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EXISTENCE RESULTS FOR A SECOND ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATION WITH STATE–DEPENDENT DELAY

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Abstract. In this paper, we study existence of mild solutions for a second order impulsive neutral functional differential equations with state-dependent delay. By using a fixed point theorem for condensing maps combined with theories of a strongly continuous cosine family of bounded linear operators, we prove the main existence theorems. As applications of these obtained results, some practical consequences are derived for the sub-linear growth cases. And an example is also given to illustrate our main results.

1. Introduction

This paper is mainly concerned with the existence of mild solutions for a second order impulsive neutral functional differential equation with state-dependent delay such as

$$\frac{d}{dt}[x'(t)-g(t,x_t)] = Ax(t)+f(t,x_{\rho(t,x_t)}), \ \ t\in I=[0,a], \tag{1.1}$$

$$x_0 = \varphi\in\mathscr{B}, \quad x'(0)=\zeta\in X, \tag{1.2}$$

$$\Delta x(t_i) = I_i(x_{t_i}), \quad i=1,2,\ldots,n, \tag{1.3}$$

$$\Delta x'(t_i) = J_i(x_{t_i}), \quad i=1,2,\ldots,n, \tag{1.4}$$

where A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operator $(C(t))_{t\in\mathbb{R}}$ defined on a Banach space X; the function $x_s : (-\infty,0]\to X$, $x_s(\theta)=x(s+\theta)$, belongs to some abstract phase space $\mathscr{B}$ described axiomatically; $0<t_1<\cdots<t_n<a$ are prefixed numbers; $f,g : I\times\mathscr{B}\to X$, $\rho : I\times\mathscr{B}\to(-\infty,a]$, $I_i(\cdot):\mathscr{B}\to X$, $J_i(\cdot):\mathscr{B}\to X$ are appropriate functions and the symbol $\Delta\xi(t)$ represents the jump of the function $\xi(\cdot)$ at t, which is defined by $\Delta\xi(t)=\xi(t^+)-\xi(t^-)$.

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The theory of impulsive differential equations has become an important area of investigation in recent years stimulated by their numerous applications to problems from mechanics, electrical engineering, medicine, biology, ecology, etc. For more details on impulsive differential equations and on its applications, we refer the reader to [3, 5, 29] and the references therein. Ordinary differential equations of first and second order with impulses have been treated in several works, see for instance [6, 13]. Abstract partial differential equations with impulses have been studied by Liu [31], Rogovchenko [34, 35], Chang et al. [8, 9, 10, 11], and Hernández et al. [18, 19, 23].

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years. The literature devoted to this subject is concerned fundamentally with first order functional differential equations for which the state belong to some finite dimensional space, see among another works [1, 4, 7, 12, 15, 16, 17].

The problem of the existence of solutions for first and second order partial functional differential with state-dependent delay have treated recently in [2, 20, 21, 22, 24, 25, 26, 30, 32, 33]. The literature relative second order impulsive differential system with state-dependent delay is very restrict, and related this matter we only cite [37] for ordinary differential system and [24] for abstract partial differential systems. To the best of our knowledge, the study of the existence of solutions for abstract impulsive second order neutral functional differential equations with state-dependent delay is an untreated topic in the literature and this fact, is the main motivation of the present work.

2. Preliminaries

In this section, we mention a few results, notations and lemmas needed to establish our main results.

Throughout this paper, A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on Banach space $(X, \|\cdot\|)$. We denote by $(S(t))_{t\in\mathbb{R}}$ the sine function associated with $(C(t))_{t\in\mathbb{R}}$ which is defined by $S(t)x = \int_0^t C(s)x ds$, for $x \in X$ and $t \in \mathbb{R}$.

The notation $[D(A)]$ stands for the domain of the operator A endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$. Moreover, in this work, E is the space formed by the vectors $x \in X$ for which $C(\cdot)x$ is of class C^1 on $\mathbb{R}$. It was proved by Kisinsky [28] that E endowed with the norm

$$\|x\|_E = \|x\| + \sup_{0\leqslant t\leqslant 1} \|AS(t)x\|, \quad x \in E,$$

is a Banach space. The operator valued function

$$G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$$

is a strongly continuous group of bounded linear operators on the space $E \times X$ generated by the operator $\mathscr{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. It follows from this that $AS(t)$:

$E \to X$ is a bounded linear operator and that $AS(t)x \to 0,\ t \to 0,$ for each $x \in E$. Furthermore, if $x : [0,\infty) \to X$ is a locally integrable function, then $z(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function. This is a consequence of the fact that

$$\int_0^t G(t-s)\begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \left[\int_0^t S(t-s)x(s)\,ds,\ \int_0^t C(t-s)x(s)\,ds\right]^T$$

defines an $E \times X$-valued continuous function.

The existence of solutions for the second order abstract Cauchy problem

$$\begin{cases} x''(t) = Ax(t) + h(t), & 0 \leqslant t \leqslant a, \\ x(0) = z, \ x'(0) = w, \end{cases} \tag{2.1}$$

where $h : I \to X$ is an integrable function has been discussed in [38]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [39]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)z + S(t)w + \int_0^t S(t-s)h(s)ds, \quad 0 \leqslant t \leqslant a, \tag{2.2}$$

is called mild solution of (2.1) and that when $z \in E,\ x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)z + C(t)w + \int_0^t C(t-s)h(s)ds, \quad 0 \leqslant t \leqslant a. \tag{2.3}$$

For additional details about cosine function theory, we refer to the reader to [38, 39].

To consider the impulsive conditions (1.3)-(1.4), it is convenient to introduce some additional concepts and notations.

A function $u : [\sigma,\tau] \to X$ is said to be a normalized piecewise continuous function on $[\sigma,\tau]$ if u is piecewise continuous and left continuous on $(\sigma,\tau]$. We denote by $\mathscr{PC}([\sigma,\tau],X)$ the space of normalized piecewise continuous functions from $[\sigma,\tau]$ into X. In particular, we introduce the space $\mathscr{PC}$ formed by all normalized piecewise continuous functions $u : [0,a] \to X$ such that u is continuous at $t \neq t_i$, $i = 1,\dots,n$. It is clear that $\mathscr{PC}$ endowed with the norm $\| u \|_{\mathscr{P}C} = \sup_{s\in I} \| u(s) \|$ is a Banach space.

In what follows, we set $t_0 = 0$, $t_{n+1} = a$, and for $u \in \mathscr{PC}$ we denote by $\tilde{u}_i$, for $i = 0,1,...,n-1$, the function $\tilde{u}_i \in C([t_i,t_{i+1}];X)$ given by $\widetilde{u}_i(t) = u(t)$ for $t \in (t_i,t_{i+1}]$ and $\widetilde{u}_i(t_i) = \lim_{t\to t_i^+} u(t)$. Moreover, for a set $B \subseteq \mathscr{PC}$, we denote by $\widetilde{B}_i$, for $i = 0,1,...,n-1$, the set $\widetilde{B}_i = \{\tilde{u}_i : u \in B\}$.

LEMMA 2.1. [21] *A set $B \subseteq \mathscr{PC}$ is relatively compact in $\mathscr{PC}$ if, and only if, each set $\widetilde{B}_i$, $i = 0,1,\cdots,n-1$, is relatively compact in $C([t_i,t_{i+1}],X)$.*

In this work we will employ an axiomatic definition of the phase space $\mathscr{B}$, which has been used in [21] and suitably modified to treat retarded impulsive differential equations. Specifically, $\mathscr{B}$ will be a linear space of functions mapping $(-\infty,0]$ into X endowed with a seminorm $\| \cdot \|_{\mathscr{B}}$ and we will assume that $\mathscr{B}$ satisfies the following axioms:

(A) If $x:(-\infty,\sigma+b]\to X$, $b>0$, is such that $x_\sigma\in\mathscr{B}$ and $x|_{[\sigma,\sigma+b]}\in\mathscr{PC}([\sigma,\sigma+b],X)$, then for every $t\in[\sigma,\sigma+b)$ the following conditions hold:

(i) x_t is in $\mathscr{B}$,

(ii) $\| x(t) \|\leqslant H \| x_t \|_{\mathscr{B}}$,

(iii) $\| x_t \|_{\mathscr{B}}\leqslant K(t-\sigma)\sup\{\| x(s) \|: \sigma\leqslant s\leqslant t\}+M(t-\sigma)\| x_\sigma \|_{\mathscr{B}}$,

where $H>0$ is a constant; $K,M:[0,\infty)\to[1,\infty)$, K is continuous, M is locally bounded, and H,K,M are independent of $x(\cdot)$.

(B) The space $\mathscr{B}$ is complete.

Next, we consider some examples of phase spaces.

EXAMPLE 2.1. (*The phase space* $\mathscr{P}C_\rho(X)$) A function $\psi:(-\infty,0]\to X$ is said to be normalized piecewise continuous if ψ is left continuous and the restriction of ψ to any interval $[-r,0]$ is piecewise continuous. Let $g:(-\infty,0]\to[1,\infty)$ be a continuous nondecreasing function which satisfies the conditions (g-1), (g-2) in the terminology of [27]. Next, we slightly modify the definition of spaces C_g, C_g^0 in [27]. We denote by $\mathscr{PC}_g(X)$ the space formed by the normalized piecewise continuous functions ψ such that ψ/g is bounded on $(-\infty,0]$, and by $\mathscr{PC}_g^0(X)$ the subspace of $\mathscr{PC}_g(X)$ consisting of functions ψ such that $[\psi(\theta)/g(\theta)]\to 0$ as $\theta\to-\infty$. It is easy to see that $\mathscr{B}=\mathscr{PC}_g(X)$ and $\mathscr{B}=\mathscr{PC}_g^0(X)$ endowed with the norm $\| \psi \|_{\mathscr{B}}:=\sup_{\theta\in(-\infty,0]}[\| \psi(\theta) \| /g(\theta)]$ are phase spaces in the sense defined above.

EXAMPLE 2.2. (*The phase space* $\mathscr{PC}_r\times L^p(g,X)$). Let $r\geqslant 0$, $1\leqslant p<\infty$ and let $g:(-\infty,-r]\to\mathbb{R}$ be a non-negative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [27]. Briefly, this means that ρ is locally integrable and there exists a non-negative, locally bounded function γ on $(-\infty,0]$ such that $g(\xi+\theta)\leqslant\gamma(\xi)\rho(\theta)$, for all $\xi\leqslant 0$ and $\theta\in(-\infty,-r)\setminus N_\xi$, where $N_\xi\subseteq(-\infty,-r)$ is a set with Lebesgue measure zero. The space $\mathscr{B}=\mathscr{PC}_r\times L^p(g,X)$ consists of all classes of Lebesgue-measurable functions $\psi:(-\infty,0]\to X$ such that $\psi|_{[-r,0]}\in\mathscr{PC}([-r,0],X)$ and $\rho\| \psi \|^p$ is Lebesgue integrable on $(-\infty,-r)$. The seminorm in this space is defined by

$$\| \psi \|_{\mathscr{B}}=\sup\{\| \psi(\theta)\|:-r\leqslant\theta\leqslant 0\}+\left(\int_{-\infty}^{-r}g(\theta)\| \psi(\theta) \|^p\,d\theta\right)^{1/p}.$$

Proceeding as in the proof of [27, Theorem 1.3.8], it follows that $\mathscr{B}$ is a space which satisfies the axioms (A) and (B). Moreover, when $r=0$ this space coincides with $C_0\times L^p(g,X)$ and if, in addition, $p=2$, we can take $H=1$, $M(t)=\gamma(-t)^{1/2}$ and $K(t)=1+\left(\int_{-t}^0 g(\theta)d\theta\right)^{1/2}$ for $t\geqslant 0$.

REMARK 2.1. Let $\psi\in\mathscr{B}$ and $t\leqslant 0$. The notation ψ_t represents the function defined by $\psi_t(\theta)=\psi(t+\theta)$. Consequently, if the function $x(\cdot)$ in axiom (A) is such that $x_0=\psi$, then $x_t=\psi_t$. We observe that ψ_t is well defined for $t<0$ since the domain of ψ is $(-\infty,0]$. We also note that in general $\psi_t\notin\mathscr{B}$; consider, for example, functions of the type $x^{\mu}(t)=(t-\mu)^{-\alpha}\mathscr{X}_{(\mu,0]}$, $\mu>0$, where $\mathscr{X}_{(\mu,0]}$ is the characteristic function of $(\mu,0]$, $\mu<-r$ and $\alpha p\in(0,1)$, in the space $\mathscr{PC}_r\times L^p(g;X)$.

Additional terminologies and notations used in the sequel are standard in functional analysis. In particular, for Banach spaces $(Z,\|\cdot\|_Z)$, $(W,\|\cdot\|_W)$, the notation $\mathscr{L}(Z,W)$ stands for the Banach space of bounded linear operators from Z into W and we abbreviate to $\mathscr{L}(Z)$ whenever $Z=W$. Additionally, $B_r(x,Z)$ denotes the closed ball with center at x and radius $r>0$ in Z.

Our main results are based upon the following well-known result.

LEMMA 2.2. [36, Sadovskii's Fixed Point Theorem] *Let G be a condensing operator on a Banach space X. If $G(D)\subset D$ for a convex, closed and bounded set D of X, then G has a fixed point in D.*

3. Existence results

In this section we discuss the existence of mild solutions for the abstract system (1.1)-(1.4). In the rest of this paper, we always assume that N and $\tilde{N}$ are positive constants such that $\|C(t)\|\leqslant N$ and $\|S(t)\|\leqslant\tilde{N}$, for every $t\in I$. We also suppose, that $\varphi\in\mathscr{B}$ and that $\rho:I\times\mathscr{B}\to(-\infty,a]$ is a continuous function. Additionally, we introduce following conditions.

(H_φ): Let $\mathscr{R}(\rho^-)=\{\rho(s,\psi):(s,\psi)\in I\times\mathscr{B},\rho(s,\psi)\leqslant 0\}$. The function $t\to\varphi_t$ is well defined from $\mathscr{R}(\rho^-)$ into $\mathscr{B}$ and there exists a continuous and bounded function $J^{\varphi}:\mathscr{R}(\rho^-)\to(0,\infty)$ such that $\|\varphi_t\|_{\mathscr{B}}\leqslant J^{\varphi}(t)\|\varphi\|_{\mathscr{B}}$ for every $t\in\mathscr{R}(\rho^-)$.

(H_1): The function $f:I\times\mathscr{B}\to X$ satisfies the following conditions:

(i) Let $x:(-\infty,a]\to X$ be such that $x_0=\varphi$ and $x|_I\in\mathscr{PC}$. The function $t\to f(t,x_{\rho(t,x_t)})$ is measurable on I and the function $t\to f(s,x_t)$ is continuous on $\mathscr{R}(\rho^-)\cup I$ for every $s\in I$.

(ii) For each $t\in I$, the function $f(t,\cdot):\mathscr{B}\to X$ is continuous.

(iii) There exist an integrable function $m:I\to[0,\infty)$ and a continuous nondecreasing function $W:[0,\infty)\to(0,\infty)$ such that for every $(t,\psi)\in I\times\mathscr{B}$

$$\|f(t,\psi)\|\leqslant m(t)W(\|\psi\|_{\mathscr{B}}),\ \liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}=\Lambda<\infty.$$

(H_2): The funtion $g:I\times\mathscr{B}\to X$ is continuous and there exists $L_g>0$ such that

$$\|g(t,\psi_1)-g(t,\psi_2)\|\leqslant L_g\|\psi_1-\psi_2\|_{\mathscr{B}},\quad (t,\psi_i)\in I\times\mathscr{B},\ i=1,2.$$

(H_3): There are positive constants L_{I_i}, L_{J_i} such that

$$\|I_i(\psi_1) - I_i(\psi_2)\| \leqslant L_{I_i}\|\psi_1 - \psi_2\|_{\mathscr{B}}, \quad \psi_j \in \mathscr{B}, \quad i = 1,2,\ldots,n,$$
$$\|J_i(\psi_1) - J_i(\psi_2)\| \leqslant L_{J_i}\|\psi_1 - \psi_2\|_{\mathscr{B}}, \quad \psi_j \in \mathscr{B}, \quad i = 1,2,\ldots,n.$$

(H_4): There exist positive constants c_1, c_2 such that $\|g(t,\psi)\| \leqslant c_1\|\psi\|_{\mathscr{B}} + c_2$, for every $(t,\psi) \in I \times \mathscr{B}$.

(H_5): The maps $I_i, J_i : \mathscr{B} \to X$, $i = 1,2,\ldots,n$ are completely continuous and there exist continuous nondecreasing functions $\Phi_i, \Psi_i : [0,\infty) \to (0,\infty)$, $i = 1,2,\ldots,n$, such that:

$$\|I_i(\psi)\| \leqslant \Phi_i(\|\psi\|_{\mathscr{B}}), \quad \liminf_{\zeta \to +\infty} \frac{\Phi_i(\zeta)}{\zeta} = \zeta_i < \infty,$$
$$\|J_i(\psi)\| \leqslant \Psi_i(\|\psi\|_{\mathscr{B}}), \quad \liminf_{\zeta \to +\infty} \frac{\Psi_i(\zeta)}{\zeta} = \eta_i < \infty.$$

REMARK 3.1. The condition (H_φ) is frequently satisfied by functions that are continuous and bounded. In fact, assume that the space of continuous and bounded functions $C_b((-\infty,0],X)$ is continuously included in $\mathscr{B}$. Then, there exists $L > 0$ such that

$$\| \varphi_t \|_{\mathscr{B}} \leqslant L \frac{\sup_{\theta \leqslant 0} \| \varphi(\theta) \|}{\| \varphi \|_{\mathscr{B}}} \| \varphi \|_{\mathscr{B}}, \qquad t \leqslant 0,\ \varphi \neq 0,\ \varphi \in C_b((-\infty,0] : X).$$

It is easy to see that the space $C_b((-\infty,0],X)$ is continuously included in $\mathscr{P}C_g(X)$ and $\mathscr{P}\mathscr{C}_g^0(X)$. Moreover, if $g(\cdot)$ verifies (g-5)-(g-6) in [27] and $g(\cdot)$ is integrable on $(-\infty,-r]$, then the space $C_b((-\infty,0],X)$ is also continuously included in $\mathscr{P}\mathscr{C}_r \times L^p(g;X)$. For complementary details related this matter, see Proposition 7.1.1 and Theorems 1.3.2 and 1.3.8 in [27].

Motivated by (2.2) we introduce the following concept of mild solutions for the system (1.1)-(1.4).

DEFINITION 3.1. A function $x : (-\infty,a] \to X$ is called a mild solution of the abstract Cauchy problem (1.1)-(1.4) if $x_0 = \varphi, x_{\rho(s,x_s)} \in \mathscr{B}$ for every $s \in I; x(\cdot)|_I \in \mathscr{P}\mathscr{C}$ and

$$\begin{aligned} x(t) = C(t)\varphi(0) &+ S(t)[\zeta - g(0,\varphi)] \\ &+ \int_0^t C(t-s)g(s,x_s)ds + \int_0^t S(t-s)f(s,x_{\rho(s,x_s)})ds \\ &+ \sum_{0<t_i<t} C(t-t_i)I_i(x_{t_i}) + \sum_{0<t_i<t} S(t-t_i)J_i(x_{t_i}), \quad t \in I. \end{aligned}$$

REMARK 3.2. In the rest of this paper, $y : (-\infty,a] \to X$ is the function defined by $y(t) = \varphi(t)$ on $(-\infty,0]$ and $y(t) = C(t)\varphi(0) + S(t)\zeta$ for $t \in I$. Also, $\| y \|_a$, M_a, K_a, and J_0^φ are the constants defined by $\| y \|_a = \sup_{s\in[0,a]} \| y(s) \|$, $M_a = \sup_{s\in[0,a]} M(s)$, $K_a = \sup_{s\in[0,a]} K(s)$ and $J_0^\varphi = \sup_{t\in\mathscr{R}(\rho^-)} J^\varphi(t)$.

LEMMA 3.1. [20, Lemma 2.1] *Let $x : (-\infty, a] \to X$ be a function such that $x_0 = \varphi$ and $x|_I \in \mathscr{PC}$. Then*

$$\| x_s \|_{\mathscr{B}} \leqslant (M_a + J_0^{\varphi}) \| \varphi \|_{\mathscr{B}} + K_a \sup\{ \| x(\theta) \|; \theta \in [0, \max\{0, s\}] \}, \quad s \in \mathscr{R}(\rho^-) \cup I.$$

Now, we prove our main existence results.

THEOREM 3.1. *Let conditions (H_φ), $(H_1) - (H_4)$ be hold and assume that $S(t)$ is compact for every $t \in \mathbb{R}$. If*

$$K_a \Big[\tilde{N}\Lambda \int_0^a m(s)ds + \sum_{i=1}^{n} (NL_{I_i} + \tilde{N}L_{J_i}) + aNL_g \Big] < 1,$$

then the problem (1.1)-(1.4) *has at least one mild solution on* $(-\infty, a]$.

Proof. On the space $Y = \{x \in \mathscr{PC} : u(0) = \varphi(0)\}$ endowed with the uniform convergence topology, we define the operator $\Gamma : Y \to Y$ by

$$\begin{aligned} \Gamma x(t) = {} & C(t)\varphi(0) + S(t)[\zeta - g(0,\varphi)] \\ & + \int_0^t C(t-s) g(s, \bar{x}_s) ds + \int_0^t S(t-s) f(s, \bar{x}_{\rho(s, \bar{x}_s)}) ds \\ & + \sum_{0<t_i<t} C(t-t_i) I_i(\bar{x}_{t_i}) + \sum_{0<t_i<t} S(t-t_i) J_i(\bar{x}_{t_i}), \quad t \in I, \end{aligned}$$

where $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From axiom (A) and our assumptions on φ, we infer that $\Gamma x \in \mathscr{PC}$.

Next, we prove that there exists $r > 0$ such that $\Gamma(B_r(y_{|I}, Y)) \subseteq B_r(y_{|I}, Y)$. If we assume this property is false, then for every $r > 0$ there exist $x^r \in B_r(y_{|I}, Y)$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r) - y(t^r)\|$. Then, from Lemma 3.1 we get

$$\begin{aligned} r < {} & \|\Gamma x^r(t^r) - y(t^r)\| \\ \leqslant {} & NH\|\varphi\|_{\mathscr{B}} + \tilde{N}[\|\zeta\| + \|g(0,\varphi)\|] + N\int_0^{t^r} \|g(s, \overline{x^r}_s) - g(s, y_s)\| ds \\ & + N\int_0^{t^r} \|g(s, y_s)\| ds + \tilde{N}\int_0^{t^r} m(s) W(\|\overline{x^r}_{\rho(s, (\overline{x^r})_s)}\|_{\mathscr{B}}) ds \\ & + \sum_{i=1}^{n} N(L_{I_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathscr{B}} + \|I_i(y_{t_i})\|) + \sum_{i=1}^{n} \tilde{N}(L_{J_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathscr{B}} + \|J_i(y_{t_i})\|) \\ \leqslant {} & NH\|\varphi\|_{\mathscr{B}} + \tilde{N}[\|\zeta\| + \|g(0,\varphi)\|] \\ & + NL_g K_a \int_0^{t^r} \|\overline{x^r} - y\|_s ds + N\int_0^{t^r} (c_1\|y_s\|_{\mathscr{B}} + c_2) ds \\ & + \tilde{N} W\big((M_a + J_0^{\varphi})\|\varphi\|_{\mathscr{B}} + K_a r + K_a \| y \|_a\big) \int_0^a m(s) ds \\ & + \sum_{i=1}^{n} N(L_{I_i} K_a r + \|I_i(y_{t_i})\|) + \sum_{i=1}^{n} \tilde{N}(L_{J_i} K_a r + \|J_i(y_{t_i})\|), \end{aligned}$$

and hence

$$1 \leqslant K_a\Big[\tilde{N}\Lambda\int_0^a m(s)ds + \sum_{i=1}^n (NL_{I_i} + \tilde{N}L_{J_i}) + aNL_g\Big],$$

which is contrary to our assumption.

Let $r > 0$ be such that $\Gamma(B_r(y_{|I}, Y)) \subset B_r(y_{|I}, Y)$. In order to prove that Γ is a condensing map on $B_r(y_{|I}, Y)$ into $B_r(y_{|I}, Y)$. We introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\begin{aligned}
\Gamma_1 x(t) &= C(t)\varphi(0) + S(t)[\zeta - g(0,\varphi)] + \int_0^t C(t-s)g(s,\bar{x}_s)ds + \sum_{0<t_i<t} C(t-t_i)I_i(\bar{x}_{t_i}) \\
&\quad + \sum_{0<t_i<t} S(t-t_i)J_i(\bar{x}_{t_i}), \\
\Gamma_2 x(t) &= \int_0^t S(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds.
\end{aligned}$$

From the proof of [24, Theorem 3.4], we know that Γ_2 is completely continuous. Moreover, from the estimate

$$\begin{aligned}
\|\Gamma_1 x - \Gamma_1 z\|_{\mathscr{PC}} &\leqslant aNL_g K_a \|x-z\|_{\mathscr{PC}} + K_a \sum_{i=1}^n (NL_{I_i} + \tilde{N}L_{J_i})\|x-z\|_{\mathscr{PC}} \\
&\leqslant K_a[aNL_g + \sum_{i=1}^n (NL_{I_i} + \tilde{N}L_{J_i})]\|x-z\|_{\mathscr{PC}}
\end{aligned}$$

it follows that Γ_1 is contraction on $B_r(y_{|I}, Y)$, which implies that Γ is a condensing operator on $B_r(y_{|I}, Y)$.

Finally, from Lemma 2.2, we infer that there exists a mild solution of (1.1)-(1.4). The proof is complete.

THEOREM 3.2. *Let conditions (H_φ), (H_1), (H_2), (H_4), (H_5) be hold and assume that $S(t)$ is compact for every $t \in \mathbb{R}$. If*

$$K_a\Big[\tilde{N}\Lambda\int_0^a m(s)ds + \sum_{i=1}^n (N\zeta_i + \tilde{N}\eta_i) + aNL_g\Big] < 1,$$

then the problem (1.1)-(1.4) *admits at least one mild solution on* $(-\infty, a]$.

Proof. On the space $Y = \{x \in \mathscr{PC} : u(0) = \varphi(0)\}$ endowed with the uniform convergence topology, we define the operator $\Gamma : Y \to Y$ by

$$\begin{aligned}
\Gamma x(t) &= C(t)\varphi(0) + S(t)[\zeta - g(0,\varphi)] \\
&\quad + \int_0^t C(t-s)g(s,\bar{x}_s)ds + \int_0^t S(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds \\
&\quad + \sum_{0<t_i<t} C(t-t_i)I_i(\bar{x}_{t_i}) + \sum_{0<t_i<t} S(t-t_i)J_i(\bar{x}_{t_i}), \quad t \in I,
\end{aligned}$$

where $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From axiom (A) and our assumptions on φ, we infer that $\Gamma x \in \mathscr{PC}$.

Next, we prove that there exists $r > 0$ such that $\Gamma(B_r(y_{|I}, Y)) \subseteq B_r(y_{|I}, Y)$. If we assume this property is false, then for every $r > 0$ there exist $x^r \in B_r(y_{|I}, Y)$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r) - y(t^r)\|$. Then, from Lemma 3.1 we get

$$\begin{aligned}
r &< \|\Gamma x^r(t^r) - y(t^r)\| \\
&\leqslant NH\|\varphi\|_{\mathscr{B}} + \tilde{N}[\|\zeta\| + \|g(0,\varphi)\|] \\
&\quad + N\int_0^{t^r} \|g(s,\overline{x^r}_s) - g(s,y_s)\|ds + N\int_0^{t^r} \|g(s,y_s)\|ds \\
&\quad + \tilde{N}\int_0^{t^r} m(s)W(\|\overline{x^r}_{\rho(s,(\overline{x^r})_s)}\|_{\mathscr{B}})ds + N\sum_{i=1}^{n} \|I_i(\bar{x}_{t_i})\| + \tilde{N}\sum_{i=1}^{n} \|J_i(\bar{x}_{t_i})\| \\
&\leqslant NH\|\varphi\|_{\mathscr{B}} + \tilde{N}[\|\zeta\| + \|g(0,\varphi)\|] \\
&\quad + NL_gK_a\int_0^{t^r} \|\overline{x^r} - y\|_s ds + N\int_0^{t^r} (c_1\|y_s\|_{\mathscr{B}} + c_2)ds \\
&\quad + \tilde{N}W((M_a + J_0^{\varphi})\|\varphi\|_{\mathscr{B}} + K_a r + K_a \parallel y \parallel_a)\int_0^a m(s)ds \\
&\quad + N\sum_{i=1}^{n} \Phi_i(\|\bar{x}_{t_i}\|_{\mathscr{B}}) + \tilde{N}\sum_{i=1}^{n} \Psi_i(\|\bar{x}_{t_i}\|_{\mathscr{B}}).
\end{aligned}$$

Since Φ_i and Ψ_i are nondecreasing operators, we have

$$\begin{aligned}
r < NH\|\varphi\|_{\mathscr{B}} &+ \tilde{N}[\|\zeta\| + \|g(0,\varphi)\|] \\
&+ NL_gK_a\int_0^{t^r} \|\overline{x^r} - y\|_s ds + N\int_0^{t^r} (c_1\|y_s\|_{\mathscr{B}} + c_2)ds \\
&+ \tilde{N}W((M_a + J_0^{\varphi})\|\varphi\|_{\mathscr{B}} + K_a r + K_a \parallel y \parallel_a)\int_0^a m(s)ds \\
&+ N\sum_{i=1}^{n} \Phi_i(r^*) + \tilde{N}\sum_{i=1}^{n} \Psi_i(r^*),
\end{aligned}$$

where $\|\bar{x}_{t_i}\|_{\mathscr{B}} \leqslant r^* = (M_a + J_0^{\phi})\|\varphi\|_{\mathscr{B}} + K_a(r + \|y\|_a)$ and hence

$$1 \leqslant K_a\Big[\tilde{N}\Lambda\int_0^a m(s)ds + \sum_{i=1}^{n}(N\zeta_i + \tilde{N}\eta_i) + aNL_g\Big],$$

which contradicts to our assumption.

Let $r > 0$ be such that $\Gamma(B_r(y_{|I}, Y)) \subset B_r(y_{|I}, Y)$. In order to prove that Γ is a condensing map on $B_r(y_{|I}, Y)$, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ where

$$\begin{aligned}
\Gamma_1 x(t) &= C(t)\varphi(0) + S(t)[\zeta - g(0,\varphi)] + \int_0^t C(t-s)g(s,\bar{x}_s)ds, \\
\Gamma_2 x(t) &= \int_0^t S(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})ds, \\
\Gamma_3 x(t) &= \sum_{0<t_i<t} C(t-t_i)I_i(\bar{x}_{t_i}) + \sum_{0<t_i<t} S(t-t_i)J_i(\bar{x}_{t_i}).
\end{aligned}$$

From the proof of [24, Theorem 3.4], we know that Γ_2 is completely continuous.

In the sequel, by using Lemma 2.1, we prove that Γ_3 is also completely continuous. Since $(C(t))_{t\in\mathbb{R}}, (S(t))_{t\in\mathbb{R}}$ are bounded by $N, \overline{N}$ and $I_i, i = 1, \cdots, n$ are completely continuous, the continuity of Γ_3 can be proved standardly by using the phase space axioms.

Let r be a positive number, $B_r = B_r(0, \mathscr{PC})$ and $r^* > 0$ be such that $\| x_t \|_{\mathscr{B}} \leqslant r^*$ for every $x \in B_r$ and all $t \in I$. Since that the functions I_i are completely continuous and the cosine function $(C(t))_{t\in\mathbb{R}}$ is strongly continuous, for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\| C(t+h)I_i(\psi) - C(t)I_i(\psi) \| \leqslant \frac{\varepsilon}{n}, \qquad i = 1, ..., n, \tag{3.1}$$

for every $t \in I$, $0 < | h | < \delta$ and all $\psi \in B_{r^*}(0, \mathscr{PC})$. Consequently, for $x \in B_r$, $t \in [t_i, t_{i+1}]$ and $0 < | h | < \delta$ with $t + h \in [t_i, t_{i+1}]$, we see that

$$\begin{aligned} \| [\Gamma_3 x]_i(t+h) - [\Gamma_3 x]_i(t) \| &\leqslant \sum_{j=1}^{n} \| (C(t+h-t_j) - C(t-t_j))I_j(u_{t_j}) \| \\ &\quad + \sum_{j=1}^{n} \| (S(t+h-t_j) - S(t-t_j))J_j(u_{t_j}) \| \\ &\leqslant \varepsilon + hN \sum_{j=1}^{n} (\Psi_i(r^*)), \end{aligned}$$

which proves the set of functions $[\Gamma_3 B_r]_i$ is equicontinuous on $[t_i, t_{i+1}]$ for all $i = 0, ..., n-1$.

On the other hand, from the relations

$$[\Gamma_3 x]_i(t) \in \sum_{j=1}^{i} C(t-t_j)I_j(B_{r^*}(0, \mathscr{B})) + \sum_{j=1}^{i} S(t-t_j)J_j(B_{r^*}(0, \mathscr{B})) \quad t \in (t_i, t_{i+1}],$$

$$[\Gamma_3 x]_i(t_i) \in \sum_{j=1}^{i-1} C(t_i-t_j)I_j(B_{r^*}(0, \mathscr{B})) + I_i(B_{r^*}(0, \mathscr{B})) + \sum_{j=1}^{i-1} S(t_i-t_j)J_j(B_{r^*}(0, \mathscr{B})),$$

we infer that the set $[\Gamma_3 x]_i(t)$ is relatively compact in X for every $t \in [t_i, t_{i+1}]$ and all $i = 0, ..., n-1$. Now, from Lemma 2.1 we assert that Γ_3 is completely continuous. Next, by using hypothesis (H_2), we prove that Γ_1 is a contraction. Moreover, from the estimate

$$\|\Gamma_1 x - \Gamma_1 z\|_{\mathscr{PC}} \leqslant aNL_g K_a \|x - z\|_{\mathscr{PC}}$$

it follows that, Γ_1 is contraction on $B_r(y_{|I}, Y)$, which implies that Γ is a condensing operator on $B_r(y_{|I}, Y)$.

Finally, from Lemma 2.2, we infer that there exists a mild solution of (1.1)-(1.4). The proof is complete.

According to Theorems (3.1)-(3.2), we can easily deduce some practical consequences for sub-linear growth cases.

COROLLARY 3.1. *If all conditions of Theorem 3.1 hold except that* (H_1)*(iii) replaced by*
$(C1)$: *there exist an integrable function* $m : J \to [0,+\infty)$ *and a constant* $\tau \in [0,1)$ *such that*

$$\|f(t,\psi)\| \leqslant m(t)(1+\|\psi\|_{\mathscr{B}}^{\tau}), \quad \textit{for each } (t,\psi) \in I \times \mathscr{B},$$

then the problem (1.1)-(1.4) *admits at least one mild solution on* $(-\infty,a]$ *provided that*

$$K_a\Big[\sum_{i=1}^{n}(NL_{I_i}+\tilde{N}L_{J_i})+aNL_g\Big]<1.$$

COROLLARY 3.2. *If all conditions of Theorem 3.2 hold except that* (H_5) *replaced by the following one,*
$(C2)$: *there exist positive constants* c_i, d_i, e_i, l_i, $i = 1,2,\dots,n$ *and constants* $\theta, \vartheta \in [0,1)$ *such that for each* $\psi \in \mathscr{B}$,

$$\|I_i(\psi)\| \leqslant c_i + d_i(\|\psi\|_{\mathscr{B}})^{\theta}, \quad i = 1,2,\dots,n,$$

and

$$\|J_i(\psi)\| \leqslant e_i + l_i(\|\psi\|_{\mathscr{B}})^{\vartheta}, \quad i = 1,2,\dots,n,$$

then the problem (1.1)-(1.4) *has at least one mild solution on* $(-\infty,a]$ *provided that*

$$K_a\left[\tilde{N}\Lambda\int_0^a m(s)ds + aNL_g\right] < 1.$$

COROLLARY 3.3. *If all conditions of Theorem 3.2 hold except that* (H_1)*(iii) and* (H_5) *replaced by* $(C1)$ *and* $(C2)$*, then the problem has at least one mild solution on* $(-\infty,a]$ *provided that*

$$K_a aNL_g < 1.$$

REMARK 3.3. According to conditions (C1)-(C2), we can see that

$$\Lambda = 0, \quad \zeta_i = 0, \quad \eta_i = 0.$$

4. An example

In this section, we consider an application of our abstract results. At first we introduce the required technical framework. In the rest of this section, $X = L^2([0,\pi]), \mathscr{B} = \mathscr{P}\mathscr{C}_0 \times L^2(g,X)$ is the space introduced in Example 2.2 and $A : D(A) \subset X \to X$ is the operator $Ax = x''$ with domain $D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}$. It is well-known that A is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ on X. Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2$, for

$n \in \mathbb{N}$, with corresponding eigenvectors $z_n(\tau) = (2/\pi)^{1/2}\sin(n\tau)$, and the following properties hold.

(a) The set $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X and $Az = -\sum_{n=1}^{\infty} n^2\langle z, z_n\rangle z_n$, for $\varphi \in D(A)$.

(b) For $z \in X$, $C(t)z = \sum_{n=1}^{\infty}\cos(nt)\langle z, z_n\rangle z_n$. It follows from this expression that $S(t)z = \sum_{n=1}^{\infty}\frac{\sin(nt)}{n}\langle z, z_n\rangle z_n$, which implies that the operator $S(t)$ is compact, for all $t \in \mathbb{R}$ and that $\|C(t)\| = \|S(t)\| = 1$, for all $t \in \mathbb{R}$.

(c) If Φ is the group of translations on X defined by $\Phi(t)x(\zeta) = \tilde{x}(\zeta + t)$, where $\tilde{x}(\cdot)$ is the extension of $x(\cdot)$ with period 2π, then $C(t) = \frac{1}{2}\left[\Phi(t) + \Phi(-t)\right]$ and $A = B^2$, where B is the infinitesimal generator of Φ and $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$ (see [14] for details). In particular, we observe that the inclusion $\imath : E \to X$ is compact.

Consider the differential system

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}u(t,\zeta) + \int_{-\infty}^{t}\int_0^{\pi} b(t-s,\eta,\zeta)u(s,\eta)d\eta ds\right] = \frac{\partial^2}{\partial\zeta^2}u(t,\zeta)$$
$$+ \int_{-\infty}^{t} a(s-t)u(s-\rho_1(t)\rho_2(\|u(t)\|),\zeta)ds, \quad t \in I,\ \zeta \in [0,\pi], \tag{4.1}$$
$$u(t,0) = u(t,\pi) = 0, \quad t \in I, \tag{4.2}$$
$$\frac{\partial}{\partial t}u(0,\zeta) = \zeta(\pi), \tag{4.3}$$
$$u(\tau,\zeta) = \varphi(\tau,\zeta), \quad \tau \leqslant 0, \quad 0 \leqslant \zeta \leqslant \pi, \tag{4.4}$$
$$\triangle u(t_i)(\zeta) = \int_{-\infty}^{t_i} b_i(t_i - s)u(s,\zeta)ds, \quad i = 1,2,\dots,n, \tag{4.5}$$
$$\triangle u'(t_i)(\zeta) = \int_{-\infty}^{t_i} \tilde{b}_i(t_i - s)u(s,\zeta)ds, \quad i = 1,2,\dots,n. \tag{4.6}$$

To treat this system, let the functions $\rho_i : [0,\infty) \to [0,\infty)$; $a : \mathbb{R} \to \mathbb{R}$ be continuous, $L_f = \left(\int_{-\infty}^{0}\frac{(a^2(s))}{g(s)}ds\right)^{\frac{1}{2}} < \infty$, and let the following conditions hold:

(a) The functions $b(s,\eta,\zeta)$, $\frac{\partial b(s,\eta,\zeta)}{\partial\zeta}$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ and

$$L_g = \max\left\{\left(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\frac{1}{g(s)}\left(\frac{\partial^i b(s,\eta,\zeta)}{\partial\zeta^i}\right)^2 d\eta ds d\zeta\right)^{\frac{1}{2}} : i = 0,1\right\} < \infty.$$

(b) The functions $b_i, \tilde{b}_i \in C(\mathbb{R},\mathbb{R})$ and $L_{I_i} := \left(\int_{-\infty}^{0}\frac{b_i^2(s)}{g(s)}ds\right)^{\frac{1}{2}}$, $L_{J_i} := \left(\int_{-\infty}^{0}\frac{\tilde{b}_i^2(s)}{g(s)}ds\right)^{\frac{1}{2}}$, $i = 1,\cdots,n$, are finite.

Under these conditions, we define the functions $g, f : J \times \mathscr{B} \to X$, $\rho : I \times \mathscr{B} \to$

X, $I_i : \mathscr{B} \to X$ and $J_i : \mathscr{B} \to X$ by

$$\begin{aligned}
g(\psi)(\zeta) &= \int_{-\infty}^{0}\int_{0}^{\pi} b(s,\nu,\zeta)\psi(s,\nu)d\nu ds,\\
f(t,\psi)(\zeta) &= \int_{-\infty}^{0} a(s)\psi(s,\zeta)ds,\\
\rho(s,\psi) &= s-\rho_1(s)\rho_2(\|\psi(0)\|),\\
I_i(\psi)(\zeta) &= \int_{-\infty}^{0} b_i(-s)\psi(s,\zeta)ds, \quad i=1,2,\ldots,n,\\
J_i(\psi)(\zeta) &= \int_{-\infty}^{0} \tilde{b}_i(-s)\psi(s,\zeta)ds, \quad i=1,2,\ldots,n,
\end{aligned}$$

which permit to transform (4.1)-(4.6) into the abstract Cauchy problem (1.1)-(1.4). Moreover, the maps $g,\ f,\ I_i,\ J_i,\ i=1,2,\ldots,n$ are bounded linear operators with

$$\|g(t,\cdot)\|_{\mathscr{L}(\mathscr{B},X)} \leqslant L_g,\ \|f(t,\cdot)\|_{\mathscr{L}(\mathscr{B},X)} \leqslant L_f,\ \|I_i\|_{\mathscr{L}(\mathscr{B},X)} \leqslant L_{I_i}, \|J_i\|_{\mathscr{L}(\mathscr{B},X)} \leqslant L_{J_i}.$$

The next results are consequence of Theorem 3.2 and Remark 3.1. We omit the proof.

PROPOSITION 4.1. *Assume $\varphi \in \mathscr{B}$, $\zeta \in X$ and that the condition (H_φ) is verified. Then there exists a mild solution $u(\cdot)$ of* (4.1)-(4.6). *Moreover, if $\varphi(0) \in H_0^1(0,\pi)$ and $I_i(u_{t_i}) \in H_0^1(0,\pi)$ for each $i=1,2,\ldots,n$, then the conditions* (4.5) *and* (4.6) *are verified.*

COROLLARY 4.1. *If φ is continuous and bounded, then there exists a mild solution of* (4.1)-(4.6).

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