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A NONOSCILLATION THEOREM FOR HALF–LINEAR DIFFERENTIAL EQUATIONS WITH DELAY NONLINEAR PERTURBATIONS

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Abstract. This paper deals with the oscillation problem for nonlinear differential equations with delay. A sufficient condition is obtained for the equation to have a nonoscillatory solution. The main result is the best possible in a certain sense. Examples are given to illustrate the main result.

1. Introduction

We consider the nonlinear differential equation with delay

$$(|x'(t)|^{\alpha-1}x'(t))' + \left(\frac{\alpha}{(\alpha+1)t}\right)^{\alpha+1}|x(t)|^{\alpha-1}x(t) + a(t)f(x(ct)) = 0, \quad ' = \frac{d}{dt} \tag{1.1}$$

where α and c are constants satisfying $\alpha > 0$ and $0 < c \leqslant 1$, respectively; $a(t)$ is nonnegative, continuous and locally of bounded variation on $(0,\infty)$; f is continuous on $\mathbb{R}$ and satisfies

$$yf(y) > 0 \qquad \text{if } y \neq 0. \tag{1.2}$$

Let $t_0 \geqslant 0$. By a *solution* of (1.1) we mean a function $x\colon [ct_0,\infty) \to \mathbb{R}$ which has the property $|x'|^{\alpha-1}x' \in C^1(t_0,\infty)$ and which satisfies (1.1) for all $t \in [t_0,\infty)$. A solution $x(t)$ of (1.1) is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to ∞ such that $x(t_n) = 0$. Otherwise, it is said to be *nonoscillatory*.

As has already been shown in [16, Theorem 2.1], under the above assumptions, all nontrivial solutions of (1.1) are continuable into the future. Hence, it is worthwhile to investigate whether or not solutions of (1.1) are oscillatory.

Let $c = 1$ and

$$a(t)f(y) = \frac{\varepsilon}{t^{\alpha+1}}|y|^{\alpha-1}y,$$

where ε is nonnegative. Then (1.1) becomes the half-linear differential equation without delay

$$(|x'|^{\alpha-1}x')' + \frac{1}{t^{\alpha+1}}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \varepsilon\right\}|x|^{\alpha-1}x = 0, \tag{1.3}$$

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which is called the generalized Euler differential equation (see [4]). It is well known that (1.3) has a nonoscillatory solution if and only if the corresponding characteristic equation

$$\alpha|z|^{\alpha+1}-\alpha|z|^{\alpha-1}z+\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}+\varepsilon=0$$

has a real root (see [1, 2, 3, 6, 7, 12, 13, 14, 16]). Thus, (1.3) has a nonoscillatory solution if $\varepsilon=0$ and all nontrivial solutions of (1.3) are oscillatory if $\varepsilon>0$. Hence, (1.1) consists of the half-linear differential equation (1.3) with $\varepsilon=0$ which has a nonoscillatory solution, and the nonlinear perturbation with delay $a(t)f(x(ct))$.

Recently, the author and Sugie [16] have presented a sufficient condition for all nontrivial solutions of (1.1) to be oscillatory.

THEOREM A. ([16, Theorem 1.1]) *Assume* (1.2) *and suppose that* $a(t)$ *satisfies*

$$a(t)\geqslant\frac{1}{t^{\alpha+1}(\log t)^{\beta}}$$

for t *sufficiently large, and that*

$$\frac{f(y)}{|y|^{\alpha-1}y}\geqslant\frac{\lambda}{(\log|y|)^{\gamma}}$$

for $|y|$ *large enough, where* β *and* γ *are nonnegative constants satisfying* $\beta+\gamma=2$. *If*

$$\lambda>\frac{1}{2c^{\alpha^2/(\alpha+1)}}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+\gamma}, \tag{1.4}$$

then all nontrivial solutions of (1.1) *are oscillatory.*

It follows from Theorem A that all nontrivial solutions of (1.1) are oscillatory if there exists λ such that (1.4) satisfies

$$\beta+\gamma=2, \qquad a(t)f(y)=\frac{\lambda}{t^{\alpha+1}(\log t)^{\beta}(\log|y|)^{\gamma}}|y|^{\alpha-1}y$$

for t and $|y|$ sufficiently large, where β and γ are nonnegative constants. However, to show that the constant given in (1.4) is the best possible, we need a nonoscillation theorem for (1.1). For this reason, we give a sufficient condition for (1.1) to have a nonoscillatory solution. Our main result can be stated as follows.

THEOREM 1.1. *Assume that* (1.2) *holds and suppose that* $a(t)$ *satisfies*

$$0<a(t)\leqslant\frac{1}{t^{\alpha+1}(\log t)^{\beta}} \tag{1.5}$$

for t *sufficiently large, and that* $f(y)$ *is nondecreasing for* $y\in\mathbb{R}$ *and satisfies*

$$\frac{f(y)}{|y|^{\alpha-1}y}\leqslant\frac{\lambda}{(\log|y|)^{\gamma}} \tag{1.6}$$

for $y > 0$ *or* $y < 0$, $|y|$ *sufficiently large, where* β *and* γ *are nonnegative constants satisfying* $\beta + \gamma = 2$. *If*

$$\lambda < \frac{1}{2c^{\alpha^2/(\alpha+1)}} \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+\gamma}, \tag{1.7}$$

then (1.1) *has a nonoscillatory solution.*

REMARK 1.1. To prove Theorem 1.1, we need to assume that the function $f(y)$ is nondecreasing for $y \in \mathbb{R}$.

REMARK 1.2. For the case of linear or half-linear differential equations without delay, we note that, by Sturm's separation theorem, all nontrivial solutions of (1.1) are nonoscillatory ([5, 8, 9, 15]).

2. Preliminary

To prove Theorem 1.1, we require two lemmas. First we present the following result concerning a positive solution of the half-linear differential equation

$$(|y'|^{\alpha-1}y')' + \frac{1}{t^{\alpha+1}} \left\{ \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} + \frac{\lambda}{(\log t)^2} \right\} |y|^{\alpha-1}y = 0. \tag{2.1}$$

LEMMA 2.1. *Suppose that* (2.1) *has a positive solution. Then the derivative of the solution is nonnegative for* t *sufficiently large and it tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a positive solution of (2.1). Then there exists $T > 0$ such that $y(t) > 0$ for $t \geqslant T$, and therefore, we have

$$(|y'(t)|^{\alpha-1}y'(t))' = -\frac{1}{t^{\alpha+1}} \left\{ \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} + \frac{\lambda}{(\log t)^2} \right\} |y(t)|^{\alpha-1}y(t) < 0. \tag{2.2}$$

We first show that $y'(t) \geqslant 0$ for $t \geqslant T$. By way of contradiction, we suppose that there exists $t_1 \geqslant T$ such that $y'(t_1) < 0$. Integrating both sides of (2.2) from t_1 to t, we obtain

$$|y'(t)|^{\alpha-1}y'(t) \leqslant |y'(t_1)|^{\alpha-1}y'(t_1) \quad \text{for } t \geqslant t_1,$$

and therefore, we have $y'(t) \leqslant y'(t_1)$ for $t \geqslant t_1$. Hence, it follows that

$$y(t) < y'(t_1)(t - t_1) + y(t_1) \to -\infty \quad \text{as } t \to \infty.$$

This contradicts the assumption that $y(t)$ is positive for $t \geqslant T$. Thus, $y'(t) \geqslant 0$ for $t \geqslant T$.

Next, we will show that $\lim_{t\to\infty} y'(t) = 0$. From (2.2), $y'(t)$ is nonincreasing for $t \geqslant T$. Then, there exists $m \geqslant 0$ such that $y'(t) \to m$ as $t \to \infty$. If $m > 0$, then we have

$y(t) \geqslant m(t-T)+y(T)$ for $t \geqslant T$. Hence, there exists $t_2 \geqslant T$ such that $y(t) \geqslant mt/2$ for $t \geqslant T$. Since $y(t)$ is a solution of (2.1), we have

$$\begin{aligned}
(|y'(t)|^{\alpha-1}y'(t))' &= -\frac{1}{t^{\alpha+1}}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}+\frac{\lambda}{(\log t)^2}\right\}|y(t)|^{\alpha-1}y(t) \\
&\leqslant -\frac{1}{t^{\alpha+1}}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}|y(t)|^{\alpha-1}y(t) \\
&\leqslant -\frac{1}{t^{\alpha+1}}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{m}{2}\right)^{\alpha}t^{\alpha} \\
&= -\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{m}{2}\right)^{\alpha}\frac{1}{t} \quad \text{for } t \geqslant t_2.
\end{aligned}$$

Integrating this inequality from t_2 to t, we obtain

$$|y'(t)|^{\alpha-1}y'(t) \leqslant -\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{m}{2}\right)^{\alpha}\log\frac{t}{T}+|x'(t_2)|^{\alpha-1}x'(t_2) \quad \text{for } t \geqslant t_2.$$

This is a contradiction to the assumption that $y'(t) \geqslant 0$ for $t \geqslant T$. This completes the proof of Lemma 2.1. □

We now consider the relation between (1.1) and the inequality

$$y'(t) \geqslant \left(\int_t^{\infty}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{s^{\alpha+1}}|y(s)|^{\alpha-1}y(s)+a(s)f(y(cs))\right\}ds\right)^{1/\alpha}. \tag{2.3}$$

LEMMA 2.2. *Assume that the function f satisfies* (1.2) *and is nondecreasing for $x \in \mathbb{R}$. Suppose that there exists a positive function $y(t)$ satisfying* (2.3) *for t sufficiently large. Then* (1.1) *has a nonoscillatory solution.*

Proof. Let t_0 be a positive number such that $y(t) > 0$ and (2.3) holds for $t \geqslant t_0$. We define the function sequences $\{x_n(t)\}$ and $\{w_n(t)\}$ as follows.

$$\begin{aligned}
w_1(t) &= y'(t) \quad \text{for } t \geqslant t_1, \\
x_1(t) &= y(t) \quad \text{for } t \geqslant t_0, \\
w_{n+1}(t) &= \left(\int_t^{\infty}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{s^{\alpha+1}}|x_n(s)|^{\alpha-1}x_n(s) + a(s)f(x_n(cs))\right\}ds\right)^{1/\alpha} \quad \text{for } t \geqslant t_1,
\end{aligned}$$

$$x_{n+1}(t) = \begin{cases} y(t) & \text{for } t_0 \leqslant t \leqslant t_1, \\ \displaystyle\int_{t_1}^{t} w_{n+1}(s)ds + y(t_1) & \text{for } t > t_1, \end{cases}$$

where $t_1 = t_0/c$. We will show that, for any $n \in \mathbb{N}$,

$$0 < w_{n+1}(t) \leqslant w_n(t) \quad \text{for } t \geqslant t_1 \quad \text{and} \quad 0 < x_{n+1}(t) \leqslant x_n(t) \quad \text{for } t \geqslant t_0 \tag{2.4}$$

by mathematical induction on n.

From condition (2.3), we obtain

$$\begin{aligned} w_2(t) &= \left(\int_t^\infty \left(\left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \frac{1}{s^{\alpha+1}} |x_1(s)|^{\alpha-1} x_1(s) + a(s) f(x_1(cs)) \right) ds \right)^{1/\alpha} \\ &= \left(\int_t^\infty \left(\left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \frac{1}{s^{\alpha+1}} |y(s)|^{\alpha-1} y(s) + a(s) f(y(cs)) \right) ds \right)^{1/\alpha} \\ &\leqslant y'(t) = w_1(t) \quad \text{for } t \geqslant t_1. \end{aligned}$$

We also have that $w_2(t) > 0$ for $t \geqslant t_1$ because $y(t) > 0$ for $t \geqslant t_0$. Hence, $x_2(t)$ is positive for $t \geqslant t_0$,

$$\begin{aligned} x_2(t) &= \int_{t_1}^t w_2(s) ds + y(t_1) \\ &\leqslant \int_{t_1}^t w_1(s) ds + y(t_1) = \int_{t_1}^t y'(s) ds + y(t_1) \\ &= y(t) - y(t_1) + y(t_1) = y(t) = x_1(t) \quad \text{for } t > t_1 \end{aligned}$$

and $x_2(t) = y(t) = x_1(t)$ for $t_0 \leqslant t \leqslant t_1$. Thus, (2.4) is true for $n = 1$.

Assume that condition (2.4) is satisfied with $n = k$. Since the function $f(y)$ is nondecreasing for $y \in \mathbb{R}$, we have

$$\begin{aligned} w_{k+2}(t) &= \left(\int_t^\infty \left(\left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \frac{1}{s^{\alpha+1}} |x_{k+1}(s)|^{\alpha-1} x_{k+1}(s) + a(s) f(x_{k+1}(cs)) \right) ds \right)^{1/\alpha} \\ &\leqslant \left(\int_t^\infty \left(\left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \frac{1}{s^{\alpha+1}} |x_k(s)|^{\alpha-1} x_k(s) + a(s) f(x_k(cs)) \right) ds \right)^{1/\alpha} \\ &= w_{k+1}(t), \quad \text{and that} \end{aligned}$$

$$\begin{aligned} x_{k+2}(t) &= \int_{t_1}^t w_{k+1}(s) ds + y(t_1) \\ &\leqslant \int_{t_1}^t w_k(s) ds + y(t_1) = x_{k+1}(t) \qquad \text{for } t \geqslant t_1. \end{aligned}$$

It is easy to check that $x_{k+2}(t) = y(t) = x_{k+1}(t)$ for $t_0 \leqslant t \leqslant t_1$, $w_{k+2}(t) \geqslant 0$ for $t \geqslant t_1$ and that $x_{k+2}(t) > 0$ for $t > t_0$. Thus, (2.4) is true for $n = k+1$.

Let $x(t) = \lim_{n\to\infty} x_n(t)$ and $w(t) = \lim_{n\to\infty} w_n(t)$. Then, using the Lebesgue monotone convergence theorem, we have

$$x(t) = \int_{t_1}^t w(s) ds + y(t_1) \qquad \text{for } t \geqslant t_1$$

and

$$w(t)=\left(\int_t^{\infty}\left(\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{s^{\alpha+1}}|x(s)|^{\alpha-1}x(s)+a(s)f(x(cs))\right)ds\right)^{1/\alpha}$$

for $t \geqslant t_1$, respectively. Moreover, we obtain

$$x(t)=\lim_{n\to\infty}x_n(t)\geqslant\lim_{n\to\infty}x_n(t_1)=y(t_1)>0 \qquad \text{for } t\geqslant t_1$$

because $x_n'(t)=w_n(t)>0$ for $t\geqslant t_1$. Thus, $x(t)$ is a nonoscillatory solution of (1.1). $\square$

3. Proof of the main theorem

We deal only with the case that condition (1.6) is satisfied for $y>0$ sufficiently large, because the other case can be proved in a similar manner.

It is known that the half-linear differential equation

$$(|y'|^{\alpha-1}y')'+\frac{1}{t^{\alpha+1}}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}+\frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{(\log t)^2}\right\}|y|^{\alpha-1}y=0 \tag{3.1}$$

has a nonoscillatory solution $y(t)$ satisfying

$$y(t)=t^{\alpha/(\alpha+1)}(\log t)^{1/(\alpha+1)}(M_1+O(1/\log t)) \quad \text{as } t\to\infty, \tag{3.2}$$

where $M_1\neq 0$ (for example, see [4, Corollary 5.2.3], [6, Corollary 1] and [10]). Since $-y(t)$ is also a solution of (3.1), we may assume that the constant M_1 is positive. Then, there exists $T>0$ such that $y(t)>0$ for $t\geqslant T$.

From (1.7), we can choose ε_0 such that

$$\lambda\leqslant\frac{1}{2c^{\alpha^2/(\alpha+1)}}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+\gamma}\left(\frac{1-\varepsilon_0}{1+\varepsilon_0}\right)^{\alpha} \quad \text{and} \quad 0<\varepsilon_0<1. \tag{3.3}$$

By (3.2), there exists $t_0>T$ such that $y(t)$ satisfies

$$\left|\frac{y(t)}{t^{\alpha/(\alpha+1)}(\log t)^{1/(\alpha+1)}}-M_1\right|<M_1\varepsilon_0 \quad \text{for } t\geqslant t_0,$$

and therefore, we have

$$\begin{aligned}\frac{\log t}{\log y(ct)}&\leqslant\frac{\log t}{\log(M_1(1-\varepsilon_0)(ct)^{\alpha/(\alpha+1)}(\log(ct))^{1/(\alpha+1)})}\\&=\frac{\log t}{\log M_1(1-\varepsilon_0)+(\alpha/(\alpha+1))(\log c+\log t)+\log(\log(ct))^{1/(\alpha+1)}}\\&\nearrow\frac{1}{\alpha/(\alpha+1)} \quad \text{as } t\to\infty.\end{aligned}$$

Hence, we can find $t_1 > t_0/c$ such that

$$\frac{1}{\log y(ct)} \leqslant \frac{1}{\alpha/(\alpha+1)\log t} \quad \text{for } t \geqslant t_1. \tag{3.4}$$

We also have

$$\frac{y(ct)}{y(t)} \leqslant \frac{M_1(1+\varepsilon_0)(ct)^{\alpha/(\alpha+1)}(\log(ct))^{1/(\alpha+1)}}{M_1(1-\varepsilon_0)t^{\alpha/(\alpha+1)}(\log t)^{1/(\alpha+1)}} \leqslant \frac{1+\varepsilon_0}{1-\varepsilon_0}c^{\alpha/(\alpha+1)} \quad \text{for } t \geqslant t_1. \tag{3.5}$$

Hence, from conditions (1.5), (1.6), (3.3), (3.4) and (3.5), we have

$$\begin{aligned} a(t)f(y(ct)) &\leqslant \frac{\lambda}{t^{\alpha+1}(\log t)^{\beta}(\log y(ct))^{\gamma}}(y(ct))^{\alpha} \\ &\leqslant \frac{\lambda}{t^{\alpha+1}(\log t)^{\beta}(\alpha/(\alpha+1))^{\gamma}(\log t)^{\gamma}}\left(\frac{y(ct)}{y(t)}\right)^{\alpha}(y(t))^{\alpha} \\ &\leqslant \frac{\lambda}{t^{\alpha+1}(\log t)^{\beta+\gamma}(\alpha/(\alpha+1))^{\gamma}}\left(\frac{1+\varepsilon_0}{1-\varepsilon_0}\right)^{\alpha} c^{\alpha^2/(\alpha+1)}(y(t))^{\alpha} \\ &\leqslant \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\frac{1}{t^{\alpha+1}(\log t)^2}|y(t)|^{\alpha-1}y(t) \quad \text{for } t \geqslant t_1. \end{aligned} \tag{3.6}$$

Integrating both sides of (3.1) from t to ∞ ($t \geqslant t_0$) and using Lemma 2.1, we obtain

$$\begin{aligned} |y'(t)|^{\alpha-1}y'(t) = \int_t^{\infty}\Bigg\{ &\frac{1}{s^{\alpha+1}}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}|y(s)|^{\alpha-1}y(s) \\ &+\frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\frac{1}{s^{\alpha+1}(\log s)^2}|y(s)|^{\alpha-1}y(s)\Bigg\}ds \quad \text{for } t \geqslant t_0. \end{aligned}$$

Hence, by (3.6), $y(t)$ satisfies the inequality

$$y'(t) \geqslant \left(\int_t^{\infty}\left\{\frac{1}{s^{\alpha+1}}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}|y(s)|^{\alpha-1}y(s) + a(s)f(y(cs))\right\}ds\right)^{1/\alpha}$$

for $t \geqslant t_1$. Thus, from Lemma 2.2, (1.1) has a nonoscillatory solution. This completes the proof of Theorem 1.1. □

4. Examples and Oscillation constants

To illustrate Theorem 1.1, consider the equation

$$(|x'(t)|^{\alpha-1}x'(t))' + \left(\frac{\alpha}{(\alpha+1)t}\right)^{\alpha+1}|x(t)|^{\alpha-1}x(t) + \lambda\, a(t)f(x(ct)) = 0, \tag{1.1$_\lambda$}$$

where λ is a nonnegative number. We introduce the following definition. The number λ_0 is called an *oscillation constant* for $(1.1)_\lambda$ if all nontrivial solutions of $(1.1)_\lambda$ are oscillatory for $\lambda > \lambda_0$, and there exists a nonoscillatory solution of $(1.1)_\lambda$ for $0 < \lambda < \lambda_0$ (for example, see [4, p.238] and [15, p.52]).

REMARK 4.1. When an oscillation constant λ_0 exists for $(1.1)_\lambda$, we cannot in general decide whether or not solutions of $(1.1)_{\lambda_0}$ are oscillatory.

A typical example of $(1.1)_\lambda$ is the half-linear differential equation without delay (2.1). Elbert and Schneider [6] discussed the oscillation problem for (2.1). They gave the following oscillation constant for (2.1).

EXAMPLE 4.1. An oscillation constant for (2.1) is $(\alpha/(\alpha+1))^\alpha/2$.

Another example of $(1.1)_\lambda$ for which there exists an oscillation constant is the linear differential equation with delay

$$x''(t) + \frac{1}{4t^2}x(t) + \frac{\lambda}{t^2(\log t)^2}x(ct) = 0. \tag{4.1}$$

The oscillation problem for this equation was examined by Sugie and Iwasaki [11], and they presented the following oscillation constant.

EXAMPLE 4.2. The value $1/(4\sqrt{c})$ is an oscillation constant for (4.1).

From Theorems 1.1 and A, an oscillation constant for $(1.1)_\lambda$ is

$$\frac{1}{2c^{\alpha^2/(\alpha+1)}}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+\gamma} \tag{4.2}$$

when the perturbation $a(t)f(y)$ satisfies

$$a(t)f(y) = \frac{1}{t^{\alpha+1}(\log t)^\beta(\log|y|)^\gamma}|y|^{\alpha-1}y \tag{4.3}$$

for t and $|y|$ sufficiently large, where β and γ are nonnegative constants satisfying $\beta+\gamma=2$. Hence, if $c=1$ and $\gamma=0$ ($\alpha=1$ and $\gamma=0$, respectively), then $(1.1)_\lambda$ with (4.3) becomes (2.1) ((4.1), respectively) and the constant (4.2) coincides with the oscillation constant of (2.1) ((4.1), respectively). Thus, the constant (4.2) is a complete generalization of the oscillation constants for (2.1) and (4.1).

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