

## OPTIMIZATION OF THE FIRST EIGENVALUE IN PROBLEMS INVOLVING THE BI-LAPLACIAN

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*Abstract.* This paper concerns minimization and maximization of the first eigenvalue in problems involving the bi-Laplacian under Dirichlet boundary conditions. Physically, in case of  $N = 2$ , our equation models the vibration of a non homogeneous plate  $\Omega$  which is clamped along the boundary. Given several materials (with different densities) of total extension  $|\Omega|$ , we investigate the location of these materials throughout  $\Omega$  so to minimize or maximize the first eigenvalue in the vibration of the clamped plate.

### 1. Introduction

Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^N$  and let  $g_0 = g_0(x)$  be a measurable function satisfying  $0 \leq g_0(x) \leq M$  in  $\Omega$ , where  $M$  is a positive constant. Define  $\mathcal{G}$  as the family of all measurable functions which are rearrangements of  $g_0$ . For  $g \in \mathcal{G}$ , consider the eigenvalue problem

$$\Delta^2 u = \Lambda g u, \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

where  $\nu$  denotes the exterior normal,  $\Lambda = \Lambda_g$  is the first eigenvalue and  $u = u_g(x)$  is a corresponding eigenfunction. The first eigenvalue  $\Lambda$  of problem (1.1) is obtained by minimizing the associate Rayleigh quotient

$$\Lambda = \inf \left\{ \frac{\int_{\Omega} (\Delta w)^2 dx}{\int_{\Omega} g w^2 dx} : w \in H_0^2(\Omega), w \not\equiv 0 \right\}. \quad (1.2)$$

It is well known [25] that a minimum  $w = u_g$  is attained in (1.2) and satisfies (1.1) in a weak sense. For the regularity of  $u_g$  we refer to [3]. In particular, this function belongs to  $H_{loc}^4(\Omega)$  and the equation (1.1) holds a.e. in  $\Omega$ . If we multiply by  $u = u_g$  in (1.1) and we integrate over  $\Omega$  we find

$$\Lambda_g = \frac{\int_{\Omega} (\Delta u_g)^2 dx}{\int_{\Omega} g u_g^2 dx}.$$

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Furthermore, if we have

$$\Lambda_g = \frac{\int_{\Omega} (\Delta v)^2 dx}{\int_{\Omega} g v^2 dx}$$

for some  $v \in H_0^2(\Omega)$  then also  $v$  satisfies problem (1.1).

We are interested in the following optimization problems

$$\min_{g \in \mathcal{G}} \Lambda_g, \quad \max_{g \in \mathcal{G}} \Lambda_g. \tag{1.3}$$

We prove existence of a minimizer in  $\mathcal{G}$ . Furthermore, if  $\bar{g}$  is a minimizer we prove that  $\bar{g} = \phi(u_{\bar{g}}^2)$ , where  $u_{\bar{g}}$  is a corresponding eigenfunction and  $\phi$  is some increasing function. This representation gives some information on the minimizer, because we know that  $u_{\bar{g}}^2$  is small close to the boundary.

The maximization problem is more difficult. We are able to prove existence of a maximizer in  $\mathcal{G}$  only for domains  $\Omega$  such that the operator  $\Delta^2 u$  is positive preserving under Dirichlet homogeneous boundary conditions; we refer to [17], [18], [19], [20] [26] for a discussion on this topic. In case  $\Omega$  has this property we prove existence of a maximizer. Furthermore, if  $\underline{g}$  is a maximizer we prove that  $\underline{g} = \psi(u_{\underline{g}}^2)$ , where  $u_{\underline{g}}$  is a corresponding eigenfunction and  $\psi$  is some decreasing function.

We observe that the optimization problems in case of Navier boundary conditions are discussed in [1] (minimization) and [14] (maximization). One difficulty that arises in the investigation of the first eigenvalue in case of Dirichlet boundary conditions is that the corresponding eigenfunction is not, in general, one signed [21]. In this paper we prove that an eigenfunction cannot vanish in a set of positive measure. Probably this result is known, but since we could not find it we include a proof in the last section of the present paper. By using this fact we are able to prove existence and a representation formula for a minimizer.

Let us give a motivation for the study of these problems in case of  $N = 2$ . Physically, our equation models the vibration of a non homogeneous plate  $\Omega$  which is clamped along the boundary  $\partial\Omega$ . Given several materials (with different densities) of total extension  $|\Omega|$ , we investigate the location of these materials throughout  $\Omega$  so to minimize or maximize the first eigenvalue in the vibration of the plate.

The corresponding problems for second order equations have been discussed in [5], [6], [7], [12], [11] and [13]. Related maximization problems are treated in [2] and [15].

## 2. Preliminaries

Denote with  $|E|$  the Lebesgue measure of the (measurable) set  $E$ . Given a function  $g_0(x)$  defined in  $\Omega$  and satisfying  $0 \leq g_0(x) \leq M$ , we say that  $g(x)$ , defined in  $\Omega$ , belongs to the class of rearrangements  $\mathcal{G} = \mathcal{G}(g_0)$  if

$$|\{g(x) \geq \beta\}| = |\{g_0(x) \geq \beta\}| \quad \forall \beta \geq 0.$$

Here and in what follows we write  $\{g(x) \geq \beta\}$  instead of  $\{x \in \Omega : g(x) \geq \beta\}$ . For a general theory on rearrangements we refer to [9].

We make use of the following results proved in [5] and [6]. For short, throughout the paper we shall write increasing instead of non-decreasing, and decreasing instead of non-increasing.

LEMMA 2.1. *Let  $g : \Omega \rightarrow \mathbb{R}$  and  $w : \Omega \rightarrow \mathbb{R}$  be measurable functions, and suppose that every level set of  $w$  has measure zero. Then there exists an increasing function  $\phi$  such that  $\phi(w)$  is a rearrangement of  $g$ . Furthermore, there exists a decreasing function  $\psi$  such that  $\psi(w)$  is a rearrangement of  $g$ .*

*Proof.* The first assertion follows from Lemma 2.9 of [6]. The second assertion follows applying the first one to  $-w$ .  $\square$

Denote with  $\overline{\mathcal{G}}$  the weak closure of  $\mathcal{G}$  in  $L^p(\Omega)$ . It is well known that  $\overline{\mathcal{G}}$  is convex and weakly sequentially compact (see for example [6], Lemma 2.2).

LEMMA 2.2. *Let  $\mathcal{G}$  be the set of rearrangements of a fixed function  $g_0 \in L^p(\Omega)$ ,  $p \geq 1$ , and let  $w \in L^q(\Omega)$ ,  $q = p/(p-1)$ . If there is an increasing function  $\phi$  such that  $\phi(w) \in \mathcal{G}$  then*

$$\int_{\Omega} g w dx \leq \int_{\Omega} \phi(w) w dx \quad \forall g \in \overline{\mathcal{G}},$$

*and the function  $\phi(w)$  is the unique maximizer relative to  $\overline{\mathcal{G}}$ . Furthermore, if there is a decreasing function  $\psi$  such that  $\psi(w) \in \mathcal{G}$  then*

$$\int_{\Omega} g w dx \geq \int_{\Omega} \psi(w) w dx \quad \forall g \in \overline{\mathcal{G}},$$

*and the function  $\psi(w)$  is the unique minimizer relative to  $\overline{\mathcal{G}}$ .*

*Proof.* The first assertion follows from Lemma 2.4 of [6]. To prove the second assertion we put  $\phi(t) = \psi(-t)$ . Since  $\phi$  is increasing, applying the previous result we have

$$\int_{\Omega} g(-w) dx \leq \int_{\Omega} \phi(-w)(-w) dx \quad \forall g \in \overline{\mathcal{G}},$$

and  $\phi(-w) = \psi(w)$  is the unique function satisfying the inequality. Equivalently, we have

$$\int_{\Omega} g w dx \geq \int_{\Omega} \psi(w) w dx \quad \forall g \in \overline{\mathcal{G}}.$$

The lemma is proved.  $\square$

LEMMA 2.3. *Let  $\mathcal{G}$  be the set of rearrangements of a fixed function  $g_0 \in L^p(\Omega)$ ,  $p \geq 1$ , and let  $w \in L^q(\Omega)$ ,  $q = p/(p-1)$ . There are  $g_1, g_2 \in \mathcal{G}$  such that*

$$\int_{\Omega} g_1 w dx \leq \int_{\Omega} g w dx \leq \int_{\Omega} g_2 w dx \quad \forall g \in \overline{\mathcal{G}}.$$

*Proof.* It follows from Lemma 2.4 of [6].  $\square$

LEMMA 2.4. Let  $\Psi : L^p(\Omega) \rightarrow \mathbb{R}$  be a convex functional, let  $\mathcal{G}$  denote the set of rearrangements of  $g_0$  on  $\Omega$ , and let  $q = p/(p - 1)$ .

(i) Suppose that  $\Psi$  is sequentially continuous in the  $L^q(\Omega)$  topology on  $L^p(\Omega)$ . Then  $\Psi$  attains a maximum value relative to  $\mathcal{G}$ .

(ii) Suppose  $\Psi$  is strictly convex, that  $g^*$  is a maximizer for  $\Psi$  relative to  $\mathcal{G}$  and that  $u$  is a member of the sub differential of  $\Psi$  at  $g^*$ . Then  $g^* = \phi(u)$  a.e. in  $\Omega$  for some increasing function  $\phi$ .

*Proof.* See Theorem 7 of [5].  $\square$

We recall that the  $L^q(\Omega)$  topology on  $L^p(\Omega)$  is the weak topology if  $1 \leq p < \infty$ , or the weak\* topology if  $p = \infty$  [5].

### 3. Main results

In all this section,  $\mathcal{G}$  is the class of rearrangements of a fixed function  $g_0(x)$  such that  $0 \leq g_0(x) \leq M$ , and  $\overline{\mathcal{G}}$  is the closure of  $\mathcal{G}$  with respect to the weak\* topology of  $L^\infty(\Omega)$ .

LEMMA 3.1. Let  $g \in \overline{\mathcal{G}}$ . If  $\Lambda_g$  is the first eigenvalue of problem (1.1) then the functional

$$g \mapsto \Lambda_g$$

is continuous with respect to the weak\* topology in  $L^\infty(\Omega)$ .

*Proof.* Let  $g_i \rightarrow g$  in the weak\* topology of  $L^\infty(\Omega)$ . Let  $\Lambda_{g_i}, \Lambda_g$  be the corresponding eigenvalues, and let  $u_{g_i}, u_g$  be corresponding eigenfunctions normalized so that

$$\int_{\Omega} (\Delta u_{g_i})^2 dx = \int_{\Omega} (\Delta u_g)^2 dx = 1.$$

Since  $\|\Delta w\|_{L^2(\Omega)}$  is equivalent to the norm  $\|w\|_{H^2_0(\Omega)}$  in  $H^2_0(\Omega)$ , a sub-sequence of  $u_{g_i}$  (denoted again  $u_{g_i}$ ) converges weakly in  $\bar{u} \in H^2_0(\Omega)$  and strongly in  $L^2(\Omega)$ . We have

$$\begin{aligned} \frac{1}{\Lambda_{g_i}} &= \int_{\Omega} g_i u_{g_i}^2 dx = \int_{\Omega} (g_i - g) u_{g_i}^2 dx + \int_{\Omega} g u_{g_i}^2 dx \\ &\leq \int_{\Omega} (g_i - g) u_{g_i}^2 dx + \int_{\Omega} g u_g^2 dx = \int_{\Omega} (g_i - g) u_{g_i}^2 dx + \frac{1}{\Lambda_g}. \end{aligned}$$

Since  $u_{g_i}$  converges in the  $L^2(\Omega)$  norm we have

$$\limsup_{i \rightarrow \infty} \frac{1}{\Lambda_{g_i}} \leq \frac{1}{\Lambda_g}. \tag{3.1}$$

On the other side, we have

$$\frac{1}{\Lambda_{g_i}} = \int_{\Omega} g_i u_{g_i}^2 dx \geq \int_{\Omega} g_i u_g^2 dx = \int_{\Omega} (g_i - g) u_g^2 dx + \int_{\Omega} g u_g^2 dx.$$

Hence,

$$\liminf_{i \rightarrow \infty} \frac{1}{\Lambda_{g_i}} \geq \int_{\Omega} g u_g^2 dx = \frac{1}{\Lambda_g}.$$

The lemma follows by (3.1) and the latter inequality.  $\square$

### 3.1. The minimum

**THEOREM 3.2.** *Let  $0 \leq g_0(x) \leq M$ , and let  $\mathcal{G}$  be the class of all rearrangements of  $g_0$ . If  $\Lambda_g$  is the first eigenvalue of problem (1.1) then there exists  $\bar{g} \in \mathcal{G}$  such that*

$$\Lambda_{\bar{g}} = \min_{g \in \mathcal{G}} \Lambda_g.$$

Moreover, if  $u_{\bar{g}}$  is an eigenfunction of (1.1) corresponding to  $g = \bar{g}$  then  $\bar{g} = \phi(u_{\bar{g}}^2)$  for some increasing functions  $\phi$ .

*Proof.* By Lemma 3.1 the functional  $g \rightarrow \Lambda_g$  is continuous with respect to the weak\* topology in  $L^\infty(\Omega)$ . We claim that  $g \mapsto \frac{1}{\Lambda_g}$  is strictly convex on  $\mathcal{G}$ . Indeed, if  $g_1, g_2 \in \mathcal{G}$ , if  $g_t = t g_1 + (1-t) g_2$  with  $0 < t < 1$ , if  $\Lambda_{g_1}, \Lambda_{g_2}, \Lambda_{g_t}$  are the corresponding eigenvalues, and if  $u_{g_1}, u_{g_2}, u_{g_t}$  are corresponding eigenfunctions then we have

$$\begin{aligned} \frac{1}{\Lambda_{g_t}} &= \frac{\int_{\Omega} g_t u_{g_t}^2 dx}{\int_{\Omega} (\Delta u_{g_t})^2 dx} = t \frac{\int_{\Omega} g_1 u_{g_t}^2 dx}{\int_{\Omega} (\Delta u_{g_t})^2 dx} + (1-t) \frac{\int_{\Omega} g_2 u_{g_t}^2 dx}{\int_{\Omega} (\Delta u_{g_t})^2 dx} \\ &\leq t \frac{\int_{\Omega} g_1 u_{g_1}^2 dx}{\int_{\Omega} (\Delta u_{g_1})^2 dx} + (1-t) \frac{\int_{\Omega} g_2 u_{g_2}^2 dx}{\int_{\Omega} (\Delta u_{g_2})^2 dx} = t \frac{1}{\Lambda_{g_1}} + (1-t) \frac{1}{\Lambda_{g_2}}. \end{aligned}$$

If equality holds in above then we must have

$$\frac{\int_{\Omega} g_1 u_{g_t}^2 dx}{\int_{\Omega} (\Delta u_{g_t})^2 dx} = \frac{1}{\Lambda_{g_1}}$$

and

$$\frac{\int_{\Omega} g_2 u_{g_t}^2 dx}{\int_{\Omega} (\Delta u_{g_t})^2 dx} = \frac{1}{\Lambda_{g_2}}.$$

Then,

$$\Delta^2 u_{g_t} = \Lambda_{g_1} g_1 u_{g_t} = \Lambda_{g_2} g_2 u_{g_t} \text{ a.e. in } \Omega.$$

By the unique continuation theorem (see next section) we have  $u_{g_t} \neq 0$  a.e. in  $\Omega$ . Therefore,

$$\Lambda_{g_1} g_1 = \Lambda_{g_2} g_2 \text{ a.e. in } \Omega. \tag{3.2}$$

Since

$$\int_{\Omega} g_1 dx = \int_{\Omega} g_2 dx,$$

by (3.2) we find  $\Lambda_{g_1} = \Lambda_{g_2}$ . Finally, (3.2) implies that  $g_1 = g_2$  a.e. in  $\Omega$ . The strict convexity of  $\frac{1}{\Lambda_g}$  is proved.

The existence of a maximizer of  $\frac{1}{\Lambda_g}$  (which is a minimizer of  $\Lambda_g$ ) follows by Lemma 2.4. To prove the second statement of the theorem, we must find a sub-gradient of  $\frac{1}{\Lambda_g}$  corresponding to a maximizer  $\bar{g}$ . Let  $\bar{g}$  be a maximizer, let  $\Lambda_{\bar{g}}$  be the corresponding eigenvalue and let  $u_{\bar{g}}$  be a corresponding eigenfunction normalized so that  $\|\Delta u_{\bar{g}}\|_{L^2(\Omega)} = 1$ . Let  $h \in \mathcal{F}$ , let  $\Lambda_h$  be the corresponding eigenvalue and let  $u_h$  be a corresponding eigenfunction normalized so that  $\|\Delta u_h\|_{L^2(\Omega)} = 1$ . Then

$$\frac{1}{\Lambda_h} = \int_{\Omega} h u_h^2 dx \geq \int_{\Omega} h u_{\bar{g}}^2 dx = \int_{\Omega} \bar{g} u_{\bar{g}}^2 dx + \int_{\Omega} (h - \bar{g}) u_{\bar{g}}^2 dx = \frac{1}{\Lambda_{\bar{g}}} + \int_{\Omega} (h - \bar{g}) u_{\bar{g}}^2 dx.$$

Therefore,  $u_{\bar{g}}^2$  is a member of the sub-gradient, and by Lemma 2.4 there is an increasing function  $\phi$  such that  $\bar{g} = \phi(u_{\bar{g}}^2)$ . The theorem is proved.  $\square$

### 3.2. The maximum

Now we investigate the problem

$$\sup_{g \in \mathcal{F}} \inf_{w \in H_0^2(\Omega), w \neq 0} \frac{\int_{\Omega} (\Delta w)^2 dx}{\int_{\Omega} g w^2 dx}. \tag{3.5}$$

We note that we cannot interchange (in general) the superior with the inferior. Following [11] (see also [14]), we give a different formulation of the problem by using the Auchmuty’s principle [4].

LEMMA 3.3. *Let  $0 \leq g(x) \leq M$ ,  $g(x) \not\equiv 0$ , and let  $\Lambda_g$  be the first eigenvalue of problem (1.1). If*

$$A(g, w) = \frac{1}{2} \int_{\Omega} (\Delta w)^2 dx - \left( \int_{\Omega} g w^2 dx \right)^{\frac{1}{2}} \tag{3.6}$$

then

$$\min_{w \in H_0^2(\Omega)} A(g, w) = -\frac{1}{2} \max_{w \in H_0^2(\Omega), w \neq 0} \frac{\int_{\Omega} g w^2 dx}{\int_{\Omega} (\Delta w)^2 dx} = -\frac{1}{2} \frac{1}{\Lambda_g}.$$

The minimum of  $A(g, w)$  is attained at an eigenfunction  $w = u_g$  normalized as

$$\frac{1}{\Lambda_g} = \left( \int_{\Omega} g u_g^2 dx \right)^{\frac{1}{2}} = \int_{\Omega} (\Delta u_g)^2 dx. \tag{3.7}$$

*Proof.* See [4] or Lemma 3.3 of [14].  $\square$

LEMMA 3.4. *If  $A(g, w)$  is defined as in (3.6) with  $w \in H^2(\Omega)$ , the function  $g \mapsto A(g, w)$  is quasi-concave in  $\mathcal{F}$ .*

*Proof.* It suffices to prove that the function  $g \mapsto \left( \int_{\Omega} g w^2 dx \right)^{\frac{1}{2}}$  is quasi-convex. Let

$$\left( \int_{\Omega} g_1 w^2 dx \right)^{\frac{1}{2}} \leq c, \quad \left( \int_{\Omega} g_2 w^2 dx \right)^{\frac{1}{2}} \leq c.$$

Then

$$\begin{aligned} \left( \int_{\Omega} (tg_1 + (1-t)g_2)w^2 dx \right)^{\frac{1}{2}} &= \left( t \int_{\Omega} g_1 w^2 dx + (1-t) \int_{\Omega} g_2 w^2 dx \right)^{\frac{1}{2}} \\ &\leq (tc^2 + (1-t)c^2)^{\frac{1}{2}} = c. \end{aligned}$$

The lemma is proved.  $\square$

From now on we restrict the class of domains  $\Omega$ . We suppose  $\Omega$  is such that the operator  $\Delta^2 u$  under homogeneous Dirichlet boundary conditions is positivity preserving. This means that any nontrivial solution of  $\Delta^2 u \geq 0$  in  $\Omega$  with  $u = \frac{\partial u}{\partial \nu} = 0$  on  $\partial\Omega$  satisfies  $u(x) > 0$  in  $\Omega$ . It is well known that the ball has this property; for other domains which enjoy such a property we refer to [17], [18], [19], [20]. By Krein-Rutman theorem [23], for this kind of domains  $\Omega$ , the first eigenvalue of problem (1.1) is simple and a corresponding eigenfunction is positive. By the unique continuation principle we know that such an eigenfunction is strictly positive.

Let  $\Omega$  be positivity preserving for  $\Delta^2 u$  under homogeneous Dirichlet boundary conditions, and let  $g \in \mathcal{G}$  fixed. If  $u_g$  is a positive solution of (1.1) normalized as in (3.7) we define

$$\Pi_g = \left\{ w \in H_0^2(\Omega) : \Lambda_g^2 \int_{\Omega} g u_g w dx > 1 \right\}. \quad (3.8)$$

LEMMA 3.5. *The function  $w \mapsto A(g, w)$  defined in (3.6) is strictly convex on  $\Pi_g$ .*

*Proof.* Let  $w \in \Pi_g$ . For  $z \in H_0^2(\Omega)$  define  $\Phi(t) = A(g, w + tz)$ . We find

$$\Phi'(t) = \int_{\Omega} (\Delta w + t\Delta z)\Delta z dx - \left( \int_{\Omega} g(w + tz)^2 dx \right)^{-\frac{1}{2}} \int_{\Omega} g(w + tz)z dx,$$

and

$$\Phi''(0) = \int_{\Omega} (\Delta z)^2 dx - \left( \int_{\Omega} g w^2 dx \right)^{-\frac{1}{2}} \int_{\Omega} g z^2 dx + \left( \int_{\Omega} g w^2 dx \right)^{-\frac{3}{2}} \left( \int_{\Omega} g w z dx \right)^2.$$

Since

$$\int_{\Omega} (\Delta z)^2 dx \geq \Lambda_g \int_{\Omega} g z^2 dx$$

we have

$$\Phi''(0) \geq \Lambda_g \int_{\Omega} g z^2 dx - \left( \int_{\Omega} g w^2 dx \right)^{-\frac{1}{2}} \int_{\Omega} g z^2 dx = \left[ \Lambda_g - \left( \int_{\Omega} g w^2 dx \right)^{-\frac{1}{2}} \right] \int_{\Omega} g z^2 dx.$$

On the other side, using Schwarz inequality and recalling the normalization of  $u_g$  given by (3.7) we have

$$1 < \Lambda_g^2 \int_{\Omega} g u_g w dx \leq \Lambda_g^2 \left( \int_{\Omega} g u_g^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} g w^2 dx \right)^{\frac{1}{2}} = \Lambda_g \left( \int_{\Omega} g w^2 dx \right)^{\frac{1}{2}}.$$

It follows that  $\Phi''(0) > 0$ . The lemma is proved.  $\square$

As the set  $\Pi_g$  over which  $w \mapsto A(g, w)$  is convex depends on  $g$  we introduce, following [11] pag.176, the indicator function

$$\pi(g, u) = \begin{cases} 0 & \text{if } u \in \Pi_g \\ \infty & \text{otherwise.} \end{cases}$$

LEMMA 3.6. *Let  $u \in H_0^2(\Omega)$ . The function  $g \mapsto A(g, u) + \pi(g, u)$  is weak\* upper semicontinuous over  $\overline{\mathcal{G}}$ .*

*Proof.* Let  $g_i \rightarrow \overline{g}$  in the weak\* topology of  $L^\infty(\Omega)$ . Since  $\int_\Omega g_i u^2 dx \rightarrow \int_\Omega \overline{g} u^2 dx$ , it follows that  $g \mapsto A(g, u)$  is weak\* upper semicontinuous over  $\mathcal{G}$ . On the other side, let us show that

$$\limsup_{i \rightarrow \infty} \pi(g_i, u) \leq \pi(\overline{g}, u) \quad \forall u \in H_0^2(\Omega).$$

This is clearly true if  $u \notin \Pi_{\overline{g}}$ . Let  $u \in \Pi_{\overline{g}}$ . By Lemma 3.1 we know that  $\Lambda_{g_i} \rightarrow \Lambda_{\overline{g}}$ . Since  $\Omega$  is positivity preserving, there is a unique positive normalized eigenfunction  $u_{g_i}$  and a unique positive normalized eigenfunction  $u_{\overline{g}}$ . Then, by the proof of Lemma 3.1 we find that  $u_{g_i} \rightarrow u_{\overline{g}}$  in  $L^s(\Omega)$  for some  $s > 2$ . It follows that

$$\Lambda_{g_i}^2 \int_\Omega g_i u_{g_i} u \, dx \rightarrow \Lambda_{\overline{g}}^2 \int_\Omega \overline{g} u_{\overline{g}} u \, dx.$$

$u \in \Pi_{\overline{g}}$  now implies that for  $i$  greater than some  $i_0$  depending only on  $\overline{g}$ ,  $u$  belongs to  $\Pi_{g_i}$ , that is,  $\pi(g_i, u) = 0 = \pi(\overline{g}, u)$ . The lemma follows.  $\square$

LEMMA 3.7. *Let  $0 \leq g_0(x) \leq M$ ,  $g_0(x) \not\equiv 0$ , and let  $A(g, w)$  be defined as in (3.6). Then the following equality holds:*

$$\sup_{g \in \overline{\mathcal{G}}} \inf_{w \in H_0^2(\Omega)} A(g, w) = \inf_{w \in H_0^2(\Omega)} \sup_{g \in \overline{\mathcal{G}}} A(g, w). \tag{3.9}$$

*Proof.* The proof uses Lemmata 3.4, 3.5 and 3.6, and is the same as that of Proposition 7.7 of [11].  $\square$

As observed in [11], Lemma 3.7 implies that  $A(g, w)$  has a saddle point. Since there is not an explicit proof of this fact in the cited paper, we enclose here a simple proof (see [14] for a similar proof in the corresponding case of Navier boundary conditions).

PROPOSITION 3.8. *There exists  $\underline{g} \in \overline{\mathcal{G}}$  such that the pair  $(\underline{g}, u_{\underline{g}})$  is a saddle point for  $A(g, w)$  in  $(\overline{\mathcal{G}}, H_0^2(\Omega))$ .*

*Proof.* For  $w \in H_0^2(\Omega)$  we define

$$B(w) = \sup_{g \in \overline{\mathcal{G}}} A(g, w).$$



We claim that there exists  $\underline{u} \in H_0^2(\Omega)$  such that

$$B(\underline{u}) = \inf_{w \in H_0^2(\Omega)} B(w). \quad (3.10)$$

Let  $u_k$  be a minimizing sequence in  $H_0^2(\Omega)$ . Since we can assume that  $B(u_k)$  is decreasing, we have  $B(u_k) \leq C_1$  for some positive constant. This estimate implies that

$$\frac{1}{2} \int_{\Omega} (\Delta u_k)^2 dx - \inf_{g \in \overline{\mathcal{G}}} \left( \int_{\Omega} g u_k^2 dx \right)^{\frac{1}{2}} \leq C_1.$$

Since  $\|g\|_{\infty} \leq M$  for all  $g \in \overline{\mathcal{G}}$  and since  $\|\Delta w\|_{L^2(\Omega)}$  is equivalent to the norm  $\|w\|_{H^2(\Omega)}$  in  $H_0^2(\Omega)$  we can write

$$\frac{1}{2} \|\Delta u_k\|_{L^2(\Omega)}^2 \leq C_1 + C_2 \|\Delta u_k\|_{L^2(\Omega)}$$

for some positive constant  $C_2$ . A straightforward calculation shows that

$$\|\Delta u_k\|_{L^2(\Omega)} \leq C_2 + \sqrt{C_2^2 + 2C_1}.$$

Then, up to a subsequence,  $u_k$  is weakly convergent in  $H^2(\Omega)$  and strongly in  $L^2(\Omega)$  to a function  $\underline{u} \in H_0^2(\Omega)$ .

By Lemma 2.3 we find  $g_1 \in \mathcal{G}$  such that

$$\inf_{g \in \overline{\mathcal{G}}} \left( \int_{\Omega} g \underline{u}^2 dx \right)^{\frac{1}{2}} = \left( \int_{\Omega} g_1 \underline{u}^2 dx \right)^{\frac{1}{2}}. \quad (3.11)$$

On the other side we have

$$\lim_{k \rightarrow \infty} \int_{\Omega} g_1 u_k^2 dx = \int_{\Omega} g_1 \underline{u}^2 dx, \quad (3.12)$$

and

$$\inf_{g \in \overline{\mathcal{G}}} \left( \int_{\Omega} g u_k^2 dx \right)^{\frac{1}{2}} \leq \left( \int_{\Omega} g_1 u_k^2 dx \right)^{\frac{1}{2}}.$$

Taking the lim sup in the last inequality and using (3.11) and (3.12) we find

$$\limsup_{k \rightarrow \infty} \inf_{g \in \overline{\mathcal{G}}} \left( \int_{\Omega} g u_k^2 dx \right)^{\frac{1}{2}} \leq \left( \int_{\Omega} g_1 \underline{u}^2 dx \right)^{\frac{1}{2}} = \inf_{g \in \overline{\mathcal{G}}} \left( \int_{\Omega} g \underline{u}^2 dx \right)^{\frac{1}{2}}. \quad (3.13)$$

Note that the weak convergence of  $u_k \rightarrow \underline{u}$  in the  $H^2(\Omega)$  norm implies

$$\liminf_{k \rightarrow \infty} \int_{\Omega} (\Delta u_k)^2 dx \geq \int_{\Omega} (\Delta \underline{u})^2 dx.$$

Using the latter inequality and (3.13) we find

$$\inf_{w \in H_0^2(\Omega)} B(w) = \lim_{k \rightarrow \infty} B(u_k) \geq B(\underline{u}) \geq \inf_{w \in H_0^2(\Omega)} B(w).$$

The claim (3.10) follows.

From (3.10) we infer the existence of  $\underline{u} \in H_0^2(\Omega)$  such that

$$\sup_{g \in \overline{\mathcal{G}}} A(g, \underline{u}) = \inf_{w \in H_0^2(\Omega)} \sup_{g \in \overline{\mathcal{G}}} A(g, w). \tag{3.14}$$

Since  $\overline{\mathcal{G}}$  is compact with respect to the weak\* topology of  $L^\infty(\Omega)$ , by using Lemma 3.1 one finds the existence of  $\underline{g} \in \overline{\mathcal{G}}$  such that

$$\sup_{g \in \overline{\mathcal{G}}} \Lambda_g = \Lambda_{\underline{g}}.$$

Hence, in view of Lemma 3.3 we have

$$\sup_{g \in \overline{\mathcal{G}}} \inf_{w \in H_0^2(\Omega)} A(g, w) = A(\underline{g}, \underline{u}_g) = \inf_{w \in H_0^2(\Omega)} A(\underline{g}, w). \tag{3.15}$$

By using (3.14), Lemma 3.7 and (3.15), we can write

$$\sup_{g \in \overline{\mathcal{G}}} A(g, \underline{u}) = \inf_{w \in H_0^2(\Omega)} \sup_{g \in \overline{\mathcal{G}}} A(g, w) = \sup_{g \in \overline{\mathcal{G}}} \inf_{w \in H_0^2(\Omega)} A(g, w) = \inf_{w \in H_0^2(\Omega)} A(\underline{g}, w) \leq A(\underline{g}, \underline{u}).$$

Hence, for any  $g \in \overline{\mathcal{G}}$  we have

$$A(g, \underline{u}) \leq A(\underline{g}, \underline{u}). \tag{3.16}$$

Similarly we find

$$A(\underline{g}, \underline{u}) \leq \sup_{g \in \overline{\mathcal{G}}} A(g, \underline{u}) = \inf_{w \in H_0^2(\Omega)} \sup_{g \in \overline{\mathcal{G}}} A(g, w) = \sup_{g \in \overline{\mathcal{G}}} \inf_{w \in H_0^2(\Omega)} A(g, w) = \inf_{w \in H_0^2(\Omega)} A(\underline{g}, w).$$

Therefore, for any  $w \in H_0^2(\Omega)$  we have

$$A(\underline{g}, \underline{u}) \leq A(\underline{g}, w). \tag{3.17}$$

By (3.17) and Lemma 3.3 we must have  $\underline{u} = \underline{u}_g$ . Hence, the proposition follows by (3.16) and (3.17).  $\square$

**THEOREM 3.9.** *Let  $\Omega$  be positivity preserving for  $\Delta^2 u$  under homogeneous Dirichlet boundary conditions. Let  $0 \leq g_0(x) \leq M$ ,  $g_0(x) \not\equiv 0$ , and let  $\mathcal{G}$  be the class of all rearrangements of  $g_0$ . If  $\Lambda_g$  is the first eigenvalue of problem (1.1) then there exists  $\underline{g} \in \mathcal{G}$  such that*

$$\Lambda_{\underline{g}} = \max_{g \in \mathcal{G}} \Lambda_g. \tag{3.18}$$

*Moreover, if  $\underline{u}_g$  is an eigenfunction of (1.1) corresponding to  $g = \underline{g}$  then  $\underline{g} = \psi(u_{\underline{g}}^2)$  for some decreasing functions  $\psi$ .*

*Proof.* By Proposition 3.8, there is a saddle point  $(\underline{g}, u_{\underline{g}})$  for  $A(g, w)$  in  $(\overline{\mathcal{G}}, H_0^2(\Omega))$ . In particular, we have

$$A(g, u_{\underline{g}}) \leq A(\underline{g}, u_{\underline{g}}) \quad \forall g \in \overline{\mathcal{G}}.$$

Recalling the definition (3.6) of  $A(g, w)$ , the latter inequality implies

$$\int_{\Omega} g u_{\underline{g}}^2 dx \geq \int_{\Omega} \underline{g} u_{\underline{g}}^2 dx \quad \forall g \in \overline{\mathcal{G}}. \tag{3.19}$$

The functions  $\underline{g}$  and  $u = u_{\underline{g}}$  satisfy

$$\Delta^2 u = \Lambda_{\underline{g}} \underline{g} u \text{ a.e. in } \Omega. \tag{3.20}$$

Recall that  $u = u_{\underline{g}}$  is either strictly positive or strictly negative. Therefore, by (3.20) it follows that the function  $u$  cannot have flat zones in the set  $F = \{x \in \Omega : \underline{g}(x) > 0\}$ . If  $|F| = |\Omega|$ , by Lemma 2.1 there is a decreasing function  $\psi(t)$  such that  $\psi(u_{\underline{g}}^2)$  is a rearrangement of  $g_0(x)$  on  $\Omega$ . By (3.19) and Lemma 2.2 we must have  $\underline{g} = \psi(u_{\underline{g}}^2) \in \mathcal{G}$ , and the theorem is proved. If  $|F| < |\Omega|$ , since  $\underline{g} \in \overline{\mathcal{G}}$ , by Lemma 2.14 of [6] we have  $|F| \geq |\{x \in \Omega : g_0(x) > 0\}|$ . Therefore there is  $g_1 \in \mathcal{G}$  such that its support is contained in  $F$ . By Lemma 2.1, there is a decreasing function  $\psi_1(t)$  such that  $\psi_1(u_{\underline{g}}^2)$  is a rearrangement of  $g_1(x)$  on  $F$ .

Define

$$\alpha = \inf_{x \in \Omega \setminus F} u_{\underline{g}}^2(x).$$

We claim that  $u_{\underline{g}}^2(x) \leq \alpha$  in  $F$ . Arguing by contradiction suppose the claim is false. Therefore there exist a number  $S_1 > \alpha$  and a subset  $A$  of  $F$  with  $|A| > 0$  such that  $u_{\underline{g}}^2(x) > S_1$  a.e. on  $A$ . Now let  $\alpha < S_2 < S_1$ . We can find a set  $D$  of positive measure contained in  $\Omega \setminus F$  such that  $u_{\underline{g}}^2(x) < S_2$  a.e. on  $D$ . We can assume  $|A| = |D|$ . Using a measure preserving  $T$  we define a particular rearrangement of  $\underline{g}$ , denoted by  $h$ , as follows.

$$h(x) = \begin{cases} \underline{g}(Tx), & x \in A \\ \underline{g}(T^{-1}x), & x \in D \\ \underline{g}(x), & x \in \Omega \setminus (A \cup D). \end{cases}$$

Thus

$$\begin{aligned} \int_{\Omega} h u_{\underline{g}}^2 dx - \int_{\Omega} \underline{g} u_{\underline{g}}^2 dx &= \int_{A \cup D} h u_{\underline{g}}^2 dx - \int_{A \cup D} \underline{g} u_{\underline{g}}^2 dx \\ &= \int_A h u_{\underline{g}}^2 dx + \int_A \underline{g} u_{\underline{g}}^2 \circ T dx - \int_A \underline{g} u_{\underline{g}}^2 dx - \int_A h u_{\underline{g}}^2 \circ T dx \\ &= \int_A (u_{\underline{g}}^2 \circ T - u_{\underline{g}}^2)(\underline{g} - h) dx < (S_2 - S_1) \int_A \underline{g} dx < 0. \end{aligned}$$

Therefore  $\int_{\Omega} h u_{\underline{g}}^2 dx < \int_{\Omega} \underline{g} u_{\underline{g}}^2 dx$ , which contradicts (3.19), and the claim follows.

By using equation (3.20) again we find that  $u_{\underline{g}}^2(x) < \alpha$  a.e. in  $F$ . Now define

$$\psi(t) = \begin{cases} \psi_1(t) & \text{if } 0 \leq t < \alpha \\ 0 & \text{if } t \geq \alpha. \end{cases}$$

The function  $\psi(t)$  is decreasing and  $\psi(u_{\underline{g}}^2)$  is a rearrangement of  $g_1(x)$  in  $\Omega$ . Indeed, the functions  $g_1$  and  $\psi(u_{\underline{g}}^2)$  have the same rearrangement on  $F$ , and both vanish on  $\Omega \setminus F$ . By (3.19) and Lemma 2.2 we must have  $\underline{g} = \psi(u_{\underline{g}}^2) \in \mathcal{G}$ . The theorem follows.  $\square$

#### 4. Unique continuation

Let  $\Omega$  be a domain in  $\mathbb{R}^N$  and let  $g \in L^\infty(\Omega)$ . Consider a weak solution  $u \in H^2(\Omega)$  of the equation

$$\Delta^2 u = g(x)u \text{ in } \Omega. \tag{4.1}$$

It is well known [3] that  $u \in H_{loc}^4(\Omega)$ .

LEMMA 4.1. *Let  $\Omega$  be a domain in  $\mathbb{R}^N$  containing the origin 0. Let  $u \in H_{loc}^4(\Omega)$  be a solution to (4.1) with  $|g(x)| \leq M$  and let for all  $n \in \mathbb{N}$*

$$\int_{|x| \leq r} u^2 = O(r^n), \quad r \rightarrow 0.$$

*Then  $u$  is identically zero in  $\Omega$ .*

*Proof.* It follows by Theorem 1.1 of [8]. See also [10] and [24].  $\square$

LEMMA 4.2. *Let  $B(x_0, r)$  and  $B(x_0, 2r)$ ,  $r \in (0, 1)$ , be two concentric balls contained in  $\Omega$ , and let  $u$  be a solution of (4.1) with  $|g(x)| \leq M$ . Then there exists a constant  $C$  independent of  $r$  such that*

$$\int_{B(x_0, r)} (\Delta u)^2 \leq C \left[ \frac{1}{r^2} \int_{B(x_0, 2r)} |\nabla u|^2 + \frac{1}{r^4} \int_{B(x_0, 2r)} u^2 \right].$$

*Proof.* Let  $\theta \in C_0^\infty(B(x_0, 2r))$  with  $0 \leq \theta \leq 1$ ,  $|\nabla \theta| \leq C/r$  and  $|\Delta \theta| \leq C/r^2$  in  $B(x_0, 2r)$ , and  $\theta = 1$  in  $B(x_0, r)$ . Here and in what follows we denote by  $C$  a generic positive constant, which may vary from line to line, and is independent of  $r$ .

Multiplying the equation (4.1) by  $\theta^4 u$  and integrating by parts on  $B(x_0, 2r)$  we find

$$\int_{B(x_0, 2r)} \Delta u \Delta(\theta^4 u) = \int_{B(x_0, 2r)} g(x) \theta^4 u^2.$$

If  $|g(x)| \leq M$  we get

$$\begin{aligned} & \int_{B(x_0, 2r)} \theta^4 (\Delta u)^2 + 4 \int_{B(x_0, 2r)} \theta^3 \Delta u \Delta \theta u \\ & + 12 \int_{B(x_0, 2r)} \theta^2 |\nabla \theta|^2 \Delta u u + 8 \int_{B(x_0, 2r)} \theta^3 \nabla \theta \cdot \nabla u \Delta u \\ & \leq M \int_{B(x_0, 2r)} u^2. \end{aligned} \tag{4.2}$$

Using the Schwarz inequality we find

$$-4 \int_{B(x_0, 2r)} \theta^3 \Delta u \Delta \theta u \leq \frac{1}{6} \int_{B(x_0, 2r)} \theta^4 (\Delta u)^2 + 24 \int_{B(x_0, 2r)} \theta^2 (\Delta \theta)^2 u^2.$$

Recalling that  $|\Delta \theta| \leq C/r^2$  and  $0 \leq \theta \leq 1$  we have

$$-4 \int_{B(x_0, 2r)} \theta^3 \Delta u \Delta \theta u \leq \frac{1}{6} \int_{B(x_0, 2r)} \theta^4 (\Delta u)^2 + \frac{C}{r^4} \int_{B(x_0, 2r)} u^2. \quad (4.3)$$

Similarly, we find

$$-8 \int_{B(x_0, 2r)} \theta^2 |\nabla \theta|^2 \Delta u u \leq \frac{1}{6} \int_{B(x_0, 2r)} \theta^4 (\Delta u)^2 + 96 \int_{B(x_0, 2r)} |\nabla \theta|^4 u^2.$$

Since  $|\nabla \theta| \leq C/r$  we have

$$-8 \int_{B(x_0, 2r)} \theta^2 |\nabla \theta|^2 \Delta u u \leq \frac{1}{6} \int_{B(x_0, 2r)} \theta^4 (\Delta u)^2 + \frac{C}{r^4} \int_{B(x_0, 2r)} u^2. \quad (4.4)$$

Finally, we find

$$\begin{aligned} -8 \int_{B(x_0, 2r)} \theta^3 \nabla \theta \cdot \nabla u \Delta u &\leq \frac{1}{6} \int_{B(x_0, 2r)} \theta^4 (\Delta u)^2 + 96 \int_{B(x_0, 2r)} \theta^2 |\nabla \theta|^2 |\nabla u|^2 \\ &\leq \frac{1}{6} \int_{B(x_0, 2r)} \theta^4 (\Delta u)^2 + \frac{C}{r^2} \int_{B(x_0, 2r)} |\nabla u|^2. \end{aligned} \quad (4.5)$$

Inserting the estimates (4.3), (4.4) and (4.5) into (4.2), after some simplification and recalling that  $\theta = 1$  in  $B(x_0, r)$ , we get the desired estimate.  $\square$

**LEMMA 4.3.** *Let  $u$  be a solution of (4.1) with  $|g(x)| \leq M$ . If  $E = \{x \in \Omega : u(x) = 0\}$  has a positive measure then there is  $x_0 \in \Omega$  such that for every  $n \in \mathbb{N}$  we have*

$$\int_{B_r} u^2 = O(r^n), \quad \int_{B_r} |\nabla u|^2 = O(r^n), \quad r \rightarrow 0, \quad (4.6)$$

where  $B_r = \{x \in \Omega : |x - x_0| < r\}$ .

*Proof.* We know that almost every point of  $E$  is a point of density. Let  $x_0$  be such a point. This means that for a given  $\varepsilon > 0$  there is  $r_0 = r_0(\varepsilon)$  such that for  $r < r_0$  we have

$$\frac{|E^c \cap B_r|}{|B_r|} < \varepsilon, \quad \frac{|E \cap B_r|}{|B_r|} > 1 - \varepsilon,$$

where  $E^c$  denotes the complement of  $E$ . We may suppose  $B_{4r} \subset \Omega$  and  $r < 1$ . For  $N \geq 3$  we have

$$\begin{aligned} \int_{B_r} u^2 &= \int_{B_r \cap E^c} u^2 \leq \left( \int_{B_r} |u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}} |B_r \cap E^c|^{\frac{2}{N}} \\ &\leq C \int_{B_r} [|\nabla u|^2 + u^2] \varepsilon^{\frac{2}{N}} r^2 \leq C \varepsilon^{\frac{2}{N}} \left[ r^2 \int_{B_r} |\nabla u|^2 + \int_{B_r} u^2 \right]. \end{aligned} \quad (4.7)$$

Let  $\tilde{E} = \{x \in B_r : \nabla u(x) = 0\}$ . Since for almost every point of  $E$  we have  $\nabla u = 0$  (see [16], Lemma 7.7), the inclusion  $E \subset \tilde{E}$  holds except possibly in a set of zero measure. Hence,  $x_0$  is a point of density also for  $\tilde{E}$ . For  $r < r_1(\varepsilon)$  (with  $r_1(\varepsilon) \leq r_0(\varepsilon)$ ) we have

$$\begin{aligned} \int_{B_r} |\nabla u|^2 &= \int_{B_r \cap \tilde{E}^c} |\nabla u|^2 \leq \left( \int_{B_r} |\nabla u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}} |B_r \cap \tilde{E}^c|^{\frac{2}{N}} \\ &\leq C \varepsilon^{\frac{2}{N}} r^2 \left[ \int_{B_r} |D^2 u|^2 + u^2 \right]. \end{aligned} \tag{4.8}$$

If  $v \in H^2(B_{2r}) \cap H_0^1(B_{2r})$  we have ([16], Theorem 8.12)

$$\int_{B_{2r}} [|D^2 v|^2 + v^2] \leq C \int_{B_{2r}} [(\Delta v)^2 + v^2].$$

Take  $v = u\theta$  with  $\theta \in C_0^\infty(B_{2r})$ ,  $\theta = 1$  in  $B_r$ ,  $|\nabla \theta| \leq C/r$  and  $|D^2 \theta| \leq C/r^2$  in  $B(x_0, 2r)$ . We find

$$\int_{B_r} [|D^2 u|^2 + u^2] \leq C \left[ \int_{B_{2r}} (\Delta u)^2 + \frac{1}{r^2} \int_{B_{2r}} |\nabla u|^2 + \frac{1}{r^4} \int_{B_{2r}} u^2 \right]. \tag{4.9}$$

Lemma 4.2 and inequality (4.9) yield

$$\int_{B_r} [|D^2 u|^2 + u^2] \leq C \left[ \frac{1}{r^2} \int_{B_{4r}} |\nabla u|^2 + \frac{1}{r^4} \int_{B_{4r}} u^2 \right].$$

From (4.8) and the latter inequality we find

$$r^2 \int_{B_r} |\nabla u|^2 \leq C \varepsilon^{\frac{2}{N}} \left[ r^2 \int_{B_{4r}} |\nabla u|^2 + \int_{B_{4r}} u^2 \right].$$

Adding this inequality to (4.7) we get

$$r^2 \int_{B_r} |\nabla u|^2 + \int_{B_r} u^2 \leq C \varepsilon^{\frac{2}{N}} \left[ (4r)^2 \int_{B_{4r}} |\nabla u|^2 + \int_{B_{4r}} u^2 \right]. \tag{4.10}$$

Let

$$f(r) = r^2 \int_{B_r} |\nabla u|^2 + \int_{B_r} u^2.$$

From (4.10) we find

$$f(r) \leq C \varepsilon^{\frac{2}{N}} f(4r).$$

Given  $n$  take  $\varepsilon$  small so that  $C \varepsilon^{\frac{2}{N}} \leq 4^{-n}$ . Then

$$f(r) \leq 4^{-n} f(4r) \text{ for } r \leq r_0(n).$$

Iterating  $k$  times we get

$$f(r) \leq 4^{-kn} f(4^k r), \text{ for } 4^{k-1} r \leq r_0(n). \tag{4.11}$$

Given  $0 < r < r_0(n)$  choose  $k \in \mathbb{N}$  such that

$$4^{-k}r_0 \leq r \leq 4^{-k+1}r_0.$$

By (4.11) we find

$$f(r) \leq 4^{-kn} f(4^k r) \leq 4^{-kn} f(4r_0).$$

Since  $4^{-k} \leq r/r_0$  we obtain

$$f(r) \leq \left(\frac{r}{r_0}\right)^n f(4r_0).$$

In particular, there is a constant  $C$  such that  $f(r) \leq Cr^n$ . Hence

$$r^2 \int_{B_r} |\nabla u|^2 + \int_{B_r} u^2 \leq Cr^n.$$

For  $N = 1$  or  $N = 2$ , with easy changes in the proof, one finds (4.10) with  $\varepsilon^\sigma$ ,  $0 < \sigma < 1$ , in place of  $\varepsilon^{\frac{2}{N}}$ . Therefore the last estimate holds for all  $N$ . The lemma follows.  $\square$

**THEOREM 4.4.** *Let  $u$  be a solution of (4.1) with  $|g(x)| \leq M$ . If the set  $E = \{x \in \Omega : u(x) = 0\}$  has a positive measure then  $u$  is identically zero in  $\Omega$ .*

*Proof.* By Lemma 4.3 we have, for some  $x_0 \in \Omega$ ,

$$\int_{|x-x_0| \leq r} u^2 = O(r^n), \quad r \rightarrow 0. \tag{4.12}$$

If we put  $x - x_0 = y$  equation (4.1) becomes

$$\Delta^2 \tilde{u} = \tilde{g}(y) \tilde{u} \quad \text{in } \tilde{\Omega},$$

where  $\tilde{\Omega}$  contains 0. Since  $|\tilde{g}(y)| \leq M$ , by (4.12) we find

$$\int_{|y| \leq r} \tilde{u}^2 = O(r^n), \quad r \rightarrow 0.$$

The theorem follows now by Lemma 4.1.  $\square$

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