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BIFURCATIONS OF PERIODIC SOLUTIONS IN FORCED ORDINARY DIFFERENTIAL INCLUSIONS

MICHAL FEČKAN

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Abstract. We are interested in periodic solutions of a coupled system of two periodically forced ordinary differential inclusions when the first differential inclusion is weakly nonlinear with respect to a small parameter while the second differential inclusion is strongly nonlinear. We investigate two cases when the second equation of the unperturbed autonomous system has either a single or a non-degenerate family of periodic solutions parameterized by the first variable. The second case usually occurs when the second unperturbed differential equation is symmetric. A combination of the topological degree approach with the averaging method is applied to find topological degree conditions for bifurcations of forced periodic solutions of the perturbed system of differential inclusions for the small parameter from the above-mentioned periodic solutions of the unperturbed equation. Concrete examples of discontinuous periodically forced differential equations are also treated to illustrate the theory.

1. Introduction

In this paper we consider the following weakly coupled and periodically forced system of ordinary differential inclusions

$$\begin{aligned} x' &\in \varepsilon f(x,y,t,\varepsilon)\,, \\ y' &\in g(x,y) + \varepsilon h(x,y,t,\varepsilon)\,, \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $g \in C^3(\mathbb{R}^{n+m},\mathbb{R}^m)$, $f : \mathbb{R}^{n+m+2} \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ and $g : \mathbb{R}^{n+m+2} \to 2^{\mathbb{R}^m} \setminus \{\emptyset\}$ are upper semi-continuous multivalued mappings with compact and convex set values, which are 1-periodic in $t \in \mathbb{R}$. For $\varepsilon = 0$, the system (1.1) becomes into an autonomous ordinary differential equation,

$$y' = g(x,y)\,, \tag{1.2}$$

parameterized by $x \in \mathbb{R}^n$. For an open subset $U \subset \mathbb{R}^n$, we consider two cases:

1. equation (1.2) has a single 1-periodic solution for any $x \in U$,
2. equation (1.2) has a nondegenerate family of 1-periodic solutions for any $x \in U$.

Our aim is to find topological degree bifurcation conditions under which (1.1) has

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forced 1-periodic solutions for $\varepsilon \neq 0$ small. For this purpose, a combination of the Lyapunov-Schmidt method together with the Leray-Schauder degree theory for multi-valued mappings is applied [7]. These results Theorems 2.1 and 3.1 are extensions of similar ones for ordinary differential equations [1, 2, 3, 7, 8] to ordinary differential inclusion (1.1). We also present Examples 4.1 and 4.4 to illustrate our theory. Averaging methods for symmetric ordinary differential equations are given in [4, 5, 6].

2. Bifurcations from single periodic solutions

In this part, we consider the case 1 from Introduction. More precisely, we suppose the condition,

(H1): the equation (1.2) has a 1-periodic solution $y=\varphi(t,x)$ for any $x\in U$, where the function $\varphi\in C^1(\mathbb{R}\times U,\mathbb{R}^m)$ is 1-periodic in $t\in\mathbb{R}$.

Certainly the function $\varphi'(t,x)$ satisfies the variational equation

$$v' = g_y(x,\varphi(t,x))v. \tag{2.1}$$

We also consider the dual variational system

$$w' = -g_y^*(x,\varphi(t,x))w. \tag{2.2}$$

Next, we suppose the condition,

(H2): there are smooth bases

$$\Big\{v_0(t,x),v_1(t,x),\cdots,v_r(t,x)\Big\} \quad \text{and} \quad \Big\{w_0(t,x),w_1(t,x),\cdots,w_r(t,x)\Big\}$$

of 1-periodic solutions of (2.1) and (2.2), respectively, for any $x\in U$, where we assume that $v_0(t,x)=\varphi'(t,x)$.

We consider the Banach spaces:

$$\begin{aligned}
X &:= \{x\in C(\mathbb{R},\mathbb{R}^n) : x(t)\text{ is 1-periodic}\},\\
Y &:= \{y\in C(\mathbb{R},\mathbb{R}^m) : y(t)\text{ is 1-periodic}\},\\
X_\infty &:= \{x\in L^\infty(\mathbb{R},\mathbb{R}^n) : x(t)\text{ is 1-periodic}\},\\
Y_\infty &:= \{y\in L^\infty(\mathbb{R},\mathbb{R}^m) : y(t)\text{ is 1-periodic}\},\\
W_n^{1,\infty} &:= \big\{x\in W^{1,\infty}(\mathbb{R},\mathbb{R}^n) : x(t)\text{ is 1-periodic}\big\},\\
W_m^{1,\infty} &:= \big\{y\in W^{1,\infty}(\mathbb{R},\mathbb{R}^m) : y(t)\text{ is 1-periodic}\big\},
\end{aligned}$$

with the usual sup-norms. Then we introduce the projections

$$P_1 : X\to X, \quad P_x : Y\to Y,$$

defined as follows:

$$P_1x := x(t) - \int_0^1 x(s)\,ds,$$

$$P_x y := y(t) - q_0 w_0(t,x) - q_1 w_1(t,x) - \cdots - q_r w_r(t,x),$$

$$(q_0, q_1, \cdots, q_r)^* := A(x)^{-1} \left(\int_0^1 (y(t), w_0(t,x))\, dt, \cdots, \int_0^1 (y(t), w_r(t,x))\, dt \right)^*,$$

where $(\cdot,\cdot)$ is the scalar product on $\mathbb{R}^m$ and $A(x) : \mathbb{R}^{r+1} \to \mathbb{R}^{r+1}$ is the matrix given by

$$A(x) := \left(\int_0^1 (w_i(t,x), w_j(t,x))\, dt \right)_{i,j=0}^{r}.$$

The meaning of these projections is the following: the nonhomogeneous variational equation of (2.1) along $\varphi(t,x)$ is given by

$$\begin{aligned} u' &= \widetilde{h}_1, \\ v' &= g_y(x, \varphi(t,x))v + \widetilde{h}_2. \end{aligned} \tag{2.3}$$

From [9, Theorem 1.2, p. 411] we know that (2.3) has a 1-periodic solution in $W_n^{1,\infty} \times W_m^{1,\infty}$ for $\widetilde{h}_1 \in X_\infty$ and $\widetilde{h}_2 \in Y_\infty$ if and only if $P_1 \widetilde{h}_1 = \widetilde{h}_1$ and $P_x \widetilde{h}_2 = \widetilde{h}_2$. Moreover this solution is unique if,

$$P_1 u = u \quad \text{and} \quad \int_0^1 (v(t), v_i(t,x))\, dt = 0, \ \text{ for } \ i = 0, 1, \cdots, r.$$

Next for any $h_1 \in X_\infty$, $h_2 \in Y_\infty$ we set for $\widetilde{h}_1 := P_1 h_1$, $\widetilde{h}_2 := P_x \widetilde{h}_2$, and we denote by $(u,v) := \mathscr{K}_x(h_1, h_2)$ the above solution of (2.3). Then

$$\mathscr{K}_x : X_\infty \times Y_\infty \to X \times Y$$

is compact and linear, since $W_n^{1,\infty} \times W_m^{1,\infty} \subset\subset X \times Y$ is a compact embedding. Moreover, a mapping

$$\mathscr{K} : U \to L(X_\infty \times Y_\infty, X \times Y)$$

defined as $\mathscr{K}(x) := \mathscr{K}_x$ is continuous.

Now we shift $t \to t + \alpha$, $\alpha \in \mathbb{R}$ and then we make in (1.1) the changes of variables

$$\varepsilon \to \varepsilon^2, \quad x = \varepsilon^2 u + x_1, \quad u \in X, \quad P_1 u = u, \quad x_1 \in \mathbb{R}^n,$$

$$y = \varphi(t, \varepsilon^2 u + x_1) + \varepsilon^2 v + \varepsilon \sum_{i=1}^{r} \beta_i v_i(t, \varepsilon^2 u + x_1),$$

$$\int_0^1 (v(t), v_i(t, x_1))\, dt = 0, \quad i = 0, 1, \cdots, r,$$

to derive

$$u' \in f\left(\varepsilon^2 u + x_1, \varphi(t, \varepsilon^2 u + x_1) + \varepsilon^2 v + \varepsilon \sum_{i=1}^{r} \beta_i v_i(t, \varepsilon^2 u + x_1), t + \alpha, \varepsilon^2 \right)$$

$$v' - g_y(x_1,\varphi(t,x_1))v \in H(u,v,x_1,\alpha,\beta,t,\varepsilon)\,, \tag{2.4}$$

where $\beta := (\beta_1,\beta_2,\cdots,\beta_r)$ and

$$\begin{aligned}
&H(u,v,x_1,\alpha,\beta,t,\varepsilon) := \\
&\quad \frac{1}{\varepsilon^2}\Bigg[g\left(\varepsilon^2 u + x_1, \varphi(t,\varepsilon^2 u + x_1) + \varepsilon^2 v + \varepsilon\sum_{i=1}^r \beta_i v_i(t,\varepsilon^2 u + x_1)\right) \\
&\qquad\qquad - g(\varepsilon^2 u + x_1, \varphi(t,\varepsilon^2 u + x_1)) \\
&\qquad\qquad - g_y(\varepsilon^2 u + x_1, \varphi(t,\varepsilon^2 u + x_1))\left(\varepsilon^2 v + \varepsilon\sum_{i=1}^r \beta_i v_i(t,\varepsilon^2 u + x_1)\right)\Bigg] \\
&\quad + \Big(g_y(\varepsilon^2 u + x_1, \varphi(t,\varepsilon^2 u + x_1)) - g_y(x_1,\varphi(t,x_1))\Big)v \\
&\quad - \left(\varepsilon\sum_{i=1}^r \beta_i v_{ix}(t,\varepsilon^2 u + x_1) + \varphi_x(t,\varepsilon^2 u + x_1)\right)u' \\
&\quad + h\left(\varepsilon^2 u + x_1, \varepsilon^2 v + \varepsilon\sum_{i=1}^r \beta_i v_i(t,\varepsilon^2 u + x_1) + \varphi(t,\varepsilon^2 u + x_1), t+\alpha, \varepsilon^2\right).
\end{aligned}$$

Next we set the following mapping

$$G : X \times Y \times U \times \mathbb{R}^{r+2} \times [0,1] \to 2^{X_\infty \times Y_\infty} \setminus \{\emptyset\}$$

given by

$$\begin{aligned}
&G(u,v,x_1,\alpha,\beta,\varepsilon,\lambda) := \\
&\quad \Bigg\{(h_1,h_2) \in X_\infty \times Y_\infty : h_1(t) \in f\Bigg(\lambda\varepsilon^2 u(t) + x_1, \varphi(t,\lambda\varepsilon^2 u(t) + x_1) \\
&\qquad\qquad + \lambda\varepsilon^2 v(t) + \lambda\varepsilon\sum_{i=1}^r \beta_i v_i(t,\varepsilon^2 u(t) + x_1), t+\alpha, \lambda\varepsilon^2\Bigg), \\
&\quad h_2(t) + \left(\lambda\varepsilon\sum_{i=1}^r \beta_i v_{ix}(t,\varepsilon^2 u(t) + x_1) + \varphi_x(t,\lambda\varepsilon^2 u(t) + x_1)\right)h_1(t) \\
&\quad \in \frac{1}{\varepsilon^2}\Bigg[g\Bigg(\lambda\varepsilon^2 u(t) + x_1, \varphi(t,\lambda\varepsilon^2 u(t) + x_1) + \lambda\varepsilon^2 v(t) \\
&\qquad + \lambda\varepsilon\sum_{i=1}^r \beta_i v_i(t,\varepsilon^2 u(t) + x_1)\Bigg) - g(\lambda\varepsilon^2 u(t) + x_1, \varphi(t,\lambda\varepsilon^2 u(t) + x_1)) \\
&\qquad - \lambda g_y(\lambda\varepsilon^2 u(t) + x_1, \varphi(t,\lambda\varepsilon^2 u(t) + x_1))\left(\varepsilon^2 v(t) + \varepsilon\sum_{i=1}^r \beta_i v_i(t,\varepsilon^2 u(t) + x_1)\right) \\
&\qquad - \frac{\lambda^2\varepsilon^2}{2} g_{yy}(\lambda\varepsilon^2 u(t) + x_1, \varphi(t,\lambda\varepsilon^2 u(t) + x_1))\left(\sum_{i=1}^r \beta_i v_i(t,\varepsilon^2 u(t) + x_1)\right)^2\Bigg]
\end{aligned}$$

$$+\frac{1}{2}g_{yy}(\lambda\varepsilon^2u(t)+x_1,\varphi(t,\lambda\varepsilon^2u(t)+x_1))\left(\sum_{i=1}^{r}\beta_iv_i(t,\lambda\varepsilon^2u(t)+x_1)\right)^2$$
$$+\lambda\Big(g_y(\varepsilon^2u(t)+x_1,\varphi(t,\varepsilon^2u(t)+x_1))-g_y(x_1,\varphi(t,x_1))\Big)v(t)$$
$$+h\Bigg(\lambda\varepsilon^2u(t)+x_1,\lambda\varepsilon^2v(t)+\lambda\varepsilon\sum_{i=1}^{r}\beta_iv_i(t,\varepsilon^2u(t)+x_1)$$
$$+\varphi(t,\lambda\varepsilon^2u(t)+x_1),t+\alpha,\lambda\varepsilon^2\Bigg)\text{ for almost each (f.a.e.) } t\in\mathbb{R}\Bigg\},$$

where the term in the brackets $[\cdots]$ is set to 0 for $\varepsilon=0$.

Now we write (2.4) as follows

$$(u,v,0,0)\in F(u,v,x_1,\alpha,\beta,\varepsilon,1), \tag{2.5}$$

where the mapping

$$F:X\times Y\times U\times\mathbb{R}^{r+2}\times[0,1]\to 2^{X\times Y\times\mathbb{R}^{n+r+1}}\setminus\{\emptyset\}$$

is defined by

$$F(u,v,x_1,\alpha,\beta,t,\varepsilon,\lambda):=\Bigg\{\Bigg(\lambda K_{x_1}(h_1,h_2),\int_0^1 h_1(t)\,dt,$$
$$\int_0^1(h_2(t),w_0(t,x_1))\,dt,\cdots,\int_0^1(h_2(t),w_r(t,x_1))\,dt\Bigg)$$
$$:(h_1,h_2)\in G(u,v,x_1,\alpha,\beta,t,\varepsilon,\lambda)\Bigg\}.$$

It is standard to verify that F is upper semicontinuous with compact and convex set values [7]. Furthermore, we set

$$M(x_1,\alpha,\beta):=$$
$$\Bigg\{\Bigg(\int_0^1 h_1(t)\,dt,\int_0^1(h_2(t),w_0(t,x_1))\,dt,\cdots,\int_0^1(h_2(t),w_r(t,x_1))\,dt\,dt\Bigg)$$
$$:h_1(t)\in f(x_1,\varphi(t,x_1),t+\alpha,0)\text{ f.a.e. } t\in\mathbb{R},$$
$$h_2(t)+\varphi_x(t,x_1)h_1(t)\in\sum_{i,j=1}^{r}\beta_i\beta_ja_{ij}(t,x_1)$$
$$+h(x_1,\varphi(t,x_1),t+\alpha,0)\text{ f.a.e. } t\in\mathbb{R}\Bigg\}, \tag{2.6}$$

where

$$a_{ij}(t,x_1) := \frac{1}{2} g_{yy}(x_1,\varphi(t,x_1))(v_i(t,x_1),v_j(t,x_1)).$$

Again the mapping

$$M : U \times \mathbb{R}^{r+1} \to 2^{\mathbb{R}^{n+r+1}} \setminus \{\emptyset\}$$

is upper semicontinuous with compact and convex set values. Summarizing, we arrive at the following result.

THEOREM 2.1. *Suppose* (H1) *and* (H2) *hold. If there is an open bounded subset* $\Omega \subset \bar{\Omega} \subset U \times \mathbb{R}^{r+1}$ *such that:*

a) $0 \notin M(x_1,\beta,\alpha)$ *on the boundary* $\partial\Omega$,

b) $\deg(M,\Omega,0) \neq 0$, *where* deg *is the Brouwer degree.*

Then the system (1.1) *has a* 1*-periodic solution for* $\varepsilon > 0$ *small.*

Proof. To solve (1.1), we need to solve (2.5), which we put in the homotopy

$$(u,v,0,0) \in F(u,v,x_1,\alpha,\beta,\varepsilon,\lambda),$$

for $\lambda \in [0,1]$. It is not difficult to find positive constants c_1 and ε_0 such that

$$(u,v,0,0) \notin F(u,v,x_1,\alpha,\beta,\varepsilon,\lambda)$$

for any $(u,v,x_1,\alpha,\beta) \in \partial\mathscr{O}$ and any $(\varepsilon,\lambda) \in (0,\varepsilon_0) \times [0,1]$, where

$$\mathscr{O} := B_{c_1} \times \Omega, \quad B_{c_1} := \{(u,v) \in X \times Y : \|u\| + \|v\| < c_1\}.$$

Hence

$$\begin{aligned}
&\deg\Big((u,v,0,0) - F(u,v,x_1,\alpha,\beta,\varepsilon,1),\mathscr{O},0\Big)\\
&\quad = \deg\Big((u,v,0,0) - F(u,v,x_1,\alpha,\beta,\varepsilon,0),\mathscr{O},0\Big)\\
&\quad = \deg\Big((u,v,-M(x_1,\alpha,\beta)),\mathscr{O},0\Big) = \deg(-M,\Omega,0) \neq 0.
\end{aligned}$$

So (2.5) is solvable for any $\varepsilon \in (0,\varepsilon_0)$. The proof is finished. □

3. Bifurcations from families of periodics

When the unperturbed equation (1.2) has some symmetry then in place of condition (H1) the following one may hold:

(C1): the equation (1.2) has a smooth family $\varphi(t,x,\theta)$ of 1-periodic solutions for any $x \in U$ and $\theta \in \Gamma$, where $U \subset \mathbb{R}^n$, $\Gamma \subset \mathbb{R}^r$ are open bounded subsets.

Then we can repeat the above procedure to (1.1) with the next modifications: First, (2.1) is replaced with

$$v' = g_y(x,\varphi(t,x,\theta))v. \tag{3.1}$$

Clearly $\varphi'(t,x,\theta)$, $\varphi_{\theta_i}(t,x,\theta)$, $i=1,2,\cdots,r$, $\theta=(\theta_1,\theta_2,\cdots,\theta_r)$ are 1-periodic solutions of (3.1). We suppose:

(C2): the family $\varphi(t,x,\theta)$ is *non-degenerate*, i.e. the functions $\widetilde{v}_0(t,x,\theta):=\varphi'(t,x,\theta)$, $\widetilde{v}_i(t,x,\theta):=\varphi_{\theta_i}(t,x,\theta)$, $i=1,2,\cdots,r$ form a basis of the space of 1-periodic solutions of (3.1).

From [9, Lemma 1.3, p. 410] we know that condition (C2) implies the existence of a smooth basis $\widetilde{w}_j(t,x,\theta)$, $j=0,1,\cdots,r$ of the space of 1-periodic solutions of the adjoint system

$$w'=-g_y^*(x,\varphi(t,x,\theta))w$$

to (3.1).

Now, in the above procedure, we keep the projection P_1, but we replace P_x with $P_{x,\theta}:Y\to Y$ defined by:

$$P_{x,\theta}y:=y(t)-\widetilde{q}_0\widetilde{w}_0(t,x,\theta)-\widetilde{q}_1\widetilde{w}_1(t,x,\theta)-\cdots-\widetilde{q}_r\widetilde{w}_r(t,x,\theta),$$

$$(\widetilde{q}_0,\widetilde{q}_1,\cdots,\widetilde{q}_r)^*:=$$

$$\widetilde{A}(x,\theta)^{-1}\left(\int_0^1(y(t),\widetilde{w}_0(t,x,\theta))\,dt,\cdots,\int_0^1(y(t),\widetilde{w}_r(t,x,\theta))\,dt\right)^*$$

where

$$\widetilde{A}(x,\theta):=\left(\int_0^1(\widetilde{w}_i(t,x,\theta),\widetilde{w}_j(t,x,\theta))\,dt\right)_{i,j=0}^{r}$$

is an $(r+1)\times(r+1)$-matrix. Then changing

$$x=\varepsilon u+x_1,\quad u\in X,\quad P_1u=u,\quad x_1\in\mathbb{R}^n,$$

$$y=\varepsilon v+\varphi(t,\varepsilon u+x_1,\theta),\int_0^1(v(t),\widetilde{v}_i(t,x_1,\theta))\,dt=0,\quad i=0,1,\cdots,r,$$

in (1.1), we derive like above

$$\begin{aligned}&u'\in f(\varepsilon u+x_1,\varepsilon v+\varphi(t,\varepsilon u+x_1,\theta),t+\alpha,\varepsilon),\\&v'-g_y(x_1,\varphi(t,x_1,\theta))v\in\widetilde{H}(u,v,\varepsilon,\alpha,\theta,t),\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}\widetilde{H}(u,v,\varepsilon,\alpha,\theta,t):=\frac{1}{\varepsilon}\Big(&g(\varepsilon u+x_1,\varepsilon v+\varphi(t,\varepsilon u+x_1,\theta))\\&-g(\varepsilon u+x_1,\varphi(t,\varepsilon u+x_1,\theta))\Big)-g_y(x_1,\varphi(t,x_1,\theta))v\\&+h(\varepsilon u+x_1,\varepsilon v+\varphi(t,\varepsilon u+x_1,\theta),t+\alpha,\varepsilon)-\varphi_x(t,\varepsilon u+x_1,\theta)u'.\end{aligned}$$

Furthermore, we set

$$\widetilde{M}(x_1,\alpha,\theta) :=$$
$$\Bigg\{ \Bigg(\int_0^1 h_1(t)\,dt, \int_0^1 (h_2(t), w_0(t,x_1))\,dt, \cdots, \int_0^1 (h_2(t), w_r(t,x_1))\,dt\,dt \Bigg)$$
$$: h_1(t) \in f(x_1, \varphi(t,x_1,\theta), t+\alpha, 0) \text{ f.a.e. } t \in \mathbb{R},$$
$$h_2(t) + \varphi_x(t,x_1,\theta)h_1(t) \in h(x_1, \varphi(t,x_1,\theta), t+\alpha, 0) \text{ f.a.e. } t \in \mathbb{R} \Bigg\}. \quad (3.3)$$

The mapping

$$\widetilde{M} : U \times \mathbb{R} \times \Gamma \to 2^{\mathbb{R}^{n+r+1}} \setminus \{\emptyset\}$$

is upper semicontinuous with compact and convex set values. Consequently, we can directly modify the method of Section 2 to derive the following result.

THEOREM 3.1. *Suppose* (C1) *and* (C2). *If there is an open bounded subset* $\Omega \subset \bar{\Omega} \subset U \times \mathbb{R} \times \Gamma$ *such that:*

a) $0 \notin M(x_1,\alpha,\theta)$ *on the boundary* $\partial\Omega$*,*

b) $\deg(M,\Omega,0) \neq 0$.

Then the system (1.1) *has a* 1*-periodic solution for* $\varepsilon \neq 0$ *small.*

4. Applications to weakly coupled discontinuous nonlinear oscillators

We present in this part two examples by applying Theorems 2.1 and 3.1, respectively.

EXAMPLE 4.1. We first apply Theorem 2.1 to the system

$$\begin{aligned} y_1' &\in y_1 - y_2 - x^2(y_1^2+y_2^2)y_1 + \varepsilon\mu_1 \operatorname{Sgn} y_2, \\ y_2' &\in y_1 + y_2 - x^2(y_1^2+y_2^2)y_2 + \varepsilon\mu_2 \operatorname{Sgn} y_1 + \varepsilon\mu_3 \cos t, \\ x' &\in \varepsilon\left(y_1 \cos t + y_2 \sin t + \operatorname{Sgn} x\right), \end{aligned} \quad (4.1)$$

where $\mu_{1,2,3}$ are positive parameters and $\operatorname{Sgn} : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is given by

$$\operatorname{Sgn} y := \begin{cases} \operatorname{sgn} y & y \neq 0, \\ [-1,1] & y = 0, \end{cases}$$

where $\operatorname{sgn} y := y/|y|$ for $y = 0$ and $\operatorname{sgn} 0 = 0$. We need to verify conditions (H1) and (H2) for the unperturbed system (4.1) of the form

$$\begin{aligned} y_1' &= y_1 - y_2 - x^2(y_1^2+y_2^2)y_1, \\ y_2' &= y_1 + y_2 - x^2(y_1^2+y_2^2)y_2, \end{aligned} \quad (4.2)$$

possessing a smooth family of 2π-periodic solutions

$$\varphi(t,x) = \frac{1}{x}(\cos t, \sin t) \tag{4.3}$$

for $x \neq 0$. The linearization of (4.2) along (4.3) is as follows

$$\begin{aligned} v_1' &= -(1+\cos 2t)v_1 - (1+\sin 2t)v_2, \\ v_2' &= (1-\sin 2t)v_1 - (1-\cos 2t)v_2, \end{aligned} \tag{4.4}$$

and its adjoint system is given by

$$\begin{aligned} w_1' &= (1+\cos 2t)w_1 - (1-\sin 2t)w_2, \\ w_2' &= (1+\sin 2t)w_1 + (1-\cos 2t)w_2\,. \end{aligned} \tag{4.5}$$

One readily verifies that (4.4) has solutions

$$\begin{aligned} v_0(t,x) &= (-\sin t, \cos t), \\ \widetilde{v}(x,t) &= \left(\mathrm{e}^{-2t}\cos t, \mathrm{e}^{-2t}\sin t\right). \end{aligned}$$

Hence $v_0(x,t)$ is a basis of 2π-periodic solutions of (4.4). Furthermore, the functions:

$$\begin{aligned} w_0(t,x) &= (\sin t, -\cos t), \\ \widetilde{w}(x,t) &= \mathrm{e}^{2t}(\cos t, \sin t), \end{aligned}$$

are solutions of (4.5), so $w_0(t,x)$ is a basis of 2π-periodic solutions of (4.5). Consequently, now we do not have parameters β. After some computations, the function M of (2.6) for this case (4.1) has the form

$$M(x_1,\alpha) = \left(\frac{2\pi}{x_1}(\cos\alpha + |x_1|), 4\,\mathrm{sgn}\,x_1(\mu_1-\mu_2) - \pi\mu_3\cos\alpha\right). \tag{4.6}$$

We immediately see that (4.6) has a simple root

$$\widetilde{x}_1 = \frac{4(\mu_2-\mu_1)}{\pi\mu_3}, \quad \widetilde{\alpha} = \arccos\left[-\frac{4|\mu_1-\mu_2|}{\pi\mu_3}\right], \tag{4.7}$$

provided

$$0 < 4|\mu_1-\mu_2| < \pi\mu_3\,. \tag{4.8}$$

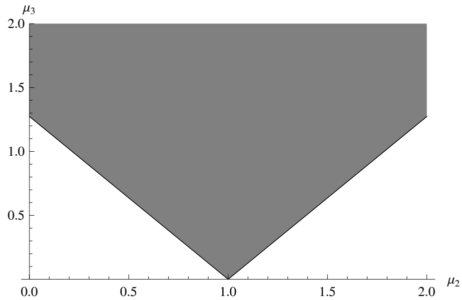
Taking a small neighborhood $\Omega \subset \mathbb{R}^2$ of the point $(\widetilde{x}_1, \widetilde{\alpha})$, Theorem 2.1 gives the following result.

THEOREM 4.2. *If positive parameters $\mu_{1,2,3}$ satisfy assumption* (4.8)*, then the system* (4.1) *has an 2π-periodic solution for any $\varepsilon \neq 0$ small which is located in an $O(|\varepsilon|)$-neighborhood of the vector function*

$$\left(\frac{1}{\widetilde{x}_1}\cos(t-\widetilde{\alpha}), \frac{1}{\widetilde{x}_1}\sin(t-\widetilde{\alpha}), \widetilde{x}_1\right),$$

where $\widetilde{x}_1$ and $\widetilde{\alpha}$ are given by (4.7)*.*

To visualize the set given by (4.8), first we consider its section for $\mu_1 = 1$ and $\mu_2 \in (0,2]$:



Similar figure is for the section for $\mu_2 = 1$ and $\mu_1 \in (0,2]$. Note if (μ_1, μ_2, μ_3) satisfies (4.8) then also $\xi(\mu_1, \mu_2, \mu_3)$ satisfies it for any $\xi > 0$.

REMARK 4.3. We can repeat the above arguments to an example when (4.2) is replaced with [8]

$$\begin{aligned} y_1' &= (x^2+1)(y_1^2+y_2^2)y_2, \\ y_2' &= -(x^2+1)(y_1^2+y_2^2)y_1, \end{aligned}$$

possessing a smooth family of 1-periodic solutions

$$\varphi(t,x) = \sqrt{\frac{2\pi k}{x^2+1}}\Big(\sin 2\pi kt, \cos 2\pi kt\Big)$$

for $k \in \mathbb{N}$. Then

$$\begin{aligned} v_0(t,x) &= (\cos 2\pi kt, -\sin 2\pi kt), \\ w_0(t,x) &= (\sin 2\pi kt, \cos 2\pi kt). \end{aligned}$$

We do not perform further computations in this paper.

EXAMPLE 4.4. Finally, we consider the system

$$\begin{aligned} y_1' &\in -y_2 + \varepsilon \frac{\mu_1}{y_2^2+1} \operatorname{Sgn} y_1 + \varepsilon \mu_2 \cos t, \\ y_2' &\in y_1 + \varepsilon \frac{\mu_3}{y_1^2+1} \operatorname{Sgn} y_2 + \varepsilon \mu_4 \cos t, \\ x' &\in \varepsilon \left(\frac{\operatorname{Sgn} x}{x^2+1} + y_1 \cos t + y_2 \sin t \right), \end{aligned} \tag{4.9}$$

where $\mu_{1,2,3,4}$ are positive parameters. We verify assumptions (C1) and (C2) for its unperturbed system

$$y_1' = -y_2, \quad y_2' = y_1, \tag{4.10}$$

which is just the harmonic oscillator. So now

$$\varphi(t,x,\theta)=\theta(\cos t,\sin t),$$
$$\widetilde{v}_0(t,x,\theta)=\theta(-\sin t,\cos t),\quad \widetilde{v}_1(t,x,\theta)=(\cos t,\sin t),$$
$$\widetilde{w}_0(t,x,\theta)=(-\sin t,\cos t),\quad \widetilde{w}_1(t,x,\theta)=(\cos t,\sin t),$$

for $\theta\neq 0$. After some computations we derive (3.3) of the form

$$\widetilde{M}(x_1,\alpha,\theta)=\left(2\pi\left(\frac{\operatorname{sgn}x_1}{x_1^2+1}+\theta\cos\alpha\right),\pi\left(\mu_2\sin\alpha+\mu_4\cos\alpha\right),\right.$$
$$\left.\pi\left(\mu_2\cos\alpha-\mu_4\sin\alpha\right)+4\frac{\arctan\theta}{|\theta|}\left(\mu_1+\mu_3\right)\right). \tag{4.11}$$

Now we need the following obvious result [7].

LEMMA 4.5. *Let $F_1\in C^1\left(\Omega_1\times\Omega_2,\mathbb{R}^n\right)$, $F_2\in C^1\left(\Omega_1\times\Omega_2,\mathbb{R}^m\right)$, $\Omega_1\subset\mathbb{R}^n$, $\Omega_2\subset\mathbb{R}^m$ be open subsets. Suppose that for any $y\in\Omega_2$ there is a $x:=f(y)\in\Omega_1$ such that $F_1(f(y),y)=0$ and $D_xF_1(f(y),y):\mathbb{R}^n\to\mathbb{R}^n$ is regular, i.e. $F_1(x,y)=0$ has a simple root $x=f(y)$ in Ω_1 for any $y\in\Omega_2$. Assume that $G(y):=F_2(f(y),y)=0$ has a simple root $y_0\in\Omega_2$, i.e. $G(y_0)=0$ and $DG(y_0)$ is regular. Then (x_0,y_0), $x_0:=f(y_0)$ is a simple root of $F=(F_1,F_2)^*$, i.e. $F(x_0,y_0)=0$ and $DF(x_0,y_0)$ is regular. Note a local uniqueness of simple roots and their smooth dependence on parameters follow from the implicit function theorem, so we suppose that $f\in C^1(\Omega_2,\Omega_1)$.*

Applying Lemma 4.5 to (4.11), we solve the system

$$\frac{\operatorname{sgn}\widetilde{x}_1}{\widetilde{x}_1^2+1}+\widetilde{\theta}\cos\widetilde{\alpha}=0, \tag{4.12}$$
$$\mu_2\sin\widetilde{\alpha}+\mu_4\cos\widetilde{\alpha}=0, \tag{4.13}$$
$$\pi\left(\mu_2\cos\widetilde{\alpha}-\mu_4\sin\widetilde{\alpha}\right)+4\frac{\arctan\widetilde{\theta}}{|\widetilde{\theta}|}\left(\mu_1+\mu_3\right)=0. \tag{4.14}$$

Clearly (4.12) and (4.13) give

$$\widetilde{\theta}=-\frac{\operatorname{sgn}\widetilde{x}_1}{\cos\widetilde{\alpha}(\widetilde{x}_1^2+1)}\quad\text{and}\quad\tan\widetilde{\alpha}=-\frac{\mu_4}{\mu_2}. \tag{4.15}$$

Then for $\widetilde{\alpha}\in(-\pi/2,\pi/2)$, by inserting (4.15) into (4.14) we obtain

$$\pi\frac{\mu_2^2+\mu_4^2}{4\mu_2(\mu_1+\mu_3)}-\operatorname{sgn}\widetilde{x}_1(\widetilde{x}_1^2+1)\arctan\frac{\sqrt{\mu_2^2+\mu_4^2}}{\mu_2(\widetilde{x}_1^2+1)}=0. \tag{4.16}$$

First, the equality (4.16) implies $\widetilde{x}_1>0$. Next, the function

$$z\to(z^2+1)\arctan\frac{\sqrt{\mu_2^2+\mu_4^2}}{\mu_2(z^2+1)}$$

is strictly increasing on $[0,\infty)$ from $\arctan \frac{\sqrt{\mu_2^2+\mu_4^2}}{\mu_2}$ to $\frac{\sqrt{\mu_2^2+\mu_4^2}}{\mu_2}$. Consequently, if

$$\arctan \frac{\sqrt{\mu_2^2+\mu_4^2}}{\mu_2} < \frac{\pi}{4}\frac{\mu_2^2+\mu_4^2}{\mu_2(\mu_1+\mu_3)} < \frac{\sqrt{\mu_2^2+\mu_4^2}}{\mu_2},$$

which is equivalent to

$$\frac{\pi}{4}\sqrt{\mu_2^2+\mu_4^2} < \mu_1+\mu_3 < \frac{\pi}{4}\frac{\mu_2^2+\mu_4^2}{\mu_2 \arctan \frac{\sqrt{\mu_2^2+\mu_4^2}}{\mu_2}}, \tag{4.17}$$

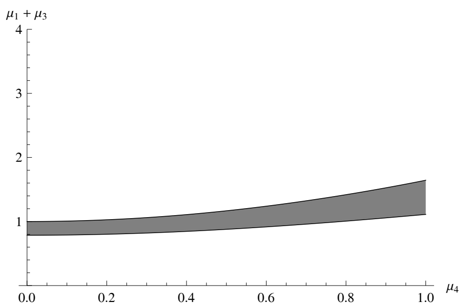
then (4.16) possesses a unique simple zero $\widetilde{x}_1$ on $(0,\infty)$. Summarizing, by Lemma 4.5 and Theorem 3.1, we obtain the following result.

THEOREM 4.6. *If positive parameters* $\mu_{1,2,3,4}$ *satisfy assumption* (4.17)*, then the system* (4.9) *has an* 2π*-periodic solution for any* $\varepsilon \neq 0$ *small which is located in an* $O(|\varepsilon|)$*-neighborhood of the vector function*

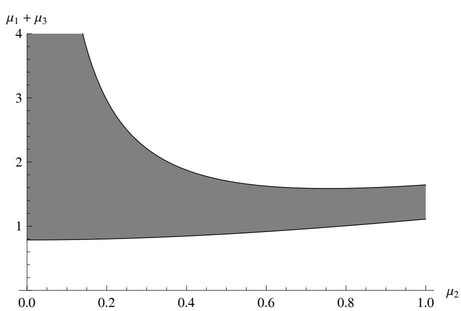
$$\left(\widetilde{\theta}\cos(t-\widetilde{\alpha}), \widetilde{\theta}\sin(t-\widetilde{\alpha}), \widetilde{x}_1\right),$$

where $\widetilde{x}_1$*,* $\widetilde{\alpha} \in (-\pi/2,\pi/2)$ *and* $\widetilde{\theta}$ *are given by* (4.15) *and* (4.16)*.*

To visualize the set given by (4.17), first we consider its section for $\mu_2 = 1$ and $\mu_4 \in (0,1]$:



then for $\mu_4 = 1$ and $\mu_2 \in (0,1]$:



Again note if $(\mu_1,\mu_2,\mu_3,\mu_4)$ satisfies (4.17), then also $\xi(\mu_1,\mu_2,\mu_3,\mu_4)$ satisfies it for any $\xi>0$.

REMARK 4.7. Finally, we can consider more complicated system than (4.9), when the unperturbed one has the form

$$\begin{aligned} y_1' &= y_2, \quad y_2' = -y_1-(x^2+1)(y_1^2+y_3^2)y_1,\\ y_3' &= y_4, \quad y_4' = -y_3-(x^2+1)(y_1^2+y_3^2)y_3. \end{aligned} \tag{4.18}$$

We note that (4.18) has the form

$$\ddot{w}+(1+(x^2+1)\|w\|^2)w=0, \tag{4.19}$$

for $w=(y_1,y_3)$ and $\|w\|=\sqrt{y_1^2+y_3^2}$. For

$$\Gamma(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},$$

we see that if $w(t)$ solves (4.19), then $\Gamma(\theta)w(t)$ is also its solution. We know [10] that

$$y_1(t)=v(t,x,k)=\frac{\sqrt{2}k}{\sqrt{(1-2k^2)(x^2+1)}}\,\mathrm{cn}\,\frac{t}{\sqrt{1-2k^2}}$$

solves $y_1'=y_2$, $y_2'=-y_1-(x^2+1)y_1^3$, where cn is the Jacobi elliptic function and k is the elliptic modulus [10]. Consequently, the system (4.18) has a smooth family of periodic solutions

$$y(t,x,\theta,k)=\Big(\cos\theta\, v(t,x,k),\cos\theta\, v(t,x,k)',\sin\theta\, v(t,x,k),\sin\theta\, v(t,x,k)'\Big). \tag{4.20}$$

The function $y(t,x,\theta,k)$ has the period $T(k)=4K(k)\sqrt{1-2k^2}$ for the complete elliptic integral $K(k)$ of the first kind. We note $T(0)=2\pi$ and $T(\sqrt{2}/2)=0$. By numerically solving the equation $T(k)=1$, we find its unique solution $k_0\doteq 0.700595$ with $T(k_0)'\neq 0$. So we fix $k=k_0$ and take

$$\varphi(t,x,\theta)=y(t,x,\theta,k_0)$$

to satisfy condition (C1). Condition (C2) is verified for this case in [8]. Again we do not carry out more computations for this example.

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Michal Fečkan
Department of Mathematical Analysis and Numerical Mathematics
Comenius University
Mlynská dolina, 842 48 Bratislava
Slovakia
e-mail: Michal.Feckan@fmph.uniba.sk

and

Mathematical Institute of Slovak Academy of Sciences
Štefánikova 49, 814 73 Bratislava
Slovakia