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ON AN EIGENVALUE PROBLEM INVOLVING THE $p(x)$–LAPLACE OPERATOR PLUS A NON–LOCAL TERM

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Abstract. We study an eigenvalue problem involving variable exponent growth conditions and a non-local term, on a bounded domain $\Omega \subset \mathbb{R}^N$. Using adequate variational techniques, mainly based on the mountain-pass theorem of A. Ambrosetti and P. H. Rabinowitz, we prove the existence of a continuous family of eigenvalues lying in a neighborhood at the right of the origin.

1. Introduction

Elliptic equations involving variable exponent growth conditions have been intensively discussed in the last decade. A strong motivation in studying such kind of problems is due to the fact that they can model with high accuracy various phenomena which arise from the study of elastic mechanics (see, V. Zhikov [27]), electrorheological fluids (see, E. Acerbi and G. Mingione [1, 2], L. Diening [5], T. C. Halsey [14], M. Ruzicka [24, 25]) or image restoration (see, Y. Chen, S. Levine and R. Rao [4]). In that context, eigenvalue problems involving variable exponent growth conditions represent a starting point in analyzing more complicated equations. A first contribution in this sense is the paper of X. Fan, Q. Zhang and D. Zhao [12] where the following eigenvalue problem has been considered

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p : \overline{\Omega} \to (1,\infty)$ is a continuous function and λ is a real number. The result in [12] establishes the existence of infinitely many eigenvalues for problem (1) by using an argument based on the Ljusternik-Schnirelmann critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, the authors showed that $\sup\Lambda = +\infty$ and they pointed out that only under special conditions, which are somehow connected with a kind of monotony of the function $p(x)$, we have $\inf\Lambda > 0$ (this is in contrast with the case when $p(x)$ is a constant; then, we always have $\inf\Lambda > 0$).

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We notice that the above discussion is in keeping with the fact that considering the Rayleigh quotient associated with problem (1), that is

$$\mu_1 := \inf_{u\in C_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^{p(x)}\,dx}{\displaystyle\int_\Omega |u|^{p(x)}\,dx},$$

we often have $\mu_1 = 0$ for general $p(x)$. An example in that sense is illustrated by X. Fan and D. Zhao in [13], pages 444-445. More exactly, letting $\Omega = (-2,2) \subset \mathbb{R}$ and defining $p(x) = 3$ if $0 \leqslant |x| \leqslant 1$, and $p(x) = 4 - |x|$ if $1 \leqslant |x| \leqslant 2$ it can be proved that $\mu_1 = 0$. A simple conclusion one can draw from this remark is that, generally, we can not establish the existence of a positive constant C such that the following Poincaré type inequality holds true,

$$\int_\Omega |u|^{p(x)}\,dx \leqslant C\int_\Omega |\nabla u|^{p(x)}\,dx, \quad \forall\, u \in C_0^1(\Omega).$$

Going further, another eigenvalue problem involving variable exponent growth conditions intensively studied is the following

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda|u|^{q(x)-2}u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a bounded domain with smooth boundary $\partial\Omega$, p, q : $\overline{\Omega} \to (1,\infty)$ are two continuous functions and λ is a real number. In the case when $p(x) \neq q(x)$ the competition between the growth rates involved in equation (2) is essential in describing the set of eigenvalues of this problem. Thus, in the case when $\min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x)$ and $q(x)$ has a subcritical growth M. Mihăilescu and V. Rădulescu [21] used Ekeland's variational principle in order to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin. This result was later extended by X. Fan in [9]. In the case when $\max_{x\in\overline{\Omega}} p(x) < \min_{x\in\overline{\Omega}} q(x)$ and $q(x)$ has a subcritical growth, a mountain-pass argument, similar with that used by Fan and Zhang [11], can be applied in order to show that any $\lambda > 0$ is an eigenvalue of problem (2). Finally, in the case when $\max_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x)$ it can be proved that the energetic functional which can be associated with the eigenvalue problem has a nontrivial minimum for any positive λ large enough (see, [11]). Clearly, in this case, the result of M. Mihăilescu and V. Rădulescu [21] can be also applied. Consequently, in this situation there exist two positive constants $\lambda^\star$ and $\lambda^{\star\star}$ such that any $\lambda \in (0,\lambda^\star) \cup (\lambda^{\star\star},\infty)$ is an eigenvalue of the problem.

All the results pointed out in relation to problem (2) were extended to the case of anisotropic elliptic equations, these are equations obtained when $\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is replaced by $\sum_{i=1}^N \partial_i(|\partial_i u|^{p_i(x)-2}\partial_i u)$, where $p_i(x) > 1$ are different continuous functions, by M. Mihăilescu, P. Pucci and V. Rădulescu [19] and M. Mihăilescu and G. Moroşanu [17, 18].

In an appropriate context we also point out the study of the eigenvalue problem

$$\begin{cases} -\mathrm{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u) = \lambda|u|^{q(x)-2}u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a bounded domain with smooth boundary $\partial\Omega$ and $p_1(x)$, $p_2(x)$, $q(x) : \overline{\Omega} \to (1,\infty)$ are continuous functions satisfying $p_2(x) < q(x) < p_1(x)$. For this problem M. Mihăilescu and V. Rădulescu [22] proved the existence of two positive constants λ_0 and λ_1 with $\lambda_0 \leqslant \lambda_1$ such that any $\lambda \in [\lambda_1,\infty)$ is an eigenvalue of problem (3) while any $\lambda \in (0,\lambda_0)$ is not an eigenvalue of problem (3).

For more information and connections regarding the study of eigenvalue problems involving variable exponent growth conditions we also refer to [15], (see, the web-site of the *Research group on variable exponent Lebesgue and Sobolev spaces*, http://www.math.helsinki.fi/analysis/varsobgroup/).

The goal of this paper is to study a new eigenvalue problem involving variable exponent growth conditions and a non-local term. With that end in view, let $\Omega \subset \mathbb{R}^N$, $(N \geqslant 3)$, be a bounded domain with smooth boundary $\partial\Omega$. We analyze the eigenvalue problem

$$\begin{cases} -\eta[u]\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x,u) & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega, \end{cases} \tag{4}$$

where $p : \overline{\Omega} \to (1,\infty)$ is a continuous function, $\eta[u]$ is a non-local term defined by the following relation

$$\eta[u] = 2 + \left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)^{\frac{\max_{\overline{\Omega}} p}{\min_{\overline{\Omega}} p}-1} + \left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)^{\frac{\min_{\overline{\Omega}} p}{\max_{\overline{\Omega}} p}-1}$$

and λ is a real number and $f = f(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is given by the relation

$$f(x,t) := \begin{cases} |t|^{p(x)-2}t, \text{ if } |t| < 1, \\ |t|^{r(x)-2}t, \text{ if } |t| \geqslant 1, \end{cases}$$

with $r : \overline{\Omega} \to (1,\infty)$ a continuous function satisfying

$$(\max_{\overline{\Omega}} p)^2 / \min_{\overline{\Omega}} p < \min_{\overline{\Omega}} r \leqslant \max_{\overline{\Omega}} r < N \min_{\overline{\Omega}} p / (N - \min_{\overline{\Omega}} p).$$

For problem (4) we will prove the existence of a continuous set of eigenvalues in a neighborhood at the right of the origin by using as main argument the mountain-pass theorem. We notice that problem (4) is connected with problem (1) since near the origin $f(x,t) = |t|^{p(x)-2}t$ and also with problem (2) since far of the origin $f(x,t) = |t|^{r(x)-2}t$, with $\min_{\overline{\Omega}} r > \max_{\overline{\Omega}} p$. On the other hand, the presence of the non-local term $\eta[u]$ balances the absence of homogeneity which occurs in the case of variable exponent growth conditions. Particularly, the presence of $\eta[u]$ will help us to formulate a Poincaré type inequality which will be essential in our variational approach (see Proposition 1 below).

2. Preliminary results

In this section we point out some basic results on the theory of Lebesgue–Sobolev spaces with variable exponent. For more details we refer to the book by J. Musielak [23] and the papers by D. E. Edmunds et al. [6, 7, 8], O. Kovacik and J. Rákosník [16], M. Mihăilescu and V. Rădulescu [20], and S. Samko and B. Vakulov [26].

Throughout this paper we assume that $p(x) > 1$, $p(x) \in C(\overline{\Omega})$. Set

$$C_+(\overline{\Omega}) = \{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \ \text{ and } \ h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u:\ u \text{ is a measurable real-valued function and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leqslant 1\right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces [16, Theorem 2.5], the Hölder inequality holds [16, Theorem 2.1], they are reflexive if and only if $1 < p^- \leqslant p^+ < \infty$ [16, Corollary 2.7] and continuous functions are dense if $p^+ < \infty$ [16, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally [16, Theorem 2.8]: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leqslant p_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

We denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv\, dx\right| \leqslant \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)} |v|_{q(x)} \tag{5}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold true

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \leqslant \rho_{p(x)}(u) \leqslant |u|_{p(x)}^{p^+} \tag{6}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \leqslant \rho_{p(x)}(u) \leqslant |u|_{p(x)}^{p^-} \tag{7}$$

$$|u_n - u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_n - u) \to 0. \tag{8}$$

Spaces with $p^+ = \infty$ have been studied by Edmunds, Lang and Nekvinda [6].

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where $p^\star(x) = Np(x)/(N-p(x))$ if $p(x) < N$ or $p^\star(x) = +\infty$ if $p(x) \geqslant N$. We refer to [7, 8, 10, 13, 16] for further properties of variable exponent Lebesgue-Sobolev spaces.

3. The main result

In this paper we seek solutions for problem (4) belonging to the space $W_0^{1,p(x)}(\Omega)$ in the sense below.

DEFINITION 1. We say $u \in W_0^{1,p(x)}(\Omega)$ is a *weak solution* for problem (4) if

$$\eta[u] \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx - \lambda \int_\Omega f(x,u) v \, dx = 0,$$

for all $v \in W_0^{1,p(x)}(\Omega)$. Moreover, we say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (4) if the weak solution u defined above is not trivial.

In order to get our main result we define

$$\nu_1 := \inf_{u \in E\setminus\{0\}} \frac{1}{\displaystyle\int_\Omega \frac{1}{p(x)} |u|^{p(x)} \, dx} \left[2 \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \frac{p^-}{p^+} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right)^{\frac{p^+}{p^-}} \right.$$
$$\left. + \frac{p^+}{p^-} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right)^{\frac{p^-}{p^+}} \right],$$

where $E = W_0^{1,p(x)}(\Omega)$. A key result regarding ν_1 is given by the following proposition.

PROPOSITION 1. *Assume that* $p : \overline{\Omega} \to (1, \infty)$ *is a continuous function. Then* $\nu_1 > 0$.

Remark. In the particular case when $p(x)$ is a constant function on $\overline{\Omega}$ say $p(x) = p > 1$ for any $x \in \overline{\Omega}$, then $\nu_1 = 4\lambda_1$, where λ_1 is defined by the relation

$$\lambda_1 := \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}. \tag{9}$$

The main result of our paper is given by the following theorem.

THEOREM 1. *Assume* $(p^+)^2/p^- < r^- \leqslant r^+ < Np^-/(N-p^-)$. *Then any* $\lambda \in (0,\nu_1)$ *is an eigenvalue of problem* (4).

In the light of the above remark, we point out the following corollary which represents a particular case of Theorem 1 obtained in the case when $p(x) = p > 1$ for any $x \in \overline{\Omega}$, where p is a constant.

COROLLARY 1. *Assume* $p(x) = p > 1$ *for any* $x \in \overline{\Omega}$, *where* p *is a constant,* $p < r^- \leqslant r^+ < Np/(N-p)$ *and* λ_1 *is defined by relation* (9). *Then any* $\lambda \in (0,4\lambda_1)$ *is an eigenvalue of problem* (4).

4. Proof of the main result

Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$ and let $\lambda \in (0,\nu_1)$ be fixed. The energy functional corresponding to problem (4) is defined as $J : E \to \mathbb{R}$,

$$J(u) = 2\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \frac{p^-}{p^+}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)^{\frac{p^+}{p^-}}$$
$$+\frac{p^+}{p^-}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)^{\frac{p^-}{p^+}} - \lambda\int_\Omega F(x,u)\,dx,$$

where $F(x,u) = \int_0^u f(x,t)\,dt$. It is known that the operator $\Lambda : E \to \mathbb{R}$,

$$\Lambda(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx$$

satisfies $\Lambda \in C^1(E,\mathbb{R})$ with

$$\langle \Lambda'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx$$

for all u, $v \in E$ (see e.g. [11]). Defining Λ_1, $\Lambda_2 : E \to \mathbb{R}$,

$$\Lambda_1(u) = \left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)^{\frac{p^+}{p^-}} \quad\text{and}\quad \Lambda_2(u) = \left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)^{\frac{p^-}{p^+}}$$

we observe that

$$\Lambda_1(u) = (\Lambda(u))^{\frac{p^+}{p^-}} \quad\text{and}\quad \Lambda_2(u) = (\Lambda(u))^{\frac{p^-}{p^+}}.$$

Thus, it is easy to verify that $\Lambda_1 \in C^1(E,\mathbb{R})$ and $\Lambda_2 \in C^0(E,\mathbb{R}) \cap C^1(E\setminus\{0\},\mathbb{R})$ with

$$\langle \Lambda_1'(u), v\rangle = \frac{p^+}{p^-}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)^{\frac{p^+}{p^-}-1}\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx$$

and

$$\langle \Lambda_2'(u), v\rangle = \frac{p^-}{p^+}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)^{\frac{p^-}{p^+}-1}\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx$$

for all $u \in E \setminus \{0\}$, $v \in E$.

We deduce that $J \in C^0(E,\mathbb{R}) \cap C^1(E \setminus \{0\},\mathbb{R})$ with

$$\langle J'(u), v\rangle = \eta[u]\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx - \lambda\int_\Omega f(x,u)v\, dx,$$

for all $u \in E \setminus \{0\}$, $v \in E$. Thus, the weak solutions of (4) are exactly the critical points of J. Our idea is to apply a mountain-pass argument in order to obtain a nontrivial weak solution for problem (4), and thus, to show that $\lambda \in (0,\nu_1)$ is an eigenvalue of (4).

In order to show that Proposition 1 holds true we prove the following lemma.

LEMMA 1. *There exists a positive constant $C > 0$ such that the following inequality holds true*

$$\int_\Omega |u|^{p(x)}\, dx \leqslant C\left[2\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx + \frac{p^-}{p^+}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\right)^{\frac{p^+}{p^-}} + \frac{p^+}{p^-}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\right)^{\frac{p^-}{p^+}}\right]$$

for any $u \in E$.

Proof. Using relations (6) and (7) we deduce that for any $u \in E$ we have

$$\int_\Omega |u|^{p(x)}\, dx \leqslant |u|_{p(x)}^{p^+} + |u|_{p(x)}^{p^-}. \tag{10}$$

The Sobolev embedding of E into $L^{p(x)}(\Omega)$ guarantees the existence of a positive constant $c_1 > 0$ such that

$$|u|_{p(x)} \leqslant c_1\|u\| \tag{11}$$

for any $u \in E$.

Relations (10) and (11) imply that there exists a positive constant $c_2 > 0$ such that

$$\int_\Omega |u|^{p(x)}\, dx \leqslant c_2(\|u\|^{p^+} + \|u\|^{p^-}), \quad \forall\, u \in E. \tag{12}$$

On the other hand, using once again relations (6) and (7), we find that for any $u \in E$

$$\|u\| \leqslant \left(\int_\Omega |\nabla u|^{p(x)}\, dx\right)^{\frac{1}{p^+}} + \left(\int_\Omega |\nabla u|^{p(x)}\, dx\right)^{\frac{1}{p^-}}. \tag{13}$$

By (12) and (13) we have

$$\int_\Omega |u|^{p(x)}\, dx \leqslant c_2\left\{\left[\left(\int_\Omega |\nabla u|^{p(x)}\, dx\right)^{\frac{1}{p^+}} + \left(\int_\Omega |\nabla u|^{p(x)}\, dx\right)^{\frac{1}{p^-}}\right]^{p^+} + \left[\left(\int_\Omega |\nabla u|^{p(x)}\, dx\right)^{\frac{1}{p^+}} + \left(\int_\Omega |\nabla u|^{p(x)}\, dx\right)^{\frac{1}{p^-}}\right]^{p^-}\right\} \tag{14}$$

for any $u \in E$.

We remember that for any $s > 0$ there exists a positive constant $c_s > 0$ such that

$$(\alpha+\beta)^s \leqslant c_s(\alpha^s+\beta^s), \quad \forall\, \alpha,\, \beta > 0.$$

Relation (14) and the above inequality assure that there exists a positive constant $c_3 > 0$ such that

$$\int_\Omega |u|^{p(x)}\,dx \leqslant c_3\left[2\int_\Omega |\nabla u|^{p(x)}\,dx + \left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{\frac{p^+}{p^-}} + \left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{\frac{p^-}{p^+}}\right]$$

for any $u \in E$. By the above inequality we conclude that Lemma 1 holds true. □

Remark. It is easy to see that Lemma 1 implies that Proposition 1 holds true.

In order to prove Theorem 1 we first point out certain properties which are satisfied by function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$.

(f1) There exists a constant $c > 0$ such that

$$|f(x,t)| \leqslant c(1+|t|^{r(x)-1}), \quad \forall\, x \in \Omega,\ t \in \mathbb{R};$$

(f2) There exist $\theta > (p^+)^2/p^-$ and $t_0 > 0$ such that

$$0 < \theta F(x,t) \leqslant f(x,t)t, \quad \forall\, x \in \overline{\Omega},\ t \in \mathbb{R} \text{ with } |t| \geqslant t_0,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$;

(f3) For any $\lambda \in (0,\nu_1)$ the following relation holds true

$$\frac{\lambda f(x,t)}{|t|^{p(x)-2}t} = \lambda < \nu_1,$$

for any $t \in \mathbb{R}$ satisfying $|t| \leqslant 1$ and $x \in \Omega$.

Remark. Conditions (f1)-(f3) are useful to show that functional J is well defined and of class C^1 on E.

The next lemma shows that the functional J possesses a mountain-pass geometry.

LEMMA 2. *1. There exist $a,\ \rho > 0$ such that*

$$J(u) \geqslant a > 0, \quad \forall\, u \in E \text{ with } \|u\| = \rho.$$

2. There exists $e \in E$ with $\|e\| > \rho$ (where ρ is given above) such that

$$J(e) < 0.$$

Proof. 1. Since f satisfies (f3) and for any $\lambda \in (0, \nu_1)$ there exists $\varepsilon > 0$ such that $\lambda \leqslant \nu_1 - \varepsilon$ we have

$$\frac{\lambda f(x,t)}{|t|^{p(x)-2}t} = \lambda \leqslant \nu_1 - \varepsilon$$

for all $x \in \Omega$ and $t \in \mathbb{R}$ with $|t| \leqslant 1$. We deduce that

$$\lambda F(x,t) \leqslant \frac{1}{p(x)}(\nu_1 - \varepsilon)|t|^{p(x)}$$

for all $x \in \Omega$ and $t \in \mathbb{R}$ with $|t| \leqslant 1$.

We define

$$\varphi[u] = 2 + \frac{p^-}{p^+}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\right)^{\frac{p^+}{p^-}-1} + \frac{p^+}{p^-}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\right)^{\frac{p^-}{p^+}-1}. \tag{15}$$

Using the above estimate and condition (f1) we have

$$\begin{aligned} J(u) &\geqslant \varphi[u]\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx - (\nu_1 - \varepsilon)\int_{\{x\in\Omega;\ |u(x)|\leqslant 1\}} \frac{1}{p(x)}|u|^{p(x)}\, dx \\ &\quad - c_4\int_{\{x\in\Omega;\ |u(x)|\geqslant 1\}} |u|^{r(x)}\, dx \\ &\geqslant \varphi[u]\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx - (\nu_1 - \varepsilon)\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\, dx - c_4\int_\Omega |u|^{r(x)}\, dx, \end{aligned}$$

where $c_4 > 0$ is a constant.

Provided $\|u\| < 1$ the above inequality, the Sobolev embedding, Proposition 1 and relation (6) yield

$$\begin{aligned} J(u) &\geqslant \frac{\nu_1 - (\nu_1 - \varepsilon)}{\nu_1}\varphi[u]\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx - c_4\int_\Omega |u|^{r^+}\, dx - c_4\int_\Omega |u|^{r^-}\, dx \\ &\geqslant 2\frac{\varepsilon}{\nu_1}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx - c_5\|u\|^{r^+} - c_6\|u\|^{r^-} \\ &\geqslant 2\frac{\varepsilon}{\nu_1}\|u\|^{p^+} - c_5\|u\|^{r^+} - c_6\|u\|^{r^-}, \end{aligned}$$

where c_5 and c_6 are two positive constants. Since $r^+ > r^- > p^+$ the above inequalities prove the first part of the lemma.

2. By condition (f2) we deduce that $F(x,t) \geqslant C|t|^\theta$, $\forall x \in \overline{\Omega}$, $t \in \mathbb{R}$ with $|t| \geqslant t_0$, where C is a positive constant.

Let $u_0 \in E$ be fixed, such that $|\{x \in \Omega;\ u_0(x) \geqslant t_0\}| > 0$. Let $t > 1$ and let M be a real number defined by $M = \sup\{|F(x,t)|;\ x \in \overline{\Omega},\ |t| \leqslant t_0\}$. We have

$$J(tu_0) = \varphi[tu_0]\int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla u_0|^{p(x)}\, dx - \lambda\int_\Omega F(x, t u_0)\, dx$$

$$
\begin{aligned}
&\leqslant \left\{2t^{p^+}\int_\Omega \frac{1}{p(x)}|\nabla u_0|^{p(x)}\,dx + \frac{p^-}{p^+}t^{(p^+)^2/p^-}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_0|^{p(x)}\,dx\right)^{\frac{p^+}{p^-}}\right.\\
&\qquad \left. + \frac{p^+}{p^-}t^{p^-}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_0|^{p(x)}\,dx\right)^{\frac{p^-}{p^+}}\right\}\\
&\qquad -\lambda C\int_{\{x\in\Omega;\, u_0(x)\geqslant t_0\}} t^\theta |u_0|^\theta\,dx + \lambda M|\Omega|,
\end{aligned}
$$

where $\varphi[u]$ was defined in relation (15). Taking into account that $\theta > (p^+)^2/p^-$ and passing to the limit as $t\to\infty$ we obtain that $\lim_{t\to\infty} J(tu_0) = -\infty$. We conclude that Lemma 2 holds true. □

Proof of Theorem 1. We set $\Gamma = \{\gamma \in C([0,1],E);\ \gamma(0)=0,\ \gamma(1)=e\}$, where $e\in E$ is given by Lemma 2, and

$$
\zeta = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)).
$$

According to the second part of Lemma 2 we know that $\|e\|>\rho$ so every path $\gamma\in\Gamma$ intersects the sphere $\|x\|=\rho$. Then Lemma 1 implies

$$
\zeta \geqslant \inf_{\|u\|=\rho} J(u) \geqslant a,
$$

with the constant $a>0$ given in Lemma 2.1, thus $\zeta>0$.

The mountain-pass theorem (see, e.g. [3]) implies the existence of a sequence $\{u_n\}\subset E$ such that

$$
J(u_n)\to\zeta \quad\text{and}\quad J'(u_n)\to 0. \tag{16}
$$

First, we show that $\{u_n\}$ is bounded in E. Assume by contradiction the contrary. Then, passing eventually to a subsequence, still denoted by $\{u_n\}$, we may assume that $\|u_n\|\to\infty$ as $n\to\infty$. Thus we may consider that $\|u_n\|>1$ for any integer n. Relations (16) and (7) imply that for n large enough it holds

$$
\begin{aligned}
1+\zeta+\|u_n\| &\geqslant J(u_n) - \frac{1}{\theta}\langle J'(u_n),u_n\rangle\\
&\geqslant 2\left(\frac{1}{p^+}-\frac{1}{\theta}\right)\|u_n\|^{p^-} + \left(\frac{1}{p^+}\|u_n\|^{p^-}\right)^{\frac{p^+}{p^-}-1}\left(\frac{p^-}{(p^+)^2}-\frac{1}{\theta}\right)\|u_n\|^{p^-}\\
&\quad + \left(\frac{1}{p^+}\|u_n\|^{p^-}\right)^{\frac{p^-}{p^+}-1}\left(\frac{1}{p^-}-\frac{1}{\theta}\right)\|u_n\|^{p^-}\\
&\quad + \lambda\int_{\{x\in\Omega;\, u_n(x)\geqslant t_0\}}\left[\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right]dx - M_1|\Omega|,
\end{aligned}
$$

where $M_1 = \nu_1\sup\{\left|\frac{1}{\theta}f(x,t)t - F(x,t)\right|;\ x\in\overline{\Omega},\ |t|\leqslant t_0\}$. Taking into account that condition (f2) holds true, dividing the above inequality by $\|u_n\|$ and passing to the limit as $n\to\infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E.

Since $\{u_n\}$ is bounded in E we deduce that there exists a subsequence, again denoted by $\{u_n\}$, and $u_0 \in E$ such that $\{u_n\}$ converges weakly to u_0 in E. We prove that $\{u_n\}$ converges strongly to u_0 in E.

To do that, first we point out that condition (f1) implies

$$|f(x,u_n)(u_n-u_0)| \leqslant c(|u_n-u_0|+|u_n|^{r(x)-1}|u_n-u_0|), \quad \forall\, x\in\Omega. \tag{17}$$

Since $r^+ < Np^-/(N-p^-)$ we deduce that E is compactly embedded in $L^{r(x)}(\Omega)$ thus $\{u_n\}$ converges strongly to u_0 in $L^{r(x)}(\Omega)$ and $L^1(\Omega)$. That fact combined with inequality (17) and Hölder's inequality yield

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u_0)\,dx = 0. \tag{18}$$

On the other hand, by relation (16) we get

$$\lim_{n\to\infty}\langle J'(u_n),u_n-u_0\rangle = 0. \tag{19}$$

Relations (18), (19) and the fact that $\{u_n\}$ is bounded in E imply that

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n-u_0)\,dx = 0.$$

The above relation and the fact that $\{u_n\}$ converges weakly to u_0 in E enable us to apply Theorem 3.1 in [11] in order to obtain that $\{u_n\}$ converges strongly to u_0 in E. Then, since $J\in C^0(E,\mathbb{R})\cap C^1(E\setminus\{0\},\mathbb{R})$ we conclude $J(u_n)\to J(u_0)$ and $J'(u_n)\to J'(u_0)$ as $n\to\infty$. We find $J(u_0)=\zeta>0$ and $J'(u_0)=0$ and thus u_0 is a nontrivial weak solution for problem (4). The proof of Theorem 1 is complete. □

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