m \right| &= \left| a_0 + \sum_{m=1}^{\infty} a_{2m} x^{2m} \right| \geqslant a_0 + \sum_{m=1}^{\infty} a_{2m} \\ &\geqslant \frac{450}{509}\frac{102}{100}\varepsilon - \frac{450}{509}\sum_{m=1}^{\infty}\frac{1}{10^m}\varepsilon = \frac{450}{509}\varepsilon\left(\frac{102}{100} - \frac{1}{9}\right) \\ &= \frac{409}{509}\varepsilon. \end{aligned}$$

Therefore, we get

$$\sum_{m=0}^{\infty} |a_m x^m| \leqslant \frac{509}{409} \left| \sum_{m=0}^{\infty} a_m x^m \right|$$

for all $x \in (-1,1)$. (That is, the constant K in Theorem 4 is given by $\frac{509}{409}$.)

Further, it follows from (10) that

$$|y(x) - y_h(x)| = \frac{450}{509}\varepsilon \left| \sum_{m=0}^{\infty} \frac{x^{2m}}{10^{2m}} \right| < \frac{450}{509}\varepsilon \sum_{m=0}^{\infty} \frac{1}{10^{2m}} < \frac{9}{10}\varepsilon$$

for all $x \in [-\rho_0, \rho_0]$, $0 < \rho_0 < 1$, which is consistent with the result of Theorem 4 if we set $\rho_1 = \rho = 1$ and $n = 1$.

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