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INTERVAL OSCILLATION THEOREMS FOR SECOND ORDER NONLINEAR PARTIAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. Using the integral averaging method and the generalized Riccati technique, we derive new interval oscillation criteria for second order nonlinear partial delay differential equations. These results are different from most known ones in the sense that they are based on information only on a sequence of subintervals of $[0,\infty)$, rather than on the whole $[0,\infty)$. Our results are of a high degree of generality and sharper than the existing results in literature.

1. Introduction

Consider the second order nonlinear partial delay differential equation

$$\frac{\partial}{\partial t}\left(r(t)\frac{\partial}{\partial t}u(x,t)\right)+p(t)\frac{\partial u(x,t)}{\partial t}=a(t)\Delta u(x,t)+\sum_{k=1}^{s}a_k(t)\Delta u(x,t-\rho_k(t))$$
$$-q(x,t)f(u(x,t))-\sum_{j=1}^{m}q_j(x,t)f_j(u(x,t-\sigma_j)),\quad (x,t)\in\Omega\times\mathbb{R}_+\equiv G \tag{1.1}$$

with the Robin boundary condition

$$\frac{\partial u(x,t)}{\partial \nu}+g(x,t)u(x,t)=0,\quad (x,t)\in\partial\Omega\times\mathbb{R}_+, \tag{1.2}$$

or the Dirichlet boundary condition

$$u(x,t)=0,\quad (x,t)\in\partial\Omega\times\mathbb{R}_+. \tag{1.3}$$

Here Δ is the Laplacian operator in $\mathbb{R}^N$, $\mathbb{R}_+=[0,\infty)$, Ω is a bounded domain in $\mathbb{R}^N$ with piecewise smooth boundary $\partial\Omega$, ν denotes the unit exterior normal vector to $\partial\Omega$, and $g(x,t)$ is a nonnegative continuous function on $\partial\Omega\times\mathbb{R}_+$.

It is assumed throughout this paper that:

(H1) $r\in C^1(\mathbb{R}_+,(0,\infty))$, $p\in C(\mathbb{R}_+,\mathbb{R})$;

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(H2) $q, q_j \in C(\bar{G}, \mathbb{R}_+)$, $q(t) = \min\limits_{x \in \bar{G}} q(x,t)$, $q_j(t) = \min\limits_{x \in \bar{G}} q_j(x,t)$, $j \in I_m = \{1, 2, \cdots, m\}$;

(H3) a, a_k, $\rho_k \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\lim\limits_{t \to \infty}(t - \rho_k(t)) = \infty$, σ_j are nonnegative constants, $j \in I_m$, $k \in I_s = \{1, 2, \cdots, s\}$;

(H4) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $f_j \in C(\mathbb{R}, \mathbb{R})$ are convex in $\mathbb{R}_+$ with $uf_j(u) > 0$, $uf(u) > 0$ and $f'(u) \geqslant \mu > 0$ for $u \neq 0, j \in I_m$.

We need the following definitions.

DEFINITION 1.1. By a solution of the problem (1.1), (1.2) (or (1.1), (1.3)) we mean a function $u(x,t) \in C^2(\bar{G} \times [t_{-1}, \infty); \mathbb{R}) \cap C(\bar{G} \times [\tilde{t}_{-1}, \infty); \mathbb{R})$ which satisfies (1.1) in the domain G and the corresponding condition, where

$$t_{-1} = \min\Big\{0, \min_{1 \leqslant k \leqslant s}\Big\{\inf_{t \geqslant 0}(t - \rho_k(t))\Big\}\Big\}, \quad \tilde{t}_{-1} = \min\Big\{0, -\max_{1 \leqslant j \leqslant m} \sigma_j\Big\}.$$

DEFINITION 1.2. The solution $u(x,t)$ of problem (1.1), (1.2) (or (1.1), (1.3)) is said to be oscillatory in the domain G if for any positive number T_0 there exists a point $(x_0, t_0) \in \Omega \times [T_0, \infty)$ such that $u(x_0, t_0) = 0$ holds.

The study of oscillatory behavior of solutions of partial functional differential equations is of both theoretical and practical importance. Some applications involving dynamics with spatial migrations, chemical reactions, control systems, combinatorics can be found in the monograph [13]. Various results on the oscillation for different types of second order partial delay differential equations have been obtained over the last couple of decades. For more details, we refer reader to the monograph [1], the papers [2-6,9-12,15] and the references therein. In particular, under the restriction

$$\int^{\infty} \frac{ds}{r(s)} = \infty, \tag{1.4}$$

recently, Wang, Meng and Liu [11] gave the Kamenev-type oscillation criteria [7] for Eq.(1.1) with $f(x) = f_j(x) = x$, $j \in I_m$, under the boundary condition (1.2). Furthermore, under the same restrict condition (1.4), Wang, Meng and Liu [12] also established the interval oscillation criteria [8] for Eq.(1.1) with $p(t) \equiv 0$ and $f(x) = f_j(x) = x$, $j \in I_m$. However, the obtained results in [11,12] can not apply to Eq.(1.1).

Motivated by the ideas in [8,11,12], employing the integral averaging method and the generalized Riccati technique, we shall derive new interval oscillation theorems for Eq.(1.1), thereby improving the main results in [11,12] with dropping the restrict condition (1.4). These results are different from most known ones in the sense that they are based on information only on a sequence of subintervals of $\mathbb{R}_+$, rather than on the whole half-linear. Our results are of a high degree of generality and sharper than the existing results in literature. In fact, by choosing appropriate functions H_1, H_2, ρ, Φ and ϕ, we shall present several easily verifiable oscillation criteria. Finally, some examples that point out the applications of our results are also included.

2. Main results

In the sequel, we say that a pair of functions (H_1,H_2) belong to a function class $\mathscr{H}$, denoted by $(H_1,H_2)\in\mathscr{H}$, if $H_1,H_2\in C(D,\mathbb{R}_+)$ satisfy

$$H_i(t,t)=0,\quad H_i(t,s)>0\quad \text{for}\quad t>s\geqslant t_0,\quad i=1,2,$$

where $D=\{(t,s):0<s\leqslant t<\infty\}$. Furthermore, the partial derivatives $\partial H_1(t,s)/\partial t$ and $\partial H_2(t,s)/\partial s$ exist on D, and there are h_1, $h_2\in C_{loc}(D,\mathbb{R})$ such that

$$\frac{\partial H_1}{\partial t}(t,s)=h_1(t,s)H_1(t,s)\quad\text{and}\quad \frac{\partial H_2}{\partial s}(t,s)=-h_2(t,s)H_2(t,s).$$

REMARK 2.1. Note that the functions H_1 and H_2 play slightly different form here from the papers in Kong [8] and Wang, Weng, Liu [12]. The reason is that we wish to establish more general interval oscillation theorems for Eq.(1.1).

Let $\rho\in C^1(\mathbb{R}_+,(0,\infty))$, $\theta\in C(\mathbb{R}_+,\mathbb{R})$, we define two integral operators $A_T^{\rho}(\theta,t)$ and $B_T^{\rho}(\theta,t)$ in term of H_i and ρ by

$$A_T^{\rho}(\theta,t)=\int_T^t H_1(s,T)\theta(s)\rho(s)\mathrm{d}s\ \text{ and }\ B_T^{\rho}(\theta,t)=\int_T^t H_2(t,s)\theta(s)\rho(s)\mathrm{d}s,\ t>T\geqslant 0.$$

To simplify notation, for functions $\rho,\Phi\in C^1(\mathbb{R}_+,(0,\infty))$ and $\phi\in C(\mathbb{R}_+,\mathbb{R})$, set, for $(t,s)\in D$,

$$\lambda_1:=\lambda_1(s,t)=\frac{\rho'(s)}{\rho(s)}+\frac{\Phi'(s)}{\Phi(s)}-\frac{p(s)}{r(s)}+2\mu\phi(s)+h_1(s,t),$$
$$\lambda_2:=\lambda_2(t,s)=\frac{\rho'(s)}{\rho(s)}+\frac{\Phi'(s)}{\Phi(s)}-\frac{p(s)}{r(s)}+2\mu\phi(s)-h_2(t,s),$$

and

$$g(s)=\frac{1}{\mu}r(s)\Phi(s),\qquad \psi(s)=\Phi(s)[q(s)+\mu r(s)\phi^2(s)-p(s)\phi(s)-(r(s)\phi(s))'].$$

Now, we state and prove the main results of this paper.

THEOREM 2.1. *If for each $T\geqslant 0$, there exist $(H_1,H_2)\in\mathscr{H}$, and the functions $\rho,\Phi\in C^1(\mathbb{R}_+,(0,\infty))$, $\phi\in C^1(\mathbb{R}_+,\mathbb{R})$, and real numbers $a,b,c\in\mathbb{R}_+$ such that $T\leqslant a<c<b$ and*

$$\frac{1}{H_1(c,a)}A_a^{\rho}\Big(\psi-\frac{1}{4}g\lambda_1^2,c\Big)+\frac{1}{H_2(b,c)}B_c^{\rho}\Big(\psi-\frac{1}{4}g\lambda_2^2,b\Big)>0,\tag{2.1}$$

then:
(i) *every solution u(x,t) of the problem* (1.1), (1.2) *is oscillatory in G;*
(ii) *every solution u(x,t) of the problem* (1.1), (1.3) *is oscillatory in G.*

Proof. (I) First, we prove the part (i). Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1.1), (1.2), which has no zero in $\Omega\times[t_0,\infty)$ for some $t_0>0$. Without loss of generality we may assume that $u(x,t)>0$, $u(x,t-\rho_k(t))>0$ and $u(x,t-\sigma_j)>0$ in $\Omega\times[t_1,\infty)$, $t_1\geqslant t_0$, $k\in I_s$, $j\in I_m$. Integrating (1.1) with respect to x over the domain Ω, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\Big(r(t)\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u(x,t)\mathrm{d}x\Big)+p(t)\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u(x,t)\mathrm{d}x\\
&\quad=a(t)\int_\Omega\Delta u(x,t)\mathrm{d}x+\sum_{k=1}^{s}a_k(t)\int_\Omega\Delta u(x,t-\rho_k(t))\mathrm{d}x\\
&\qquad-\int_\Omega q(x,t)f(u(x,t))\mathrm{d}x-\sum_{j=1}^{m}\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))\mathrm{d}x,\quad t\geqslant t_1.
\end{aligned}\tag{2.2}$$

It follows from Green's formula and the boundary condition (1.2) that

$$\int_\Omega\Delta u(x,t)\mathrm{d}x=\int_{\partial\Omega}\frac{\partial u(x,t)}{\partial\nu}\mathrm{d}s=-\int_{\partial\Omega}g(x,t)u(x,t)\mathrm{d}s\leqslant 0,\tag{2.3}$$

and

$$\begin{aligned}
\int_\Omega\Delta u(x,t-\rho_k(t))\mathrm{d}x&=\int_{\partial\Omega}\frac{\partial u(x,t-\rho_k(t))}{\partial\nu}\mathrm{d}s\\
&=-\int_{\partial\Omega}g(x,t-\rho_k(t))u(x,t-\rho_k(t))\mathrm{d}s\leqslant 0,
\end{aligned}\tag{2.4}$$

where $\mathrm{d}s$ denotes the surface element on $\partial\Omega$. Moreover, from (H2), (H4), and Jensen's inequality, we have

$$\int_\Omega q(x,t)f(u(x,t))\mathrm{d}x\geqslant q(t)\int_\Omega f(u(x,t))\mathrm{d}x\geqslant|\Omega|q(t)f\Big(\frac{1}{|\Omega|}\int_\Omega u(x,t)\mathrm{d}x\Big),\tag{2.5}$$

and

$$\begin{aligned}
\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))\mathrm{d}x&\geqslant q_j(t)\int_\Omega f_j(u(x,t-\sigma_j))\mathrm{d}x\\
&\geqslant|\Omega|q_j(t)f_j\Big(\frac{1}{|\Omega|}\int_\Omega u(x,t-\sigma_j)\mathrm{d}x\Big),
\end{aligned}\tag{2.6}$$

where $|\Omega|=\int_\Omega\mathrm{d}x$. Define

$$v(t)=\frac{1}{|\Omega|}\int_\Omega u(x,t)\mathrm{d}x,\quad t\geqslant t_1.\tag{2.7}$$

In view of (2.3)-(2.7), (2.2) yields that

$$(r(t)v'(t))'+p(t)v'(t)+q(t)f(v(t))+\sum_{j=1}^{m}q_j(t)f_j(v(t-\sigma_j))\leqslant 0,\quad t\geqslant t_1.\tag{2.8}$$

Note that (H4), it follows from (2.8) that

$$(r(t)v'(t))' + p(t)v'(t) + q(t)f(v(t)) \leqslant 0, \quad t \geqslant t_1. \tag{2.9}$$

Put

$$w(t) = \Phi(t)\Big[\frac{r(t)v'(t)}{f(v(t))} + r(t)\phi(t)\Big], \quad t \geqslant t_1. \tag{2.10}$$

Then, by (2.9),

$$\begin{aligned} w'(t) &\leqslant \frac{\Phi'(s)}{\Phi(s)}w(t) - \Phi(t)\Big[q(t) + \frac{r(t)f'(v(t))v'^2(t)}{f^2(v(t))} + \frac{p(t)v'(t)}{f(v(t))} - (r(t)\phi(t))'\Big] \\ &\leqslant \frac{\Phi'(s)}{\Phi(s)}w(t) - \Phi(t)\Big[q(t) + \mu r(t)\Big(\frac{w(t)}{r(t)\Phi(t)} - \phi(t)\Big)^2 + \frac{p(t)}{r(t)\Phi(t)}w(t) \\ &\quad - p(t)\phi(t) - (r(t)\phi(t))'\Big] \\ &= -\psi(t) + \Big(\frac{\Phi'(s)}{\Phi(s)} - \frac{p(s)}{r(s)} + 2\mu\phi(s)\Big)w(t) - \frac{1}{g(t)}w^2(t), \end{aligned}$$

i.e.,

$$\psi(s) \leqslant -w'(s) + \Big(\frac{\Phi'(s)}{\Phi(s)} - \frac{p(s)}{r(s)} + 2\mu\phi(s)\Big)w(s) - \frac{1}{g(s)}w^2(s). \tag{2.11}$$

Applying the operator $A_t^{\rho}(\cdot,c)$ to (2.11), $t \in [a,c)$, we get

$$\begin{aligned} A_t^{\rho}(\psi,c) &\leqslant -H_1(c,t)\rho(c)w(c) + A_t^{\rho}(\lambda_1 w,c) - A_t^{\rho}(g^{-1}w^2,c) \\ &= -H_1(c,t)\rho(c)w(c) - A_t^{\rho}\Big(g^{-1}\Big(w - \frac{1}{2}g\lambda_1\Big)^2,c\Big) + A_t^{\rho}\Big(\frac{1}{4}g\lambda_1^2,c\Big) \\ &\leqslant -H_1(c,t)\rho(c)w(c) + A_t^{\rho}\Big(\frac{1}{4}g\lambda_1^2,c\Big). \end{aligned}$$

Letting $t \to a^+$ in above, and dividing by $H_1(c,a)$, we obtain

$$\frac{1}{H_1(c,a)}A_a^{\rho}\Big(\psi - \frac{1}{4}g\lambda_1^2,c\Big) \leqslant -\rho(c)w(c). \tag{2.12}$$

Similarly, applying the operator $B_c^{\rho}(\cdot,t)$ to (2.11), $t \in [c,b)$, and proceeding as in the proof of (2.12), we have

$$\frac{1}{H_2(b,c)}B_c^{\rho}\Big(\psi - \frac{1}{4}g\lambda_2^2,b\Big) \leqslant \rho(c)w(c). \tag{2.13}$$

We now claim that every nontrivial solution of the differential inequality (2.8) has at least one zero in (a,b).

Suppose the contrary. Without loss of generality we may assume that there is a solution of (2.8) that $v(t) > 0$ for $t \in (a,b)$. Adding (2.12) and (2.13), we get the inequality which contradicts the assumption (2.1). Thus the claim holds.

Pick up a sequence $\{T_i\} \subset [t_0,\infty)$ such that $T_i \to \infty$ as $i \to \infty$. By the assumptions, for each $i \in \mathbb{N}$, there exist $a_i,\ b_i, c_i \in \mathbb{R}_+$ such that $T_i \leqslant a_i < c_i < b_i$, and (2.1) holds with a,b,c replaced by a_i,b_i,c_i, respectively. From that, every nontrivial solution $v(t)$ of (2.8) has at least one zero $t_i \in (a_i,b_i)$. Noting that $t_i > a_i \geqslant T_i,\ i \in \mathbb{N}$, we see that every solution $v(t)$ has arbitrarily large zero. This contradicts the fact that $v(t)$ is nonoscillatory by (2.8) and the assumption $u(x,t) > 0$ in $\Omega \times [t_0,\infty)$ for some $t_0 > 0$. Hence, every solution of problem (1.1), (1.2) is oscillatory in G.

(II) To prove the part (ii), the following fact will be used, see [14]. The smallest eigenvalue λ of the Dirichlet problem,

$$\begin{cases} \Delta u(x) + \lambda u(x) = 0,\ x \in \Omega, \\ u(x) = 0,\ x \in \partial\Omega, \end{cases} \tag{2.14}$$

is positive and the corresponding eigenfunction $\varphi(x)$ is positive in Ω.

Next we prove the part (ii). Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1.1), (1.3), which has no zero in $\Omega \times [t_0,\infty)$ for some $t_0 > 0$. Without loss of generality we may assume that $u(x,t) > 0, u(x,t-\rho_k(t)) > 0$ and $u(x,t-\sigma_j) > 0$ in $\Omega \times [t_1,\infty)$, $t_1 \geqslant t_0$, $k \in I_s$, $j \in I_m$. Multiplying (1.1) by $\varphi(x) > 0$ and integrating with respect to x over the domain Ω, we have, for $t \geqslant t_1$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}\Big(r(t)\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u(x,t)\varphi(x)\mathrm{d}x\Big) + p(t)\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u(x,t)\varphi(x)\mathrm{d}x \\ &\quad = a(t)\int_\Omega \Delta u(x,t)\varphi(x)\mathrm{d}x + \sum_{k=1}^{s} a_k(t)\int_\Omega \Delta u(x,t-\rho_k(t))\varphi(x)\mathrm{d}x \\ &\qquad - \int_\Omega q(x,t)f(u(x,t))\varphi(x)\mathrm{d}x - \sum_{j=1}^{m}\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))\varphi(x)\mathrm{d}x. \end{aligned} \tag{2.15}$$

It follows from Green's formula and the boundary condition (1.3) that

$$\int_\Omega \Delta u(x,t)\varphi(x)\mathrm{d}x = \int_\Omega u(x,t)\Delta\varphi(x)\mathrm{d}x = -\lambda\int_\Omega u(x,t)\varphi(x)\mathrm{d}x \leqslant 0, \tag{2.16}$$

and

$$\begin{aligned} \int_\Omega \Delta u(x,t-\rho_k(t))\varphi(x)\mathrm{d}x &= \int_\Omega u(x,t-\rho_k(t))\Delta\varphi(x)\mathrm{d}x \\ &= -\lambda\int_\Omega u(x,t-\rho_k(t))\varphi(x)\mathrm{d}x \leqslant 0. \end{aligned} \tag{2.17}$$

Moreover, from (H2), (H4), and Jensen's inequality, we have

$$\begin{aligned} &\int_\Omega q(x,t)f(u(x,t))\varphi(x)\mathrm{d}x \geqslant q(t)\int_\Omega f(u(x,t))\varphi(x)\mathrm{d}x \\ &\quad \geqslant \Big[q(t)\int_\Omega \varphi(x)\mathrm{d}x\Big]\Big[f\Big(\int_\Omega u(x,t)\varphi(x)\mathrm{d}x\Big(\int_\Omega\varphi(x)\mathrm{d}x\Big)^{-1}\Big)\Big], \end{aligned} \tag{2.18}$$

and

$$\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))\varphi(x)\mathrm{d}x \geqslant q_j(t)\int_\Omega f_j(u(x,t-\sigma_j))\varphi(x)\mathrm{d}x$$
$$\geqslant \left[q_j(t)\int_\Omega \varphi(x)\mathrm{d}x\right]\left[f_j\Big(\int_\Omega u(x,t-\sigma_j)\varphi(x)\mathrm{d}x\Big(\int_\Omega \varphi(x)\mathrm{d}x\Big)^{-1}\Big)\right]. \tag{2.19}$$

Define

$$v(t) = \Big(\int_\Omega u(x,t)\varphi(x)\mathrm{d}x\Big)\Big(\int_\Omega \varphi(x)\mathrm{d}x\Big)^{-1}, \quad t \geqslant t_1. \tag{2.20}$$

In view of (2.16)-(2.20), (2.15) yields that

$$(r(t)v'(t))' + p(t)v'(t) + q(t)f(v(t)) + \sum_{j=1}^m q_j(t)f_j(v(t-\sigma_j)) \leqslant 0, \quad t \geqslant t_1. \tag{2.21}$$

From (H4) and (2.21), we obtain

$$(r(t)v'(t))' + p(t)v'(t) + q(t)f(v(t)) \leqslant 0, \quad t \geqslant t_1. \tag{2.22}$$

The remainder of the proof is similar to that of the part (I), we omit the details. Hence, the proof of Theorem 2.1 is complete. □

REMARK 2.2. As far as the author knows, most relevant results in literature suppose $H_1(t,s) = H_2(t,s)$, i.e., the same weighting function is used in the obtained oscillation criteria, see [8,12]. Hence, the results established in current paper is more general than the existing results [12]. Moreover, our theorems also take the Dirichlet boundary condition (1.3) into account.

As an immediate consequence of Theorem 2.1 we get the following oscillation criterion.

THEOREM 2.2. *For some functions* $(H_1,H_2) \in \mathscr{H}$, $\rho, \Phi \in C^1(\mathbb{R}_+,(0,\infty))$, $\phi \in C^1(\mathbb{R}_+,\mathbb{R})$, *and for each* $T \geqslant 0$, *if*

$$\limsup_{t\to\infty} A_T^\rho\Big(\psi - \frac{1}{4}g\lambda_1^2, t\Big) > 0, \tag{2.23}$$

and

$$\limsup_{t\to\infty} B_T^\rho\Big(\psi - \frac{1}{4}g\lambda_2^2, t\Big) > 0, \tag{2.24}$$

then the conclusions of Theorem 2.1 hold.

Proof. For any $T \geqslant 0$, let $a = T$. In (2.23), we choose $T = a$. Then there exists $c > a$ such that

$$A_a^\rho\Big(\psi - \frac{1}{4}g\lambda_1^2, c\Big) > 0. \tag{2.25}$$

In (2.24), we choose $T = c$. Then there exists $b > c$ such that

$$B_c^{\rho}\Big(\psi - \frac{1}{4}g\lambda_2^2, b\Big) > 0. \tag{2.26}$$

Combining (2.25) and (2.26), we obtain (2.1) holds. The conclusions thus come from Theorem 2.1, and the proof is completed. □

The case when $(H_1,H_2) \in \mathscr{H}$ with $H_i := H_i(t-s)$, $i = 1,2$, we define the subclass of the class $\mathscr{H}$ containing such (H_1,H_2) by $\mathscr{H}_0$. Applying Theorem 2.1 to $(H_1,H_2) \in \mathscr{H}_0$, we obtain the following result.

THEOREM 2.3. *If for each $T \geqslant 0$, there exist functions $(H_1,H_2) \in \mathscr{H}_0$, $\Phi \in C^1(\mathbb{R}^+,(0,\infty))$, and $\phi \in C^1(\mathbb{R}_+,\mathbb{R})$ such that $T \leqslant a < c$ and*

$$\begin{aligned}&\frac{1}{H_1(c-a)}\int_a^c \rho(s)H_1(s-a)\Big[\psi(s) - \frac{1}{4}g(s)h_1^2(s-a)\Big]\mathrm{d}s\\ &\quad + \frac{1}{H_2(c-a)}\int_a^c \rho(2c-s)H_2(s-a)\Big[\psi(2c-s) - \frac{1}{4}g(2c-s)h_2^2(s-a)\Big]\mathrm{d}s > 0,\end{aligned} \tag{2.27}$$

where

$$\rho(s)\Phi(s) = \exp\Big(\int_{t_0}^s \Big(\frac{p(t)}{r(t)} - 2\mu\phi(t)\Big)\mathrm{d}t\Big),$$

then the conclusions of Theorem 2.1 hold.

Proof. In view of the definition of $\rho(t)$, we get

$$\lambda_1(s,t) = h_1(s,t) \quad \text{and} \quad \lambda_2(t,s) = -h_2(t,s).$$

On the other hand, let $b = 2c - a$. Then

$$H_i(b-c) = H_i(c-a) = H_i\Big(\frac{1}{2}(b-a)\Big), \quad i = 1,2.$$

So that for any $W \in L[a,b]$, we have

$$\int_a^c W(s)\mathrm{d}s = \int_c^b W(2c-s)\mathrm{d}s.$$

Thus, that (2.27) holds implies that (2.1) hold for $(H_1,H_2) \in \mathscr{H}_0$. Hence, the conclusion of Theorem 2.3 follows from Theorem 2.1. □

COROLLARY 2.1. *If there exists two constants $\alpha, \beta > 1$ such that for all $T \geqslant 0$,*

$$\limsup_{t\to\infty}\int_T^t \exp\Big(\int_{t_0}^s \frac{p(\tau)}{r(\tau)}\mathrm{d}\tau\Big)\Big[(s-T)^{\alpha}q(s) - \frac{\alpha^2}{4\mu}r(s)(s-T)^{\alpha-2}\Big]\mathrm{d}s > 0, \tag{2.28}$$

and

$$\limsup_{t\to\infty}\int_T^t \exp\Big(\int_{t_0}^s \frac{p(\tau)}{r(\tau)}\mathrm{d}\tau\Big)\Big[(t-s)^{\beta}q(s) - \frac{\beta^2}{4\mu}r(s)(t-s)^{\beta-2}\Big]\mathrm{d}s > 0, \tag{2.29}$$

then the conclusions of Theorem 2.1 hold.

Proof. In Theorem 2.2, let

$$\rho(t) = \exp\left(\int_{t_0}^{t} \frac{p(\tau)}{r(\tau)} \mathrm{d}\tau\right),\ \Phi(t) = 1,\ \phi(t) = 0,$$
$$H_1(t,s) = (t-s)^{\alpha},\ \text{ and }\ H_2(t,s) = (t-s)^{\beta}.$$

Then

$$\psi(t) = q(t),\ g(t) = \frac{1}{\mu} r(t),\ h_1(t,s) = \frac{\alpha}{t-s},\ \text{ and }\ h_2(t,s) = \frac{\beta}{t-s}.$$

Therefore, by Theorem 2.2, the conclusions of Corollary 2.1 hold. □

Next, we define:

$$\rho(t) = \exp\left(\int_{t_0}^{t} \frac{p(\tau)}{r(\tau)} \mathrm{d}\tau\right),\ R(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r(s)\rho(s)},$$
$$H_1(t,s) = [R(t) - R(s)]^{\alpha},\ \text{ and }\ H_2(t,s) = [R(t) - R(s)]^{\beta}.$$

COROLLARY 2.2. *Let* $\lim_{t\to\infty} R(t) = \infty$. *If there exist constants* $\alpha, \beta > 1$ *such that for all* $T \geqslant 0$,

$$\limsup_{t\to\infty} \frac{1}{R^{\alpha-1}(t)} \int_T^t q(s)[R(s) - R(T)]^{\alpha} \exp\left(\int_{t_0}^{s} \frac{p(\tau)}{r(\tau)} \mathrm{d}\tau\right) \mathrm{d}s > \frac{\alpha^2}{4\mu(\alpha-1)}, \tag{2.30}$$

and

$$\limsup_{t\to\infty} \frac{1}{R^{\beta-1}(t)} \int_T^t q(s)[R(t) - R(s)]^{\beta} \exp\left(\int_{t_0}^{s} \frac{p(\tau)}{r(\tau)} \mathrm{d}\tau\right) \mathrm{d}s > \frac{\beta^2}{4\mu(\beta-1)}, \tag{2.31}$$

then the conclusions of Theorem 2.1 hold.

Proof. In Theorem 2.2, let $\Phi(t) = 1$, $\phi(t) = 0$. Then $\psi(t) = q(t)$, $g(t) = r(t)/\mu$, and

$$h_1(t,s) = \frac{\alpha}{[R(t) - R(s)] r(t)\rho(t)},\ h_2(t,s) = \frac{\beta}{[R(t) - R(s)] r(s)\rho(s)}.$$

Note that

$$A_T^{\rho}(r\lambda_1^2, t) = \frac{\alpha^2}{\alpha-1}[R(t) - R(T)]^{\alpha-1},\ B_T^{\rho}(r\lambda_2^2, t) = \frac{\beta^2}{\beta-1}[R(t) - R(T)]^{\beta-1}.$$

So that, in view of $\lim_{t\to\infty} R(t) = \infty$, it follows that

$$\lim_{t\to\infty} \frac{1}{R^{\alpha-1}(t)} A_T^{\rho}(r\lambda_1^2, t) = \frac{\alpha^2}{\alpha-1}, \tag{2.32}$$

and

$$\lim_{t\to\infty} \frac{1}{R^{\beta-1}(t)} B_T^{\rho}(r\lambda_2^2, t) = \frac{\beta^2}{\beta-1}. \tag{2.33}$$

From (2.30) and (2.32), we have that

$$\limsup_{t\to\infty}\frac{1}{R^{\alpha-1}(t)}A_T^{\rho}\Big(\psi-\frac{1}{4}g\lambda_1^2,t\Big)$$
$$=\limsup_{t\to\infty}\frac{1}{R^{\alpha-1}(t)}\int_T^t q(s)[R(s)-R(T)]^{\alpha}\exp\Big(\int_{t_0}^s\frac{p(\tau)}{r(\tau)}\mathrm{d}\tau\Big)\mathrm{d}s-\frac{\alpha^2}{4\mu(\alpha-1)}>0,$$

i.e., (2.23) holds. Similarly, (2.31) implies that (2.24) holds. By Theorem 2.2, the conclusion of Corollary 2.2 holds. The proof is complete. □

REMARK 2.3. The theorems above presented in the form of a high degree of generality. They extend, improve, and complement a number of existing results in [11,12] and handle some case not covered by the known criteria. They also give rather wide possibilities of deriving different explicit oscillation criteria for (1.1), (1.2) (or (1.1), (1.3)) with appropriate choices of the functions H_i, ρ, Φ and ϕ.

REMARK 2.4. We drop the restriction "$\int^{\infty} 1/r(s)ds=\infty$" in [11,12].

3. Examples

In final section, we give three examples to illustrate the applications of the main results established in precede section.

EXAMPLE 3.1. Consider the equation

$$\frac{\partial}{\partial t}\Big(\frac{1}{t}\frac{\partial}{\partial t}u(x,t)\Big)+\frac{2}{t^2}\frac{\partial u(x,t)}{\partial t}=\frac{1}{t}\Delta u(x,t)+\frac{1}{t^2}\Delta u(x,t-\frac{3}{2}\pi)$$
$$+\frac{2}{t}\Delta u(x,t-\pi)-\frac{2}{t}u(x,t)-\frac{2}{t^2}u(x,t-\frac{\pi}{2}),\quad (x,t)\in(0,\pi)\times[1,\infty)\equiv G, \tag{3.1}$$

with the boundary condition

$$u(0,t)=u(\pi,t)=0,\ \ t\geqslant 1, \tag{3.2}$$

where

$$N=1,\ s=2,\ m=1,\ r(t)=\frac{1}{t},\ p(t)=\frac{2}{t^2},\ q(t)=\frac{2}{t},$$
$$f'(u)=1=\mu,\ a(t)=\frac{1}{t},\ a_1(t)=\frac{3}{t^2},\ a_2(t)=\frac{2}{t},\ q_1(x,t)=\frac{2}{t^2},$$

and $\rho_1(t)=3\pi/2$, $\rho_2(t)=\pi$, and $\sigma_1=\pi/2$. If we take $\rho(s)=s^2$, $\Phi(t)=1$, $\phi(s)=0$, and $H_1(t,s)=H_2(t,s)=(t-s)^2$, then

$$g(s)=\frac{1}{s},\quad \psi(s)=\frac{2}{s},\quad \lambda_1(t,s)=-\lambda_2(t,s)=\frac{2}{t-s}.$$

It is easily computed that

$$\begin{aligned}\limsup_{t\to\infty} A_T^{\rho}(\psi-\frac{1}{4}g\lambda_1^2,t) &= \limsup_{t\to\infty}\int_T^t \Big[2s(s-T)^2-s\Big]\mathrm{d}s\\ &= \limsup_{t\to\infty}\Big[\frac{1}{2}t^4-\frac{4}{3}t^3T+\Big(T^2-\frac{1}{2}\Big)t^2+\frac{1}{2}T^2-\frac{1}{6}T^4\Big]>0,\end{aligned}$$

and

$$\begin{aligned}\limsup_{t\to\infty} B_T^{\rho}(\psi-\frac{1}{4}g\lambda_2^2,t) &= \limsup_{t\to\infty}\int_T^t \Big[2s(t-s)^2-s\Big]\mathrm{d}s\\ &= \limsup_{t\to\infty}\Big[\frac{1}{6}t^4-\Big(T^2+\frac{1}{2}\Big)t^2+\frac{4}{3}tT^3+\frac{1}{2}T^2-\frac{1}{2}T^4\Big]>0.\end{aligned}$$

Thus, (2.23) and (2.24) hold for any $T\geqslant 1$. Therefore, by Theorem 2.2, every solution $u(x,t)$ of the problem (3.1),(3.2) is oscillatory in G. For example, $u(x,t)=\sin x\cos t$ is such a solution.

EXAMPLE 3.2. Consider the equation

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial t^2} &= \Delta u(x,t)+\frac{1}{t}\Delta u(x,t-\frac{\pi}{2})+\frac{1}{t}\Delta u(x,t-\frac{3}{2}\pi)-q(t)u(x,t)[1+\varepsilon u^2(x,t)]\\ &\quad -q(t)u(x,t-\pi),\quad (x,t)\in(0,\pi)\times[1,\infty)\equiv G,\end{aligned}\tag{3.3}$$

with the boundary condition

$$u(0,t)=u(\pi,t)=0,\ \ t\geqslant 1,\tag{3.4}$$

where

$$\begin{gathered}N=1,\ s=2,\ m=1,\ r(t)=1,\ p(t)=0,\\ f'(u)=1+3\varepsilon u^2\geqslant 1=\mu,\ \varepsilon\geqslant 0,\ a(t)=1,\ a_1(t)=a_2(t)=\frac{1}{t},\\ q(x,t)=q_1(x,t)=q(t),\ \rho_1(t)=\frac{\pi}{2},\ \rho_2(t)=\frac{3}{2}\pi,\ \sigma_1=\pi,\end{gathered}$$

and for $n\in\mathbb{N}$, $q(t)$ is defined by

$$q(t)=\begin{cases}5(t-3n), & 3n\leqslant t\leqslant 3n+1,\\ 5(-t+3n+2), & 3n+1<t\leqslant 3n+2,\\ |\sin t|, & 3n+2<t\leqslant 3n+3.\end{cases}$$

For any $T\geqslant 1$, there exists $n\in\mathbb{N}$, such that $3n>T$. Let $a=3n, c=3n+1, \Phi(s)=1, \phi(s)=0$, and $H_1(t,s)=H_2(t,s)=(t-s)^2$. Then

$$\rho(s)=1,\ g(s)=1,\ \psi(s)=q(s),\ h_1(t,s)=-h_2(t,s)=\frac{2}{t-s}.$$

Thus, the left-hand side of (2.27) takes the form

$$\begin{aligned}&\int_{3n}^{3n+1}(s-3n)^2\Big[q(s)+q(6n+2-s)-\frac{2}{(s-3n)^2}\Big]\mathrm{d}s\\&\quad=\int_{3n}^{3n+1}[10(s-3n)^3-2]\mathrm{d}s=\frac{1}{2}>0,\end{aligned}$$

i.e., (2.27) holds. Hence, every solution $u(x,t)$ of the problem (3.3),(3.4) is oscillatory in G by Theorem 2.3. For example, if $\varepsilon=0$, $u(x,t)=\sin x\cos t$ is such a solution.

EXAMPLE 3.3. Consider the equation

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{1}{t}\frac{\partial u(x,t)}{\partial t}&=\Delta u(x,t)+\frac{1}{t}\Delta u(x,t-\frac{3}{2}\pi)-\frac{\kappa}{t^2\ln^2 t}u(x,t)[1+\varepsilon u^2(x,t)]\\&\quad-\frac{\kappa}{t^2\ln^2 t}u(x,t-\pi),\quad (x,t)\in(0,\pi)\times(1,\infty)\equiv G,\end{aligned}\tag{3.5}$$

with the boundary condition

$$u_x(0,t)=u_x(\pi,t)=0,\ \ t>1,\tag{3.6}$$

where

$$\begin{aligned}&N=s=m=1,\ r(t)=1,\ p(t)=\frac{1}{t},\ q(t)=q(x,t)=q_1(x,t)=\frac{\kappa}{t^2\ln^2 t},\ \kappa>0,\\&f'(u)=1+3\varepsilon u^2\geqslant 1=\mu,\ \varepsilon\geqslant 0,\ a(t)=1,\ a_1(t)=\frac{1}{t},\ \rho_1(t)=\frac{3}{2}\pi,\ \sigma_1=\pi.\end{aligned}$$

Note that $\rho(t)=t$ and $R(t)=\log t$. For $\alpha>1$, the left-hand side of (2.30) takes the form

$$\limsup_{t\to\infty}\frac{\kappa}{R^{\alpha-1}(t)}\int_T^t[R(s)-R(T)]^{\alpha}\frac{1}{s\ln^2 s}\mathrm{d}s=\frac{\kappa}{(\alpha-1)}\tag{3.7}$$

According to Lemma 3.1 in [8], we have

$$\int_T^t[R(t)-R(s)]^{\alpha}\frac{1}{s\ln^2 s}\mathrm{d}s\geqslant\int_T^t[R(s)-R(T)]^{\alpha}\frac{1}{s\ln^2 s}\mathrm{d}s.\tag{3.8}$$

So, the left-hand side of (2.31) takes the form

$$\begin{aligned}&\limsup_{t\to\infty}\frac{\kappa}{R^{\beta-1}(t)}\int_T^t[R(t)-R(s)]^{\beta}\frac{1}{s\ln^2 s}\mathrm{d}s\\&\quad\geqslant\limsup_{t\to\infty}\frac{\kappa}{R^{\beta-1}(t)}\int_T^t[R(s)-R(T)]^{\beta}\frac{1}{s\ln^2 s}\mathrm{d}s=\frac{\kappa}{(\beta-1)}.\end{aligned}\tag{3.9}$$

By (3.7) and (3.9), for any $\kappa>1$, there are $\alpha>1$, $\beta>1$ such that

$$\frac{\kappa}{(\alpha-1)}>\frac{\alpha^2}{4(\alpha-1)}\quad\text{and}\quad\frac{\kappa}{(\beta-1)}>\frac{\alpha^2}{4(\beta-1)}.$$

Thus, (2.30) and (2.31) hold for any $T\geqslant 1$. Therefore, by Corollary 2.2, every solution $u(x,t)$ of the problem (3.5), (3.6) is oscillatory in G for $\kappa>1$. For example, if $\varepsilon=0$, $u(x,t)=\cos x\sin t$ is such a solution.

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