OPTIMIZATION OF THE FIRST EIGENVALUE IN PROBLEMS INVOLVING THE BI–LAPLACIAN

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Abstract. This paper concerns minimization and maximization of the first eigenvalue in problems involving the bi-Laplacian under Dirichlet boundary conditions. Physically, in case of $N = 2$, our equation models the vibration of a non homogeneous plate $\Omega$ which is clamped along the boundary. Given several materials (with different densities) of total extension $|\Omega|$, we investigate the location of these materials throughout $\Omega$ so to minimize or maximize the first eigenvalue in the vibration of the clamped plate.

1. Introduction

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$ and let $g_0 = g_0(x)$ be a measurable function satisfying $0 \leq g_0(x) \leq M$, where $M$ is a positive constant. Define $\mathcal{G}$ as the family of all measurable functions which are rearrangements of $g_0$. For $g \in \mathcal{G}$, consider the eigenvalue problem

$$\Delta^2 u = \Lambda gu, \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega,$$

where $\nu$ denotes the exterior normal, $\Lambda = \Lambda_g$ is the first eigenvalue and $u = u_g(x)$ is a corresponding eigenfunction. The first eigenvalue $\Lambda$ of problem (1.1) is obtained by minimizing the associate Rayleigh quotient

$$\Lambda = \inf \left\{ \frac{\int_{\Omega} (\Delta w)^2 \, dx}{\int_{\Omega} gw^2 \, dx} : w \in H^2_0(\Omega), \ w \neq 0 \right\}. \quad (1.2)$$

It is well known [25] that a minimum $w = u_g$ is attained in (1.2) and satisfies (1.1) in a weak sense. For the regularity of $u_g$ we refer to [3]. In particular, this function belongs to $H^4_{loc}(\Omega)$ and the equation (1.1) holds a.e. in $\Omega$. If we multiply by $u = u_g$ in (1.1) and we integrate over $\Omega$ we find

$$\Lambda_g = \frac{\int_{\Omega} (\Delta u_g)^2 \, dx}{\int_{\Omega} gu_g^2 \, dx}.$$
Furthermore, if we have
\[ \Lambda_g = \frac{\int_{\Omega} (\Delta v)^2 \, dx}{\int_{\Omega} g v^2 \, dx} \]
for some \( v \in H^2_0(\Omega) \) then also \( v \) satisfies problem (1.1).

We are interested in the following optimization problems
\[ \min_{g \in \mathcal{G}} \Lambda_g, \quad \max_{g \in \mathcal{G}} \Lambda_g. \quad (1.3) \]

We prove existence of a minimizer in \( \mathcal{G} \). Furthermore, if \( g \) is a minimizer we prove that \( g = \phi(u^2_g) \), where \( u_g \) is a corresponding eigenfunction and \( \phi \) is some increasing function. This representation gives some information on the minimizer, because we know that \( u^2_g \) is small close to the boundary.

The maximization problem is more difficult. We are able to prove existence of a maximizer in \( \mathcal{G} \) only for domains \( \Omega \) such that the operator \( \Delta^2 u \) is positive preserving under Dirichlet homogeneous boundary conditions; we refer to [17], [18], [19], [20] [26] for a discussion on this topic. In case \( \Omega \) has this property we prove existence of a maximizer. Furthermore, if \( g \) is a maximizer we prove that \( g = \psi(u^2_g) \), where \( u_g \) is a corresponding eigenfunction and \( \psi \) is some decreasing function.

We observe that the optimization problems in case of Navier boundary conditions are discussed in [1] (minimization) and [14] (maximization). One difficulty that arises in the investigation of the first eigenvalue in case of Dirichlet boundary conditions is that the corresponding eigenfunction is not, in general, one signed [21]. In this paper we prove that an eigenfunction cannot vanish in a set of positive measure. Probably this result is known, but since we could not find it we include a proof in the last section of the present paper. By using this fact we are able to prove existence and a representation formula for a minimizer.

Let us give a motivation for the study of these problems in case of \( N = 2 \). Physically, our equation models the vibration of a non homogeneous plate \( \Omega \) which is clamped along the boundary \( \partial \Omega \). Given several materials (with different densities) of total extension \( |\Omega| \), we investigate the location of these materials throughout \( \Omega \) so to minimize or maximize the first eigenvalue in the vibration of the plate.

The corresponding problems for second order equations have been discussed in [5], [6], [7], [12], [11] and [13]. Related maximization problems are treated in [2] and [15].

2. Preliminaries

Denote with \( |E| \) the Lebesgue measure of the (measurable) set \( E \). Given a function \( g_0(x) \) defined in \( \Omega \) and satisfying \( 0 \leq g_0(x) \leq M \), we say that \( g(x) \), defined in \( \Omega \), belongs to the class of rearrangements \( \mathcal{G} = \mathcal{G}(g_0) \) if
\[ |\{g(x) \geq \beta\}| = |\{g_0(x) \geq \beta\}| \quad \forall \beta \geq 0. \]

Here and in what follows we write \( \{g(x) \geq \beta\} \) instead of \( \{x \in \Omega : g(x) \geq \beta\} \). For a general theory on rearrangements we refer to [9].
We make use of the following results proved in [5] and [6]. For short, throughout the paper we shall write increasing instead of non-decreasing, and decreasing instead of non-increasing.

**Lemma 2.1.** Let $g : \Omega \rightarrow \mathbb{R}$ and $w : \Omega \rightarrow \mathbb{R}$ be measurable functions, and suppose that every level set of $w$ has measure zero. Then there exists an increasing function $\phi$ such that $\phi(w)$ is a rearrangement of $g$. Furthermore, there exists a decreasing function $\psi$ such that $\psi(w)$ is a rearrangement of $g$.

**Proof.** The first assertion follows from Lemma 2.9 of [6]. The second assertion follows applying the first one to $-w$. □

Denote with $\overline{\mathcal{G}}$ the weak closure of $\mathcal{G}$ in $L^p(\Omega)$. It is well known that $\overline{\mathcal{G}}$ is convex and weakly sequentially compact (see for example [6], Lemma 2.2).

**Lemma 2.2.** Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^p(\Omega)$, $p \geq 1$, and let $w \in L^q(\Omega)$, $q = p/(p-1)$. If there is an increasing function $\phi$ such that $\phi(w) \in \mathcal{G}$ then

$$\int_{\Omega} g \, dw \leq \int_{\Omega} \phi(w) \, dw \quad \forall \, g \in \overline{\mathcal{G}},$$

and the function $\phi(w)$ is the unique maximizer relative to $\overline{\mathcal{G}}$. Furthermore, if there is a decreasing function $\psi$ such that $\psi(w) \in \mathcal{G}$ then

$$\int_{\Omega} g \, dw \geq \int_{\Omega} \psi(w) \, dw \quad \forall \, g \in \overline{\mathcal{G}},$$

and the function $\psi(w)$ is the unique minimizer relative to $\overline{\mathcal{G}}$.

**Proof.** The first assertion follows from Lemma 2.4 of [6]. To prove the second assertion we put $\phi(t) = \psi(-t)$. Since $\phi$ is increasing, applying the previous result we have

$$\int_{\Omega} g(-w) \, dw \leq \int_{\Omega} \phi(-w)(-w) \, dw \quad \forall \, g \in \overline{\mathcal{G}},$$

and $\phi(-w) = \psi(w)$ is the unique function satisfying the inequality. Equivalently, we have

$$\int_{\Omega} g \, dw \geq \int_{\Omega} \psi(w) \, dw \quad \forall \, g \in \overline{\mathcal{G}}.$$

The lemma is proved. □

**Lemma 2.3.** Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^p(\Omega)$, $p \geq 1$, and let $w \in L^q(\Omega)$, $q = p/(p-1)$. There are $g_1, g_2 \in \mathcal{G}$ such that

$$\int_{\Omega} g_1 \, dw \leq \int_{\Omega} g \, dw \leq \int_{\Omega} g_2 \, dw \quad \forall \, g \in \overline{\mathcal{G}}.$$
Lemma 2.4. Let $\Psi : L^p(\Omega) \to \mathbb{R}$ be a convex functional, let $\mathcal{G}$ denote the set of rearrangements of $g_0$ on $\Omega$, and let $q = p/(p-1)$.

(i) Suppose that $\Psi$ is sequentially continuous in the $L^q(\Omega)$ topology on $L^p(\Omega)$. Then $\Psi$ attains a maximum value relative to $\mathcal{G}$.

(ii) Suppose $\Psi$ is strictly convex, that $g^*$ is a maximizer for $\Psi$ relative to $\mathcal{G}$ and that $u$ is a member of the sub differential of $\Psi$ at $g^*$. Then $g^* = \phi(u)$ a.e. in $\Omega$ for some increasing function $\phi$.

Proof. See Theorem 7 of [5]. □

We recall that the $L^q(\Omega)$ topology on $L^p(\Omega)$ is the weak topology if $1 \leq p < \infty$, or the weak* topology if $p = \infty$ [5].

3. Main results

In all this section, $\mathcal{G}$ is the class of rearrangements of a fixed function $g_0(x)$ such that $0 \leq g_0(x) \leq M$, and $\overline{\mathcal{G}}$ is the closure of $\mathcal{G}$ with respect to the weak* topology of $L^\infty(\Omega)$.

Lemma 3.1. Let $g \in \overline{\mathcal{G}}$. If $\Lambda_g$ is the first eigenvalue of problem (1.1) then the functional

$g \mapsto \Lambda_g$

is continuous with respect to the weak* topology in $L^\infty(\Omega)$.

Proof. Let $g_i \to g$ in the weak* topology of $L^\infty(\Omega)$. Let $\Lambda_{g_i}$, $\Lambda_g$ be the corresponding eigenvalues, and let $u_{g_i}$, $u_g$ be corresponding eigenfunctions normalized so that

$$\int_{\Omega} (\Delta u_{g_i})^2 dx = \int_{\Omega} (\Delta u_{g})^2 dx = 1.$$ 

Since $||\Delta w||_{L^2(\Omega)}$ is equivalent to the norm $||w||_{H^2(\Omega)}$ in $H^2_0(\Omega)$, a sub-sequence of $u_{g_i}$ (denoted again $u_{g_i}$) converges weakly in $H^2_0(\Omega)$ and strongly in $L^2(\Omega)$. We have

$$\frac{1}{\Lambda_{g_i}} = \int_{\Omega} g_i u_{g_i}^2 dx = \int_{\Omega} (g_i - g)u_{g_i}^2 dx + \int_{\Omega} g u_{g_i}^2 dx \\
\leq \int_{\Omega} (g_i - g)u_{g_i}^2 dx + \int_{\Omega} g u_{g_i}^2 dx = \int_{\Omega} (g_i - g)u_{g_i}^2 dx + \frac{1}{\Lambda_g}.$$ 

Since $u_{g_i}$ converges in the $L^2(\Omega)$ norm we have

$$\limsup_{i \to \infty} \frac{1}{\Lambda_{g_i}} \leq \frac{1}{\Lambda_g}. \quad (3.1)$$

On the other side, we have

$$\frac{1}{\Lambda_{g_i}} = \int_{\Omega} g_i u_{g_i}^2 dx \geq \int_{\Omega} g_i u_g^2 dx = \int_{\Omega} (g_i - g)u_g^2 dx + \int_{\Omega} g u_g^2 dx.$$
Hence,
\[
\liminf_{i \to \infty} \frac{1}{\Lambda_{g_i}} \geq \int_\Omega g \, u_g^2 \, dx = \frac{1}{\Lambda_g}.
\]
The lemma follows by (3.1) and the latter inequality. \(\square\)

### 3.1. The minimum

**Theorem 3.2.** Let \(0 \leq g_0(x) \leq M\), and let \(\mathcal{G}\) be the class of all rearrangements of \(g_0\). If \(\Lambda_g\) is the first eigenvalue of problem (1.1) then there exists \(\overline{g} \in \mathcal{G}\) such that
\[
\Lambda_{\overline{g}} = \min_{g \in \mathcal{G}} \Lambda_g.
\]
Moreover, if \(u_g\) is an eigenfunction of (1.1) corresponding to \(g = \overline{g}\) then \(\overline{g} = \phi(u_g^2)\) for some increasing functions \(\phi\).

**Proof.** By Lemma 3.1 the functional \(g \to \Lambda_g\) is continuous with respect to the weak* topology in \(L^\infty(\Omega)\). We claim that \(g \mapsto \frac{1}{\Lambda_g}\) is strictly convex on \(\overline{\mathcal{G}}\). Indeed, if \(g_1, g_2 \in \mathcal{G}\), if \(g_t = tg_1 + (1-t)g_2\) with \(0 < t < 1\), if \(\Lambda_{g_1}, \Lambda_{g_2}, \Lambda_{g_t}\) are the corresponding eigenvalues, and if \(u_{g_1}, u_{g_2}, u_{g_t}\) are corresponding eigenfunctions then we have
\[
\frac{1}{\Lambda_{g_t}} = \frac{\int_\Omega g_t \, u_{g_t}^2 \, dx}{\int_\Omega (\Delta u_{g_t})^2 \, dx} = t \left( \frac{\int_\Omega g_1 \, u_{g_1}^2 \, dx}{\int_\Omega (\Delta u_{g_1})^2 \, dx} + (1-t) \frac{\int_\Omega g_2 \, u_{g_2}^2 \, dx}{\int_\Omega (\Delta u_{g_2})^2 \, dx} \right) \leq t \frac{\int_\Omega g_1 \, u_{g_1}^2 \, dx}{\int_\Omega (\Delta u_{g_1})^2 \, dx} + (1-t) \frac{\int_\Omega g_2 \, u_{g_2}^2 \, dx}{\int_\Omega (\Delta u_{g_2})^2 \, dx} = t \frac{1}{\Lambda_{g_1}} + (1-t) \frac{1}{\Lambda_{g_2}}.
\]
If equality holds in above then we must have
\[
\frac{\int_\Omega g_1 \, u_{g_1}^2 \, dx}{\int_\Omega (\Delta u_{g_1})^2 \, dx} = \frac{1}{\Lambda_{g_1}},
\]
and
\[
\frac{\int_\Omega g_2 \, u_{g_2}^2 \, dx}{\int_\Omega (\Delta u_{g_2})^2 \, dx} = \frac{1}{\Lambda_{g_2}}.
\]
Then,
\[
\Lambda_{g_t}^2 u_{g_t} = \Lambda_{g_1} g_1 u_{g_1} = \Lambda_{g_2} g_2 u_{g_2}, \ a.e. \ in \ \Omega.
\]
By the unique continuation theorem (see next section) we have \(u_{g_t} \neq 0\) a.e. in \(\Omega\). Therefore,
\[
\Lambda_{g_1} g_1 = \Lambda_{g_2} g_2, \ a.e. \ in \ \Omega. \quad (3.2)
\]
Since
\[
\int_\Omega g_1 \, dx = \int_\Omega g_2 \, dx,
\]
by (3.2) we find \(\Lambda_{g_1} = \Lambda_{g_2}\). Finally, (3.2) implies that \(g_1 = g_2\) a.e. in \(\Omega\). The strict convexity of \(\frac{1}{\Lambda_g}\) is proved.
The existence of a maximizer of $\frac{1}{\Lambda_g}$ (which is a minimizer of $\Lambda_g$) follows by Lemma 2.4. To prove the second statement of the theorem, we must find a sub-gradient of $\frac{1}{\Lambda_g}$ corresponding to a maximizer $\overline{g}$. Let $\overline{g}$ be a maximizer, let $\Lambda_{\overline{g}}$ be the corresponding eigenvalue and let $u_{\overline{g}}$ be a corresponding eigenfunction normalized so that $\|\Delta u_{\overline{g}}\|_{L^2(\Omega)} = 1$. Let $h \in \overline{g}$, let $\Lambda_h$ be the corresponding eigenvalue and let $u_h$ be a corresponding eigenfunction normalized so that $\|\Delta u_h\|_{L^2(\Omega)} = 1$. Then

$$\frac{1}{\Lambda_h} = \int_\Omega h u_h^2 dx \geq \int_\Omega h u_{\overline{g}}^2 dx = \int_\Omega \overline{g} u_{\overline{g}}^2 dx + \int_\Omega (h - \overline{g}) u_{\overline{g}}^2 dx = \frac{1}{\Lambda_{\overline{g}}} + \int_\Omega (h - \overline{g}) u_{\overline{g}}^2 dx.$$  

Therefore, $u_{\overline{g}}^2$ is a member of the sub-gradient, and by Lemma 2.4 there is an increasing function $\phi$ such that $\overline{g} = \phi(u_{\overline{g}}^2)$. The theorem is proved. \hfill \Box

3.2. The maximum

Now we investigate the problem

$$\sup_{g \in \overline{g}} \inf_{w \in H_0^1(\Omega), w \neq 0} \frac{\int_\Omega (\Delta w)^2 dx}{\int_\Omega g w^2 dx}. \quad (3.5)$$

We note that we cannot interchange (in general) the superior with the inferior. Following [11] (see also [14]), we give a different formulation of the problem by using the Auchmuty’s principle [4].

**Lemma 3.3.** Let $0 \leq g(x) \leq M$, $g(x) \not\equiv 0$, and let $\Lambda_g$ be the first eigenvalue of problem (1.1). If

$$A(g, w) = \frac{1}{2} \int_\Omega (\Delta w)^2 dx - \left( \int_\Omega g w^2 dx \right)^{\frac{1}{2}} \quad (3.6)$$

then

$$\min_{w \in H_0^1(\Omega)} A(g, w) = -\frac{1}{2} \max_{w \in H_0^1(\Omega), w \neq 0} \frac{\int_\Omega g w^2 dx}{\int_\Omega (\Delta w)^2 dx} = -\frac{1}{2} \frac{1}{\Lambda_g}. \nonumber$$

The minimum of $A(g, w)$ is attained at an eigenfunction $w = u_g$ normalized as

$$\frac{1}{\Lambda_g} = \left( \int_\Omega g u_g^2 dx \right)^{\frac{1}{2}} = \int_\Omega (\Delta u_g)^2 dx. \quad (3.7)$$

**Proof.** See [4] or Lemma 3.3 of [14]. \hfill \Box

**Lemma 3.4.** If $A(g, w)$ is defined as in (3.6) with $w \in H^2(\Omega)$, the function $g \mapsto A(g, w)$ is quasi-concave in $\overline{g}$.

**Proof.** It suffices to prove that the function $g \mapsto \left( \int_\Omega g w^2 dx \right)^{\frac{1}{2}}$ is quasi-convex. Let

$$\left( \int_\Omega g_1 w^2 dx \right)^{\frac{1}{2}} \leq c, \quad \left( \int_\Omega g_2 w^2 dx \right)^{\frac{1}{2}} \leq c.$$
Then
\[
\left( \int_\Omega (tg_1 + (1-t)g_2)w^2dx \right)^{\frac{1}{2}} = \left( t \int_\Omega g_1w^2dx + (1-t) \int_\Omega g_2w^2dx \right)^{\frac{1}{2}} \\
\leq \left( tc^2 + (1-t)c^2 \right)^{\frac{1}{2}} = c.
\]

The lemma is proved.  □

From now on we restrict the class of domains \( \Omega \). We suppose \( \Omega \) is such that the operator \( \Delta^2u \) under homogeneous Dirichlet boundary conditions is positivity preserving. This means that any nontrivial solution of \( \Delta^2u \geq 0 \) in \( \Omega \) with \( u = \frac{\partial u}{\partial \nu} = 0 \) on \( \partial \Omega \) satisfies \( u(x) > 0 \) in \( \Omega \). It is well known that the ball has this property; for other domains which enjoy such a property we refer to [17], [18], [19], [20]. By Krein-Rutman theorem [23], for this kind of domains \( \Omega \), the first eigenvalue of problem (1.1) is simple and a corresponding eigenfunction is positive. By the unique continuation principle we know that such an eigenfunction is strictly positive.

Let \( \Omega \) be positivity preserving for \( \Delta^2u \) under homogeneous Dirichlet boundary conditions, and let \( g \in \mathcal{G} \) fixed. If \( u_g \) is a positive solution of (1.1) normalized as in (3.7) we define
\[
\Pi_g = \left\{ w \in H^2_0(\Omega) : \Lambda_g^2 \int_\Omega g u_g w dx > 1 \right\}.
\]

**Lemma 3.5.** The function \( w \mapsto A(g, w) \) defined in (3.6) is strictly convex on \( \Pi_g \).

**Proof.** Let \( w \in \Pi_g \). For \( z \in H^2_0(\Omega) \) define \( \Phi(t) = A(g, w + tz) \). We find
\[
\Phi'(t) = \int_\Omega (\Delta w + t\Delta z)\Delta z \, dx - \left( \int_\Omega g(w + tz)^2 \, dx \right)^{\frac{1}{2}} \int_\Omega g(w + tz)z \, dx,
\]
and
\[
\Phi''(0) = \int_\Omega (\Delta z)^2 \, dx - \left( \int_\Omega g w^2 \, dx \right)^{\frac{1}{2}} \int_\Omega g z^2 \, dx + \left( \int_\Omega g w^2 \, dx \right)^{-\frac{1}{2}} \left( \int_\Omega g w z \, dx \right)^2.
\]

Since
\[
\int_\Omega (\Delta z)^2 \, dx \geq \Lambda_g \int_\Omega g z^2 \, dx
\]
we have
\[
\Phi''(0) \geq \Lambda_g \int_\Omega g z^2 \, dx - \left( \int_\Omega g w^2 \, dx \right)^{\frac{1}{2}} \int_\Omega g z^2 \, dx = [\Lambda_g - \left( \int_\Omega g w^2 \, dx \right)^{-\frac{1}{2}}] \int_\Omega g z^2 \, dx.
\]

On the other side, using Schwarz inequality and recalling the normalization of \( u_g \) given by (3.7) we have
\[
1 < \Lambda_g^2 \int_\Omega g u_g w \, dx \leq \Lambda_g^2 \left( \int_\Omega g u_g^2 \, dx \right)^{\frac{1}{2}} \left( \int_\Omega g w^2 \, dx \right)^{\frac{1}{2}} = \Lambda_g \left( \int_\Omega g w^2 \right)^{\frac{1}{2}}.
\]

It follows that \( \Phi''(0) > 0 \). The lemma is proved. □
As the set $\Pi_g$ over which $w \mapsto A(g,w)$ is convex depends on $g$ we introduce, following [11] pag.176, the indicator function

$$
\pi(g,u) = \begin{cases} 
0 & \text{if } u \in \Pi_g \\
\infty & \text{otherwise}.
\end{cases}
$$

**Lemma 3.6.** Let $u \in H^2_0(\Omega)$. The function $g \mapsto A(g,u) + \pi(g,u)$ is weak* upper semicontinuous over $\mathcal{G}$.

**Proof.** Let $g_i \rightharpoonup g$ in the weak* topology of $L^\infty(\Omega)$. Since $\int_\Omega g_i u^2 dx \to \int_\Omega g u^2 dx$, it follows that $g \mapsto A(g,u)$ is weak* upper semicontinuous over $\mathcal{G}$. On the other side, let us show that

$$
limitsup_{i \to \infty} \pi(g_i,u) \leq \pi(g,u) \quad \forall u \in H^2_0(\Omega).
$$

This is clearly true if $u \notin \Pi_g$. Let $u \in \Pi_g$. By Lemma 3.1 we know that $\Lambda_{g_i} \to \Lambda_g$. Since $\Omega$ is positivity preserving, there is a unique positive normalized eigenfunction $u_{g_i}$ and a unique positive normalized eigenfunction $u_g$. Then, by the proof of Lemma 3.1 we find that $u_{g_i} \rightharpoonup u_g$ in $L^s(\Omega)$ for some $s > 2$. It follows that

$$
\Lambda_{g_i}^2 \int_\Omega g_i u_{g_i} u dx \to \Lambda_g^2 \int_\Omega g u_g u dx.
$$

$u \in \Pi_g$ now implies that for $i$ greater than some $i_0$ depending only on $g$, $u$ belongs to $\Pi_{g_i}$, that is, $\pi(g_i,u) = 0 = \pi(g,u)$. The lemma follows. \qed

**Lemma 3.7.** Let $0 \leq g_0(x) \leq M$, $g_0(x) \neq 0$, and let $A(g,w)$ be defined as in (3.6). Then the following equality holds:

$$
sup_{g \in \mathcal{G}} \inf_{w \in H^2_0(\Omega)} A(g,w) = \inf_{w \in H^2_0(\Omega)} \sup_{g \in \mathcal{G}} A(g,w). \quad (3.9)
$$

**Proof.** The proof uses Lemmata 3.4, 3.5 and 3.6, and is the same as that of Proposition 7.7 of [11]. \qed

As observed in [11], Lemma 3.7 implies that $A(g,w)$ has a saddle point. Since there is not an explicit proof of this fact in the cited paper, we enclose here a simple proof (see [14] for a similar proof in the corresponding case of Navier boundary conditions).

**Proposition 3.8.** There exists $g \in \mathcal{G}$ such that the pair $(g,u_g)$ is a saddle point for $A(g,w)$ in $(\mathcal{G},H^2_0(\Omega))$.

**Proof.** For $w \in H^2_0(\Omega)$ we define

$$
B(w) = \sup_{g \in \mathcal{G}} A(g,w).
$$
We claim that there exists \( u \in H^2_0(\Omega) \) such that

\[
B(u) = \inf_{w \in H^2_0(\Omega)} B(w). \tag{3.10}
\]

Let \( u_k \) be a minimizing sequence in \( H^2_0(\Omega) \). Since we can assume that \( B(u_k) \) is decreasing, we have \( B(u_k) \leq C_1 \) for some positive constant. This estimate implies that

\[
\frac{1}{2} \int_{\Omega} (\Delta u_k)^2 \, dx - \inf_{g \in G} \left( \int_{\Omega} g u_k^2 \, dx \right)^{\frac{1}{2}} \leq C_1.
\]

Since \( \|g\|_\infty \leq M \) for all \( g \in G \) and since \( \|\Delta w\|_{L^2(\Omega)} \) is equivalent to the norm \( \|w\|_{H^2(\Omega)} \) in \( H^2_0(\Omega) \) we can write

\[
\frac{1}{2} \|\Delta u_k\|_{L^2(\Omega)}^2 \leq C_1 + C_2 \|\Delta u_k\|_{L^2(\Omega)}
\]

for some positive constant \( C_2 \). A straightforward calculation shows that

\[
\|\Delta u_k\|_{L^2(\Omega)} \leq C_2 + \sqrt{C_2^2 + 2C_1}.
\]

Then, up to a subsequence, \( u_k \) is weakly convergent in \( H^2(\Omega) \) and strongly in \( L^2(\Omega) \) to a function \( u \in H^2_0(\Omega) \).

By Lemma 2.3 we find \( g_1 \in G \) such that

\[
\inf_{g \in G} \left( \int_{\Omega} g u_k^2 \, dx \right)^{\frac{1}{2}} = \left( \int_{\Omega} g_1 u_k^2 \, dx \right)^{\frac{1}{2}}. \tag{3.11}
\]

On the other side we have

\[
\lim_{k \to \infty} \int_{\Omega} g_1 u_k^2 \, dx = \int_{\Omega} g_1 u^2 \, dx, \tag{3.12}
\]

and

\[
\inf_{g \in G} \left( \int_{\Omega} g u_k^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\Omega} g_1 u_k^2 \, dx \right)^{\frac{1}{2}}. \]

Taking the limsup in the last inequality and using (3.11) and (3.12) we find

\[
\limsup_{k \to \infty} \inf_{g \in G} \left( \int_{\Omega} g u_k^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\Omega} g_1 u^2 \, dx \right)^{\frac{1}{2}} = \inf_{g \in G} \left( \int_{\Omega} g u^2 \, dx \right)^{\frac{1}{2}}. \tag{3.13}
\]

Note that the weak convergence of \( u_k \to u \) in the \( H^2(\Omega) \) norm implies

\[
\liminf_{k \to \infty} \int_{\Omega} (\Delta u_k)^2 \, dx \geq \int_{\Omega} (\Delta u)^2 \, dx.
\]

Using the latter inequality and (3.13) we find

\[
\inf_{w \in H^2_0(\Omega)} B(w) = \lim_{k \to \infty} B(u_k) \geq B(u) \geq \inf_{w \in H^2_0(\Omega)} B(w).
\]
The claim (3.10) follows.

From (3.10) we infer the existence of \( u \in H^2_0(\Omega) \) such that

\[
\sup_{g \in \mathcal{G}} A(g, u) = \inf_{w \in H^2_0(\Omega)} \sup_{g \in \mathcal{G}} A(g, w). \tag{3.14}
\]

Since \( \mathcal{G} \) is compact with respect to the weak* topology of \( L^\infty(\Omega) \), by using Lemma 3.1 one finds the existence of \( g^* \in \mathcal{G} \) such that

\[
\sup_{g \in \mathcal{G}} A(g) = A(g^*). \tag{3.15}
\]

By using (3.14), Lemma 3.7 and (3.15), we can write

\[
\sup_{g \in \mathcal{G}} A(g, u) = \inf_{w \in H^2_0(\Omega)} \sup_{g \in \mathcal{G}} A(g, w) = \sup_{g \in \mathcal{G}} \inf_{w \in H^2_0(\Omega)} A(g, w) = \inf_{w \in H^2_0(\Omega)} A(g, w) \leq A(g, u). \tag{3.16}
\]

Hence, for any \( g \in \mathcal{G} \) we have

\[
A(g, u) \leq A(g^*, u). \tag{3.17}
\]

Similarly we find

\[
A(g^*, u) \leq \sup_{g \in \mathcal{G}} A(g, u) = \inf_{w \in H^2_0(\Omega)} \sup_{g \in \mathcal{G}} A(g, w) = \sup_{g \in \mathcal{G}} \inf_{w \in H^2_0(\Omega)} A(g, w) = \inf_{w \in H^2_0(\Omega)} A(g, w). \tag{3.18}
\]

Therefore, for any \( w \in H^2_0(\Omega) \) we have

\[
A(g, u) \leq A(g, w). \tag{3.19}
\]

By (3.17) and Lemma 3.3 we must have \( u = u^* \). Hence, the proposition follows by (3.16) and (3.17).

\[ \square \]

**Theorem 3.9.** Let \( \Omega \) be positivity preserving for \( \Delta^2 u \) under homogeneous Dirichlet boundary conditions. Let \( 0 \leq g_0(x) \leq M \), \( g_0(x) \neq 0 \), and let \( \mathcal{G} \) be the class of all rearrangements of \( g_0 \). If \( \Lambda_g \) is the first eigenvalue of problem (1.1) then there exists \( g^* \in \mathcal{G} \) such that

\[
\Lambda_g = \max_{g \in \mathcal{G}} \Lambda_g. \tag{3.20}
\]

Moreover, if \( u^*_g \) is an eigenfunction of (1.1) corresponding to \( g = g^* \) then \( g = \psi(u^2_g) \) for some decreasing functions \( \psi \).
Proof. By Proposition 3.8, there is a saddle point \((g, u_g)\) for \(A(g, w)\) in \((\mathcal{G}, H^1_0(\Omega))\). In particular, we have

\[ A(g, u_g) \leq A(g, u_g^G) \quad \forall g \in \mathcal{G}. \]

Recalling the definition (3.6) of \(A(g, w)\), the latter inequality implies

\[
\int_{\Omega} g u^2_g dx \geq \int_{\Omega} g u^2_g^G dx \quad \forall g \in \mathcal{G}.
\]

(3.19)

The functions \(g\) and \(u = u_g\) satisfy

\[
\Delta^2 u = \Lambda u \quad \text{a.e. in } \Omega.
\]

(3.20)

Recall that \(u = u_g\) is either strictly positive or strictly negative. Therefore, by (3.20) it follows that the function \(u\) cannot have flat zones in the set \(F = \{x \in \Omega : g(x) > 0\}\).

If \(|F| = |\Omega|\), by Lemma 2.1 there is a decreasing function \(\psi(t)\) such that \(\psi(u^2_g)\) is a rearrangement of \(g_0(x)\) on \(\Omega\). By (3.19) and Lemma 2.2 we must have \(g = \psi(u^2_g) \in \mathcal{G}\), and the theorem is proved. If \(|F| < |\Omega|\), since \(g \in \mathcal{G}\), by Lemma 2.14 of [6] we have \(|F| > \{|x \in \Omega : g_0(x) > 0\}|\).

Therefore there is \(g_1 \in \mathcal{G}\) such that its support is contained in \(F\). By Lemma 2.1, there is a decreasing function \(\psi_1(t)\) such that \(\psi_1(u^2_g)\) is a rearrangement of \(g_1(x)\) on \(F\).

Define

\[
\alpha = \inf_{x \in \Omega \setminus F} u^2_g(x).
\]

We claim that \(u^2_g(x) \leq \alpha\) in \(F\). Arguing by contradiction suppose the claim is false. Therefore there exist a number \(S_1 > \alpha\) and a subset \(A\) of \(F\) with \(|A| > 0\) such that \(u^2_g(x) > S_1\) a.e. on \(A\). Now let \(\alpha < S_2 < S_1\). We can find a set \(D\) of positive measure contained in \(\Omega \setminus F\) such that \(u^2_g(x) < S_2\) a.e. on \(D\). We can assume \(|A| = |\Omega|\).

Using a measure preserving \(T\) we define a particular rearrangement of \(g\), denoted by \(h\), as follows.

\[
h(x) = \begin{cases} g(Tx), & x \in A \\ g(T^{-1}x), & x \in D \\ g(x), & x \in \Omega \setminus (A \cup D). \end{cases}
\]

Thus

\[
\int_{\Omega} hu^2_g dx - \int_{\Omega} g u^2_g dx = \int_{A \cup D} hu^2_g dx - \int_{A \cup D} g u^2_g dx = \int_A hu^2_g dx + \int_A gu^2_g \circ T dx - \int_A gu^2_g dx - \int_A hu^2_g \circ T dx = \int_A (u^2_g \circ T - u^2_g) (g-h) dx < (S_2 - S_1) \int_A g dx < 0.
\]

Therefore \(\int_{\Omega} hu^2_g dx < \int_{\Omega} gu^2_g dx\), which contradicts (3.19), and the claim follows.

By using equation (3.20) again we find that \(u^2_g(x) < \alpha\) a.e. in \(F\). Now define

\[
\psi(t) = \begin{cases} \psi_1(t) & \text{if } 0 \leq t < \alpha \\ 0 & \text{if } t \geq \alpha. \end{cases}
\]
The function \( \psi(t) \) is decreasing and \( \psi(u_2^2) \) is a rearrangement of \( g_1(x) \) in \( \Omega \). Indeed, the functions \( g_1 \) and \( \psi(u_2^2) \) have the same rearrangement on \( F \), and both vanish on \( \Omega \setminus F \). By (3.19) and Lemma 2.2 we must have \( \underline{c} = \psi(u_2^2) \in \mathcal{G} \). The theorem follows. \( \square \)

4. Unique continuation

Let \( \Omega \) be a domain in \( \mathbb{R}^N \) and let \( g \in L^\infty(\Omega) \). Consider a weak solution \( u \in H^2(\Omega) \) of the equation

\[
\Delta^2 u = g(x) \text{ in } \Omega.
\]

(4.1)

It is well known [3] that \( u \in H^{4}_{\text{loc}}(\Omega) \).

**Lemma 4.1.** Let \( \Omega \) be a domain in \( \mathbb{R}^N \) containing the origin 0. Let \( u \in H^{4}_{\text{loc}}(\Omega) \) be a solution to (4.1) with \( |g(x)| \leq M \) and let for all \( n \in \mathbb{N} \)

\[
\int_{|x| \leq r} u^2 = O(r^n), \quad r \to 0.
\]

Then \( u \) is identically zero in \( \Omega \).

**Proof.** If follows by Theorem 1.1 of [8]. See also [10] and [24]. \( \square \)

**Lemma 4.2.** Let \( B(x_0, r) \) and \( B(x_0, 2r) \), \( r \in (0, 1) \), be two concentric balls contained in \( \Omega \), and let \( u \) be a solution of (4.1) with \( |g(x)| \leq M \). Then there exists a constant \( C \) independent of \( r \) such that

\[
\int_{B(x_0, r)} (\Delta u)^2 \leq C \left[ \frac{1}{r^2} \int_{B(x_0, 2r)} |\nabla u|^2 + \frac{1}{r^4} \int_{B(x_0, 2r)} u^2 \right].
\]

**Proof.** Let \( \theta \in C^\infty_0(B(x_0, 2r)) \) with \( 0 \leq \theta \leq 1 \), \( |\nabla \theta| \leq C/r \) and \( |\Delta \theta| \leq C/r^2 \) in \( B(x_0, 2r) \), and \( \theta = 1 \) in \( B(x_0, r) \). Here and in what follows we denote by \( C \) a generic positive constant, which may vary from line to line, and is independent of \( r \).

Multiplying the equation (4.1) by \( \theta^4 u \) and integrating by parts on \( B(x_0, 2r) \) we find

\[
\int_{B(x_0, 2r)} \Delta u \Delta (\theta^4 u) = \int_{B(x_0, 2r)} g(x) \theta^4 u^2.
\]

If \( |g(x)| \leq M \) we get

\[
\int_{B(x_0, 2r)} \theta^4 (\Delta u)^2 + 4 \int_{B(x_0, 2r)} \theta^3 \Delta u \Delta \theta u
\]

\[
+ 12 \int_{B(x_0, 2r)} \theta^2 |\nabla \theta|^2 \Delta u u + 8 \int_{B(x_0, 2r)} \theta^3 \nabla \theta \cdot \nabla u \Delta u
\]

\[
\leq M \int_{B(x_0, 2r)} u^2.
\]

(4.2)
Using the Schwarz inequality we find
\[-4 \int_{B(x_0,2r)} \theta^3 \Delta u \Delta \theta \, u \leq \frac{1}{6} \int_{B(x_0,2r)} \theta^4(\Delta u)^2 + 24 \int_{B(x_0,2r)} \theta^2(\Delta \theta)^2 u^2.\]

Recalling that $|\Delta \theta| \leq C/r^2$ and $0 \leq \theta \leq 1$ we have
\[-4 \int_{B(x_0,2r)} \theta^3 \Delta u \Delta \theta \, u \leq \frac{1}{6} \int_{B(x_0,2r)} \theta^4(\Delta u)^2 + \frac{C}{r^2} \int_{B(x_0,2r)} u^2. \tag{4.3}\]

Similarly, we find
\[-8 \int_{B(x_0,2r)} \theta^2|\nabla \theta|^2 \Delta u \, u \leq \frac{1}{6} \int_{B(x_0,2r)} \theta^4(\Delta u)^2 + 96 \int_{B(x_0,2r)} |\nabla \theta|^4 u^2. \tag{4.4}\]

Finally, we find
\[-8 \int_{B(x_0,2r)} \theta^3 \nabla \theta \cdot \nabla u \Delta u \, u \leq \frac{1}{6} \int_{B(x_0,2r)} \theta^4(\Delta u)^2 + 96 \int_{B(x_0,2r)} \theta^2|\nabla \theta|^2|\nabla u|^2 \leq \frac{1}{6} \int_{B(x_0,2r)} \theta^4(\Delta u)^2 + \frac{C}{r^2} \int_{B(x_0,2r)} |\nabla u|^2. \tag{4.5}\]

Inserting the estimates (4.3), (4.4) and (4.5) into (4.2), after some simplification and recalling that $\theta = 1$ in $B(x_0,r)$, we get the desired estimate. \qed

**Lemma 4.3.** Let $u$ be a solution of (4.1) with $|g(x)| \leq M$. If $E = \{x \in \Omega : u(x) = 0\}$ has a positive measure then there is $x_0 \in \Omega$ such that for every $n \in \mathbb{N}$ we have
\[
\int_{B_r} u^2 = O(r^n), \quad \int_{B_r} |\nabla u|^2 = O(r^n), \quad r \to 0,
\]
where $B_r = \{x \in \Omega : |x-x_0| < r\}$.

**Proof.** We know that almost every point of $E$ is a point of density. Let $x_0$ be such a point. This means that for a given $\varepsilon > 0$ there is $r_0 = r_0(\varepsilon)$ such that for $r < r_0$ we have
\[
\frac{|E^c \cap B_r|}{|B_r|} < \varepsilon, \quad \frac{|E \cap B_r|}{|B_r|} > 1 - \varepsilon,
\]
where $E^c$ denotes the complement of $E$. We may suppose $B_{4r} \subset \Omega$ and $r < 1$. For $N \geq 3$ we have
\[
\int_{B_r} u^2 = \int_{B_{r}\cap E^c} u^2 \leq \left( \int_{B_r} |u|^{2N} \right)^{\frac{N-2}{N}} |B_r \cap E^c| \frac{2}{N} \leq C \int_{B_r} |\nabla u|^2 + u^2 \leq C \varepsilon^2 \left[ r^2 \int_{B_r} |\nabla u|^2 + \int_{B_r} u^2 \right]. \tag{4.7}\]
Let $\tilde{E} = \{ x \in B_r : \nabla u(x) = 0 \}$. Since for almost every point of $E$ we have $\nabla u = 0$ (see [16], Lemma 7.7), the inclusion $E \subset \tilde{E}$ holds except possibly in a set of zero measure. Hence, $x_0$ is a point of density also for $\tilde{E}$. For $r < r_1(\varepsilon)$ (with $r_1(\varepsilon) \leq r_2(\varepsilon)$) we have

$$\int_{B_r} |\nabla u|^2 = \int_{B_r \cap \tilde{E}^c} |\nabla u|^2 \leq \left( \int_{B_r} |\nabla u|^\frac{2n}{n-2} \right)^\frac{n-2}{n} |B_r \cap \tilde{E}^c|^{\frac{2}{n}}$$

$$\leq C \varepsilon^{\frac{n}{n-2}} r^2 \left[ \int_{B_r} |D^2u|^2 + u^2 \right]. \tag{4.8}$$

If $v \in H^2(B_{2r}) \cap H_0^1(B_{2r})$ we have ([16], Theorem 8.12)

$$\int_{B_{2r}} |D^2v|^2 + v^2 \leq C \int_{B_{2r}} (\Delta v)^2 + v^2.$$  

Take $v = u\theta$ with $\theta \in C_0^\infty(B_{2r})$, $\theta = 1$ in $B_r$, $|\nabla \theta| \leq C/r$ and $|D^2\theta| \leq C/r^2$ in $B(x_0, 2r)$. We find

$$\int_{B_r} |D^2u|^2 + u^2 \leq C \left[ \int_{B_{2r}} (\Delta u)^2 + \frac{1}{r^2} \int_{B_{2r}} |\nabla u|^2 + \frac{1}{r^2} \int_{B_{2r}} u^2 \right]. \tag{4.9}$$

Lemma 4.2 and inequality (4.9) yield

$$\int_{B_r} |D^2u|^2 + u^2 \leq C \left[ \frac{1}{r^2} \int_{B_{4r}} |\nabla u|^2 + \frac{1}{r^2} \int_{B_{4r}} u^2 \right].$$

From (4.8) and the latter inequality we find

$$r^2 \int_{B_r} |\nabla u|^2 \leq C \varepsilon^{\frac{n}{n-2}} \left[ r^2 \int_{B_{4r}} |\nabla u|^2 + \int_{B_{4r}} u^2 \right].$$

Adding this inequality to (4.7) we get

$$r^2 \int_{B_r} |\nabla u|^2 + \int_{B_r} u^2 \leq C \varepsilon^{\frac{n}{n-2}} \left[ (4r)^2 \int_{B_{4r}} |\nabla u|^2 + \int_{B_{4r}} u^2 \right]. \tag{4.10}$$

Let

$$f(r) = r^2 \int_{B_r} |\nabla u|^2 + \int_{B_r} u^2.$$  

From (4.10) we find

$$f(r) \leq C \varepsilon^{\frac{n}{n-2}} f(4r).$$

Given $n$ take $\varepsilon$ small so that $C \varepsilon^{\frac{n}{n-2}} < 4^{-n}$. Then

$$f(r) \leq 4^{-n} f(4r) \quad \text{for} \quad r \leq r_0(n).$$

Iterating $k$ times we get

$$f(r) \leq 4^{-kn} f(4^kr), \quad \text{for} \quad 4^{k-1}r \leq r_0(n). \tag{4.11}$$
Given $0 < r < r_0(n)$ choose $k \in \mathbb{N}$ such that
\[ 4^{-k}r_0 \leq r \leq 4^{-(k+1)}r_0. \]

By (4.11) we find
\[ f(r) \leq 4^{-kn} f(4^k r) \leq 4^{-kn} f(4r_0). \]

Since $4^{-k} \leq r/r_0$ we obtain
\[ f(r) \leq \left( \frac{r}{r_0} \right)^n f(4r_0). \]

In particular, there is a constant $C$ such that $f(r) \leq C r^n$. Hence
\[ r^2 \int_{B_r} |\nabla u|^2 + \int_{B_r} u^2 \leq C r^n. \]

For $N = 1$ or $N = 2$, with easy changes in the proof, one finds (4.10) with $\varepsilon \sigma$, $0 < \sigma < 1$, in place of $\varepsilon^2$. Therefore the last estimate holds for all $N$. The lemma follows. \[ \square \]

**THEOREM 4.4.** Let $u$ be a solution of (4.1) with $|g(x)| \leq M$. If the set $E = \{ x \in \Omega : u(x) = 0 \}$ has a positive measure then $u$ is identically zero in $\Omega$.

**Proof.** By Lemma 4.3 we have, for some $x_0 \in \Omega$,
\[ \int_{|x-x_0| \leq r} u^2 = O(r^n), \quad r \to 0. \] (4.12)

If we put $x - x_0 = y$ equation (4.1) becomes
\[ \Delta^2 \tilde{u} = \tilde{g}(y) \tilde{u} \text{ in } \tilde{\Omega}, \]
where $\tilde{\Omega}$ contains 0. Since $|\tilde{g}(y)| \leq M$, by (4.12) we find
\[ \int_{|y| \leq r} \tilde{u}^2 = O(r^n), \quad r \to 0. \]

The theorem follows now by Lemma 4.1. \[ \square \]
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