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GENERALIZED TIME–PERIODIC SOLUTIONS TO THE EULER EQUATIONS OF COMPRESSIBLE FLUIDS

SVETLIN GEORGIEV AND PHILIPPE G. LEFLOCH

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Abstract. We introduce a notion of generalized time-periodic solutions to first-order hyperbolic systems in one space dimension and we establish the existence of such solutions to the Euler equations of compressible fluid dynamics for a large class of equations of state.

1. Introduction

We consider first-order systems of partial differential equations

$$\partial_t G(u) + \partial_x F(u) = 0, \tag{1.1}$$

where $t \geqslant 0$ and $x \in \mathbb{R}$ are the dependent variables and $u = (u_j(t,x))_{1 \leqslant j \leqslant p}$ is the unknown, while $F, G : \mathbb{R}^p \to \mathbb{R}^p$ are smooth given maps. Such systems arise in many areas of continuum physics and mathematical physics. Our primary interest in this paper is in the Euler equations of compressible fluid dynamics, discussed below.

We are interested in constructing time-periodic solutions to such systems. Rather than looking for solutions understood in a standard sense, which would lead to a rather narrow class of periodic solutions, we introduce here a concept of "generalized periodic solutions", which are motivated by the theory of quasi-periodic solutions and should provide approximations of arbitrary solutions to (1.1).

Recall that in the past forty years, there has been an active research about periodic solutions to partial differential equations, using various techniques and methods. Research on this problem began in the 40's and was carried out mostly by physicists. Eventually, in the 60's mathematicians began a systematic study of the existence and qualitative properties of periodic solutions; see [1]–[4], [7,10] and the references therein. Formal power series were introduced in order to approximate general periodic solutions. The present work contributes to this investigation and describes a new technique which applies to the Euler equations.

To tackle the above problem, we rely on the fixed point theory for compact operators developed by Krasnosel'skii and Zabrejko [8]. This theory was originally introduced to establish the existence of time-periodic solutions to the Korteweg-de Vries equation [5] and was later used for a class of nonlinear parabolic equations [6].

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The notion of generalized periodic solutions to (1.1) is defined as follows. We consider vector-valued mappings of the form $u(t,x) = (a_j(x)b_j(t,x))$, in which $a = (a_j(x))$ is a given continuously differentiable, vector-valued map and $b = (b_j(t,x))$ is the unknown. The mapping u is called a *generalized time-periodic solution* to (1.1) if for some period $\omega > 0$ the functions $t \mapsto b_j(t,x)$ are ω-periodic and satisfy the following ordinary differential equations in the variable t,

$$D_u G(ab)\, b_t\, a + D_u F(ab)\, a_x\, b = 0$$

with $ab = (a_j b_j), \ldots$, where x is a real parameter. Actual solutions should be obtained as suitable series based on such generalized solutions. The above notion is motivated by the earlier work [9] where semilinear wave equations and almost periodic breathers are dealt with.

In this paper, we focus attention on the Euler equations of compressible fluid dynamics

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = 0. \end{aligned} \tag{1.2}$$

Here, the main unknowns are the density $\rho \geqslant 0$ and the velocity u of the fluid, while the pressure $p = p(\rho) \geqslant 0$ is a given smooth function.

Based on the above definition we search for solutions of the form $(\rho,u)(t,x) = \big(a(x)\,b(t,x), f(x)\,g(t,x)\big)$. The functions a, f being given, we need to determine two periodic functions $b = b(t,x)$, $g = g(t,x)$ that satisfy the coupled ordinary differential system

$$\begin{aligned} \frac{b_t}{bg} &= -\frac{a_x f + a f_x}{a}, \\ (bg)_t &= -bg^2\left(\frac{a_x}{a} f + 2 f_x\right) - p_\rho(ab)\,\frac{a_x b}{af}. \end{aligned} \tag{1.3}$$

Here, the independent variable is $t \in [0,\omega]$ for some $\omega > 0$, and x is a fixed parameter.

DEFINITION 1.1. A pair of functions $a, f : \mathbb{R} \to \mathbb{R}$ is called an admissible data if there exist constants $\omega > 0$, $m_1 \in (M_1, 1)$, and $m_2 \in (M_2, 1)$ satisfying

$$m_1\,\omega\,\frac{e^{(m_1 - M_1)\omega}}{1 - e^{-M_1\omega}} < 1, \qquad m_2\,\omega\,\frac{e^{(m_2 - M_2)\omega}}{1 - e^{-M_2\omega}} < 1,$$

and if f, a are continuously differentiable functions satisfying the positivity and growth conditions

$$\begin{aligned} &a > 0, \quad a_x < 0, \qquad f > 0, \quad f_x < 0, \\ &M_1 \leqslant -\frac{a_x}{a} \leqslant m_1, \qquad M_2 \leqslant -\frac{f_x}{f} \leqslant m_2. \end{aligned} \tag{1.4}$$

Our main result is as follows.

THEOREM 1.2. (Time-periodic generalized solutions) *Consider the Euler equations of isentropic fluid dynamics, in which the pressure function* $p : [0,\infty) \to \mathbb{R}$ *is continuously differentiable,* p_ρ *is concave, and*

$$k\rho^\gamma \leqslant p_\rho(\rho) \leqslant l\rho^\gamma \quad (\rho \geqslant 0),$$

where $k,l > 0$ *and* $\gamma \in (0,\infty)$ *are constants. Let* a,f *be admissible data associated with the constants* ω, m_1, m_2. *Then, provided* m_1, m_2 *are sufficiently small (Precise conditions are given in the proof below.), the system* (1.3) *admits precisely one non-trivial, continuously differentiable, generalized periodic solution of the form*

$$(\rho,u) = \big(a(x)\,b(t,x), f(x)\,g(t,x)\big), \qquad t \geqslant 0, x \in \mathbb{R}, \tag{1.5}$$

in which the functions b,g *are positive, continuously differentiable in each variable, and* ω*-periodic in time.*

In particular, our theorem covers the case of linear equations of state, describing isothermal fluids. The paper is organized as follows. In Sections 2 and 3, we formulate the problem under consideration as a fixed-point problem. In Section 4 we provide a proof of Theorem 1.2.

2. Formulation of the problem

From now on, the functions a and f are given admissible data. They depend on the variable x which we regard as a fixed parameter and, in the notation, we often suppress the dependence in x. We are also given a period $\omega > 0$. Concerning the pressure function p, at this stage we only need to assume that

$$p \in \mathscr{C}^1(\mathbb{R}_+), \qquad p_\rho(\rho) > 0 \quad (\rho > 0). \tag{2.1}$$

Suppose that (1.5) is a positive solution to the system (1.3) and is continuous, ω-periodic in the time variable t. Then, it is straighforward to check that

$$\rho = \chi_1(\rho,u), \qquad u = \chi_2(\rho,u), \tag{2.2}$$

where, for every fixed $x \in \mathbb{R}$,

$$\chi_1(\rho,u)(t,x) := \frac{e^{\frac{a'(x)}{a(x)}\omega}}{1-e^{\frac{a'(x)}{a(x)}\omega}} \int_0^\omega e^{-\frac{a'(x)}{a(x)}y} X_1(t+y,x)\,dy,$$

$$X_1(s,x) := -\frac{a'(x)}{a(x)}\rho(s,x) - \frac{a'(x)}{a(x)}u(s,x)\rho(s,x) - \frac{f'(x)}{f(x)}u(s,x)\rho(s,x)$$

and

$$\chi_2(\rho,u)(t,x) := \frac{e^{\frac{f'(x)}{f(x)}\omega}}{1-e^{\frac{f'(x)}{f(x)}\omega}} \int_0^\omega e^{-\frac{f'(x)}{f(x)}y} X_2(t+y,x)\,dy,$$

$$X_2(s,x) := -\frac{f'(x)}{f(x)}u(s,x) - \frac{f'(x)}{f(x)}\,u(s,x)^2 - p_\rho(\rho)\,\frac{a'(x)}{a(x)}.$$

Note that a and f are given functions of x, while the integrals above are performed with respect to the time variable.

LEMMA 2.1. *The function* $(\rho,u) = (a(x)b(t,x), f(x)g(t,x))$ *is a generalized periodic solution to the Euler equations* (1.1) *if and only if the functions* b,g *satisfy the system* (2.2)*.*

Proof. Fix $x \in \mathbb{R}$ and suppose that the function $(\rho,u) = (a(x)b(t), f(x)g(t))$ satisfies (1.3) for every $t \in [0,\omega]$. Then, for every t we have

$$\begin{aligned}
b'(t) =& \frac{a'(x)}{a(x)} b(t) + \widetilde{X}_1(t,x),\\
\widetilde{X}_1(t,x) :=& -\frac{a'(x)}{a(x)} b(t) - \frac{a'(x)}{a^2(x)} \rho(t,x)u(t,x) - \frac{1}{a(x)} \frac{f'(x)}{f(x)} \rho(t,x)u(t,x),\\
g'(t) =& \frac{f'(x)}{f(x)} g(t) + \widetilde{X}_2(t,x),\\
\widetilde{X}_2(t,x) :=& -\frac{f'(x)}{f(x)} g(t) - \frac{f'(x)}{f^2(x)} u^2(t,x) - p_\rho(\rho) \frac{a'(x)}{a(x)} \frac{1}{f(x)} \rho(t,x).
\end{aligned}$$

The variable x being fixed, we consider the above equations as a system of ordinary differential equations and we rewrite it in the form

$$\begin{aligned}
b(t) &= e^{\frac{a'(x)}{a(x)}t} \Big(C_1 + \textstyle\int_0^t e^{-\frac{a'(x)}{a(x)}y} \widetilde{X}_1(y,x)\, dy \Big),\\
g(t) &= e^{\frac{f'(x)}{f(x)}t} \Big(C_2 + \textstyle\int_0^t e^{-\frac{f'(x)}{f(x)}y} \widetilde{X}_2(y,x)\, dy \Big),
\end{aligned}$$

where C_1 and C_2 are constants. From our assumptions on the function a, we see that

$$\lim_{t \to -\infty} e^{\frac{a'(x)}{a(x)}t} = +\infty$$

and, therefore,

$$C_1 = \int_{-\infty}^0 e^{-\frac{a'(x)}{a(x)}y} \widetilde{X}_1(y,x)\, dy.$$

It follows that

$$\begin{aligned}
b(t) =& e^{\frac{a'(x)}{a(x)}t} \int_{-\infty}^t e^{-\frac{a'(x)}{a(x)}y} \widetilde{X}_1(y,x)\, dy\\
=& e^{\frac{a'(x)}{a(x)}t} \int_{t-\omega}^t e^{-\frac{a'(x)}{a(x)}y} \widetilde{X}_1(y,x)\, dy + e^{\frac{a'(x)}{a(x)}t} \int_{t-2\omega}^{t-\omega} e^{-\frac{a'(x)}{a(x)}y} \widetilde{X}_1(y,x)\, dy + \ldots
\end{aligned}$$

By setting

$$J_1 := e^{\frac{a'(x)}{a(x)}t} \int_{t-\omega}^t e^{-\frac{a'(x)}{a(x)}y} \widetilde{X}_1(y,x)\, dy,$$

we obtain (after making the change of variable $y = s - \omega$ and using the periodicity property)

$$b = J_1 + e^{\frac{a'(x)}{a(x)}\omega} J_1 + e^{2\frac{a'(x)}{a(x)}\omega} J_1 + \cdots = \frac{1}{1 - e^{\frac{a'(x)}{a(x)}\omega}} J_1.$$

A further change of variable $y = s + t - \omega$ finally leads us to

$$\rho(t,x) = \chi_1(\rho,u)(t,x)$$

for all $x \in \mathbb{R}$ and $t \in [0,\omega]$. Similarly, we can check that $u(t,x) = \chi_2(\rho,u)$. □

3. Fixed-point argument

It will be convenient to introduce the following quantities

$$D_1^- := \min_{\substack{x\in\mathbb{R}\\ 0\leqslant y\leqslant\omega}} \frac{e^{\frac{a'(x)}{a(x)}(\omega-y)}}{1 - e^{\frac{a'(x)}{a(x)}\omega}}, \qquad D_1^+ := \max_{\substack{x\in\mathbb{R}\\ 0\leqslant y\leqslant\omega}} \frac{e^{\frac{a'(x)}{a(x)}(\omega-y)}}{1 - e^{\frac{a'(x)}{a(x)}\omega}},$$

$$D_2^- := \min_{\substack{x\in\mathbb{R}\\ 0\leqslant y\leqslant\omega}} \frac{e^{\frac{f'(x)}{f(x)}(\omega-y)}}{1 - e^{\frac{f'(x)}{f(x)}\omega}}, \qquad D_2^+ := \max_{\substack{x\in\mathbb{R}\\ 0\leqslant y\leqslant\omega}} \frac{e^{\frac{f'(x)}{f(x)}(\omega-y)}}{1 - e^{\frac{f'(x)}{f(x)}\omega}},$$

and $D^- = \min\{D_1^-, D_2^-\}$, $D^+ = \max\{D_1^+, D_2^+\}$. Let $\mathscr{C}(\omega)$ be the space of real continuous ω-periodic functions defined on the real line $\mathbb{R}$, and let $\mathscr{C}_+(\omega) \subset \mathscr{C}(\omega)$ be the subset of all positive functions.

For $v = (v_1, v_2) \in \mathscr{C}(\omega) \times \mathscr{C}(\omega)$ we use the notation

$$||v|| := \max\{\max_{t\in[0,\omega]} |v_1(t)|, \max_{t\in[0,\omega]} |v_2(t)|\},$$

$$\max_{t\in[0,\omega]} v := \max\{\max_{t\in[0,\omega]} v_1(t), \max_{t\in[0,\omega]} v_2(t)\},$$

$$\min_{t\in[0,\omega]} v := \min\{\min_{t\in[0,\omega]} v_1(t), \min_{t\in[0,\omega]} v_2(t)\}.$$

We write in short $v \leqslant w$ with $v = (v_1, v_2)$, $w = (w_1, w_2)$ whenever we have both $v_1 \leqslant w_1$ and $v_2 \leqslant w_2$. The functional space of interest in the present problem is

$$\mathscr{C}_+^\circ(\omega) := \Big\{ v \in \mathscr{C}_+(\omega) \times \mathscr{C}_+(\omega) : \min_{t\in[0,\omega]} v(t) \geqslant \frac{D^-}{D^+} \max_{t\in[0,\omega]} v(t) \Big\}.$$

It is straightforward to check the following properties.

LEMMA 3.1. *Suppose that the functions a, f satisfy the conditions* (1.4) *and the pressure function satisfies the condition* (2.1)*. Then the operator $\chi := (\chi_1, \chi_2)$ introduced in Section 2 maps $\mathscr{C}_+^\circ(\omega)$ into itself, $\chi : \mathscr{C}_+^\circ(\omega) \to \mathscr{C}_+^\circ(\omega)$ and, moreover, is a compact operator.*

Recall that a compact operator transforms a bounded subset of $\mathscr{C}(\omega)$ into a compact one.

Our proof of existence of non-trivial time-periodic solutions to the system (1.3) will be based on a theory of compact operators due to Krasnosel'skii and Zabrejko [8].

The basic notion of that theory is the notion of index. For illustration recall the definition in finite dimension. Suppose that L is a continuous vector field defined on the closure $\overline{\Omega}$ of a bounded open set $\Omega \subset \mathbb{R}^n$. A point $x_0 \in \overline{\Omega}$ is called a *singular point* for the vector field L if $Lx_0 = 0$. On the other hand, if $Lx \neq 0$ for every $x \in \overline{\Omega}$ one says that the vector field is *non-degenerate*. For every vector field we can define the *rotation number* $\gamma(L,\Omega)$ as an integer satisfying the following properties:

1. If L_1 and L_2 are homotopic vector fields on $\partial\Omega$, then $\gamma(L_1,\Omega) = \gamma(L_2,\Omega)$.

2. If L is continuous and non-degenerate and is defined on a set $\overline{\Omega} \setminus \cup_{j=1}^{\infty} \Omega_j$, with $\Omega_j \cap \Omega_i = \emptyset$ $(i \neq j)$, $\Omega_j \subset \Omega$, then one has $\gamma(Ł,\Omega_j) \neq 0$ and

$$\gamma(L,\Omega) = \gamma(L,\Omega_1) + \gamma(L,\Omega_2) + \dots$$

3. If $Lx = x - x_0$ for some $x_0 \in \Omega$, then one has $\gamma(L,\Omega) = 1$.

Roughly speaking, $\gamma(L,\Omega)$ measures the "amount of rotation" of the vector field L along $\partial\Omega$. Observe that if L is a non-degenerate vector field on $\overline{\Omega}$ then $\gamma(L,\Omega) = 0$.

If x_0 is an isolated singular point for a vector field L, then the rotation number of L on the spheres $||x - x_0|| = \rho$, provided $\rho > 0$ is small enough, is a constant. This rotation number is also called the *index of the point* x_0 (with respect to the vector field L) and is denoted by $ind(x_0,L)$.

In our present context, Ω is actually a infinite-dimensional Banach space Y endowed with a cone $Q \subset Y$. The cone Q naturally generates a semi-ordering $\overset{\circ}{\leqslant}$, defined by $v \overset{\circ}{\leqslant} w$ if and only if $w - v \in Q$. The index of an element $v \in Y$ can be defined in this situation (by suitably generalizing the definition in finite dimension) and is denoted by $ind(v,L;Q)$.

We will need the following two theorems whose proof can be found in [8].

THEOREM 3.2. (Existence of fixed points. I) *Let Y be a real Banach space endowed with a cone Q, and $L: Y \to Y$ be a compact operator which is assumed to be positive for the semi-ordering associated with Q. Then, the following properties hold:*

i) If $L(0) = 0$ and, for all sufficiently small $r > 0$, there exists no element $y \in Q$ such that $||y||_Y = r$ and $y \overset{\circ}{\leqslant} L(y)$, then, one has $ind(0,L;Q) = 1$.

ii) If, for all sufficiently large $R > 0$, there exists no element $y \in Q$ such that $||y||_Y = R$ and $L(y) \overset{\circ}{\leqslant} y$, then one has $ind(\infty,L;Q) = 0$.

iii) If $L(0) = 0$ and $ind(0,L;Q) \neq ind(\infty,L;Q)$, then the operator L has a non-trivial fixed point in Q.

THEOREM 3.3. (Existence of fixed points. II) *Let Y be a real Banach space endowed with a cone Q and $L : Y \to Y$ be a compact operator which is positive with respect to Q. Then, the following properties hold:*

i) If for all sufficiently large $R > 0$ there exists no element $y \in Q$ such that $||y||_Y = R$ and $L(y) \overset{\circ}{\geqslant} y$, then one has $ind(\infty, L; Q) = 1$.

ii) If $L(0) \neq 0$ and $ind(\infty, L; Q) \neq 0$, then the operator L admit a non-trivial fixed point in Q.

The uniqueness follows from the following observation.

LEMMA 3.4. (Uniqueness of fixed points) *Let Y be a real Banach space endowed with a cone Q and $L : Y \to Y$ be a compact operator which is positive with respect to Q. Fix some element $v_0 \in Q$ and suppose that the operator L satisfies the following conditions:*

i) $\alpha(w)v_0 \leqslant Lw \leqslant \beta(w)v_0$ for every $w \in Q$ and for some reals $\alpha(w)$, $\beta(w) > 0$.

ii) $L(\lambda w) \leqslant \lambda Lw$ for all $\lambda \in [0,1]$ and $w \in Q$.

iii) $L(\lambda w) \neq \lambda Lw$ for all $\lambda \in (0,1)$ and $w > \gamma(w)v_0$ for some $\gamma(w) > 0$.

iv) from $w \leqslant z$ with $w \neq z$, it follows that $Lw \leqslant Lz - \varepsilon_0 v_0$ for some $\varepsilon_0 > 0$.

Then, the operator L has at most one fixed point in Q.

Proof. By contradiction, suppose that the operator L has two fixed points w_1 and w_2 and, without loss of generality, $w_1 \leqslant w_2$. Let $\lambda_0 \in (0,1)$ be such that $w_1 \leqslant \lambda_0 w_2$ and $w_1 \geqslant \lambda w_2$ for every $\lambda \leqslant \lambda_0$. Then, choose $\varepsilon_0 > 0$ small enough so that $\lambda_0 - \frac{\varepsilon_0}{\beta(w_2)} > 0$, where $\beta(w_2)$ is given by our assumptions. We can then distinguish between the following two cases:

1) If $w_1 = \lambda_0 w_2$ then we write

$$w_1 = Lw_1 = L(\lambda_0 w_2) \neq \lambda_0 L w_2 = \lambda_0 w_2,$$

which is a contradiction.

2) If $w_1 < \lambda_0 w_2$, then we write

$$\begin{aligned} w_1 = Lw_1 &\leqslant L(\lambda_0 w_2) - \varepsilon_0 u_0 \\ &\leqslant \lambda_0 L w_2 - \varepsilon_0 u_0 \leqslant \lambda_0 L w_2 - \frac{\varepsilon_0}{\beta(w_2)} L w_2 \\ &= \left(\lambda_0 - \frac{\varepsilon_0}{\beta(w_2)}\right) w_2, \end{aligned}$$

which is also a contradiction since $\lambda_0 - \frac{\varepsilon_0}{\beta(w_2)} < \lambda_0$. □

4. Time-periodic solutions

We are now in a position to provide a proof of Theorem 1.2. For the interval $\gamma \in (0,1)$, we will need the condition

$$\frac{e^{(m_2-M_2)\omega}}{1-e^{-M_2\omega}}\,\omega\left(m_2+m_1Bl\,2^{1-\gamma}\Big(\frac{1-e^{-m_1\omega}}{1-e^{-M_1\omega}}\Big)^{\gamma}\Big(\frac{1-e^{-m_2\omega}}{1-e^{-M_2\omega}}\Big)\right)<1,$$
$$B:=e^{(2m_1-M_1)\gamma\omega}e^{(2m_2-M_2)\omega}.$$

Throughout the spatial variable $x\in\mathbb{R}$ is fixed. We consider the whole range $\gamma\in(0,\infty)$.

Let $1>r>0$ be sufficiently small so that

$$\Big((m_1+m_2)\frac{D_2^+}{D_2^-}r+m_1\Big)D_1^+\omega<1, \tag{4.1}$$

$$D_2^+\omega\Big(m_2+m_2\frac{D_2^+}{D_2^-}r+m_1l2^{1-\gamma}\frac{D_1^{+\gamma}}{D_1^{-\gamma}}\frac{D_2^+}{D_2^-}\Big)<1 \quad \text{for } \gamma\in(0,1], \tag{4.2}$$

$$D_2^+\omega\Big(m_2+m_2\frac{D_2^+}{D_2^-}r+m_1l\frac{r^{\gamma-1}}{2^{\gamma-1}}\frac{D_1^{+\gamma-1}}{D_1^{-\gamma-1}}\frac{D_2^+}{D_2^-}\Big)<1 \quad \text{for } \gamma>1, \tag{4.3}$$

which impose that m_1,m_2 are sufficienty small.

Step 1. Suppose that there exists a function $(\rho,u)\in\mathscr{C}_+^\circ(\omega)$ such that $(\rho,u)\leqslant(\chi_1(\rho,u),\chi_2(\rho,u))$ and $||(\rho,u)||=r$. All functions under consideration are functions of the time variable t (while the dependence in the spatial variable is omitted throughout).

If $\max_{t\in[0,\omega]}\rho(t)\geqslant r/2$, then $\max_{t\in[0,\omega]}u(t)\leqslant r/2$ and for every $t\in[0,\omega]$

$$u(t)\leqslant\max_{t\in[0,\omega]}u(t)\frac{D_2^+}{D_2^-}\leqslant\frac{D_2^+}{D_2^-}r.$$

From this it follows that, for $t\in[0,\omega]$,

$$\begin{aligned}\rho(t)&\leqslant\frac{e^{\frac{a'}{a}\omega}}{1-e^{\frac{a'}{a}\omega}}\int_0^\omega e^{-\frac{a'}{a}y}\Big(-\frac{a'}{a}\rho(t+y)-\frac{a'}{a}u(t+y)\rho(t+y)-\frac{f'}{f}u(t+y)\rho(t+y)\Big)\,dy\\&\leqslant D_1^+\int_0^\omega(m_1\rho(t+y)+m_1u(t+y)\rho(t+y)+m_2u(t+y)\rho(t+y))\,dy\\&\leqslant D_1^+\int_0^\omega\Big(m_1+m_1\frac{D_2^+}{D_2^-}r+m_2\frac{D_2^+}{D_2^-}r\Big)\rho(y)\,dy,\end{aligned}$$

i.e.

$$\rho(t)\leqslant D_1^+\int_0^\omega\Big(m_1+m_1\frac{D_2^+}{D_2^-}r+m_2\frac{D_2^+}{D_2^-}r\Big)\rho(y)\,dy.$$

Now, we integrate the last inequality over the interval $[0,\omega]$ and get

$$\int_0^\omega \rho(t)\,dt \leqslant D_1^+\Big(m_1 + m_1\frac{D_2^+}{D_2^-}r + m_2\frac{D_2^+}{D_2^-}r\Big)\omega\int_0^\omega \rho(y)\,dy,$$

which is a contradiction in view of (4.1).

Next, if $\max_{y\in[0,\omega]} u(y) \geqslant r/2$, then $\max_{y\in[0,\omega]}\rho(y) \leqslant r/2$ and for every $t\in[0,\omega]$

$$u(t) \geqslant \frac{D_2^-}{D_2^+}\max_{s\in[0,\omega]} u(s) \geqslant \frac{D_2^-}{D_2^+}\frac{r}{2},$$

from which it follows that

$$\frac{r}{2} \leqslant \frac{D_2^+}{D_2^-}u(t), \quad t\in[0,\omega], \tag{4.4}$$

$$\int_0^\omega \rho(t)\,dt \leqslant \int_0^\omega \frac{r}{2}\,dt \leqslant \frac{D_2^+}{D_2^-}\int_0^\omega u(t)\,dt, \tag{4.5}$$

and

$$\begin{aligned}\rho(t) &\leqslant \frac{D_1^+}{D_1^-}\max_{s\in[0,\omega]}\rho(s) \leqslant \frac{r}{2}\frac{D_1^+}{D_1^-}\\ &\leqslant \frac{D_1^+}{D_1^-}\frac{D_2^+}{D_2^-}u(t).\end{aligned} \tag{4.6}$$

Now, we need to distinguish between several cases, depending on the value of $\gamma\in(0,\infty)$.

1. Case $\gamma\in(0,1)$. Then, we have $\frac{1-\gamma}{\gamma} > 0$ and $r^{\frac{1-\gamma}{\gamma}} < 1$, and since p_ρ is a decreasing function by assumption, we obtain

$$\begin{aligned}p_\rho(\rho) &\leqslant p_\rho(r^{\frac{1-\gamma}{\gamma}}\rho) \leqslant l r^{1-\gamma}\rho^\gamma\\ &\leqslant l2^{1-\gamma}\Big(\frac{r}{2}\Big)^{1-\gamma}\rho^\gamma\end{aligned}$$

and, using (4.4),

$$p_\rho(\rho) \leqslant l2^{1-\gamma}\frac{D_2^{+\,1-\gamma}}{D_2^{-\,1-\gamma}}u^{1-\gamma}(t,x)\rho^\gamma(t).$$

Therefore, taking (4.6) into account, we find

$$\begin{aligned}p_\rho(\rho) &\leqslant l2^{1-\gamma}\frac{D_2^{+\,1-\gamma}}{D_2^{-\,1-\gamma}}u^{1-\gamma}(t)\frac{D_1^{+\,\gamma}}{D_1^{-\,\gamma}}\frac{D_2^{+\,\gamma}}{D_2^{-\,\gamma}}u^\gamma(t)\\ &= l2^{1-\gamma}\frac{D_1^{+\,\gamma}}{D_1^{-\,\gamma}}\frac{D_2^+}{D_2^-}u(t)\end{aligned}$$

for every $t \in [0,\omega]$, i.e.

$$p_\rho(\rho) \leqslant l 2^{1-\gamma} \frac{D_1^{+\gamma}}{D_1^{-\gamma}} \frac{D_2^+}{D_2^-} u(t). \tag{4.7}$$

Then, we obtain

$$\begin{aligned}
u(t) &\leqslant \frac{e^{\frac{f'}{f}\omega}}{1-e^{\frac{f'}{f}\omega}} \int_0^\omega e^{-\frac{f'}{f}y} \Big(-\frac{f'}{f} u(t+y) - \frac{f'}{f} u^2(t+y) - p_\rho(\rho) \frac{a'}{a} \Big)\, dy \\
&\leqslant D_2^+ \int_0^\omega \Big(m_2 u(t+y) + m_2 u^2(t+y) + m_1\, p_\rho(\rho) \Big)\, dy \\
&\leqslant D_2^+ \int_0^\omega \Big(m_2 u(t+y) + m_2 \frac{D_2^+}{D_2^-} r u(t+y) + m_1\, p_\rho(\rho) \Big)\, dy.
\end{aligned}$$

So, using (4.7)

$$\begin{aligned}
u(t) &\leqslant D_2^+ \int_0^\omega \Big(m_2 u(t+y) + m_2 \frac{D_2^+}{D_2^-} r u(t+y) + m_1\, l 2^{1-\gamma} \frac{D_1^{+\gamma}}{D_1^{-\gamma}} \frac{D_2^+}{D_2^-} u(y) \Big)\, dy \\
&= D_2^+ \Big(m_2 + m_2 \frac{D_2^+}{D_2^-} r + m_1 l 2^{1-\gamma} \frac{D_1^{+\gamma}}{D_1^{-\gamma}} \frac{D_2^+}{D_2^-} \Big) \int_0^\omega u(y)\, dy,
\end{aligned}$$

i.e. for every $y \in [0,\omega]$

$$u(y) \leqslant D_2^+ \Big(m_2 + m_2 \frac{D_2^+}{D_2^-} r + m_1 l 2^{1-\gamma} \frac{D_1^{+\gamma}}{D_1^{-\gamma}} \frac{D_2^+}{D_2^-} \Big) \int_0^\omega u(y)\, dy.$$

It remains to integrate over the interval $[0,\omega]$ and obtain

$$\int_0^\omega u(y)\, dy \leqslant D_2^+ \omega \Big(m_2 + m_2 \frac{D_2^+}{D_2^-} r + m_1 l 2^{1-\gamma} \frac{D_1^{+\gamma}}{D_1^{-\gamma}} \frac{D_2^+}{D_2^-} \Big) \int_0^\omega u(y)\, dy,$$

which is a contradiction with (4.2). Combining our conclusion with item i) of Theorem 2.5, we conclude that

$$ind(0,\chi;\mathscr{C}_+^\circ(\omega)) = 1. \tag{4.8}$$

2. *Case* $\gamma = 1$. In this case, we have for every $t \in [0,\omega]$

$$\begin{aligned}
u(t) &\leqslant D_2^+ \int_0^\omega \Big(m_2 u(t+y) + m_2 \frac{D_2^+}{D_2^-} r u(t+y) + m_1 l \rho(t+y) \Big)\, dy \\
&\leqslant D_2^+ \int_0^\omega \Big(m_2 u(t+y) + m_2 \frac{D_2^+}{D_2^-} r u(t+y) + m_1 l \frac{D_1^+}{D_1^-} \frac{D_2^+}{D_2^-} u(t+y) \Big)\, dy \\
&= D_2^+ \Big(m_2 + m_2 \frac{D_2^+}{D_2^-} r + m_1 l \frac{D_1^+}{D_1^-} \frac{D_2^+}{D_2^-} \Big) \int_0^\omega u(y)\, dy,
\end{aligned}$$

where we have used (4.6). From this inequality, we deduce that

$$\int_0^\omega u(y)\,dy \leqslant D_2^+\omega\Big(m_2+m_2\frac{D_2^+}{D_2^-}r+m_1 l\frac{D_1^+}{D_1^-}\frac{D_2^+}{D_2^-}\Big)\int_0^\omega u(y)\,dy,$$

which is a contradiction with (4.2). From this and in view of item i) of Theorem 2.5, we deduce that

$$ind(0,\chi;\mathscr{C}_+^\circ(\omega))=1. \tag{4.9}$$

3. Case $\gamma>1$. Then, we have

$$\begin{aligned} u(t) &\leqslant D_2^+\int_0^\omega\Big(m_2u(t+y)+m_2\frac{D_2^+}{D_2^-}ru(t+y)+m_1l\rho^\gamma(t+y)\Big)\,dy \\ &\leqslant D_2^+\int_0^\omega\Big(m_2u(t+y)+m_2\frac{D_2^+}{D_2^-}ru(t+y)+m_1l\frac{r^{\gamma-1}}{2^{\gamma-1}}\frac{D_1^{+\gamma-1}}{D_1^{-\gamma-1}}\rho(y)\Big)\,dy, \end{aligned}$$

where we have used (4.6). In view of (4.5), we find

$$u(t)\leqslant D_2^+\int_0^\omega\Big(m_2u(t+y)+m_2\frac{D_2^+}{D_2^-}ru(t+y)+m_1l\frac{r^{\gamma-1}}{2^{\gamma-1}}\frac{D_1^{+\gamma-1}}{D_1^{-\gamma-1}}\frac{D_2^+}{D_2^-}u(y)\Big)\,dy,$$

from which it follows

$$\int_0^\omega u(y)\,dy\leqslant D_2^+\omega\Big(m_2+m_2\frac{D_2^+}{D_2^-}r+m_1l\frac{r^{\gamma-1}}{2^{\gamma-1}}\frac{D_1^{+\gamma-1}}{D_1^{-\gamma-1}}\frac{D_2^+}{D_2^-}\Big)\int_0^\omega u(y)\,dy.$$

Again, this is a contradiction with (4.3) and, therefore, from item i) of Theorem 2.5 we deduce that

$$ind(0,\chi;\mathscr{C}_+^\circ(\omega))=1. \tag{4.10}$$

In conclusion, in view of (4.8), (4.9), and (4.10) we see that

$$ind(0,\chi;\mathscr{C}_+^\circ(\omega))=1 \qquad \text{for any } \gamma\in(0,\infty). \tag{4.11}$$

Step 2. Next, fix $R>2$ large enough so that

$$R>\frac{2D_2^+}{D_2^{-2}\omega M_2}, \tag{4.12}$$

$$R>\frac{1}{D_2^-M_1k\omega} \quad \text{for} \quad \gamma\in(0,1), \tag{4.13}$$

$$R>\frac{D_1^+}{k\omega D_1^-D_2^-M_1} \quad \text{for} \quad \gamma=1, \tag{4.14}$$

$$R > 2\Big(\frac{D_1^{+\gamma}}{M_1 D_1^{-\gamma} D_2^{-} \omega k}\Big)^{\frac{1}{\gamma-1}} \quad \text{for} \quad \gamma > 1. \tag{4.15}$$

Let us suppose that there exists $(\rho,u)\in \mathscr{C}_+^{\circ}(\omega)$ such that $(\rho,u)\geqslant(\chi_1(\rho,u),\chi_2(\rho,u))$ and $||(\rho,u)||=R$.

If $\max_t u(t)\geqslant R/2$, then we get

$$\begin{aligned} u(t) &\geqslant \frac{e^{\frac{f'}{f}\omega}}{1-e^{\frac{f'}{f}\omega}}\int_0^\omega e^{-\frac{f'}{f}y}\Big(-\frac{f'}{f}u(t+y)-\frac{f'}{f}u^2(t+y)-p_\rho(\rho)\frac{a'}{a}\Big)\,dy \\ &\geqslant \frac{e^{\frac{f'}{f}\omega}}{1-e^{\frac{f'}{f}\omega}}\int_0^\omega e^{-\frac{f'}{f}y}\Big(-\frac{f'}{f}u^2(t+y)\Big)\,dy \\ &\geqslant \frac{D_2^{-2}}{D_2^{+}}M_2\frac{R}{2}\int_0^\omega u(y)\,dy, \end{aligned}$$

from which we deduce

$$\int_0^\omega u(y)\,dy \geqslant \frac{D_2^{-2}}{D_2^{+}}M_2\frac{R}{2}\omega\int_0^\omega u(y)\,dy,$$

which is a contradiction with (4.12).

If, now, $\max_{t\in[0,\omega]}\rho(t)\geqslant R/2$, then we have $\max_{t\in[0,\omega]}u(t)\leqslant R/2$. If $u(t)\equiv 0$ then we obtain $\int_0^\omega p_\rho(\rho)\,dt\leqslant 0$, which is again a contradiction. Therefore, we have $\int_0^\omega u(t)\,dt>0$ and

$$u(t)\geqslant D_2^- M_1\int_0^\omega p_\rho(\rho)\,dy. \tag{4.16}$$

We now distinguish between three cases.

1. Case $\gamma\in(0,1)$. Then, $R^{1/\gamma}>1$. Since $p_\rho(\rho)$ is a decreasing function of ρ we have

$$\begin{aligned} p_\rho(\rho) &\geqslant p_\rho\Big(R^{\frac{1}{\gamma}}\max_{t\in[0,\omega]}\rho(t)\Big)\geqslant p_\rho\Big(R^{1/\gamma}\Big(\max_{t\in[0,\omega]}\rho(t)\Big)^{1/\gamma}\Big) \\ &\geqslant kR\max_{t\in[0,\omega]}\rho(t)\geqslant kR\max_{t\in[0,\omega]}u(t) \\ &\geqslant kRu(t). \end{aligned}$$

From this and in view of (4.16) we obtain

$$\int_0^\omega u(t)\,dt\geqslant D_2^- M_1 kR\omega\int_0^\omega u(t)\,dt,$$

which is a contradiction with (4.13).

2. Case $\gamma = 1$. Since $p_\rho(\rho)$ is a decreasing function of ρ, we have

$$\begin{aligned} p_\rho(\rho) &\geqslant p_\rho(R\rho) \geqslant kR\rho \\ &\geqslant kR\frac{D_1^-}{D_1^+}\max_{t\in[0,\omega]}\rho(t) \\ &\geqslant kR\frac{D_1^-}{D_1^+}\max_{t\in[0,\omega]}u(t) \geqslant kR\frac{D_1^-}{D_1^+}u(t). \end{aligned}$$

In view of (4.16) we obtain

$$\int_0^\omega u(t)\,dt \geqslant kD_2^- M_1\omega\frac{D_1^-}{D_1^+}R\int_0^\omega u(y)\,dy,$$

which is a contradiction with (4.14).

3. Case $\gamma > 1$. Then, we have

$$\begin{aligned} \int_0^\omega u(t)\,dt &\geqslant D_2^- M_1\omega\int_0^\omega p_\rho(\rho)\,dt \geqslant D_2^- M_1\omega k\int_0^\omega \rho^\gamma\,dt \\ &\geqslant D_2^- M_1\omega k\Big(\frac{R}{2}\Big)^{\gamma-1}\Big(\frac{D_1^-}{D_1^+}\Big)^\gamma\int_0^\omega \max_{t\in[0,\omega]}\rho(t)\,dt \\ &\geqslant D_2^- M_1\omega k\Big(\frac{R}{2}\Big)^{\gamma-1}\Big(\frac{D_1^-}{D_1^+}\Big)^\gamma\int_0^\omega \max_{t\in[0,\omega]}u(t)\,dt \\ &\geqslant D_2^- M_1\omega k\Big(\frac{R}{2}\Big)^{\gamma-1}\Big(\frac{D_1^-}{D_1^+}\Big)^\gamma\int_0^\omega u(t)\,dt, \end{aligned}$$

which is a contradiction with (4.15). Consequently, we have $ind(\infty,\chi;\mathscr{C}_+^\circ(\omega))$ and

$$ind(\infty,\chi;\mathscr{C}_+^\circ(\omega)) = 0.$$

From this and (4.11) and in view of Theorem 3.2, it follows that the operator χ has a non-trivial fixed point in the cone $\mathscr{C}_+^\circ(\omega)$. Therefore, the system (1.3) has a non-trivial solution $(\rho(t,x),u(t,x)) = (a(x)b(t,x), f(x)g(t,x))$ which is positive, continuous, ω-periodic with respect to the time variable t.

The operator χ satisfies the condition of item $i)$ of Lemma 3.4 for $v_0 = 1$, since

$$\begin{aligned} D_1^-\int_0^\omega X_1\,dy \leqslant \chi_1(\rho,u) \leqslant D_1^+\int_0^\omega X_1\,dy, \\ D_2^-\int_0^\omega X_2\,dy \leqslant \chi_2(\rho,u) \leqslant D_2^+\int_0^\omega X_2\,dy. \end{aligned}$$

The operator χ satisfies the conditions of items $ii)$, $iii)$, since $p_\rho(\lambda\rho) \leqslant \lambda p_\rho(\rho)$ for every $\lambda \in (0,1)$ and for every fixed $\rho \geqslant 0$. Indeed, setting $g(\lambda) = p_\rho(\lambda\rho) - \lambda p_\rho(\rho)$, we find $g'(\lambda) = \rho p'_\rho(\lambda\rho) - p_\rho(\rho) \leqslant 0$ for $\lambda \in (0,1)$ and $\rho \geqslant 0$. Hence, $g(\lambda) \leqslant g(0) = 0$ for every $\lambda \in (0,1)$.

Clearly, the operator χ satisfies the item *iv*) of Lemma 3.4 and, therefore, the system (1.3) admits exactly one non-trivial solution

$$(\rho(t,x),u(t,x)) = (a(x)b(t,x), f(x)g(t,x)),$$

which is positive, continuous, ω-periodic.

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Svetlin Georgiev
University of Sofia
Faculty of Mathematics and Informatics
Department of Differential Equations
15 Blvd. Tzar Osvoboditel
Sofia 1000
Bulgaria
e-mail: sgg2000bg@yahoo.com

Philippe G. LeFloch
Laboratoire Jacques-Louis Lions & Centre National de la Recherche Scientifique, Université Pierre et Marie Curie (Paris 6)
4 Place Jussieu
75252 Paris
France
e-mail: pgLeFloch@gmail.com