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ASYMPTOTIC BEHAVIOR OF RADIAL MINIMIZERS OF A p–ENERGY FUNCTIONAL WITH NONVANISHING DIRICHLET BOUNDARY CONDITION

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Abstract. This paper is concerned with a p-energy functional with nonvanishing Dirichlet boundary condition. The authors prove the $W^{1,p}_{loc}$ convergence of the radial minimizer, and discuss the location of the zeros of this minimizer. In addition, an estimate of the convergence rate of the minimizer is given by means of the iterative approach.

1. Introduction

Let $B=\{x\in \mathbf{R}^n : |x|<1\}$. Denote

$$S^{n-1}=\{x\in \mathbf{R}^{n+1} : x_1^2+x_2^2+...+x_n^2=1, x_{n+1}=0\},$$
$$S^n=\{x\in \mathbf{R}^{n+1} : x_1^2+x_2^2+...+x_n^2+x_{n+1}^2=1\}.$$

Let $g(x)=(M\frac{x}{|x|},\sqrt{1-M^2})$ where $x\in\partial B, M\in(0,1)$. We are concerned with the minimizer of a p-energy functional

$$E_\varepsilon(u,B)=\frac{1}{p}\int_B |\nabla u|^p dx+\frac{1}{2\varepsilon^p}\int_B u_{n+1}^2 dx$$

in the function class $W=\{u(x)=(\frac{x}{|x|}\sin f(r),\cos f(r))\in W^{1,p}(B,S^n) : u|_{\partial B}=g, r=|x|\}$. Sometimes we write $u_\varepsilon(x)=(u'_\varepsilon(x),u_{n+1}(x))$. By the direct method in the calculus of variations, the minimizer u_ε exists and is often called the *radial minimizer*.

When $p=2$, the functional $E_\varepsilon(u,B)$ was introduced in the study of some simplified model of high-energy physics, which controls the statics of planar ferromagnets and antiferromagnets (see [6] and [10]). The asymptotic behavior of minimizer of $E_\varepsilon(u,B)$ has been considered in [3].

If the term $\frac{u_{n+1}^2}{\varepsilon^p}$ is replaced by $\frac{(1-|u|^2)^2}{2\varepsilon^p}$, the functional is the well-known *Ginzburg-Landau* functional. When $n\geqslant 3$, the problem on the asymptotic behavior of minimizers was introduced in [1], which was studied in [2] and [5] independently. The paper

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[9] studied the asymptotic behavior of the radial minimizer when $p \in (n-1,n)$. Their work shows that the study of minimizers of n-Ginzburg-Landau functional is connected tightly with the corresponding properties of the n-harmonic map.

In this paper, we always assume $n > 2$ and $p \in (n-1,n)$. We are interested in the asymptotic behavior of minimizer of p-energy functional with nonvanishing Dirichlet boundary condition as $\varepsilon \to 0$. Based on this result, we will establish the relation between the radial minimizer and the map $x/|x|$.

If we denote

$$V = \{f \in W^{1,p}_{loc}(0,1] : r^{(n-1)/p} f_r, r^{(n-1-p)/p} \sin f \in L^p(0,1), f(r) \geqslant 0, f(1) = \arcsin M\}$$

then

$$V = \{f(r) : u(x) = (\frac{x}{|x|} \sin f(r), \cos f(r)) \in W\}$$

and it is a subset of $C[0,1]$. Substituting $u(x) = (\frac{x}{|x|} \sin f(r), \cos f(r))$ into $E_\varepsilon(u,B)$, we obtain $E_\varepsilon(u,B) = |S^{n-1}| E_\varepsilon(f,[0,1])$, where

$$E_\varepsilon(f,[0,1]) = \frac{1}{p}\int_0^1 \left(f_r^2 + (n-1)r^{-2}\sin^2 f\right)^{p/2} r^{n-1} dr + \frac{1}{2\varepsilon^p}\int_0^1 r^{n-1}\cos^2 f dr.$$

This shows that $u_\varepsilon(x) = (\frac{x}{|x|} \sin f_\varepsilon(r), \cos f_\varepsilon(r)) \in W$ is the minimizer of $E_\varepsilon(u,B)$ if and only if $f_\varepsilon(r) \in V$ is the minimizer of $E_\varepsilon(f,[0,1])$. Inspecting the expression of $E_\varepsilon(f,[0,1])$, we may assume $0 \leqslant f \leqslant \pi/2$. We will prove the following results in this paper.

THEOREM 1.1. ($W^{1,p}$ convergence) *Let $u_\varepsilon(x)$ be a radial minimizer of $E_\varepsilon(u,B)$. Then as $\varepsilon \to 0$,*

$$u_\varepsilon \to (\frac{x}{|x|}, 0) \quad in \quad W^{1,p}_{loc}(B, S^n).$$

THEOREM 1.2. (Location of zeros) *Let $u_\varepsilon(x)$ be a radial minimizer of $E_\varepsilon(u,B)$. If we denote $u_\varepsilon = (u'_\varepsilon, u_{\varepsilon n+1})$, then as $\varepsilon \to 0$, the zeros of $u'_\varepsilon(x)$ are located near the origin 0 and the boundary ∂B.*

THEOREM 1.3. (Convergence rate) *Assume that $u_\varepsilon(x)$ is a radial minimizer of $E_\varepsilon(u,B)$. Then for any $T \in (0,1/4)$,there exists a positive constant C which is independent of $\varepsilon \in (0,1)$, such that*

$$\int_T^{1-T} r^{n-1}[(f'_\varepsilon)^p + \frac{1}{\varepsilon^p}\cos^2 f_\varepsilon] dr \leqslant C\varepsilon^p.$$

2. $W^{1,p}$ convergence

Assume u_ε be a radial minimizer of $E_\varepsilon(u,B)$. Theorem 1.1 can be proved by setting up several propositions as follows.

By the direct method in the calculus of variations, it is not difficult to prove

PROPOSITION 2.1. *The minimizer u_ε satisfies*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = u|\nabla u|^p + \frac{1}{\varepsilon^p}(uu_{n+1}^2 - u_{n+1}e_{n+1}), \tag{2.1}$$

where $e_{n+1} = (0,0,\cdots,0,1)$.

PROPOSITION 2.2. *Assume that u_ε satisfies (2.1). For any $R > 0$, there exists a constant $C(R) > 0$ which is independent of $\varepsilon \in (0,1)$, such that for any $x_0 \in B$,*

$$\|\nabla u\|_{L^\infty(B(x_0,R\varepsilon))} \leqslant C(R)\varepsilon^{-1}.$$

Proof. Without loss of generality, assume that $x_0 = 0$. Let $y = x\varepsilon^{-1}$ and $v(y) = u(x)$ in (2.1), we can derive

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v)n = v|\nabla v|^p + (v_{n+1}^2 v - v_{n+1}e_{n+1}).$$

By the Theorem 2.2 in [8], we have

$$\int_{B(0,2R)} |\nabla v|^p dx \leqslant C(R).$$

Applying the same idea of §3 in [4], we also have

$$\|\nabla v\|_{L^\infty(B(0,R))} \leqslant C(R).$$

Since $v(y) = v(x\varepsilon^{-1}) = u(x)$, we obtain

$$\|\nabla u\|_{L^\infty(B(0,R\varepsilon))} \leqslant C(R)\varepsilon^{-1}.$$

If $x_0 \neq 0$, moving the coordinate center to x_0 and using the same method, we can also complete the proof.

PROPOSITION 2.3. *Let $u_\varepsilon \in W$ be a radial minimizer of $E_\varepsilon(u,B)$. Then there exists a positive constant C which is independent of $\varepsilon \in (0,1)$ such that*

$$E_\varepsilon(u,B) \leqslant C\varepsilon^{1-p}. \tag{2.2}$$

Proof. Suppose that f_5 is a function such that $(\frac{y}{|y|}\sin f_5(s), \cos f_5(s))$ is the minimizer of the functional

$$F(u,B) = \frac{1}{p}\int_B |\nabla u|^p dy + \frac{1}{2}\int_B u_{n+1}^2 dy$$

in the class $Y = \{(\frac{y}{|y|}\sin f(s), \cos f(s)) \in W^{1,p}(B) : u|_{\partial B} = g, s = |y|\}$. Define

$$f_1(r) = \begin{cases} \arcsin M, & r \in [1-\varepsilon, 1]; \\ \frac{\pi}{2} - \frac{1}{\varepsilon}[r - (1-2\varepsilon)](\frac{\pi}{2} - \arcsin M), & r \in [1-2\varepsilon, 1-\varepsilon]; \\ \frac{\pi}{2}, & r \in [\varepsilon, 1-2\varepsilon]; \\ f_5(r), & r \in [0,\varepsilon]. \end{cases}$$

Let $x=y\varepsilon$, then $f_5(r)=f_5(s\varepsilon)$. Thus

$$E_\varepsilon((\frac{x}{|x|}\sin f_1(r),\cos f_1(r)),B(0,\varepsilon))$$
$$=F((\frac{y}{|y|}\sin f_5(s),\cos f_5(s)),B)\varepsilon^{n-p}\leqslant C\varepsilon^{n-p}, \tag{2.3}$$

while we also have

$$E_\varepsilon(f_1,[\varepsilon,1-2\varepsilon])=\frac{(n-1)^{p/2}}{p}\int_\varepsilon^{1-2\varepsilon}r^{n-1-p}dr\leqslant C. \tag{2.4}$$

When $r\in[1-2\varepsilon,1-\varepsilon]$,

$$\begin{aligned}
&E_\varepsilon((\frac{x}{|x|}\sin f_1(r),\cos f_1(r)),B(0,1-\varepsilon)\backslash B(0,1-2\varepsilon))\\
&=\frac{|S^{n-1}|}{p}\int_{1-2\varepsilon}^{1-\varepsilon}[\frac{1}{\varepsilon^2}(\frac{\pi}{2}-\arcsin M)^2+(n-1)r^{-2}\sin^2 f_1]^{p/2}r^{n-1}dr\\
&+\frac{|S^{n-1}|}{2\varepsilon^p}\int_{1-2\varepsilon}^{1-\varepsilon}r^{n-1}\cos f_1 dr\leqslant C\varepsilon^{-p}\int_{1-2\varepsilon}^{1-\varepsilon}r^{n-1}dr+C\int_{1-2\varepsilon}^{1-\varepsilon}r^{n-1-p}dr\\
&\leqslant C\varepsilon^{1-p}+C\varepsilon\leqslant C\varepsilon^{1-p},
\end{aligned} \tag{2.5}$$

and when $r\in[1-\varepsilon,1]$, it is also easy to obtain

$$\begin{aligned}
E_\varepsilon(f_1,[1-\varepsilon,1])&=\frac{1}{p}\int_{1-\varepsilon}^1(n-1)^{p/2}M^p r^{n-1-p}dr+\frac{1}{2\varepsilon^p}\int_{1-\varepsilon}^1(1-M^2)r^{n-1}dr\\
&\leqslant C\varepsilon+C\varepsilon^{1-p}\leqslant C\varepsilon^{1-p}.
\end{aligned}$$

Since u_ε is a radial minimizer, it is not difficult to deduce from the result above and (2.3)-(2.5) that

$$E_\varepsilon(u_\varepsilon,B)\leqslant E_\varepsilon((\frac{x}{|x|}\sin f_1,\cos f_1),B)\leqslant C\varepsilon^{1-p}.$$

Thus the proposition is proved.

REMARK 2.4. From (2.2) it follows that for some $R>0$, there exists $C=C(R)>0$ (independent of $\varepsilon\in(0,1)$) such that

$$\int_R^1|f_\varepsilon'|^p dr\leqslant C\varepsilon^{1-p}.$$

Combining this with $f_\varepsilon\leqslant\pi/2$ on $[R,1]$ yields $\|f_\varepsilon\|_{W^{1,p}(R,1)}\leqslant C\varepsilon^{(1-p)/p}$. Using the embedding theorem, we see that for any $r\in[R,1]$,

$$|f_\varepsilon(r)-f_\varepsilon(1)|\leqslant C\varepsilon^{(1-p)/p}|r-1|^{1-1/p}.$$

Thus

$$|f_\varepsilon(r)|\geqslant\arcsin M-C\rho^{1-1/p}=\frac{1}{2}\arcsin M,\ \ r\in(1-\rho\varepsilon,1), \tag{2.6}$$

where $\rho=(\frac{\arcsin M}{2C})^{p/(p-1)}$.

PROPOSITION 2.5. *Let u_ε be a radial minimizer of $E_\varepsilon(u,B)$. Then for any $\zeta \in (0,1/3)$, there exists constant $C>0$ which is independent of $\varepsilon \in (0,1)$, such that*

$$E_\varepsilon(u_\varepsilon, B(0,1-\zeta)) \leqslant C. \tag{2.7}$$

Proof. We will prove that, for any $\zeta \in (0,1/3)$, there exists $\xi_i \in (\zeta/3, \zeta/2)$ such that

$$E_\varepsilon(f_\varepsilon,[0,1-\xi_i]) \leqslant C(\varepsilon+\varepsilon^{n-p}+\varepsilon^{i-p}) + \frac{(n-1)^{p/2}}{p}\int_\varepsilon^{1-\xi_i-\varepsilon} r^{n-1-p}dr \tag{2.8}$$

for $i \leqslant n$. Obviously, (2.8) with $i=1$ holds by virtue of Proposition 2.3. Suppose that (2.8) holds for $i=k<n$, namely

$$E_\varepsilon(f_\varepsilon,[0,1-\xi_k]) \leqslant C(\varepsilon+\varepsilon^{n-p}+\varepsilon^{k-p}) + \frac{(n-1)^{p/2}}{p}\int_\varepsilon^{1-\xi_k-\varepsilon} r^{n-1-p}dr. \tag{2.9}$$

We will prove (2.8) still true when $i=k+1$.

By (2.9) and the mean value theorem, there exists $\xi_{k+1} \in (\xi_k, \zeta/2) \subset (\zeta/3, \zeta/2)$, such that

$$E_\varepsilon(u_\varepsilon, \partial B(0,1-\xi_{k+1})) \leqslant CE_\varepsilon(u_\varepsilon,B) \leqslant C\varepsilon^{k-p}, \tag{2.10}$$

where $C>0$ only depends on ξ,ρ. Write $f=f_\varepsilon$, and define

$$f_2(r)=f(r), \quad r\in[1-\xi_{k+1},1];$$

$$f_2(r)=\tfrac{\pi}{2}-\tfrac{1}{\varepsilon}[r-(1-\xi_{k+1}-\varepsilon)](\tfrac{\pi}{2}-f(1-\xi_{k+1})), \quad r\in[1-\xi_{k+1}-\varepsilon, 1-\xi_{k+1}];$$

$$f_2(r)=\tfrac{\pi}{2} \quad r\in[\varepsilon,1-\xi_{k+1}-\varepsilon];$$

$$f_2(r)=f_5(r) \quad r\in[0,\varepsilon].$$

Here f_5 is the function in the proof of Proposition 2.3. If $x=y\varepsilon$, then $f_5(r)=f_5(s\varepsilon)$. Similar to the prove of Proposition 2.3, it is easy to obtain

$$E_\varepsilon((\frac{x}{|x|}\sin f_2(r), \cos f_2(r)), B(0,\varepsilon) \leqslant C\varepsilon^{n-p}. \tag{2.11}$$

where C dose not depend on ε.

In addition, we also have

$$\begin{aligned} &E_\varepsilon((\frac{x}{|x|}\sin f_2, \cos f_2), B(0,1-\xi_{k+1}-\varepsilon)\backslash B(0,\varepsilon)) \\ &\qquad = \frac{(n-1)^{p/2}|S^{n-1}|}{p}\int_\varepsilon^{1-\xi_{k+1}-\varepsilon} r^{n-1-p}dr. \end{aligned} \tag{2.12}$$

From (2.10), we obtain

$$\frac{1}{\varepsilon^p}\cos^2 f(1-\xi_{k+1}) \leqslant C\varepsilon^{k-p}, \tag{2.13}$$

and hence

$$\frac{1}{\varepsilon^p}\sin^2[\frac{\pi}{2}-f(1-\xi_{k+1})]\leqslant C\varepsilon^{k-p}. \tag{2.14}$$

On the other hand, we have

$$\begin{aligned}\cos\{\frac{\pi}{2}-\frac{1}{\varepsilon}[r-(1-\xi_{k+1}-\varepsilon)][\frac{\pi}{2}-f(1-\xi_{k+1})]\}\\ \leqslant\sin[\frac{\pi}{2}-f(1-\xi_{k+1})]=\cos f(1-\xi_{k+1}).\end{aligned} \tag{2.15}$$

Applying the mean value theorem, from (2.14) we can deduce that

$$\sin[\frac{\pi}{2}-f(1-\xi_{k+1})]\geqslant\frac{2}{\pi}[\frac{\pi}{2}-f(1-\xi_{k+1})],$$

as long as ε is sufficiently small. Combining this with (2.14) yields

$$\frac{1}{\varepsilon^p}[\frac{\pi}{2}-f(1-\xi_{k+1})]^2\leqslant\frac{\pi}{2}\frac{1}{\varepsilon^p}\sin^2[\frac{\pi}{2}-f(1-\xi_{k+1})]\leqslant C\varepsilon^{k-p}. \tag{2.16}$$

Therefore, when $r\in[1-\xi_{k+1}-\varepsilon,1-\xi_{k+1}]$, using (2.13)-(2.16), we can deduce

$$\begin{aligned}&E_\varepsilon((\frac{x}{|x|}\sin f_2,\cos f_2),B(0,1-\xi_{k+1})\backslash B(0,1-\xi_{k+1}-\varepsilon))\\ &=\frac{|S^{n-1}|}{p}\int_{1-\xi_{k+1}-\varepsilon}^{1-\xi_{k+1}}\left[(f_2')^2+(n-1)r^{-2}\sin^2 f_2\right]^{p/2}r^{n-1}dr\\ &\quad+\frac{|S^{n-1}|}{2\varepsilon^p}\int_{1-\xi_{k+1}-\varepsilon}^{1-\xi_{k+1}}r^{n-1}\cos^2 f_2 dr\\ &\leqslant\frac{C}{p}\int_{1-\xi_{k+1}-\varepsilon}^{1-\xi_{k+1}}\frac{1}{\varepsilon^p}\left[\frac{\pi}{2}-f(1-\xi_{k+1})\right]^p r^{n-1}dr+\frac{C}{p}\int_{1-\xi_{k+1}-\varepsilon}^{1-\xi_{k+1}}r^{n-1-p}dr\\ &\quad+C\int_{1-\xi_{k+1}-\varepsilon}^{1-\xi_{k+1}}r^{n-1}\frac{\cos^2 f(1-\xi_{k+1})}{\varepsilon^p}dr\leqslant C(\varepsilon+\varepsilon^{k+1-p}).\end{aligned} \tag{2.17}$$

Since u_ε is the radial minimizer, we have

$$E_\varepsilon(u_\varepsilon,B)\leqslant E_\varepsilon((\frac{x}{|x|}\sin f_2,\cos f_2),B). \tag{2.18}$$

Substituting (2.11), (2.12) and (2.17) into (2.18) yields

$$\begin{aligned}E_\varepsilon(u_\varepsilon,B)&\leqslant C(\varepsilon+\varepsilon^{n-p}+\varepsilon^{k+1-p})\\ &\quad+\frac{(n-1)^{p/2}|S^{n-1}|}{p}\int_\varepsilon^{1-\xi_{k+1}-\varepsilon}r^{n-1-p}dr+E_\varepsilon(u_\varepsilon,B\backslash B(0,1-\xi_{k+1})).\end{aligned}$$

This means that (2.8) holds for $i=k+1$. In particular, the conclusion is true for $i=n$ by induction. Noticing

$$E_\varepsilon(u_\varepsilon,B\backslash B(0,1-\xi_n))\leqslant E_\varepsilon(u_\varepsilon,B\backslash B(0,1-\zeta)),$$

we can derive the conclusion at last.

Proof of Theorem 1.1. Assume that K is a compact subset of B. By Proposition 2.4, there exists a subsequence u_{ε_k} of u_ε and a function $u_* \in W^{1,p}(K,S^n)$, such that

$$\lim_{\varepsilon_k\to 0} u_{\varepsilon_k} = u_* \quad \text{weakly in} \quad W^{1,p}(K,S^n); \tag{2.19}$$

$$\lim_{\varepsilon_k\to 0} u_{\varepsilon_k} = u_* \quad \text{in} \quad L^p(K,S^n). \tag{2.20}$$

When $\varepsilon \to 0$, $u_{n+1} \to 0$ in $L^p(K)$. In addition,

$$u_\varepsilon(x) = (\frac{x}{|x|}\sin f_\varepsilon(r), \cos f_\varepsilon(r)) \text{ and } |u'_\varepsilon(x)| = \sin f_\varepsilon(r) \to 1 \text{ in } L^p(K),$$

as $\varepsilon \to 0$. Hence, $u'_* = x/|x|$, that is to say $u_* = (x/|x|, 0)$ a.e. on K.

Since each subsequence of u_ε has a convergent subsequence and the limit is always u_*, we know that (2.19) and (2.20) are still true not only for a subsequence, but also for u_ε itself. Thus, (2.8), together with (2.19) and the weakly lower semi-continuity of $\int_K |\nabla u_\varepsilon|^p$, implies

$$\int_K \left|\nabla \frac{x}{|x|}\right|^p dx \leqslant \underline{\lim}_{\varepsilon\to 0} \int_K |\nabla u_\varepsilon|^p dx \leqslant \overline{\lim}_{\varepsilon\to 0} \int_K |\nabla u_\varepsilon|^p dx$$
$$\leqslant C(\varepsilon + \varepsilon^{n-p}) + (n-1)^{p/2}|S^{n-1}| \int_\varepsilon^{1-\xi_n-\varepsilon} r^{n-1-p} dr.$$

It is easy to see

$$\int_K \left|\nabla \frac{x}{|x|}\right|^p dx = (n-1)^{p/2}|S^{n-1}| \int_0^{1-\xi_n} r^{n-1-p} dr.$$

Then we have

$$\lim_{\varepsilon\to 0} \int_K |\nabla u_\varepsilon|^p dx = \int_K \left|\nabla \frac{x}{|x|}\right|^p dx.$$

Combining this with (2.19) and (2.20), we complete the proof.

PROPOSITION 2.6. *Assume that* $u_\varepsilon = (\frac{x}{|x|}\sin f_\varepsilon(r), \cos f_\varepsilon(r))$ *is a radial minimizer of* $E_\varepsilon(u,B)$. *Then for any* $R \in [0,1/4]$, *there holds*

$$\lim_{\varepsilon\to 0} f_\varepsilon(r) = \frac{\pi}{2} \quad \textit{uniformly in} \quad (R, 1-R).$$

Proof. Write $f = f_\varepsilon$. Using the mean value theorem ,we can derive that, for any $r_0 \in (R, 1-R)$,

$$|\cos f(r) - \cos f(r_0)| \leqslant |\sin \xi||f(r) - f(r_0)|, \tag{2.21}$$

where ξ is a function which value is between $f(r)$ and $f(r_0)$. Applying Proposition 2.2, we have

$$|f(r) - f(r_0)| \leqslant C\varepsilon^{-1}|r - r_0| \leqslant C\varepsilon^{-1}\frac{1}{N}\varepsilon = \frac{C}{N}, \quad r \in [r_0 - \frac{1}{N}\varepsilon, r_0]$$

where N is a sufficiently large positive integer. Therefore, by (2.21), there exists a sufficiently large number $N = N_0$ such that

$$|\cos f(r)| \geqslant |\cos f(r_0)| - |\sin\xi||f(r) - f(r_0)| \geqslant \frac{1}{2}\cos f(r_0).$$

By (2.7), we also have

$$\frac{C'\varepsilon}{nN_0}\frac{1}{4}\cos^2 f(r_0) \leqslant \int_{r-\frac{\varepsilon}{N_0}}^{r} r^{n-1}\cos^2 f(r)dr \leqslant C\varepsilon^n$$

where C' is a positive constant. So we obtain

$$\cos^2 f(r_0) \leqslant C\varepsilon^{n-1}.$$

Obviously, when $\varepsilon \to 0$, $\cos^2 f(r_0) \to 0$, hence we have $f(r_0) \to \pi/2$. Since r_0 is arbitrary in $(R, 1-R)$, thus the proposition is complete.

3. Location of zeros

To discuss the location of zeros, we will first establish the definition of bad ball and good ball.

PROPOSITION 3.1. *Let u_ε be a radial minimizer of $E_\varepsilon(u,B)$. For any given $\eta \in (0,1)$, there exist the positive constants λ, μ independent of $\varepsilon \in (0,1)$, such that if*

$$\frac{1}{\varepsilon^n}\int_{A\cap B^{2l\varepsilon}} u_{n+1}^2 dx \leqslant \mu,$$

where $A = B(0, 1-\zeta)$, and $B^{2l\varepsilon}$ is some ball of radius $2l\varepsilon$ with $l \geqslant \lambda$, then

$$|u_\varepsilon'(x)| \geqslant 1-\eta, \quad \forall x \in A\cap B^{l\varepsilon}.$$

Proof. By Proposition 2.2 and using the same argument of Proposition 2.4 in [7], it is not difficult to prove this proposition.

Let λ, μ be constants in Proposition 3.1. If

$$\frac{1}{\varepsilon^n}\int_{B(x^\varepsilon, 2\lambda\varepsilon)\cap A} u_{n+1}^2 dx \leqslant \mu,$$

then $B(x^\varepsilon, \lambda\varepsilon)$ is called a good ball. Otherwise $B(x^\varepsilon, \lambda\varepsilon)$ is called a bad ball.

Now suppose that $\{B(x_i^\varepsilon, \lambda\varepsilon), i \in I\}$ is a family of balls satisfying

$$(i)\ x_i^\varepsilon \in A, \quad i \in I; \quad (ii)\ A \subset \cup_{i\in I} B(x_i^\varepsilon, \lambda\varepsilon);$$
$$(iii)\ B(x_i^\varepsilon, \lambda\varepsilon/4) \cap B(x_j^\varepsilon, \lambda\varepsilon/4) = \phi, \quad i \neq j. \tag{3.1}$$

PROPOSITION 3.2. *Let u_ε be a radial minimizer of $E_\varepsilon(u,B)$. Then there exist a sufficiently small constant $\varepsilon_0 > 0$, and a positive constant C independent of $\varepsilon \in (0,\varepsilon_0)$, such that*

$$\frac{1}{\varepsilon^n}\int_{B(0,1-\zeta)} u_{n+1}^2 dx \leqslant C.$$

Proof. Substituting (2.11) and (2.17) with $\xi_n = \zeta$ into (2.18) yields

$$E_\varepsilon(u_\varepsilon, B(0,1-\zeta)) \leqslant C(\varepsilon^{n-p}+\varepsilon) + E_\varepsilon\Big(\frac{x}{|x|}\sin f_2, \cos f_2), B(0,1-\zeta-\varepsilon)\backslash B(0,\varepsilon)\Big).$$

In view of $p > n-1$, we have $\varepsilon \leqslant \varepsilon^{n-p}$. Therefore

$$E_\varepsilon(u_\varepsilon, B(0,1-\zeta)) \leqslant C\varepsilon^{n-p} + \frac{1}{p}\int_{B(0,1-\zeta-\varepsilon)\backslash B(0,\varepsilon)} |\nabla \frac{x}{|x|}|^p dx.$$

Noting

$$\int_{B(0,1-\zeta)} |\nabla u|^p dx \geqslant (n-1)^{p/2}|S^{n-1}|\int_\varepsilon^{1-\zeta} r^{n-1-p}\sin^p f_\varepsilon dr,$$

we have that, for any $\delta \in (0,1)$,

$$\begin{aligned}
&\frac{1}{2\varepsilon^p}\int_0^{1-\zeta} r^{n-1}\cos^2 f_\varepsilon dr\\
&\leqslant C\varepsilon^{n-p} + \frac{(n-1)^{p/2}}{p}\int_\varepsilon^{1-\zeta-\varepsilon} r^{n-1-p}(1-\sin^p f_\varepsilon)dr\\
&\leqslant C\varepsilon^{n-p} + C\int_\varepsilon^{1-\zeta-\varepsilon} r^{n-1-p}(1-\sin^2 f_\varepsilon)dr\\
&\leqslant C\varepsilon^{n-p} + C(\delta)\varepsilon^p\int_\varepsilon^{1-\zeta} r^{n-1-2p}dr + \delta\varepsilon^{-p}\int_\varepsilon^{1-\zeta} r^{n-1}(1-\sin^2 f_\varepsilon)^2 dr\\
&\leqslant C\varepsilon^{n-p} + C(\delta)\varepsilon^{n-p} + \delta\varepsilon^{-p}\int_0^{1-\zeta} r^{n-1}\cos^2 f_\varepsilon dr.
\end{aligned}$$

Choosing δ sufficiently small yields

$$\frac{1}{2\varepsilon^p}\int_0^{1-\zeta} r^{n-1}\cos^2 f_\varepsilon dr \leqslant C\varepsilon^{n-p}.$$

Thus we may obtain the conclusion by multiplying with ε^{p-n}.

Similar to the prove of Proposition 2.5 in [7], applying Proposition 3.2 and the definition of bad balls, we can derive

PROPOSITION 3.3. *Write $J_\varepsilon = \{i \in I; B(x_i^\varepsilon, \lambda\varepsilon)$ is a bad ball $\}$. There exists a positive integer N_0 which is independent of $\varepsilon \in (0,\varepsilon_0)$, such that the number of bad balls Card $J_\varepsilon \leqslant N_0$.*

Based on this proposition, and by an analogous argument of Theorem IV.1 in [1], we also have the following consequence.

PROPOSITION 3.4. *Let λ be the constant in Proposition 3.1. Then there exist a subset $J \subset J_\varepsilon$ and a constant $h \in [\lambda, \lambda 9^{N_0}]$ such that*

$$\cup_{i\in J_\varepsilon} B(x_i^\varepsilon, \lambda\varepsilon) \subset \cup_{i\in J} B(x_j^\varepsilon, h\varepsilon), \quad |x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon, \qquad i,j \in J, \quad i \neq j.$$

Applying Proposition 3.4, we may modify the family of bad balls, such that the new one, denoted by $\{B(x_i^\varepsilon, h\varepsilon); i \in J\}$, satisfies

$$\cup_{i\in J_\varepsilon} B(x_i^\varepsilon, \lambda\varepsilon) \subset \cup_{i\in J} B(x_i^\varepsilon, h\varepsilon), \quad \lambda \leqslant h, \quad Card J \leqslant Card J_\varepsilon;$$
$$|x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon, \quad i,j \in J, \quad i \neq j.$$

The last condition above implies that every two balls in the new family are disjoint.

PROPOSITION 3.5. *Let u_ε be a radial minimizer of $E_\varepsilon(u,B)$. Then for any $\eta \in (0,1)$, there is a constant $h = h(\eta)$, such that the set $\{x \in A; |u_\varepsilon'(x)| < 1-\eta\} \subset \overline{B(0,h\varepsilon)}$ as $\varepsilon \in (0,\varepsilon_0)$.*

Proof. Suppose there exists a point $x_0 \in A \setminus B(0,h\varepsilon)$ such that $|u_\varepsilon'(x_0)| < 1-\eta$. Then all points on the set $S_0 = \{x \in A \setminus B(0,h\varepsilon); x = |x_0|\}$ satisfy $|u_\varepsilon'(x)| < 1-\eta$ and hence by virtue of Proposition 3.1, all points on S_0 are contained in bad discs. On the other hand, if $|x| > h\varepsilon$, S_0 cannot be covered by a single bad disc, i.e., S_0 is covered by at least two bad discs (they are not intersected). However, this is impossible. It implies our conclusion.

Proof of Theorem 1.2. Proposition 3.5 implies that $|u_\varepsilon'(x)| \geqslant 1-\eta$, as $x \in A \setminus B(0,h\varepsilon)$. Combining this with (2.6), we can see

$$\{x \in B; |u_\varepsilon'(x)| < min(2M\sqrt{1-M^2}, 1-\eta)\} \subset B(0,h\varepsilon) \cup [B(0,1-\rho\varepsilon) \setminus B(0,1-\zeta)].$$

Thus, the zeros of u_ε are located near 0 and ∂B when $\varepsilon \to 0$.

4. Estimate of the convergence rate

Proposition 3.2 shows a convergence rate of f_ε to $\pi/2$ as $\varepsilon \to 0$

$$\int_K r^{n-1} \cos^2 f_\varepsilon dr \leqslant C\varepsilon^n. \tag{4.1}$$

Obviously, the estimate of Theorem 1.3 is better than (4.1) when $\varepsilon \to 0$. To prove Theorem 1.3, some propositions will be given.

PROPOSITION 4.1. *For any $T \in (0,1/4)$, there is $C > 0$ such that*

$$E_\varepsilon(u_\varepsilon, B(0,1-T) \setminus B(0,T)) \leqslant C\varepsilon^{n-p} + \frac{1}{p}\int_{B(0,1-T)\setminus B(0,T)} |\nabla \frac{x}{|x|}|^p dx.$$

Proof. By the mean value theorem and Proposition 2.4, for any $T \in (0,1/4)$, there exists $T^i \in (0,T)\,(i=1,2)$ such that

$$\frac{1}{\varepsilon^p}[\cos^2 f(T^1)+\cos^2 f(1-T^2)] \leqslant C.$$

Define new function f_3 by the following

$$f_3(r) = f(r) \quad r \in [0,T^1]\cup[1-T^2,1];$$
$$f_3(r) = \frac{\pi}{2}-\frac{1}{\varepsilon}[\frac{\pi}{2}-f(1-T^2)][r-(1-T^2-\varepsilon)] \quad r \in [1-T^2-\varepsilon, 1-T^2];$$
$$f_3(r) = \frac{\pi}{2} \quad r \in [T^1+\varepsilon, 1-T^2-\varepsilon];$$
$$f_3(r) = \frac{\pi}{2}-\frac{1}{\varepsilon}[\frac{\pi}{2}-f(T^1)][(T^1+\varepsilon)-r] \quad r \in [T^1, T^1+\varepsilon].$$

We can obtain

$$E_\varepsilon(u_\varepsilon, B(0,1-T^2)\setminus B(0,T^1)) \leqslant C(\varepsilon+\varepsilon^{n-p})+\frac{(n-1)^{p/2}|S^{n-1}|}{p}\int_{T^1}^{1-T^2} r^{n-1-p}dr. \tag{4.2}$$

Proposition 4.1 is completed if we notice $T \geqslant T^i$.

It follows from $Jensen's$ inequality that

$$E(f_\varepsilon,[T,1-T]) \geqslant \frac{1}{p}\int_T^{1-T}(f_\varepsilon')^p r^{n-1}dr + \frac{1}{2\varepsilon^p}\int_T^{1-T} r^{n-1}\cos^2 f_\varepsilon dr + \frac{1}{p}\int_T^{1-T}\frac{(n-1)^{p/2}}{r^{p-n+1}}\sin^p f_\varepsilon dr.$$

Combining this with (4.1) and (4.2), we obtain

$$\begin{aligned}\frac{1}{p}\int_T^{1-T}(f_\varepsilon')^p r^{n-1}dr &+ \frac{1}{2\varepsilon^p}\int_T^{1-T} r^{n-1}\cos^2 f_\varepsilon dr\\ &\leqslant \frac{1}{p}\int_T^{1-T}\frac{(n-1)^{p/2}}{r^p}(1-\sin^p f_\varepsilon)r^{n-1}dr + C\varepsilon^{n-p}\\ &\leqslant C\varepsilon^n + C\varepsilon^{n-p} \leqslant C\varepsilon^{n-p}.\end{aligned} \tag{4.3}$$

Applying the integral mean value theorem and (4.3), we can find $T_1^i \in (0,T]\,(i=1,2)$ such that

$$\frac{1}{\varepsilon^p}[\cos^2 f_\varepsilon(T_1^1)+\cos^2 f_\varepsilon(1-T_1^2)] \leqslant C\varepsilon^{n-p}. \tag{4.4}$$

Consider the functional

$$E(\rho,[T_1^1,1-T_1^2]) = \frac{1}{p}\int_{T_1^1}^{1-T_1^2}(\rho_r^2+1)^{p/2}dr + \frac{1}{2\varepsilon^p}\int_{T_1^1}^{1-T_1^2}\cos^2\rho dr.$$

Clearly, the minimizer ρ_1 of $E(\rho,[T_1^1,1-T_1^2])$ in $W^{1,p}_{f_\varepsilon}([T_1^1,1-T_1^2],R^+\cup\{0\})$ exists.

PROPOSITION 4.2. *Write* $\rho = \rho_1$ *and* ρ_1 *is the minimizer of* $E(\rho, [T_1^1, 1-T_1^2])$. *Thus*

$$\int_{T_1^1}^{1-T_1^2} (\rho_r^2+1)^{(p-2)/2}\rho_r^2 dr + \frac{1}{\varepsilon^p}\int_{T_1^1}^{1-T_1^2} \cos^2\rho dr \leqslant C\varepsilon^{n/2}.$$

Proof. Step 1. Obviously, the minimizer ρ_1 solves the classical problem

$$-(w^{(p-2)/2}\rho_r)_r = \frac{1}{\varepsilon^p}\cos\rho\sin\rho \quad \text{on } [T_1^1, 1-T_1^2], \tag{4.5}$$

$$\rho(T_1^1) = f_\varepsilon(T_1^1), \quad \rho(1-T_1^2) = f_\varepsilon(1-T_1^2), \tag{4.6}$$

where $w = \rho_r^2 + 1$. Using the maximum value principle, we can see $\rho \leqslant \pi/2$. In view of (2.6) and Proposition 3.4, we can obtain

$$\begin{aligned} &\sin\rho(r) \geqslant \min(\sin\rho(T_1^1), \\ &\sin\rho(1-T_1^2)) = \min(\sin f_\varepsilon(T_1^1), \\ &\sin f_\varepsilon(1-T_1^2)) \geqslant \min(1-\eta, 2M\sqrt{1-M^2}) := C_1 > 0. \end{aligned} \tag{4.7}$$

Since ρ_1 is a minimizer, it is led from Proposition 2.4 that

$$E(\rho_1, [T_1^1, 1-T_1^2]) \leqslant E(f_\varepsilon, [T_1^1, 1-T_1^2]) \leqslant CE_\varepsilon(f_\varepsilon, [T_1^1, 1-T_1^2]) \leqslant C. \tag{4.8}$$

Step 2. Take the function $\zeta \in C^\infty((0,1];[0,1])$ such that:

$$\zeta = 1 \text{ on } (0, T_1^1],\ \zeta = 0 \text{ in } [1-T_1^2, 1],\ |\zeta_r| \leqslant C(T_1^1, T_1^2).$$

Multiplying (4.5) with $\zeta\rho_r$ and integrating over $[T_1^1, 1-T_1^2]$, we have

$$w^{(p-2)/2}\rho_r^2|_{r=T_1^1} + \int_{T_1^1}^{1-T_1^2} w^{(p-2)/2}\rho_r(\zeta_r\rho_r + \zeta\rho_{rr})dr = \frac{1}{\varepsilon^p}\int_{T_1^1}^{1-T_1^2} \zeta\rho_r\cos\rho\sin\rho dr. \tag{4.9}$$

First we will estimate the second term on the left-hand side

$$\begin{aligned} &\left|\int_{T_1^1}^{1-T_1^2} w^{(p-2)/2}\rho_r(\zeta_r\rho_r + \zeta\rho_{rr})dr\right| \\ &\leqslant \int_{T_1^1}^{1-T_1^2} w^{(p-2)/2}|\zeta_r|\rho_r^2 dr + \frac{1}{p}\left|\int_{T_1^1}^{1-T_1^2} [(w^{p/2}\zeta)r - w^{p/2}\zeta_r]dr\right| \\ &\leqslant C\int_{T_1^1}^{1-T_1^2} w^{p/2}dr + \frac{1}{p}w^{p/2}\Big|_{r=T_1^1} \leqslant C + \frac{1}{p}w^{p/2}\Big|_{r=T_1^1}. \end{aligned} \tag{4.10}$$

Next, combining (4.8), (4.6) and (4.4), we derive

$$\begin{aligned}\frac{1}{\varepsilon^p}\Big|\int_{T_1^1}^{1-T_1^2}\zeta\rho_r\cos\rho\sin\rho dr\Big| &\leqslant \frac{1}{\varepsilon^p}\Big|\int_{T_1^1}^{1-T_1^2}\zeta\rho_r\sin\rho dr\Big| \\ &= \frac{1}{\varepsilon^p}\Big|-\int_{T_1^1}^{1-T_1^2}(\zeta\cos\rho)_r dr+\int_{T_1^1}^{1-T_1^2}\zeta_r\cos\rho dr\Big| \\ &\leqslant \frac{1}{\varepsilon^p}\cos\rho\,\Big|_{r=T_1^1}+\frac{c}{\varepsilon^p}\int_{T_1^1}^{1-T_1^2}\cos^2\rho dr\leqslant C.\end{aligned}$$

Substituting this and (4.10) into (4.9) yields

$$w^{(p-2)/2}\rho_r^2\,\Big|_{r=T_1^1}\leqslant C+\frac{1}{p}w^{p/2}\,\Big|_{r=T_1^1}.$$

This result, together with $w^{p/2}=w^{(p-2)/2}(\rho_r^2+1)$, implies

$$w^{p/2}\,\Big|_{r=T_1^1}\leqslant C. \tag{4.11}$$

Step 3. Take $\zeta\in C^\infty((0,1];[0,1])$, $\zeta=0$ on $(0,T_1^1]$, $\zeta=1$ in $[1-T_1^2,1]$ and $|\zeta_r|\leqslant C(T_1^1,T_1^2)$. Similar to the argument of step 2, using (4.4)-(4.8), we also obtain

$$w^{p/2}\,\Big|_{r=1-T_1^2}\leqslant C. \tag{4.12}$$

Step 4. Multiplying (4.5) with $\cos\rho$ and integrating over $[T_1^1,1-T_1^2]$, we have

$$\int_{T_1^1}^{1-T_1^2}w^{(p-2)/2}\rho_r^2\sin\rho dr+\frac{1}{\varepsilon^p}\int_{T_1^1}^{1-T_1^2}\cos^2\rho\sin\rho dr=-\int_{T_1^1}^{1-T_1^2}(w^{(p-2)/2}\rho_r\cos\rho)_r dr.$$

Thus, using (4.11), (4.12) and (4.5)-(4.7), we can deduce that

$$\begin{aligned}&\int_{T_1^1}^{1-T_1^2}w^{\frac{(p-2)}{2}}\rho_r^2 dr+\frac{1}{\varepsilon^p}\int_{T_1^1}^{1-T_1^2}\cos^2\rho dr \\ &\quad\leqslant C\Big(\int_{T_1^1}^{1-T_1^2}w^{\frac{(p-2)}{2}}\rho_r^2\sin\rho dr+\frac{1}{\varepsilon^p}\int_{T_1^1}^{1-T_1^2}\cos^2\rho\sin\rho dr\Big) \\ &\quad\leqslant C\Big|\int_{T_1^1}^{1-T_1^2}(w^{(p-2)/2}\rho_r\cos\rho)_r dr\Big|\leqslant C\Big|(w^{(p-2)/2}\rho_r\cos\rho)\,\Big|_{T_1^1}^{1-T_1^2}\Big|\leqslant C\varepsilon^{n/2}.\end{aligned}$$

Proposition 4.2 is proved.

PROPOSITION 4.3. *There holds*

$$E_\varepsilon(f_\varepsilon,[T_1^1,1-T_1^2])\leqslant C\varepsilon^{n/2}+\frac{(n-1)^{p/2}}{p}\int_{T_1^1}^{1-T_1^2}r^{n-1-p}dr.$$

Proof. Define

$$f_4(r)=\begin{cases} f_\varepsilon, & r\in[0,T_1^1]\cup[1-T_1^2,1];\\ \rho_1, & r\in[T_1^1,1-T_1^2]. \end{cases}$$

Since u_ε is a minimizer, we have

$$E_\varepsilon(u_\varepsilon,B)\leqslant E_\varepsilon\Big(\big(\frac{x}{|x|}\sin f_4(r),\cos f_4(r)\big),B\Big).$$

Thus

$$\begin{aligned} &E_\varepsilon(f_\varepsilon,[T_1^1,1-T_1^2])\\ &\quad\leqslant\frac{1}{p}\int_{T_1^1}^{1-T_1^2}\Big(\rho_r^2+\frac{n-1}{r^2}\sin^2\rho\Big)^{p/2}r^{n-1}dr+\frac{1}{2\varepsilon^p}\int_{T_1^1}^{1-T_1^2}r^{n-1}\cos^2\rho dr. \end{aligned}$$

Combining this with

$$\begin{aligned} &\int_{T_1^1}^{1-T_1^2}\Big[\Big(\rho_r^2+\frac{n-1}{r^2}\sin^2\rho\Big)^{p/2}-\Big(\frac{n-1}{r^2}\sin^2\rho\Big)^{p/2}\Big]r^{n-1}dr\\ &\quad=\frac{p}{2}\int_{T_1^1}^{1-T_1^2}\Big\{\int_0^1\Big[(\rho_r^2+\frac{n-1}{r^2}\sin^2\rho)s+(\frac{n-1}{r^2}\sin^2\rho)(1-s)\Big]^{(p-2)/2}ds\Big\}\rho_r^2r^{n-1}dr\\ &\quad\leqslant C\int_{T_1^1}^{1-T_1^2}(\rho_r^2+1)^{(p-2)/2}\rho_r^2dr\leqslant C\varepsilon^{n/2}, \end{aligned}$$

as well as Proposition 4.2, we get

$$\begin{aligned} E_\varepsilon(f_\varepsilon,[T_1^1,1-T_1^2])&\leqslant\frac{1}{p}\int_{T_1^1}^{1-T_1^2}\Big(\frac{n-1}{r^2}\sin^2\rho\Big)^{p/2}r^{n-1}dr\\ &\quad+C\varepsilon^{n/2}+\frac{1}{2\varepsilon^p}\int_{T_1^1}^{1-T_1^2}r^{n-1}\cos^2\rho\,dr\leqslant C\varepsilon^{n/2}+\frac{(n-1)^{p/2}}{p}\int_{T_1^1}^{1-T_1^2}r^{n-1-p}dr. \end{aligned}$$

Thus we complete Proposition 4.3.

REMARK 4.4. Similar to the derivation of (4.3), using (4.1) and Proposition 4.3, we may get

$$\int_{T_1^1}^{1-T_1^2}(f_\varepsilon')^pr^{n-1}dr+\frac{1}{\varepsilon^p}\int_{T_1^1}^{1-T_1^2}r^{n-1}\cos^2f_\varepsilon dr\leqslant C\varepsilon^{n/2}. \tag{4.13}$$

Comparing this with (4.3), we see that the exponent of ε is improved from $n-p$ to $n/2$. Thus, this estimate is better as $\varepsilon\to0$.

Proof of Theorem 1.3. Similar to the derivation of (4.4), by (4.13) and the mean value theorem, there exists $T_2^i \in (T_1^i, T]\,(i=1,2)$ such that

$$\frac{1}{\varepsilon^p}[\cos^2 f_\varepsilon(T_2^1)+\cos^2 f_\varepsilon(1-T_2^2)] \leqslant C\varepsilon^{n/2}. \tag{4.14}$$

The minimizer ρ_2 of the functional

$$E(\rho,[T_2^1,1-T_2^2]) = \frac{1}{p}\int_{T_2^1}^{1-T_2^2}(\rho_r^2+1)^{p/2}dr + \frac{1}{2\varepsilon^p}\int_{T_2^1}^{1-T_2^2}\cos^2\rho\, dr$$

in $W_{f_\varepsilon}^{1,p}([T_2^1,1-T_2^2],\mathbf{R}^+\cup\{0\})$ also exists. Similar to the proof of Proposition 4.2, using (4.14), we can also derive

$$\int_{T_2^1}^{1-T_2^2}(\rho_r^2+1)^{(p-2)/2}\rho_r^2 dr + \frac{1}{\varepsilon^p}\int_{T_2^1}^{1-T_2^2}\cos^2\rho dr \leqslant C\varepsilon^{G[1]}$$

where $G[j] = \frac{n/2}{2^j}+\frac{(2^j-1)p}{2^j}, j=0,1,2,\cdots$. By an argument of Proposition 4.3, we also obtain

$$E_\varepsilon(f_\varepsilon,[T_2^1,1-T_2^2]) \leqslant C\varepsilon^{G[1]} + \frac{(n-1)^{p/2}}{p}\int_{T_2^1}^{1-T_2^2} r^{n-1-p}dr.$$

Furthermore, similar to the derivation of (4.3), using (4.13), we may get

$$\int_{T_2^1}^{1-T_2^2}(f_\varepsilon')^p r^{n-1}dr + \frac{1}{\varepsilon^p}\int_{T_2^1}^{1-T_2^2} r^{n-1}\cos^2 f_\varepsilon dr \leqslant C\varepsilon^{G[1]}.$$

Comparing with (4.13), we have improved the exponent of ε from $G[0]=n/2$ to $G[1]$.

Proceeding in the way above (its idea is to improve the exponent of ε from $G[k]$ to $G[k+1]$ by induction), we can see that for any $k\in N$

$$\int_{T_{k+1}^1}^{1-T_{k+1}^2}(f_\varepsilon')^p r^{n-1}dr + \frac{1}{\varepsilon^p}\int_{T_{k+1}^1}^{1-T_{k+1}^2} r^{n-1}\cos^2 f_\varepsilon dr \leqslant C\varepsilon^{\frac{n/2}{2^k}+\frac{(2^k-1)p}{2^k}}$$

where $T_{k+1}^i \in (T_k^i, T], i=1,2, k=0,1,2,\cdots$. Letting $k\to\infty$ and noting $T \geqslant T_k^i$, we get our conclusion.

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