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EXISTENCE THEORY FOR QUADRATIC PERTURBATIONS OF ABSTRACT MEASURE INTEGRO–DIFFERENTIAL EQUATIONS

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Abstract. In this paper, an existence theorem for quadratic perturbations of a nonlinear abstract measure integro-differential equation is proved via a nonlinear alternative of Leray-Schauder type. An existence result is also proved for the extremal solutions for Carathéodory as well as discontinuous cases of the nonlinearities involved in the equations.

1. Introduction

In what follows, let X be a real Banach space with a convenient norm $\|\cdot\|$. Let $x, y \in X$. Then the line segment $\overline{xy}$ in X is defined by

$$\overline{xy} = \{z \in X \mid z = x + r(y - x), 0 \leqslant r \leqslant 1\}. \tag{1.1}$$

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$, we define the sets S_x and $\overline{S}_x$ in X by:

$$S_x = \{rx| -\infty < r < 1\} \quad \text{and} \quad \overline{S}_x = \{rx| -\infty < r \leqslant 1\}. \tag{1.2}$$

Let $x_1, x_2 \in \overline{xy}$ be arbitrarily given. We say $x_1 < x_2$ if $S_{x_1} \subset S_{x_2}$, or equivalently $\overline{x_0 x_1} \subset \overline{x_0 x_2}$. In this case we also write $x_2 > x_1$.

Let μ be a σ-finite positive measure on X and let $p \in \mathrm{ca}(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for some $E \in M$. In this case we also write $p << \mu$.

Let $x_0 \in X$ be fixed and let M_0 denote the σ- algebra on S_{x_0}. Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M_0 and the sets of the form $\overline{S}_x$, $x \in \overline{x_0 z}$.

Given a $p \in \mathrm{ca}(X, M)$ with $p << \mu$, we consider the abstract measure integro-differential equation of the form

$$\frac{d}{d\mu}\left(\frac{p(\overline{S}_x)}{f(x, p(\overline{S}_x))}\right) = g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p(\overline{S}_t)\big)\, d\mu\right) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z}. \tag{1.3}$$

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and

$$p(E) = q(E), \quad E \in M_0, \tag{1.4}$$

where q is a given known vector measure,

$$\lambda(\overline{S}_x) = \frac{p(\overline{S}_x)}{f(x, p(\overline{S}_x))}$$

is a signed measure such that $\lambda << \mu$; $d\lambda/d\mu$ is a Radon-Nikodym derivative of λ with respect to μ, $f : S_x \times \mathbb{R} \to \mathbb{R} - \{0\}$, $g : S_z \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the map,

$$x \mapsto g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p(\overline{S}_t)\big)\, d\mu\right)$$

is μ-integrable for each $p \in \mathrm{ca}(X, M_z)$. The details about the Radon-Nikodym derivative are given in Rudin [10].

DEFINITION 1.1. Given an initial real measure q on M_0, a vector measure $p \in \mathrm{ca}(S_z, M_z)$ $(z > x_0)$ is said to be a *solution* of (1.3)-(1.4), if:

(i) $p(E) = q(E)$, $E \in M_0$,
(ii) $p << \mu$ on $\overline{x_0 z}$, and
(iii) p satisfies (1.1) a.e. $[\mu]$ on $\overline{x_0 z}$.

A solution p of (1.3)-(1.4) in $\overline{x_0 z}$ will be denoted by $p(\overline{S}_{x_0}, q)$.

REMARK 1.1. Note that (1.3)-(1.4) is equivalent to the following abstract measure integral equation:

$$p(E) = \big[f(x, p(E))\big]\left(\int_E g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p(\overline{S}_t)\big)\, d\mu\right) d\mu\right), \text{ if } E \in M_z,\ E \subset \overline{x_0 z}, \tag{1.5}$$

and

$$p(E) = q(E) \quad \text{if } E \in M_0. \tag{1.6}$$

As a generalization of ordinary integro-differential equations, there is a series of papers dealing with the abstract measure integro-differential equations in which ordinary derivative is replaced by the derivative of set functions, namely, the Radon-Nikodym derivative of a measure with respect to another measure. See Dhage [2, 3], Dhage and Bellale [6] and the references therein. In the special case, when $f(x,y) = 1$ for all $x \in \overline{x_0 z}$ and $y \in \mathbb{R}$, our (1.3)-(1.4) includes the following abstract measure differential equation considered in Dhage and Bellale [6],

$$\frac{dp}{d\mu} = g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p(\overline{S}_t)\big)\, d\mu\right) \quad \text{a.e. } [\mu] \text{ on } \overline{x_0 z}, \tag{1.7}$$

and

$$p(E) = q(E),\ E \in M_0. \tag{1.8}$$

The above mentioned (1.7)-(1.8) again includes some already known abstract measure differential equations those considered in Sharma [11, 12], Shendge and Joshi [13] and Dhage *et al.* [5] as special cases. Thus, our (1.3)-(1.4) is more general and we claim that results of the present study are new and original contribution to the theory of nonlinear differential equations and include some of the earlier results as special cases. Here, we apply a nonlinear alternative of Leray-Schauder type due to Dhage [4] to (1.3)-(1.4) for proving the existence results under some weaker conditions than that given in Dhage and O'Regan [7], while the existence of extremal solutions is obtained using the fixed point theorems of Dhage [4] in ordered Banach algebras.

2. Auxiliary Results

Let M denote the σ-algebra of all subsets of X such that (X,M) is a measurable space. Let $\mathrm{ca}(X,M)$ be the space of all vector measures (real signed measures) and define a norm $|\cdot|$ on $\mathrm{ca}(X,M)$ by

$$\|p\| = |p|(X), \tag{2.1}$$

where $|p|$ is a total variation measure of p given by

$$|p|(X) = \sup_{\sigma} \sum_{i=1}^{\infty} |p(E_i)|, \; E_i \subset X, \tag{2.2}$$

where supremum is taken over all possible partition $\sigma = \{E_i : i \in \mathbb{N}\}$ of X. It is known that $\mathrm{ca}(X,M)$ is a Banach space with respect to the norm $\|\cdot\|$ given by (2.1). For any non-empty subset S of X, let $L^1_\mu(S,\mathbb{R})$ denote the space of μ-integrable real-valued functions on S which is equipped with the norm $\|\cdot\|_{L^1_\mu}$ given by

$$\|\phi\|_{L^1_\mu} = \int_S |\phi(x)|\, d\mu. \tag{2.3}$$

Let $p_1, p_2 \in \mathrm{ca}(X,M)$ and define a multiplication composition $\circ$ in $\mathrm{ca}(X,M)$ by

$$(p_1 \circ p_2)(E) = p_1(E) p_2(E) \tag{2.4}$$

for all $E \in M$. Then we have:

LEMMA 2.1. *The Banach space* $\mathrm{ca}(X,M)$ *is a Banach algebra with respect to the multiplication* " $\circ$ " *defined by* (2.4) *in it.*

Proof. Let $p_1, p_2 \in \mathrm{ca}(X,M)$ be any two elements. Let $\sigma = \{E_1, \cdots, E_n, \cdots\}$ be a disjoint partition of X. Then by (2.3)-(2.4),

$$\|p_1 p_2\| = |p_1 p_2|(X) = \sup_{\sigma} \sum_{i=1}^{\infty} |(p_1 \circ p_1)(E_i)| = \sup_{\sigma} \sum_{i=1}^{\infty} |p_1(E_1)||p_2(E_1)|$$

$$
\begin{aligned}
&\leqslant \sup_{\sigma}\left(\sum_{i=1}^{\infty}|p_1(E_i)|\right)\left(\sum_{i=1}^{\infty}|p_2(E_i)|\right)\\
&\leqslant \left(\sup_{\sigma}\sum_{i=1}^{\infty}|p_1(E_i)|\right)\left(\sup_{\sigma}\sum_{i=1}^{\infty}|p_2(E_i)|\right)\\
&=|p_1|(X)\,|p_2|(X)=\|p_1\|\|p_2\|.
\end{aligned}
$$

Hence, $\mathrm{ca}(X,M)$ is a Banach algebra and lemma is proved.

The study of hybrid fixed point theorems in Banach algebras is initiated by Dhage [1]. Below we state a hybrid fixed point theorem from Banach algebra which will be used in what follows. Let X be a Banach algebra and let $T : X \to X$. T is called *compact* if $\overline{T(X)}$ is a compact subset of X. T is called *totally bounded* if for any bounded subset S of X, $T(S)$ is a totally bounded subset of X. T is called *completely continuous* if T is continuous and totally bounded on X. Every compact operator is totally bounded, but the converse may not be true, however, two notions are equivalent on a bounded subset of X. The details about the completely continuous operators may be found in Granas and Dugundji [8].

An operator $T : X \to X$ is called *$\mathscr{D}$-Lipschitz* if there exists a continuous and nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|Tx - Ty\| \leqslant \psi(\|x - y\|) \tag{2.5}$$

for all $x, y \in X$, where $\psi(0) = 0$. The function ψ is called a *$\mathscr{D}$-function* of T on X. In particular, if $\psi(r) = \alpha r, \alpha > 0$, T is called a *Lipschitz* with the Lipschitz constant α. Further if $\alpha < 1$, then T is called a *contraction* with contraction constant α. Again if $\psi(r) < r$ for $r > 0$, then T is called a *nonlinear contraction* on X with $\mathscr{D}$-function ψ.

Now we are ready to state a hybrid nonlinear alternative which is useful in the sequel.

THEOREM 2.1. (Dhage [4]) *Let U and $\overline{U}$ denote respectively the open and closed bounded subset of a Banach algebra X such that $0 \in U$. Let $A, B : \overline{U} \to X$ be two operators such that:*

(a) *A is D-Lipschitz,*

(b) *B is completely continuous, and*

(c) *$M\phi(r) < r,\ r > 0$, where $M = \|B(\overline{U})\|$.*

Then either:

(i) *the equation $AxBx = x$ has a solution in $\overline{U}$, or*

(ii) *there is a point $u \in \partial U$ such that $u = \lambda AuBu$ for some $0 < \lambda < l$, where ∂U is a boundary of U in X.*

An interesting corollary to Theorem 2.1 in the applicable form is the following one.

COROLLARY 2.1. *Let $\mathscr{B}_r(0)$ and $\overline{\mathscr{B}}_r(0)$ denote respectively the open and closed balls in a Banach algebra centered at origin 0 of radius r for some real number $r > 0$. Let $A,\ B : \overline{\mathscr{B}}_r(0) \to X$ be two operators such that:*

(a) *A is Lipschitz with Lipschitz constant α,*

(b) *B is compact and continuous, and*

(c) *$\alpha M < 1$, where $M = \|B(\overline{\mathscr{B}}_r(0))\|$.*

Then either:

(i) *the operator equation $AxBx = x$ has a solution x in X with $\|x\| \leqslant r$, or*

(ii) *there is an $u \in X$ with $\|u\| = r$ such that $\lambda AuBu = u$ for some $0 < \lambda < 1$.*

We need a few order theoretic fixed point theorems in what follows. An extensive study of such type of fixed point theorems appears in Heikkilä and Lakshmikantham [9]. We define an order relation $\leqslant$ in $\mathrm{ca}(S_z, M_z)$ with the help of the cone K in $\mathrm{ca}(S_z, M_z)$ given by

$$K = \{p \in \mathrm{ca}(S_z, M_z) \mid p(E) \geqslant 0 \text{ for all } E \in M_z\}. \tag{2.6}$$

Thus for any $p_1, p_2 \in \mathrm{ca}(s_z, M_z)$ we have

$$p_1 \leqslant p_2 \text{ if and only if } p_2 - p_1 \in K \tag{2.7}$$

or, equivalently

$$p_1 \leqslant p_2 \Longleftrightarrow p_1(E) \leqslant p_2(E) \tag{2.8}$$

for all $E \in M_z$.

Obviously the cone K is positive in $\mathrm{ca}(S_z, M_z)$. To see this, let $p_1, p_2 \in K$. Then $p_1(E) \geqslant 0$ and $p_2(E) \geqslant 0$ for all $E \in M_z$. By the multiplication composition,

$$(p_1 \circ p_2)(E) = p_1(E)p_2(E) \geqslant 0$$

for all $E \in M_z$. As a result $p_1 \circ p_2 \in K$, and so K is a positive cone in $\mathrm{ca}(S_z, M_z)$.

The following lemmas follow immediately from the definition of the positive cone K in $\mathrm{ca}(S_z, M_z)$.

LEMMA 2.2. (Dhage [4]) *Let K be a positive cone. If $u_1, u_2, v_1, v_2 \in K$ are such that $u_1 \leqslant v_1$ and $u_2 \leqslant v_2$, then $u_1 u_2 \leqslant v_1 v_2$.*

LEMMA 2.3. *The cone K is normal in* $\mathrm{ca}(S_z, M_z)$.

Proof. To finish, it is enough to prove that the norm $\|\cdot\|$ is semi- monotone on K. Let $p_1, p_2 \in K$ be such that $p_1 \leqslant p_2$ on M_z . Then we have $0 \leqslant p_1(E) \leqslant p_2(E)$ for all $E \in M_z$.

Now for a countable partition $\sigma = \{E_n : n \in \mathbb{N}\}$ of S_z, one has

$$\|p_1\| = |p_1|(S_z) = \sup_{\sigma} \sum_{i=1}^{\infty} |p_1(E_i)| \leqslant \sup_{\sigma} \sum_{i=1}^{\infty} |p_2(E_i)| = |p_2|(S_z) = \|p_2\|.$$

This shows that $\|\cdot\|$ is a semi-monotone on K and consequently the cone K is normal in $\mathrm{ca}(S_z, M_z)$. The proof of the lemma is complete.

An operator $T : X \to X$ is called *positive* if the range $r(T)$ of T is contained in the cone K in X.

THEOREM 2.2. (Dhage [4]) *Let $[u,v]$ be an order interval in the real Banach algebra X and let $A,B : [u,v] \to X$ be positive and nondecreasing operators such that:*

(a) *A is Lipschitz with the Lipschitz constant α,*
(b) *B is compact and continuous, and*
(c) *the elements $u,v \in X$ with $u \leqslant v$ satisfy $u \leqslant AuBu$ and $AvBv \leqslant v$.*

Further, if the cone K is positive and normal, then the operator equation $AxBx = x$ has a least and a greatest positive solution in $[u,v]$, whenever $\alpha M < 1$, where

$$M = \|B([u,v])\| = \sup\{\|Bx\| : x \in [u,v]\}.$$

THEOREM 2.3. (Dhage [4]) *Let K be a positive cone in a real Banach algebra X and let $A,B : K \to K$ be nondecreasing operators such that:*

(a) *A is Lipschitz with the Lipschitz constant α,*
(b) *B is totally bounded, and*
(c) *there exist elements $u,v \in K$ such that $u \leqslant v$ satisfying $u \leqslant AuBu$ and $AvBv \leqslant v$.*

Further, if the cone K is positive and normal, then the operator equation $AxBx = x$ has a least and a greatest positive solution in $[u,v]$, whenever $\alpha M < 1$, where

$$M = \|B([u,v])\| = \sup\{\|Bx\| : x \in [u,v]\}.$$

In the following section we prove our main existence results of this paper.

3. Existence Result

We need the following definition in the sequel.

DEFINITION 3.1. A function $\beta : S_z \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called *Carathéodory* if

(i) $x \to \beta(x, y_1, y_2)$ is μ-measurable for each $y_1, y_2 \in \mathbb{R}$, and
(ii) the function $(y_1, y_2) \mapsto \beta(x, y_1, y_2)$ is continuous almost everywhere $[\mu]$ on $\overline{x_0 z}$.

A Carathéodory function β on $S_z \times \mathbb{R} \times \mathbb{R}$ is called L^1_μ*-Carathéodory* if

(iii) for each real number $r > 0$ there exists a function $h_r \in L^l_\mu(S_z, \mathbb{R}_+)$ such that

$$|\beta(x, y_1, y_2)| \leqslant h_r(x) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z}.$$

for all $y_1, y_2 \in \mathbb{R}$ with $|y_1| \leqslant r$ and $|y_2| \leqslant r$.

A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called *submultiplicative* if $\psi(\lambda r) \leqslant \lambda \psi(r)$ for all real number $\lambda > 0$. Let Ψ denote the class of functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that ψ is continuous, nondecreasing, and submultiplicative.

A member $\psi \in \Psi$ is called a $\mathscr{D}$-function on $\mathbb{R}_+$. There do exist $\mathscr{D}$-functions, in fact, the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\psi(\lambda) = \lambda r$, $\lambda > 0$ is a $\mathscr{D}$-function on $\mathbb{R}_+$.

We consider the following set of assumptions:

(A_0) For any $z > x_0$, the σ-algebra M_z is compact with respect to the topology generated by the pseudo-metric d defined on M_z by

$$D(E_1,E_2) = |\mu|(E_1 \Delta E_2), E_1,E_2 \in M_z.$$

(A_1) The function $x \mapsto |f(x,0)|$ is bounded with $F_0 = \sup_{x \in S_z} |f(x,0)|$.
(A_2) The function f is continuous and there exists a bounded function $\alpha : S_z \to \mathbb{R}^+$ with bound $\|\alpha\|$ such that

$$|f(x,y_1) - f(x,y_2)| \leqslant \alpha(x)|y_1 - y_2| \quad \text{a.e. } [\mu], \quad x \in \overline{x_0 z}$$

for all $y_1,y_2 \in \mathbb{R}$.
(B_0) q is continuous on M_z with respect to the pseudo-metric d defined in (A_0).
(B_1) The function $x \mapsto k(x,p(\overline{S}_x))$ is μ-integrable and there is a function $\gamma \in L^1_\mu(S_z,\mathbb{R}_+)$ satisfying

$$|k(t,y)| \leqslant \gamma(x)|y| \text{ a.e. } [\mu] \text{ on } \overline{x_0 z}$$

for all $y \in \mathbb{R}$.
(B_2) The function $g(x,y_1,y_2)$ is Carathéodory.
(B_3) There exists a function $\phi \in L^1_\mu(S_z,\mathbb{R}_+)$ such that $\phi(x) > 0$ a.e. $[\mu]$ on $\overline{x_0 z}$ and a $\mathscr{D}$-function $\psi : [0,\infty) \to (0,\infty)$ such that

$$|g(x,y_1,y_2)| \leqslant \phi(x)\psi(|y_1| + |y_2|) \quad \text{a.e. } [\mu] \text{ on } \overline{x_0 z}$$

for all $y_1,y_2 \in \mathbb{R}$.

We frequently use the following estimate of the function g in the subsequent part of the paper. If the hypotheses (B_1) and (B_3) hold, then for any $p \in \mathrm{ca}(S_z,M_z)$, one has

$$\begin{aligned}
\left| g\Big(x, p(\overline{S}_x), \int_{\overline{S}_x} k(t,p(\overline{S}_t))\,d\mu\Big) \right| &\leqslant \phi(x)\,\psi\Big(|p(S_x)| + \int_{\overline{S}_x} |k(t,p(\overline{S}_t))|\,d\mu\Big) \\
&\leqslant \phi(x)\,\psi\Big(|p|(S_z) + \int_{\overline{S}_z} \gamma(x)|p(\overline{S}_z)|\,d\mu\Big) \leqslant \phi(x)\,\psi\Big(\|p\| + \int_{\overline{S}_z} \gamma(x)\|p\|\,d\mu\Big) \\
&\leqslant \phi(x)\,\psi\left(\|p\| + \|\gamma\|_{L^1_\mu}\|p\|\right) \leqslant \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p\|).
\end{aligned}$$

Our first existence result is the following.

THEOREM 3.1. *Suppose that the assumptions* (A_0)-(A_2) *and* (B_0)-(B_3) *hold. Suppose that there exists a real number* $r > 0$ *such that*

$$r > \frac{\|q\| + F_0\left[\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(r)\right]}{1-\|\alpha\|\left[\|q\| + \|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(r)\right]}, \tag{3.1}$$

where $\|\alpha\|\left[\|q\| + \|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(r)\right] < 1$. *Then* (1.3)-(1.4) *has a solution defined on* $\overline{x_0 z}$.

Proof. Consider an open ball $\overline{\mathscr{B}}_r(0)$ in $\mathrm{ca}(S_z, M_z)$ centered at the origin 0 and of radius r, where r satisfies the inequalities in (3.1). Define two operators

$$A, B : \overline{\mathscr{B}}_r(0) \to \mathrm{ca}(S_z, M_z)$$

by

$$Ap(E) = \begin{cases} 1, & \text{if } E \in M_0, \\ f(x, p(E)), & \text{if } E \in M_z, E \subset \overline{x_0 z}, \end{cases} \tag{3.2}$$

and

$$Bp(E) = \begin{cases} q(E), & \text{if } E \in M_0, \\ \displaystyle\int_E g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p(\overline{S}_t)\big)\, d\mu\right) d\mu, & \text{if } E \in M_z, E \subset \overline{x_0 z}. \end{cases} \tag{3.3}$$

We shall show that the operators A and B satisfy all the conditions of Corollary 2.1 on $\overline{\mathscr{B}}_r(0)$.

Step I. First, we show that A is a Lipschitz on $\overline{\mathscr{B}}_r(0)$. Let $p_1, p_2 \in \overline{\mathscr{B}}_r(0)$ be arbitrary. Then by assumption (A_2),

$$\begin{aligned} |Ap_1(E) - Ap_2(E)| &= |f(x, p_1(E)) - f(x, p_2(E))| \\ &\leqslant \alpha(x)|p_1(E) - p_2(E)| \\ &\leqslant \|\alpha\||p_1 - p_2|(E) \end{aligned}$$

for all $E \in M_z$. Hence by definition of the norm in $\mathrm{ca}(S_z, M_z)$ one has

$$\|Ap_1 - Ap_2\| \leqslant \|\alpha\|\|p_1 - p_2\|$$

for all $p_1, p_2 \in \mathrm{ca}(S_z, M_z)$. As a result, we have that A is a Lipschitz operator on $\overline{\mathscr{B}}_r(0)$ with the Lipschitz constant $\|\alpha\|$.

Step II. We show that B is continuous on $\overline{\mathscr{B}}_r(0)$. Let $\{p_n\}$ be a sequence of vector measures in $\overline{\mathscr{B}}_r(0)$ converging to a vector measure p. Then by dominated convergence theorem,

$$\lim_{n\to\infty} \overline{B}p_n(E) = \lim_{n\to\infty} \int_E g\left(x, p_n(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p_n(\overline{S}_t)\big)\, d\mu\right) d\mu$$

$$= \int_E g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p(\overline{S}_t)\big)\, d\mu\right) d\mu = \overline{B}p(E),$$

for all $E \in M_z$, $E \subset \overline{x_0 z}$. Similarly, if $E \in M_0$, then

$$\lim_{n\to\infty} \overline{B}p_n(E) = q(E) = Bp(E),$$

and so B is a continuous operator on $\overline{\mathscr{B}}_r(0)$.

Step III. Next, we show that B is a totally bounded operator on $\overline{\mathscr{B}}_r(0)$. Let $\{p_n\}$ be a sequence of vector measures in $\overline{\mathscr{B}}_r(0)$. Then we have $\|p_n\| \leqslant r$ for all $n \in \mathbb{N}$. We shall show that the set $\{Bp_n : n \in \mathbb{N}\}$ is uniformly bounded and equi-continuous set in $\mathrm{ca}(S_z, M_z)$. In this step, we first show that $\{Bp_n\}$ is uniformly bounded.

Let $E \in M_z$. Then there exists two subsets $F \in M_0$ and $G \in M_z$, $G \subset \overline{x_0 z}$ such that $E = F \cup G$ and $F \cap G = \emptyset$. Hence by definition of B,

$$\begin{aligned}
|Bp_n(E)| &\leqslant |q(F)| + \int_G \left| g\left(x, p_n(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p_n(\overline{S}_t)\big)\, d\mu\right)\right| d\mu \\
&\leqslant \|q\| + \int_G \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p_n\|)\, d\mu \\
&\leqslant \|q\| + \int_E \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p_n\|)\, d\mu \\
&= \|q\| + \|\phi\|_{L^l_\mu}\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p_n\|)
\end{aligned}$$

for all $E \in M_z$.

From (2.1) and the above inequality, it follows that

$$\begin{aligned}
\|Bp_n\| = |Bp_n|(S_z) = \sup_\sigma \sum_{i=1}^\infty |Bp_n(E_i)| \\
&= \|q\| + \|\phi\|_{L^l_\mu}\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p_n\|) \\
&\leqslant \|q\| + \|\phi\|_{L^l_\mu}\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(r)
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence the sequence $\{Bp_n\}$ is uniformly bounded in $B(\overline{\mathscr{B}}_r(0))$.

Step IV. Next we show that $\{Bp_n : n \in \mathbb{N}\}$ is a equi- continuous set in $\mathrm{ca}(\mathrm{S_z}, \mathrm{M_z})$. Let $E_1, E_2 \in M_z$. Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_z$, $G_1 \subset \overline{x_0 z}$, $G_2 \subset \overline{x_0 z}$ such that

$$E_1 = F_1 \cup G_1 \text{ with } F_1 \cap G_1 = \emptyset$$

and

$$E_2 = F_2 \cup G_2 \text{ with } F_2 \cap G_2 = \emptyset.$$

We know the identities

$$\left.\begin{aligned}
G_1 &= (G_1 - G_2) \cup (G_2 \cap G_1), \\
G_2 &= (G_2 - G_1) \cup (G_1 \cap G_2).
\end{aligned}\right\} \tag{3.4}$$

Therefore, we have

$$Bp_n(E_1) - Bp_n(E_2) \leqslant q(F_1) - q(F_2) + \int\limits_{G_1 - G_2} g\left(x, p_n(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p_n(\overline{S}_t)\big)\, d\mu\right) d\mu$$
$$- \int\limits_{G_2 - G_1} g\left(x, p_n(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p_n(\overline{S}_t)\big)\, d\mu\right) d\mu.$$

Since g is Carathéodory and satisfies (B_3), we have that

$$\begin{aligned}|Bp_n(E_1) - Bp_n(E_2)| &\leqslant |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} \left| g\left(x, p_n(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p_n(\overline{S}_t)\big)\, d\mu\right) \right| d\mu \\ &\leqslant |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right) \psi(\|p_n\|)\, d\mu.\end{aligned}$$

Assume that

$$d(E_1, E_2) = |\mu|(E_1 \Delta E_2) \to 0.$$

Then we have $E_1 \to E_2$. As a result $F_1 \to F_2$ and $|\mu|(G_1 \Delta G_2) \to 0$. As q is continuous on compact M_z, it is uniformly continuous and so, as $E_1 \to E_2$,

$$\begin{aligned}&|Bp_n(E_1) - Bp_n(E_2)| \\ &\qquad \leqslant |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right) \psi(\|p_n\|)\, d\mu \to 0\end{aligned}$$

uniformly for all $E_1, E_2 \in M_z$ and $n \in \mathbb{N}$. This shows that $\{Bp_n : n \in \mathbb{N}\}$ is an equi-continuous set in $\mathrm{ca}(S_z, M_z)$. Now an application of the Arzela-Ascoli theorem yields that B is a totally bounded operator on $\overline{\mathscr{B}}_r(0)$. Now B is continuous and totally bounded operator on $\overline{\mathscr{B}}_r(0)$, so it is completely continuous operator on $\overline{\mathscr{B}}_r(0)$.

Step V. Finally, we show that hypothesis (c) of Corollary 2.1 is satisfied. The Lipschitz constant of A is $\|\alpha\|$. Here, the number M in the hypothesis (c) is given by

$$M = \|B(\overline{\mathscr{B}}_r(0)\| = \sup\{\|Bp\| : p \in \overline{\mathscr{B}}_r(0)\} = \sup\{|Bp|(S_z) : p \in \overline{\mathscr{B}}_r(0)\}. \tag{3.5}$$

Now let $E \in M_z$. Then, there are sets $F \in M_0$ and $G \in M_z,\ G \subset \overline{x_0 z}$ such that

$$E = F \cup G \quad \text{and} \quad F \cap G = \emptyset.$$

From the definition of B it follows that

$$Bp(E) = q(F) + \int_G g\Big(x, p(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p(\overline{S}_t)\big)\, d\mu\Big)\, d\mu.$$

Therefore,

$$|Bp(E)| \leqslant |q(F)| + \int_G \left| g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p(\overline{S}_t)\big)\, d\mu\right) \right| d\mu$$

$$
\begin{aligned}
&\leqslant \|q\| + \int_G \phi(x)\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|p\|)\,d\mu \\
&\leqslant \|q\| + \int_{\overline{x_0 z}} \phi(x)\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|p\|)\,d\mu \\
&= \|q\| + \|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|p\|).
\end{aligned}
$$

Hence, from (2.1) it follows that

$$\|Bp\| \leqslant \|q\| + \|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|p\|)$$

for all $p \in \overline{\mathscr{B}}_r(0)$. As a result, we have

$$M = \|B(\overline{\mathscr{B}}_r(0))\| \leqslant \|q\| + \|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|p\|).$$

Now

$$\alpha M \leqslant \|\alpha\|\left[\|q\| + \|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(r)\right] < 1$$

and so, hypothesis (c) of Corollary 2.1 is satisfied.

Now an application of Corollary 2.1 yields that either the operator $AxBx = x$ has a solution, or there is a $u \in \mathrm{ca}(\mathrm{S_z}, \mathrm{M_z})$ such that $\|u\| = r$ satisfying $u = \lambda AxBx$ for some $0 < \lambda < 1$. We show that this latter assertion does not hold. Assume the contrary. Then we have

$$u(E) = \begin{cases} \lambda\left[f(x,u(E))\right]\left(\int_E g\left(x, u(\overline{S}_x), \int_{\overline{S}_x} k\left(t, u(\overline{S}_t)\right)d\mu\right)d\mu\right), \text{ if } E \in M_z,\ E \subset \overline{x_0 z}, \\ \lambda q(E), \text{ if } E \in M_0 \end{cases}$$

for some $0 < \lambda < 1$.

If $E \in M_z$, then there sets $F \in M_0$ and $G \in \overline{M_z},\ G \subset \overline{x_0 z}$ such that $E = F \cup G$ and $F \cap G = \emptyset$. Then, we have

$$
\begin{aligned}
u(E) &= \lambda Au(E)\,Bu(E) \\
&= \lambda q(F) + \lambda\left[f(x,u(G))\right]\left(\int_G g\left(x, u(\overline{S}_x), \int_{\overline{S}_x} k\left(t, u(\overline{S}_t)\right)d\mu\right)d\mu\right) \\
&= \lambda q(F) + \lambda\left[f(x,u(G)) - f(x,0)\right]\left(\int_G g\left(x, u(\overline{S}_x), \int_{\overline{S}_x} k\left(t, u(\overline{S}_t)\right)d\mu\right)d\mu\right) \\
&\qquad + \lambda f(x,0)\left(\int_G g\left(x, u(\overline{S}_x), \int_{\overline{S}_x} k\left(t, u(\overline{S}_t)\right)d\mu\right)d\mu\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|u(E)| \leqslant \lambda|q(F)| &+ \lambda\left(|f(x,u(G)) - f(x,0)|\right)\left(\int_G \left|g\left(x, u(\overline{S}_x), \int_{\overline{S}_x} k\left(t, u(\overline{S}_t)\right)d\mu\right)\right|d\mu\right) \\
&+ |f(x,0)|\left(\int_G \left|g\left(x, u(\overline{S}_x), \int_{\overline{S}_x} k\left(t, u(\overline{S}_t)\right)d\mu\right)\right|d\mu\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \|q\| + \lambda\left[\alpha(x)|u(G)| + F_0\right]\left(\int_G \phi(x)\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|u\|)\,d\mu\right)\\
&\leqslant \|q\| + \left[\|\alpha\|\,|u|(E)| + F_0\right]\left(\int_{\overline{x_0z}} \phi(x)\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|u\|)\,d\mu\right)\\
&\leqslant \|q\| + \left[\|\alpha\|\|u\| + F_0\right]\left[\|\phi\|_{L^1_\mu}(1+\|\gamma\|_{L^1_\mu})\psi(\|u\|)\right]
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
\|u\| &\leqslant \|q\| + \left(\|\alpha\|\|u\|\left[\|\phi\|_{L^1_\mu}(1+\|\gamma\|_{L^1_\mu})\psi(\|u\|)\right]\right)\\
&\qquad + F_0\left[\|\phi\|_{L^1_\mu}(1+\|\gamma\|_{L^1_\mu})\psi(\|u\|)\right]\\
&\leqslant \frac{\|q\| + F_0\left[\|\phi\|_{L^1_\mu}(1+\|\gamma\|_{L^1_\mu})\psi(\|u\|)\right]}{1-\|\alpha\|\left[\|\phi\|_{L^1_\mu}(1+\|\gamma\|_{L^1_\mu})\psi(\|u\|)\right]}\\
&\leqslant \frac{\|q\| + F_0\left[\|\phi\|_{L^1_\mu}(1+\|\gamma\|_{L^1_\mu})\psi(\|u\|)\right]}{1-\|\alpha\|\left[\|q\| + \|\phi\|_{L^1_\mu}(1+\|\gamma\|_{L^1_\mu})\psi(\|u\|)\right]}.
\end{aligned}
$$

Substituting $\|u\| = r$ in the above inequality yields

$$
r \leqslant \frac{\|q\| + F_0\left[\|\phi\|_{L^1_\mu}(1+\|\gamma\|_{L^1_\mu})\psi(r)\right]}{1-\|\alpha\|\left[\|q\| + \|\phi\|_{L^1_\mu}(1+\|\gamma\|_{L^1_\mu})\psi(r)\right]} \tag{3.6}
$$

which is a contraction to the first inequality in (3.1). In consequence, the operator equation $p(E) = Ap(E)Bp(E)$ has a solution $u(\overline{S}_{x_0}, q)$ in $\mathrm{ca}(S_z, M_z)$ with $\|u\| \leqslant r$. This further implies that (1.3)-(1.4) has a solution on $\overline{x_0z}$. This completes the proof.

EXAMPLE 3.1. Given $p \in \mathrm{ca}(S_z, M_z)$ with $p << \mu$, consider the equation,

$$
\frac{d}{d\mu}\left(\frac{p(\overline{S}_x)}{1+|p(\overline{S}_x)|}\right) = \frac{\phi(x)p(\overline{S}_x)}{1+p^2(\overline{S}_x)} \text{ a.e. } [\mu] \text{ on } \overline{x_0z}, \tag{3.7}
$$

$$
p(\overline{S}_{x_0}) = q \in \mathbb{R}, \tag{3.8}
$$

where $\phi : \overline{x_0z} \to \mathbb{R}^+$ is μ-integrable. Define the functions $f : S_z \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $g : S_z \times \mathbb{R} \to \mathbb{R}$ by $f(x,y) = 1+|y|$ and

$$
g(x,y) = \frac{\phi(x)y}{1+y^2}
$$

respectively. Below we shall show that the functions f and g satisfy all the conditions of Theorem 3.1. Obviously f is continuous on the domain of its definition. Let $y_1, y_2 \in \mathbb{R}$. Then we have

$$
|f(x,y_1) - f(x,y_2)| = |1+|y_1| - 1 - |y_2|| = ||y_1| - |y_2|| \leqslant |y_1 - y_2|q,
$$

which shows that $f(x,y)$ satisfies the Lipschitz condition in y with the Lipschitz constant $\alpha = 1$. Obviously the function $g(x,y)$ is Carathéodory on $\overline{x_0 z}$. To see this, note that the function $x \to \phi(x)y/(1+y^2)$ is obviously μ-measurable for all $y \in \mathbb{R}$ and the function $y \to \phi(x)y/(1+y^2)$ is continuous for all $x \in \overline{x_0 z}$. Again,

$$g(x,y_1,y_2) = g(x,y_1) = \left| \frac{\phi(x)y_1}{1+y_1^2} \right| \leqslant |\phi(x)| = \phi(x)\psi(|y_1|),$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $\psi(r) = 1$.

Thus, if $\|q\| + \|\phi\|_{L^1} < 1$, then all the assumptions (A_0)-(A_2) and (B_0)-(B_3) of Theorem 3.1 are satisfied. Hence, (3.7) has a solution $p(\overline{S}_{x_0}, q)$ defined on $\overline{x_0 z}$.

4. Existence of Extremal Solutions

In this section, we shall prove the existence of a minimal and a maximal solutions for (1.3)-(1.4) on $\overline{x_0 z}$ between the given upper and lower solutions under Carathédory as well as discontinuous case of nonlinearity g involved in the equation.

4.1. Carathédory case

We need the following definitions in the sequel.

DEFINITION 4.1. A vector measure $u \in \mathrm{ca}(S_z, M_z)$ is called a *lower solution* of (1.3)-(1.4) if

$$\frac{d}{d\mu}\left(\frac{u(\overline{S}_x)}{f(x,u(\overline{S}_x))}\right) \leqslant g\Big(x, u(\overline{S}_x), \int_{\overline{S}_x} k\big(t, u(\overline{S}_t)\big)\, d\mu\Big) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z}$$

and

$$u(E) \leqslant q(E), \quad E \in M_0.$$

Similarly, a vector measure $v \in \mathrm{ca}(S_z, M_z)$ is called an *upper solution* to (1.3)-(1.4) if

$$\frac{d}{d\mu}\left(\frac{v(\overline{S}_x)}{f(x,v(\overline{S}_x))}\right) \geqslant g\Big(x, v(\overline{S}_x), \int_{\overline{S}_x} k\big(t, v(\overline{S}_t)\big)\, d\mu\Big) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z}$$

and

$$u(E) \geqslant q(E), \quad E \in M_0.$$

A vector measure $p \in \mathrm{ca}(S_z, M_z)$ is a solution to (1.3)-(1.4) if is upper as well as lower solution to (1.3)-(1.4) on $\overline{x_0 z}$.

DEFINITION 4.2. A solution p_M is called a *maximal solution* to (1.3)-(1.4) if for any other solution $p(\overline{S}_{x_0}, q)$ for the (1.3)-(1.4) we have that

$$p(E) \leqslant p_M(E) \quad \forall E \in M_z.$$

Similarly, a *minimal solution* $p_m(\overline{S}_{x_0}, q)$ of (1.3)-(1.4) is defined on $\overline{x_0 z}$.

We consider the following assumptions:

(C_0) f and g define the functions $f:\overline{x_0z}\times\mathbb{R}\to\mathbb{R}^+-\{0\}$ and $g:\overline{x_0z}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$.

(C_1) The functions $f(x,y_1)$, $k(x,y_1)$ and $g(x,y_1,y_2)$ are nondecreasing in y_1,y_2 for each $x\in\overline{x_0z}$.

(C_2) The equation (1.1)-(1.2) has a lower solution u and an upper solution v such that $u\leqslant v$ on M_z.

(C_3) The function $g(x,y_1,y_2)$ is L^1_μ-Carathéodory.

THEOREM 4.1. *Suppose that the assumptions* (A_0)-(A_2), (B_0)-(B_2) *and* (C_0)-(C_3) *hold. Further, suppose that*

$$\|\alpha\|\left(\|q\|+\|h_r\|_{L^l_\mu}\right)<1, \tag{4.1}$$

where $r=\|u\|+\|v\|$ *and the function* h_r *is defined in hypothesis* (C_3). *Then* (1.3)-(1.4) *has a minimal and a maximal solution defined on* $\overline{x_0z}$.

Proof. The equation (1.1)-(1.2) is equivalent to the abstract measure integral equation (1.3) and (1.4). Define the operators $A,B:\mathrm{ca}(S_z,M_z)\to\mathrm{ca}(S_z,M_z)$ by (3.2) and (3.3) respectively. Then (1.3)-(1.4) is equivalent to the operator equation

$$p(E)=Ap(E)Bp(E),\quad E\in M_z. \tag{4.2}$$

We shall show that the operators A and B satisfy all the conditions of Theorem 2.2 on $\mathrm{ca}(S_z,M_z)$. Since μ is a positive measure, from assumption (C_0) it follows that A and B are positive operators on $\mathrm{ca}(S_z,M_z)$. We show that they are also nondecreasing on $\mathrm{ca}(S_z,M_z)$. To show this, let $p_1,p_2\in\mathrm{ca}(S_z,M_z)$ be such that $p_1\leqslant p_2$ on M_z. From (C_2) it follows that

$$Ap_1(E)=f(x,p_1(E))\leqslant f(x,p_2(E))=Ap_2(E)$$

for all $E\in M_z,\ E\subset\overline{x_0z}$ and

$$Ap_1(E)=1=Ap_2(E)$$

for $E\in M_0$. Hence A is nondecreasing on $\mathrm{ca}(S_z,M_z)$.

Similarly, we have

$$\begin{aligned}Bp_1(E)&=\int_E g\Big(x,p_1(\overline{S}_x),\int_{\overline{S}_x}k\big(t,p_1(\overline{S}_t)\big)\,d\mu\Big)\,d\mu\\&\leqslant\int_E g\Big(x,p_2(\overline{S}_x),\int_{\overline{S}_x}k\big(t,p_2(\overline{S}_t)\big)\,d\mu\Big)\,d\mu=Bp_2(E)\end{aligned}$$

for all $E\in M_z,\ E\subset\overline{x_0z}$. Again if $E\in M_0$, then

$$Bp_1(E)=q(E)=Bp_2(E).$$

Therefore, the operator B is also nondecreasing on $\mathrm{ca}(S_z,M_z)$. Now it can be shown that as in the proof of Theorem 3.1 that A is a Lipschitz operator on $[u,v]$ with the Lipschitz constant $\|\alpha\|$. Since the cone K is normal in X, the order interval $[u,v]$ is norm-bounded. Hence, there is a real number $r>0$ such that $\|x\|\leqslant\|u\|+\|v\|=r$ for all $x\in[u,v]$. As g is L^1_μ-Carathéodory, there is a function $h_r: L^1_\mu(S_z,\mathbb{R}^+)$ such that $|g(x,y_1,y_2)|\leqslant h_r(x)$ on $\overline{x_0z}$ for all $y_1,y_2\in\mathbb{R}$. Now proceeding with the arguments as in the proof of Theorem 4.1 with $\overline{\mathscr{B}}_r(0)=[u,v]$, $\gamma(x)=h_r(x)$ and $\psi(r)=1$, it can be proved that B is compact and continuous operator on $[u,v]$. Since u is a lower solution of (1.3)-(1.4) we have

$$u(E)\leqslant[f(x,u(E))]\left(\int_E g\Big(x,u(\overline{S}_x),\int_{\overline{S}_x}k\big(t,u(\overline{S}_t)\big)\,d\mu\Big)\,d\mu\right),\quad E\in M_z,\ E\subset\overline{x_0z}$$

and

$$u(E)\leqslant q(E),\quad \text{if } E\in M_0.$$

From the above inequalities, it follows that

$$u(E)\leqslant Au(E)\,Bu(E),\quad \text{if } E\in M_z$$

and so $u\leqslant AuBu$. Similarly, since $v\in\mathrm{ca}(S_z,M_z)$ is an upper solution of (1.3)-(1.4) , it can be proved that $Av(E)Bv(E)\leqslant v(E)$ for all $E\in M_z$ and consequently $AvBv\leqslant v$ on M_z. Thus, hypothesis (a)-(c) of Theorem 2.2 are satisfied. Now, from definition of the norm, it follows that

$$\begin{aligned}M&=\|B([u,v])\|=\sup\{\|Bp\|:p\in[u,v]\}\\&=\sup\{|Bp|(S_z):p\in[u,v]\}=\sup_{p\in[u,v]}\left\{\sup_{\sigma}\sum_{i=1}^{\infty}|B|p(E_i)\right\},\end{aligned}$$

for any partition $\sigma=\{E_i:i\in\mathbb{N}\}$ of S_z such that $S_z=\cup_{i=1}^{\infty}E_i,\ E_i\cap E_j=\emptyset\ \forall i,j\in\mathbb{N}$.

Let $E\in M_z$, $E\subset\overline{x_0z}$. Then, for any $p\in[u,v]$, one has

$$\begin{aligned}|Bp|(E)&\leqslant\sup_{\sigma}\sum_{i=1}^{\infty}\int_{E_i}\Big|g\Big(x,v(\overline{S}_x),\int_{\overline{S}_x}k\big(t,v(\overline{S}_t)\big)\,d\mu\Big)\Big|\,d\mu\\&\leqslant\sup_{\sigma}\sum_{i=1}^{\infty}\int_{E_i}h_r(x)\,d\mu=\int_E h_r(u)\,d\mu=\|h_r\|_{L^l_\mu}.\end{aligned}$$

Therefore, for any $E\in M_z$, there are sets $F\in M_0$ and $G\subset\overline{x_0z}$ such that $E=F\cup G$, $F\cap G=\emptyset$. Hence, we obtain

$$M=\|B([u,v])\|\leqslant\|q\|+\|h_r\|_{L^l_\mu}.$$

Notice that $\alpha M\leqslant\|\alpha\|(\|q\|+\|h_r\|_{L^l_\mu})<1$. Thus, the operators A and B satisfy all the conditions of Theorem 2.2 and so an application of it yields that the operator equation $ApBp=p$ has a maximal and a minimal solution in $[u,v]$. This further implies that (1.3)-(1.4) has a maximal and a minimal solution on $\overline{x_0z}$. This completes the proof.

4.2. Discontinuous case

Next, we obtain an existence result for extremal solutions for (1.3)-(1.4) when the nonlinearity g is a discontinuous function in all its three variables. We consider the following assumption:

(C_4) the function $h : \overline{x_0 z} \to \mathbb{R}^+$ defined by

$$h(x) = g\Big(x, v(\overline{S}_x), \int_{\overline{S}_x} k\big(t, v(\overline{S}_t)\big)\, d\mu\Big)$$

is μ-integrable for every upper solution v of (1.3)-(1.4) on $\overline{x_0 z}$.

REMARK 4.1. Assume that the hypotheses (C_2) and (C_4) hold. Then,

$$\Big|g\Big(x, p(\overline{S}_x), \int_{\overline{S}_x} k\big(t, p(\overline{S}_t)\big)\, d\mu\Big)\Big| \leqslant h(x)$$

for all $p \in [u, v]$.

THEOREM 4.2. *Suppose that the assumptions* (A_0)-(A_2), (B_0)-(B_2) *and* (C_0)-(C_2), (C_4) *hold. Further, suppose that*

$$\|\alpha\|\left(\|q\| + \|h\|_{L^1_\mu}\right) < 1. \tag{4.3}$$

Then (1.3)-(1.4) *has a minimal and a maximal solution defined on* $\overline{x_0 z}$.

Proof. The proof is similar to Theorem 4.2 with appropriate modifications. Here, the function h plays the role of h_r while showing the totally boundedness of the operator B on $[u, v]$. Now the desired conclusion follows by an application of Theorem 2.3.

Notice that we do not need any type of continuity of the nonlinear function g in above Theorem 4.2 for guaranteeing the existence of extremal solutions for (1.3)-(1.4) on $\overline{x_0 z}$, instead we assumed the monotonicity condition on it.

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