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OPTIMIZATION OF THE FIRST EIGENVALUE IN PROBLEMS INVOLVING THE BI–LAPLACIAN

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Abstract. This paper concerns minimization and maximization of the first eigenvalue in problems involving the bi-Laplacian under Dirichlet boundary conditions. Physically, in case of $N = 2$, our equation models the vibration of a non homogeneous plate Ω which is clamped along the boundary. Given several materials (with different densities) of total extension $|\Omega|$, we investigate the location of these materials throughout Ω so to minimize or maximize the first eigenvalue in the vibration of the clamped plate.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and let $g_0 = g_0(x)$ be a measurable function satisfying $0 \leqslant g_0(x) \leqslant M$ in Ω, where M is a positive constant. Define $\mathscr{G}$ as the family of all measurable functions which are rearrangements of g_0. For $g \in \mathscr{G}$, consider the eigenvalue problem

$$\Delta^2 u = \Lambda g u, \quad \text{in} \quad \Omega, \quad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

where ν denotes the exterior normal, $\Lambda = \Lambda_g$ is the first eigenvalue and $u = u_g(x)$ is a corresponding eigenfunction. The first eigenvalue Λ of problem (1.1) is obtained by minimizing the associate Rayleigh quotient

$$\Lambda = \inf \left\{ \frac{\int_\Omega (\Delta w)^2 dx}{\int_\Omega g w^2 dx} : \ w \in H_0^2(\Omega), \ w \not\equiv 0 \right\}. \tag{1.2}$$

It is well known [25] that a minimum $w = u_g$ is attained in (1.2) and satisfies (1.1) in a weak sense. For the regularity of u_g we refer to [3]. In particular, this function belongs to $H^4_{loc}(\Omega)$ and the equation (1.1) holds a.e. in Ω. If we multiply by $u = u_g$ in (1.1) and we integrate over Ω we find

$$\Lambda_g = \frac{\int_\Omega (\Delta u_g)^2 dx}{\int_\Omega g u_g^2 dx}.$$

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Furthermore, if we have

$$\Lambda_g = \frac{\int_\Omega (\Delta v)^2 dx}{\int_\Omega g v^2 dx}$$

for some $v \in H_0^2(\Omega)$ then also v satisfies problem (1.1).

We are interested in the following optimization problems

$$\min_{g\in\mathcal{G}} \Lambda_g, \quad \max_{g\in\mathcal{G}} \Lambda_g. \tag{1.3}$$

We prove existence of a minimizer in $\mathcal{G}$. Furthermore, if $\overline{g}$ is a minimizer we prove that $\overline{g} = \phi(u_{\overline{g}}^2)$, where $u_{\overline{g}}$ is a corresponding eigenfunction and ϕ is some increasing function. This representation gives some information on the minimizer, because we know that $u_{\overline{g}}^2$ is small close to the boundary.

The maximization problem is more difficult. We are able to prove existence of a maximizer in $\mathcal{G}$ only for domains Ω such that the operator $\Delta^2 u$ is positive preserving under Dirichlet homogeneous boundary conditions; we refer to [17], [18], [19], [20] [26] for a discussion on this topic. In case Ω has this property we prove existence of a maximizer. Furthermore, if $\underline{g}$ is a maximizer we prove that $\underline{g} = \psi(u_{\underline{g}}^2)$, where $u_{\underline{g}}$ is a corresponding eigenfunction and ψ is some decreasing function.

We observe that the optimization problems in case of Navier boundary conditions are discussed in [1] (minimization) and [14] (maximization). One difficulty that arises in the investigation of the first eigenvalue in case of Dirichlet boundary conditions is that the corresponding eigenfunction is not, in general, one signed [21]. In this paper we prove that an eigenfunction cannot vanish in a set of positive measure. Probably this result is known, but since we could not find it we include a proof in the last section of the present paper. By using this fact we are able to prove existence and a representation formula for a minimizer.

Let us give a motivation for the study of these problems in case of $N = 2$. Physically, our equation models the vibration of a non homogeneous plate Ω which is clamped along the boundary $\partial\Omega$. Given several materials (with different densities) of total extension $|\Omega|$, we investigate the location of these materials throughout Ω so to minimize or maximize the first eigenvalue in the vibration of the plate.

The corresponding problems for second order equations have been discussed in [5], [6], [7], [12], [11] and [13]. Related maximization problems are treated in [2] and [15].

2. Preliminaries

Denote with $|E|$ the Lebesgue measure of the (measurable) set E. Given a function $g_0(x)$ defined in Ω and satisfying $0 \leqslant g_0(x) \leqslant M$, we say that $g(x)$, defined in Ω, belongs to the class of rearrangements $\mathcal{G} = \mathcal{G}(g_0)$ if

$$|\{g(x) \geqslant \beta\}| = |\{g_0(x) \geqslant \beta\}| \quad \forall \beta \geqslant 0.$$

Here and in what follows we write $\{g(x) \geqslant \beta\}$ instead of $\{x \in \Omega : g(x) \geqslant \beta\}$. For a general theory on rearrangements we refer to [9].

We make use of the following results proved in [5] and [6]. For short, throughout the paper we shall write increasing instead of non-decreasing, and decreasing instead of non-increasing.

LEMMA 2.1. *Let $g:\Omega\to\mathbb{R}$ and $w:\Omega\to\mathbb{R}$ be measurable functions, and suppose that every level set of w has measure zero. Then there exists an increasing function ϕ such that $\phi(w)$ is a rearrangement of g. Furthermore, there exists a decreasing function ψ such that $\psi(w)$ is a rearrangement of g.*

Proof. The first assertion follows from Lemma 2.9 of [6]. The second assertion follows applying the first one to $-w$. □

Denote with $\overline{\mathscr{G}}$ the weak closure of $\mathscr{G}$ in $L^p(\Omega)$. It is well known that $\overline{\mathscr{G}}$ is convex and weakly sequentially compact (see for example [6], Lemma 2.2).

LEMMA 2.2. *Let $\mathscr{G}$ be the set of rearrangements of a fixed function $g_0\in L^p(\Omega)$, $p\geqslant 1$, and let $w\in L^q(\Omega)$, $q=p/(p-1)$. If there is an increasing function ϕ such that $\phi(w)\in\mathscr{G}$ then*

$$\int_\Omega g\,w\,dx\leqslant\int_\Omega\phi(w)\,w\,dx\quad\forall g\in\overline{\mathscr{G}},$$

and the function $\phi(w)$ is the unique maximizer relative to $\overline{\mathscr{G}}$. Furthermore, if there is a decreasing function ψ such that $\psi(w)\in\mathscr{G}$ then

$$\int_\Omega g\,w\,dx\geqslant\int_\Omega\psi(w)\,w\,dx\quad\forall g\in\overline{\mathscr{G}},$$

and the function $\psi(w)$ is the unique minimizer relative to $\overline{\mathscr{G}}$.

Proof. The first assertion follows from Lemma 2.4 of [6]. To prove the second assertion we put $\phi(t)=\psi(-t)$. Since ϕ is increasing, applying the previous result we have

$$\int_\Omega g\,(-w)\,dx\leqslant\int_\Omega\phi(-w)\,(-w)\,dx\quad\forall g\in\overline{\mathscr{G}},$$

and $\phi(-w)=\psi(w)$ is the unique function satisfying the inequality. Equivalently, we have

$$\int_\Omega g\,w\,dx\geqslant\int_\Omega\psi(w)\,w\,dx\quad\forall g\in\overline{\mathscr{G}}.$$

The lemma is proved. □

LEMMA 2.3. *Let $\mathscr{G}$ be the set of rearrangements of a fixed function $g_0\in L^p(\Omega)$, $p\geqslant 1$, and let $w\in L^q(\Omega)$, $q=p/(p-1)$. There are $g_1,\ g_2\in\mathscr{G}$ such that*

$$\int_\Omega g_1\,w\,dx\leqslant\int_\Omega g\,w\,dx\leqslant\int_\Omega g_2\,w\,dx\quad\forall g\in\overline{\mathscr{G}}.$$

Proof. It follows from Lemma 2.4 of [6]. □

LEMMA 2.4. *Let $\Psi : L^p(\Omega) \to \mathbb{R}$ be a convex functional, let $\mathscr{G}$ denote the set of rearrangements of g_0 on Ω, and let $q = p/(p-1)$.*

(i) Suppose that Ψ is sequentially continuous in the $L^q(\Omega)$ topology on $L^p(\Omega)$. Then Ψ attains a maximum value relative to $\mathscr{G}$.

(ii) Suppose Ψ is strictly convex, that g^* is a maximizer for Ψ relative to $\mathscr{G}$ and that u is a member of the sub differential of Ψ at g^*. Then $g^* = \phi(u)$ a.e. in Ω for some increasing function ϕ.

Proof. See Theorem 7 of [5]. □

We recall that the $L^q(\Omega)$ topology on $L^p(\Omega)$ is the weak topology if $1 \leqslant p < \infty$, or the weak* topology if $p = \infty$ [5].

3. Main results

In all this section, $\mathscr{G}$ is the class of rearrangements of a fixed function $g_0(x)$ such that $0 \leqslant g_0(x) \leqslant M$, and $\overline{\mathscr{G}}$ is the closure of $\mathscr{G}$ with respect to the weak* topology of $L^\infty(\Omega)$.

LEMMA 3.1. *Let $g \in \overline{\mathscr{G}}$. If Λ_g is the first eigenvalue of problem (1.1) then the functional*

$$g \mapsto \Lambda_g$$

is continuous with respect to the weak topology in $L^\infty(\Omega)$.*

Proof. Let $g_i \to g$ in the weak* topology of $L^\infty(\Omega)$. Let Λ_{g_i}, Λ_g be the corresponding eigenvalues, and let u_{g_i}, u_g be corresponding eigenfunctions normalized so that

$$\int_\Omega (\Delta u_{g_i})^2 dx = \int_\Omega (\Delta u_g)^2 dx = 1.$$

Since $\|\Delta w\|_{L^2(\Omega)}$ is equivalent to the norm $\|w\|_{H^2(\Omega)}$ in $H_0^2(\Omega)$, a sub-sequence of u_{g_i} (denoted again u_{g_i}) converges weakly in $\overline{u} \in H_0^2(\Omega)$ and strongly in $L^2(\Omega)$. We have

$$\begin{aligned} \frac{1}{\Lambda_{g_i}} &= \int_\Omega g_i u_{g_i}^2 dx = \int_\Omega (g_i - g) u_{g_i}^2 dx + \int_\Omega g u_{g_i}^2 dx \\ &\leqslant \int_\Omega (g_i - g) u_{g_i}^2 dx + \int_\Omega g u_g^2 dx = \int_\Omega (g_i - g) u_{g_i}^2 dx + \frac{1}{\Lambda_g}. \end{aligned}$$

Since u_{g_i} converges in the $L^2(\Omega)$ norm we have

$$\limsup_{i\to\infty} \frac{1}{\Lambda_{g_i}} \leqslant \frac{1}{\Lambda_g}. \tag{3.1}$$

On the other side, we have

$$\frac{1}{\Lambda_{g_i}} = \int_\Omega g_i u_{g_i}^2 dx \geqslant \int_\Omega g_i u_g^2 dx = \int_\Omega (g_i - g) u_g^2 dx + \int_\Omega g\, u_g^2 dx.$$

Hence,

$$\liminf_{i\to\infty} \frac{1}{\Lambda_{g_i}} \geqslant \int_\Omega g\, u_g^2 dx = \frac{1}{\Lambda_g}.$$

The lemma follows by (3.1) and the latter inequality. $\square$

3.1. The minimum

THEOREM 3.2. *Let* $0 \leqslant g_0(x) \leqslant M$*, and let* $\mathscr{G}$ *be the class of all rearrangements of* g_0*. If* Λ_g *is the first eigenvalue of problem* (1.1) *then there exists* $\overline{g} \in \mathscr{G}$ *such that*

$$\Lambda_{\overline{g}} = \min_{g\in\mathscr{G}} \Lambda_g.$$

Moreover, if $u_{\overline{g}}$ *is an eigenfunction of* (1.1) *corresponding to* $g = \overline{g}$ *then* $\overline{g} = \phi(u_{\overline{g}}^2)$ *for some increasing functions* ϕ.

Proof. By Lemma 3.1 the functional $g \to \Lambda_g$ is continuous with respect to the weak* topology in $L^\infty(\Omega)$. We claim that $g \mapsto \frac{1}{\Lambda_g}$ is strictly convex on $\overline{\mathscr{G}}$. Indeed, if $g_1,\ g_2 \in \overline{\mathscr{G}}$, if $g_t = tg_1 + (1-t)g_2$ with $0 < t < 1$, if $\Lambda_{g_1},\ \Lambda_{g_2},\ \Lambda_{g_t}$ are the corresponding eigenvalues, and if $u_{g_1},\ u_{g_2},\ u_{g_t}$ are corresponding eigenfunctions then we have

$$\begin{aligned}\frac{1}{\Lambda_{g_t}} &= \frac{\int_\Omega g_t\, u_{g_t}^2 dx}{\int_\Omega (\Delta u_{g_t})^2 dx} = t\frac{\int_\Omega g_1\, u_{g_t}^2 dx}{\int_\Omega (\Delta u_{g_t})^2 dx} + (1-t)\frac{\int_\Omega g_2\, u_{g_t}^2 dx}{\int_\Omega (\Delta u_{g_t})^2 dx}\\ &\leqslant t\frac{\int_\Omega g_1\, u_{g_1}^2 dx}{\int_\Omega (\Delta u_{g_1})^2 dx} + (1-t)\frac{\int_\Omega g_2\, u_{g_2}^2 dx}{\int_\Omega (\Delta u_{g_2})^2 dx} = t\frac{1}{\Lambda_{g_1}} + (1-t)\frac{1}{\Lambda_{g_2}}.\end{aligned}$$

If equality holds in above then we must have

$$\frac{\int_\Omega g_1\, u_{g_t}^2 dx}{\int_\Omega (\Delta u_{g_t})^2 dx} = \frac{1}{\Lambda_{g_1}}$$

and

$$\frac{\int_\Omega g_2\, u_{g_t}^2 dx}{\int_\Omega (\Delta u_{g_t})^2 dx} = \frac{1}{\Lambda_{g_2}}.$$

Then,

$$\Delta^2 u_{g_t} = \Lambda_{g_1} g_1\, u_{g_t} = \Lambda_{g_2} g_2\, u_{g_t} \ \text{ a.e. in } \Omega.$$

By the unique continuation theorem (see next section) we have $u_{g_t} \neq 0$ a.e. in Ω. Therefore,

$$\Lambda_{g_1} g_1 = \Lambda_{g_2} g_2 \ \text{ a.e. in } \Omega. \tag{3.2}$$

Since

$$\int_\Omega g_1 dx = \int_\Omega g_2 dx,$$

by (3.2) we find $\Lambda_{g_1} = \Lambda_{g_2}$. Finally, (3.2) implies that $g_1 = g_2$ a.e. in Ω. The strict convexity of $\frac{1}{\Lambda_g}$ is proved.

The existence of a maximizer of $\frac{1}{\Lambda_g}$ (which is a minimizer of Λ_g) follows by Lemma 2.4. To prove the second statement of the theorem, we must find a sub-gradient of $\frac{1}{\Lambda_g}$ corresponding to a maximizer $\overline{g}$. Let $\overline{g}$ be a maximizer, let $\Lambda_{\overline{g}}$ be the corresponding eigenvalue and let $u_{\overline{g}}$ be a corresponding eigenfunction normalized so that $\|\Delta u_{\overline{g}}\|_{L^2(\Omega)} = 1$. Let $h \in \overline{\mathscr{G}}$, let Λ_h be the corresponding eigenvalue and let u_h be a corresponding eigenfunction normalized so that $\|\Delta u_h\|_{L^2(\Omega)} = 1$. Then

$$\frac{1}{\Lambda_h} = \int_\Omega h\, u_h^2 dx \geqslant \int_\Omega h\, u_{\overline{g}}^2 dx = \int_\Omega \overline{g}\, u_{\overline{g}}^2 dx + \int_\Omega (h-\overline{g})\, u_{\overline{g}}^2 dx = \frac{1}{\Lambda_{\overline{g}}} + \int_\Omega (h-\overline{g})\, u_{\overline{g}}^2 dx.$$

Therefore, $u_{\overline{g}}^2$ is a member of the sub-gradient, and by Lemma 2.4 there is an increasing function ϕ such that $\overline{g} = \phi(u_{\overline{g}}^2)$. The theorem is proved. $\square$

3.2. The maximum

Now we investigate the problem

$$\sup_{g\in\overline{\mathscr{G}}} \inf_{w\in H_0^2(\Omega),\ w\neq 0} \frac{\int_\Omega (\Delta w)^2 dx}{\int_\Omega g\, w^2 dx}. \tag{3.5}$$

We note that we cannot interchange (in general) the superior with the inferior. Following [11] (see also [14]), we give a different formulation of the problem by using the Auchmuty's principle [4].

LEMMA 3.3. *Let $0 \leqslant g(x) \leqslant M$, $g(x) \not\equiv 0$, and let Λ_g be the first eigenvalue of problem (1.1). If*

$$A(g,w) = \frac{1}{2}\int_\Omega (\Delta w)^2 dx - \left(\int_\Omega g\, w^2 dx\right)^{\frac{1}{2}} \tag{3.6}$$

then

$$\min_{w\in H_0^2(\Omega)} A(g,w) = -\frac{1}{2} \max_{w\in H_0^2(\Omega), w\neq 0} \frac{\int_\Omega g\, w^2 dx}{\int_\Omega (\Delta w)^2 dx} = -\frac{1}{2}\frac{1}{\Lambda_g}.$$

The minimum of $A(g,w)$ is attained at an eigenfunction $w = u_g$ normalized as

$$\frac{1}{\Lambda_g} = \left(\int_\Omega g\, u_g^2 dx\right)^{\frac{1}{2}} = \int_\Omega (\Delta u_g)^2 dx. \tag{3.7}$$

Proof. See [4] or Lemma 3.3 of [14]. $\square$

LEMMA 3.4. *If $A(g,w)$ is defined as in (3.6) with $w \in H^2(\Omega)$, the function $g \mapsto A(g,w)$ is quasi-concave in $\overline{\mathscr{G}}$.*

Proof. It suffices to prove that the function $g \mapsto \left(\int_\Omega g\, w^2 dx\right)^{\frac{1}{2}}$ is quasi-convex. Let

$$\left(\int_\Omega g_1 w^2 dx\right)^{\frac{1}{2}} \leqslant c, \qquad \left(\int_\Omega g_2 w^2 dx\right)^{\frac{1}{2}} \leqslant c.$$

Then

$$\left(\int_\Omega \big(tg_1+(1-t)g_2\big)w^2dx\right)^{\frac{1}{2}} = \left(t\int_\Omega g_1w^2dx+(1-t)\int_\Omega g_2w^2dx\right)^{\frac{1}{2}}$$
$$\leqslant \left(tc^2+(1-t)c^2\right)^{\frac{1}{2}} = c.$$

The lemma is proved. □

From now on we restrict the class of domains Ω. We suppose Ω is such that the operator $\Delta^2 u$ under homogeneous Dirichlet boundary conditions is positivity preserving. This means that any nontrivial solution of $\Delta^2 u \geqslant 0$ in Ω with $u = \frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ satisfies $u(x) > 0$ in Ω. It is well known that the ball has this property; for other domains which enjoy such a property we refer to [17], [18], [19], [20]. By Krein-Rutman theorem [23], for this kind of domains Ω, the first eigenvalue of problem (1.1) is simple and a corresponding eigenfunction is positive. By the unique continuation principle we know that such an eigenfunction is strictly positive.

Let Ω be positivity preserving for $\Delta^2 u$ under homogeneous Dirichlet boundary conditions, and let $g \in \mathscr{G}$ fixed. If u_g is a positive solution of (1.1) normalized as in (3.7) we define

$$\Pi_g = \left\{ w \in H_0^2(\Omega) : \ \Lambda_g^2 \int_\Omega g\, u_g w\, dx > 1 \right\}. \tag{3.8}$$

LEMMA 3.5. *The function $w \mapsto A(g,w)$ defined in (3.6) is strictly convex on Π_g.*

Proof. Let $w \in \Pi_g$. For $z \in H_0^2(\Omega)$ define $\Phi(t) = A(g, w+tz)$. We find

$$\Phi'(t) = \int_\Omega (\Delta w + t\Delta z)\Delta z\, dx - \left(\int_\Omega g(w+tz)^2 dx\right)^{-\frac{1}{2}} \int_\Omega g(w+tz)z\, dx,$$

and

$$\Phi''(0) = \int_\Omega (\Delta z)^2 dx - \left(\int_\Omega g\, w^2 dx\right)^{-\frac{1}{2}} \int_\Omega g\, z^2 dx + \left(\int_\Omega g\, w^2 dx\right)^{-\frac{3}{2}} \left(\int_\Omega g\, w\, z\, dx\right)^2.$$

Since

$$\int_\Omega (\Delta z)^2 dx \geqslant \Lambda_g \int_\Omega g\, z^2 dx$$

we have

$$\Phi''(0) \geqslant \Lambda_g \int_\Omega g\, z^2 dx - \left(\int_\Omega g\, w^2 dx\right)^{-\frac{1}{2}} \int_\Omega g\, z^2 dx = \left[\Lambda_g - \left(\int_\Omega g\, w^2 dx\right)^{-\frac{1}{2}}\right] \int_\Omega g\, z^2 dx.$$

On the other side, using Schwarz inequality and recalling the normalization of u_g given by (3.7) we have

$$1 < \Lambda_g^2 \int_\Omega g\, u_g\, w\, dx \leqslant \Lambda_g^2 \left(\int_\Omega g\, u_g^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega g\, w^2 dx\right)^{\frac{1}{2}} = \Lambda_g \left(\int_\Omega g\, w^2\right)^{\frac{1}{2}}.$$

It follows that $\Phi''(0) > 0$. The lemma is proved. □

As the set Π_g over which $w \mapsto A(g,w)$ is convex depends on g we introduce, following [11] pag.176, the indicator function

$$\pi(g,u) = \begin{cases} 0 & \text{if } u \in \Pi_g \\ \infty & \text{otherwise.} \end{cases}$$

LEMMA 3.6. *Let $u \in H_0^2(\Omega)$. The function $g \mapsto A(g,u) + \pi(g,u)$ is weak* upper semicontinuous over $\overline{\mathscr{G}}$.*

Proof. Let $g_i \to \overline{g}$ in the weak* topology of $L^\infty(\Omega)$. Since $\int_\Omega g_i u^2 dx \to \int_\Omega \overline{g} u^2 dx$, it follows that $g \mapsto A(g,u)$ is weak* upper semicontinuous over $\overline{\mathscr{G}}$. On the other side, let us show that

$$\limsup_{i\to\infty} \pi(g_i,u) \leqslant \pi(\overline{g},u) \quad \forall u \in H_0^2(\Omega).$$

This is clearly true if $u \notin \Pi_{\overline{g}}$. Let $u \in \Pi_{\overline{g}}$. By Lemma 3.1 we know that $\Lambda_{g_i} \to \Lambda_{\overline{g}}$. Since Ω is positivity preserving, there is a unique positive normalized eigenfunction u_{g_i} and a unique positive normalized eigenfunction $u_{\overline{g}}$. Then, by the proof of Lemma 3.1 we find that $u_{g_i} \to u_{\overline{g}}$ in $L^s(\Omega)$ for some $s > 2$. It follows that

$$\Lambda_{g_i}^2 \int_\Omega g_i\, u_{g_i}\, u\, dx \to \Lambda_{\overline{g}}^2 \int_\Omega \overline{g}\, u_{\overline{g}}\, u\, dx.$$

$u \in \Pi_{\overline{g}}$ now implies that for i greater than some i_0 depending only on $\overline{g}$, u belongs to Π_{g_i}, that is, $\pi(g_i,u) = 0 = \pi(\overline{g},u)$. The lemma follows. □

LEMMA 3.7. *Let $0 \leqslant g_0(x) \leqslant M$, $g_0(x) \not\equiv 0$, and let $A(g,w)$ be defined as in* (3.6). *Then the following equality holds:*

$$\sup_{g\in\overline{\mathscr{G}}} \inf_{w\in H_0^2(\Omega)} A(g,w) = \inf_{w\in H_0^2(\Omega)} \sup_{g\in\overline{\mathscr{G}}} A(g,w). \tag{3.9}$$

Proof. The proof uses Lemmata 3.4, 3.5 and 3.6, and is the same as that of Proposition 7.7 of [11]. □

As observed in [11], Lemma 3.7 implies that $A(g,w)$ has a saddle point. Since there is not an explicit proof of this fact in the cited paper, we enclose here a simple proof (see [14] for a similar proof in the corresponding case of Navier boundary conditions).

PROPOSITION 3.8. *There exists $\underline{g} \in \overline{\mathscr{G}}$ such that the pair $(\underline{g}, u_{\underline{g}})$ is a saddle point for $A(g,w)$ in $(\overline{\mathscr{G}}, H_0^2(\Omega))$.*

Proof. For $w \in H_0^2(\Omega)$ we define

$$B(w) = \sup_{g\in\overline{\mathscr{G}}} A(g,w).$$

We claim that there exists $\underline{u} \in H_0^2(\Omega)$ such that

$$B(\underline{u}) = \inf_{w \in H_0^2(\Omega)} B(w). \tag{3.10}$$

Let u_k be a minimizing sequence in $H_0^2(\Omega)$. Since we can assume that $B(u_k)$ is decreasing, we have $B(u_k) \leqslant C_1$ for some positive constant. This estimate implies that

$$\frac{1}{2}\int_\Omega (\Delta u_k)^2 dx - \inf_{g \in \overline{\mathscr{G}}} \Big(\int_\Omega g u_k^2 dx\Big)^{\frac{1}{2}} \leqslant C_1.$$

Since $\|g\|_\infty \leqslant M$ for all $g \in \overline{\mathscr{G}}$ and since $\|\Delta w\|_{L^2(\Omega)}$ is equivalent to the norm $\|w\|_{H^2(\Omega)}$ in $H_0^2(\Omega)$ we can write

$$\frac{1}{2}\|\Delta u_k\|^2_{L^2(\Omega)} \leqslant C_1 + C_2 \|\Delta u_k\|_{L^2(\Omega)}$$

for some positive constant C_2. A straightforward calculation shows that

$$\|\Delta u_k\|_{L^2(\Omega)} \leqslant C_2 + \sqrt{C_2^2 + 2C_1}.$$

Then, up to a subsequence, u_k is weakly convergent in $H^2(\Omega)$ and strongly in $L^2(\Omega)$ to a function $\underline{u} \in H_0^2(\Omega)$.

By Lemma 2.3 we find $g_1 \in \mathscr{G}$ such that

$$\inf_{g \in \overline{\mathscr{G}}} \Big(\int_\Omega g \underline{u}^2 dx\Big)^{\frac{1}{2}} = \Big(\int_\Omega g_1 \underline{u}^2 dx\Big)^{\frac{1}{2}}. \tag{3.11}$$

On the other side we have

$$\lim_{k \to \infty} \int_\Omega g_1 u_k^2 dx = \int_\Omega g_1 \underline{u}^2 dx, \tag{3.12}$$

and

$$\inf_{g \in \overline{\mathscr{G}}} \Big(\int_\Omega g u_k^2 dx\Big)^{\frac{1}{2}} \leqslant \Big(\int_\Omega g_1 u_k^2 dx\Big)^{\frac{1}{2}}.$$

Taking the $\limsup$ in the last inequality and using (3.11) and (3.12) we find

$$\limsup_{k \to \infty} \inf_{g \in \overline{\mathscr{G}}} \Big(\int_\Omega g u_k^2 dx\Big)^{\frac{1}{2}} \leqslant \Big(\int_\Omega g_1 \underline{u}^2 dx\Big)^{\frac{1}{2}} = \inf_{g \in \overline{\mathscr{G}}} \Big(\int_\Omega g \underline{u}^2 dx\Big)^{\frac{1}{2}}. \tag{3.13}$$

Note that the weak convergence of $u_k \to \underline{u}$ in the $H^2(\Omega)$ norm implies

$$\liminf_{k \to \infty} \int_\Omega (\Delta u_k)^2 dx \geqslant \int_\Omega (\Delta \underline{u})^2 dx.$$

Using the latter inequality and (3.13) we find

$$\inf_{w \in H_0^2(\Omega)} B(w) = \lim_{k \to \infty} B(u_k) \geqslant B(\underline{u}) \geqslant \inf_{w \in H_0^2(\Omega)} B(w).$$

The claim (3.10) follows.

From (3.10) we infer the existence of $\underline{u} \in H_0^2(\Omega)$ such that

$$\sup_{g \in \overline{\mathscr{G}}} A(g, \underline{u}) = \inf_{w \in H_0^2(\Omega)} \sup_{g \in \overline{\mathscr{G}}} A(g, w). \tag{3.14}$$

Since $\overline{\mathscr{G}}$ is compact with respect to the weak* topology of $L^\infty(\Omega)$, by using Lemma 3.1 one finds the existence of $\underline{g} \in \overline{\mathscr{G}}$ such that

$$\sup_{g \in \overline{\mathscr{G}}} \Lambda_g = \Lambda_{\underline{g}}.$$

Hence, in view of Lemma 3.3 we have

$$\sup_{g \in \overline{\mathscr{G}}} \inf_{w \in H_0^2(\Omega)} A(g, w) = A(\underline{g}, u_{\underline{g}}) = \inf_{w \in H_0^2(\Omega)} A(\underline{g}, w). \tag{3.15}$$

By using (3.14), Lemma 3.7 and (3.15), we can write

$$\sup_{g \in \overline{\mathscr{G}}} A(g, \underline{u}) = \inf_{w \in H_0^2(\Omega)} \sup_{g \in \overline{\mathscr{G}}} A(g, w) = \sup_{g \in \overline{\mathscr{G}}} \inf_{w \in H_0^2(\Omega)} A(g, w) = \inf_{w \in H_0^2(\Omega)} A(\underline{g}, w) \leqslant A(\underline{g}, \underline{u}).$$

Hence, for any $g \in \overline{\mathscr{G}}$ we have

$$A(g, \underline{u}) \leqslant A(\underline{g}, \underline{u}). \tag{3.16}$$

Similarly we find

$$A(\underline{g}, \underline{u}) \leqslant \sup_{g \in \overline{\mathscr{G}}} A(g, \underline{u}) = \inf_{w \in H_0^2(\Omega)} \sup_{g \in \overline{\mathscr{G}}} A(g, w) = \sup_{g \in \overline{\mathscr{G}}} \inf_{w \in H_0^2(\Omega)} A(g, w) = \inf_{w \in H_0^2(\Omega)} A(\underline{g}, w).$$

Therefore, for any $w \in H_0^2(\Omega)$ we have

$$A(\underline{g}, \underline{u}) \leqslant A(\underline{g}, w). \tag{3.17}$$

By (3.17) and Lemma 3.3 we must have $\underline{u} = u_{\underline{g}}$. Hence, the proposition follows by (3.16) and (3.17). $\square$

THEOREM 3.9. *Let Ω be positivity preserving for $\Delta^2 u$ under homogeneous Dirichlet boundary conditions. Let $0 \leqslant g_0(x) \leqslant M$, $g_0(x) \not\equiv 0$, and let $\mathscr{G}$ be the class of all rearrangements of g_0. If Λ_g is the first eigenvalue of problem (1.1) then there exists $\underline{g} \in \mathscr{G}$ such that*

$$\Lambda_{\underline{g}} = \max_{g \in \mathscr{G}} \Lambda_g. \tag{3.18}$$

Moreover, if $u_{\underline{g}}$ is an eigenfunction of (1.1) corresponding to $g = \underline{g}$ then $\underline{g} = \psi(u_{\underline{g}}^2)$ for some decreasing functions ψ.

Proof. By Proposition 3.8, there is a saddle point $(\underline{g}, u_{\underline{g}})$ for $A(g,w)$ in $(\overline{\mathscr{G}}, H_0^2(\Omega))$. In particular, we have

$$A(g, u_{\underline{g}}) \leqslant A(\underline{g}, u_{\underline{g}}) \quad \forall g \in \overline{\mathscr{G}}.$$

Recalling the definition (3.6) of $A(g,w)$, the latter inequality implies

$$\int_\Omega g u_{\underline{g}}^2 dx \geqslant \int_\Omega \underline{g} u_{\underline{g}}^2 dx \quad \forall g \in \overline{\mathscr{G}}. \tag{3.19}$$

The functions $\underline{g}$ and $u = u_{\underline{g}}$ satisfy

$$\Delta^2 u = \Lambda_{\underline{g}} \, \underline{g} \, u \ \text{ a.e. in } \ \Omega. \tag{3.20}$$

Recall that $u = u_{\underline{g}}$ is either strictly positive or strictly negative. Therefore, by (3.20) it follows that the function u cannot have flat zones in the set $F = \{x \in \Omega : \underline{g}(x) > 0\}$. If $|F| = |\Omega|$, by Lemma 2.1 there is a decreasing function $\psi(t)$ such that $\psi(u_{\underline{g}}^2)$ is a rearrangement of $g_0(x)$ on Ω. By (3.19) and Lemma 2.2 we must have $\underline{g} = \psi(u_{\underline{g}}^2) \in \mathscr{G}$, and the theorem is proved. If $|F| < |\Omega|$, since $\underline{g} \in \overline{\mathscr{G}}$, by Lemma 2.14 of [6] we have $|F| \geqslant |\{x \in \Omega : g_0(x) > 0\}|$. Therefore there is $g_1 \in \mathscr{G}$ such that its support is contained in F. By Lemma 2.1, there is a decreasing function $\psi_1(t)$ such that $\psi_1(u_{\underline{g}}^2)$ is a rearrangement of $g_1(x)$ on F.

Define

$$\alpha = \inf_{x \in \Omega \setminus F} u_{\underline{g}}^2(x).$$

We claim that $u_{\underline{g}}^2(x) \leqslant \alpha$ in F. Arguing by contradiction suppose the claim is false. Therefore there exist a number $S_1 > \alpha$ and a subset A of F with $|A| > 0$ such that $u_{\underline{g}}^2(x) > S_1$ a.e. on A. Now let $\alpha < S_2 < S_1$. We can find a set D of positive measure contained in $\Omega \setminus F$ such that $u_{\underline{g}}^2(x) < S_2$ a.e. on D. We can assume $|A| = |D|$. Using a measure preserving T we define a particular rearrangement of $\underline{g}$, denoted by h, as follows.

$$h(x) = \begin{cases} \underline{g}(Tx), & x \in A \\ \underline{g}(T^{-1}x), & x \in D \\ \underline{g}(x), & x \in \Omega \setminus (A \cup D). \end{cases}$$

Thus

$$\begin{aligned} \int_\Omega h u_{\underline{g}}^2 \, dx - \int_\Omega \underline{g} u_{\underline{g}}^2 \, dx &= \int_{A \cup D} h u_{\underline{g}}^2 \, dx - \int_{A \cup D} \underline{g} u_{\underline{g}}^2 \, dx \\ &= \int_A h u_{\underline{g}}^2 \, dx + \int_A \underline{g} u_{\underline{g}}^2 \circ T \, dx - \int_A \underline{g} u_{\underline{g}}^2 \, dx - \int_A h u_{\underline{g}}^2 \circ T \, dx \\ &= \int_A \left(u_{\underline{g}}^2 \circ T - u_{\underline{g}}^2\right)(\underline{g} - h) \, dx < (S_2 - S_1) \int_A \underline{g} \, dx < 0. \end{aligned}$$

Therefore $\int_\Omega h u_{\underline{g}}^2 \ dx < \int_\Omega \underline{g} u_{\underline{g}}^2 \, dx$, which contradicts (3.19), and the claim follows.

By using equation (3.20) again we find that $u_{\underline{g}}^2(x) < \alpha$ a.e. in F. Now define

$$\psi(t) = \begin{cases} \psi_1(t) & \text{if } \ 0 \leqslant t < \alpha \\ 0 & \text{if } \ t \geqslant \alpha. \end{cases}$$

The function $\psi(t)$ is decreasing and $\psi(u_{\underline{g}}^2)$ is a rearrangement of $g_1(x)$ in Ω. Indeed, the functions g_1 and $\psi(u_{\underline{g}}^2)$ have the same rearrangement on F, and both vanish on $\Omega \setminus F$. By (3.19) and Lemma 2.2 we must have $\underline{g} = \psi(u_{\underline{g}}^2) \in \mathscr{G}$. The theorem follows. $\square$

4. Unique continuation

Let Ω be a domain in $\mathbb{R}^N$ and let $g \in L^\infty(\Omega)$. Consider a weak solution $u \in H^2(\Omega)$ of the equation

$$\Delta^2 u = g(x)u \quad \text{in} \quad \Omega. \tag{4.1}$$

It is well known [3] that $u \in H^4_{loc}(\Omega)$.

LEMMA 4.1. *Let Ω be a domain in $\mathbb{R}^N$ containing the origin 0. Let $u \in H^4_{loc}(\Omega)$ be a solution to (4.1) with $|g(x)| \leqslant M$ and let for all $n \in \mathbb{N}$*

$$\int_{|x| \leqslant r} u^2 = O(r^n), \quad r \to 0.$$

Then u is identically zero in Ω.

Proof. If follows by Theorem 1.1 of [8]. See also [10] and [24]. $\square$

LEMMA 4.2. *Let $B(x_0,r)$ and $B(x_0,2r)$, $r \in (0,1)$, be two concentric balls contained in Ω, and let u be a solution of (4.1) with $|g(x)| \leqslant M$. Then there exists a constant C independent of r such that*

$$\int_{B(x_0,r)} (\Delta u)^2 \leqslant C\left[\frac{1}{r^2}\int_{B(x_0,2r)} |\nabla u|^2 + \frac{1}{r^4}\int_{B(x_0,2r)} u^2\right].$$

Proof. Let $\theta \in C_0^\infty(B(x_0,2r))$ with $0 \leqslant \theta \leqslant 1$, $|\nabla\theta| \leqslant C/r$ and $|\Delta\theta| \leqslant C/r^2$ in $B(x_0,2r)$, and $\theta = 1$ in $B(x_0,r)$. Here and in what follows we denote by C a generic positive constant, which may vary from line to line, and is independent of r.

Multiplying the equation (4.1) by $\theta^4 u$ and integrating by parts on $B(x_0,2r)$ we find

$$\int_{B(x_0,2r)} \Delta u \Delta(\theta^4 u) = \int_{B(x_0,2r)} g(x)\theta^4 u^2.$$

If $|g(x)| \leqslant M$ we get

$$\begin{aligned} &\int_{B(x_0,2r)} \theta^4(\Delta u)^2 + 4\int_{B(x_0,2r)} \theta^3 \Delta u \Delta\theta\, u \\ &+12\int_{B(x_0,2r)} \theta^2|\nabla\theta|^2 \Delta u\, u + 8\int_{B(x_0,2r)} \theta^3 \nabla\theta\cdot\nabla u \Delta u \\ &\leqslant M\int_{B(x_0,2r)} u^2. \end{aligned} \tag{4.2}$$

Using the Schwarz inequality we find

$$-4\int_{B(x_0,2r)}\theta^3\Delta u\Delta\theta\, u \leqslant \frac{1}{6}\int_{B(x_0,2r)}\theta^4(\Delta u)^2+24\int_{B(x_0,2r)}\theta^2(\Delta\theta)^2u^2.$$

Recalling that $|\Delta\theta|\leqslant C/r^2$ and $0\leqslant\theta\leqslant 1$ we have

$$-4\int_{B(x_0,2r)}\theta^3\Delta u\Delta\theta\, u \leqslant \frac{1}{6}\int_{B(x_0,2r)}\theta^4(\Delta u)^2+\frac{C}{r^4}\int_{B(x_0,2r)}u^2. \tag{4.3}$$

Similarly, we find

$$-8\int_{B(x_0,2r)}\theta^2|\nabla\theta|^2\Delta u\, u \leqslant \frac{1}{6}\int_{B(x_0,2r)}\theta^4(\Delta u)^2+96\int_{B(x_0,2r)}|\nabla\theta|^4u^2.$$

Since $|\nabla\theta|\leqslant C/r$ we have

$$-8\int_{B(x_0,2r)}\theta^2|\nabla\theta|^2\Delta u\, u \leqslant \frac{1}{6}\int_{B(x_0,2r)}\theta^4(\Delta u)^2+\frac{C}{r^4}\int_{B(x_0,2r)}u^2. \tag{4.4}$$

Finally, we find

$$\begin{aligned}-8\int_{B(x_0,2r)}\theta^3\nabla\theta\cdot\nabla u\Delta u &\leqslant \frac{1}{6}\int_{B(x_0,2r)}\theta^4(\Delta u)^2+96\int_{B(x_0,2r)}\theta^2|\nabla\theta|^2|\nabla u|^2\\ &\leqslant \frac{1}{6}\int_{B(x_0,2r)}\theta^4(\Delta u)^2+\frac{C}{r^2}\int_{B(x_0,2r)}|\nabla u|^2.\end{aligned} \tag{4.5}$$

Inserting the estimates (4.3), (4.4) and (4.5) into (4.2), after some simplification and recalling that $\theta=1$ in $B(x_0,r)$, we get the desired estimate. □

LEMMA 4.3. *Let u be a solution of* (4.1) *with* $|g(x)|\leqslant M$. *If* $E=\{x\in\Omega : u(x)=0\}$ *has a positive measure then there is $x_0\in\Omega$ such that for every $n\in\mathbb{N}$ we have*

$$\int_{B_r}u^2=O(r^n),\quad \int_{B_r}|\nabla u|^2=O(r^n),\ \ r\to 0, \tag{4.6}$$

where $B_r=\{x\in\Omega : |x-x_0|<r\}$.

Proof. We know that almost every point of E is a point of density. Let x_0 be such a point. This means that for a given $\varepsilon>0$ there is $r_0=r_0(\varepsilon)$ such that for $r<r_0$ we have

$$\frac{|E^c\cap B_r|}{|B_r|}<\varepsilon,\quad \frac{|E\cap B_r|}{|B_r|}>1-\varepsilon,$$

where E^c denotes the complement of E. We may suppose $B_{4r}\subset\Omega$ and $r<1$. For $N\geqslant 3$ we have

$$\begin{aligned}\int_{B_r}u^2=\int_{B_r\cap E^c}u^2 &\leqslant \Big(\int_{B_r}|u|^{\frac{2N}{N-2}}\Big)^{\frac{N-2}{N}}|B_r\cap E^c|^{\frac{2}{N}}\\ &\leqslant C\int_{B_r}[|\nabla u|^2+u^2]\varepsilon^{\frac{2}{N}}r^2\leqslant C\varepsilon^{\frac{2}{N}}\Big[r^2\int_{B_r}|\nabla u|^2+\int_{B_r}u^2\Big].\end{aligned} \tag{4.7}$$

Let $\tilde{E}=\{x\in B_r:\nabla u(x)=0\}$. Since for almost every point of E we have $\nabla u=0$ (see [16], Lemma 7.7), the inclusion $E\subset\tilde{E}$ holds except possibly in a set of zero measure. Hence, x_0 is a point of density also for $\tilde{E}$. For $r<r_1(\varepsilon)$ (with $r_1(\varepsilon)\leqslant r_0(\varepsilon)$) we have

$$\begin{aligned}\int_{B_r}|\nabla u|^2&=\int_{B_r\cap\tilde{E}^c}|\nabla u|^2\leqslant\Big(\int_{B_r}|\nabla u|^{\frac{2N}{N-2}}\Big)^{\frac{N-2}{N}}|B_r\cap\tilde{E}^c|^{\frac{2}{N}}\\&\leqslant C\varepsilon^{\frac{2}{N}}r^2\Big[\int_{B_r}|D^2u|^2+u^2\Big].\end{aligned}\tag{4.8}$$

If $v\in H^2(B_{2r})\cap H_0^1(B_{2r})$ we have ([16], Theorem 8.12)

$$\int_{B_{2r}}[|D^2v|^2+v^2]\leqslant C\int_{B_{2r}}[(\Delta v)^2+v^2].$$

Take $v=u\theta$ with $\theta\in C_0^\infty(B_{2r})$, $\theta=1$ in B_r, $|\nabla\theta|\leqslant C/r$ and $|D^2\theta|\leqslant C/r^2$ in $B(x_0,2r)$. We find

$$\int_{B_r}[|D^2u|^2+u^2]\leqslant C\Big[\int_{B_{2r}}(\Delta u)^2+\frac{1}{r^2}\int_{B_{2r}}|\nabla u|^2+\frac{1}{r^4}\int_{B_{2r}}u^2\Big].\tag{4.9}$$

Lemma 4.2 and inequality (4.9) yield

$$\int_{B_r}[|D^2u|^2+u^2]\leqslant C\Big[\frac{1}{r^2}\int_{B_{4r}}|\nabla u|^2+\frac{1}{r^4}\int_{B_{4r}}u^2\Big].$$

From (4.8) and the latter inequality we find

$$r^2\int_{B_r}|\nabla u|^2\leqslant C\varepsilon^{\frac{2}{N}}\Big[r^2\int_{B_{4r}}|\nabla u|^2+\int_{B_{4r}}u^2\Big].$$

Adding this inequality to (4.7) we get

$$r^2\int_{B_r}|\nabla u|^2+\int_{B_r}u^2\leqslant C\varepsilon^{\frac{2}{N}}\Big[(4r)^2\int_{B_{4r}}|\nabla u|^2+\int_{B_{4r}}u^2\Big].\tag{4.10}$$

Let

$$f(r)=r^2\int_{B_r}|\nabla u|^2+\int_{B_r}u^2.$$

From (4.10) we find

$$f(r)\leqslant C\varepsilon^{\frac{2}{N}}f(4r).$$

Given n take ε small so that $C\varepsilon^{\frac{2}{N}}\leqslant 4^{-n}$. Then

$$f(r)\leqslant 4^{-n}f(4r)\ \text{ for }\ r\leqslant r_0(n).$$

Iterating k times we get

$$f(r)\leqslant 4^{-kn}f(4^kr),\ \text{ for }\ 4^{k-1}r\leqslant r_0(n).\tag{4.11}$$

Given $0 < r < r_0(n)$ choose $k \in \mathbb{N}$ such that

$$4^{-k} r_0 \leqslant r \leqslant 4^{-k+1} r_0.$$

By (4.11) we find

$$f(r) \leqslant 4^{-kn} f(4^k r) \leqslant 4^{-kn} f(4r_0).$$

Since $4^{-k} \leqslant r/r_0$ we obtain

$$f(r) \leqslant \left(\frac{r}{r_0}\right)^n f(4r_0).$$

In particular, there is a constant C such that $f(r) \leqslant Cr^n$. Hence

$$r^2 \int_{B_r} |\nabla u|^2 + \int_{B_r} u^2 \leqslant Cr^n.$$

For $N = 1$ or $N = 2$, with easy changes in the proof, one finds (4.10) with ε^σ, $0 < \sigma < 1$, in place of $\varepsilon^{\frac{2}{N}}$. Therefore the last estimate holds for all N. The lemma follows. $\square$

THEOREM 4.4. *Let u be a solution of* (4.1) *with* $|g(x)| \leqslant M$. *If the set* $E = \{x \in \Omega : u(x) = 0\}$ *has a positive measure then u is identically zero in* Ω.

Proof. By Lemma 4.3 we have, for some $x_0 \in \Omega$,

$$\int_{|x-x_0| \leqslant r} u^2 = O(r^n), \quad r \to 0. \tag{4.12}$$

If we put $x - x_0 = y$ equation (4.1) becomes

$$\Delta^2 \tilde{u} = \tilde{g}(y)\tilde{u} \ \text{ in } \ \tilde{\Omega},$$

where $\tilde{\Omega}$ contains 0. Since $|\tilde{g}(y)| \leqslant M$, by (4.12) we find

$$\int_{|y| \leqslant r} \tilde{u}^2 = O(r^n), \quad r \to 0.$$

The theorem follows now by Lemma 4.1. $\square$

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