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MULTIPLICITY OF 2–NODAL SOLUTIONS FOR A SEMILINEAR ELLIPTIC EQUATION

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Abstract. In this paper, we consider the multiplicity of 2-nodal solutions of semilinear elliptic equations. Using the generalized barycenter map, we prove that existence of multiple 2-nodal solutions for semilinear elliptic equations in some domains with hole.

1. Introduction

In this paper, we study the multiplicity of 2-nodal solutions of semilinear elliptic equations of the form

$$\begin{cases} -\Delta u+u=|u|^{p-2}u^{+}+|u|^{q-2}u^{-} & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{$E_{p,q}$}$$

where Ω is a domain in $\mathbb{R}^N$, $2<p,q<2^*$ $\left(2^*=\frac{2N}{N-2} \text{ if } N\geqslant 3, 2^*=\infty \text{ if } N=2\right)$, $u^+=\max\{0,u\}$ and $u^-=\min\{u,0\}$. Associated with equation $(E_{p,q})$, we consider the energy functional J in the Sobolev space $H_0^1(\Omega)$,

$$J(u)=\frac{1}{2}\|u\|^2-\frac{1}{p}\int_\Omega \left|u^+\right|^p dx-\frac{1}{q}\int_\Omega \left|u^-\right|^q dx,$$

where

$$\|u\|=\left(\int_\Omega \left(|\nabla u|^2+u^2\right)dx\right)^{1/2}$$

is a standard norm in $H_0^1(\Omega)$. It is well-known that the functional $J\in C^2\left(H_0^1(\Omega),\mathbb{R}\right)$ and the solutions of equation $(E_{p,q})$ in Ω are the critical points of the energy functional J in $H_0^1(\Omega)$ (see Ambrosetti-Rabinowitz [1] and Willem [23]).

Generally, a standard technique to find the one sign solutions of equation $(E_{p,q})$ in Ω is using the Nehari minimization problems:

$$\alpha^{\pm}(\Omega)=\inf_{v\in\mathbf{M}^{\pm}(\Omega)} J(v),$$

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where

$$\mathbf{M}^{\pm}(\Omega)=\left\{u\in H_0^1(\Omega)\backslash\{0\}\mid\left\langle J'(u),u\right\rangle=0,\pm u\geqslant 0\right\}.$$

Note that $\alpha^{\pm}(\Omega)$ are positive numbers and $\alpha^{\pm}(\Omega_1)\geqslant\alpha^{\pm}(\Omega_2)$ if $\Omega_1\subset\Omega_2$ (see Willem [23]). Furthermore, we call a nonzero critical point u_0 of J is a least energy positive (or negative) solution of equation $(E_{p,q})$ in Ω if $u_0>0$ (or <0) and $J(u_0)=\alpha^{+}(\Omega)$ (or $\alpha^{-}(\Omega)$).

That the existence of one sign solutions of equation $(E_{p,q})$ is affected by the shape of the domain Ω has been the focus of a great deal of research in recent years. By the Rellich compactness theorem, it is easy to obtain a one sign solution of equation $(E_{p,q})$ in bounded domains. For general unbounded domains Ω, because of the lack of compactness, the issue of existence of one sign solutions of equation $(E_{p,q})$ in Ω is very difficult and unclear. Indeed, a by now classical result of Esteban-Lions [14] states that for unbounded domains satisfying the condition: there exists $\chi\in\mathbb{R}^N,||\chi||=1$ such that $n(z)\cdot\chi\geqslant 0$ and $n(z)\cdot\chi\not\equiv 0$ on $\partial\Omega$, where $n(z)$ is the unit outward normal vector to $\partial\Omega$ at the point z, equation $(E_{p,q})$ does not admit any nontrivial solution. Recently, there has benn some progress for the existence and multiplicity of one sign solutions of equation $(E_{p,q})$ in unbounded domains (see Ambrosetti-Rabinowitz [1], Bahri-Lions [2], Benci-Cerami [5], Berestycki-Lions [6], Lions [20], Lien-Tzeng-Wang [19], Del Pino-Felmer [12, 13] and Wu [24, 25], etc.). Furthermore, if $\Omega=\mathbb{R}^N$, then equation $(E_{p,q})$ has a unique positive solution (see Kwong [18]). For the equation $(E_{p,q})$ in exterior domain Ω, we can see that the Mountain Pass value is equal to the first level of breaking down of Palais-Smale condition (see Benci-Cerami [5]) and we cannot get a positive solution through the Mountain Pass Theorem (i.e. equation $(E_{p,q})$ does not admit any least energy solution). However, Benci-Cerami [5] and Bahri-Lions [2] showed the existence of at least one positive solution of equation $(E_{p,q})$ in exterior domain Ω.

In the works mentioned above, the authors considered one sign solutions. For other situations, Wang [21] proved the existence of a nodal (sign-changing) solution if the domain Ω is bounded and the operator is $-\Delta$ rather than $-\Delta+1$ for equation $(E_{p,q})$. Bartsch [3] obtained infinite nodal solutions for equation $(E_{p,q})$ in bounded domains. Furtado [15, 16] used the Ljusternik-Schnirelmann category and showed that the number of 2-nodal solutions of equation $(E_{p,q})$ depends on the topology and the symmetries of a symmetric bounded domain Ω. A 2-nodal solution is a nontrivial solution u such that the set $\{x\in\Omega:u(x)\neq 0\}$ has exactly two connected components, and u is positive in one of them and negative in the other (see Castro-Clapp [8] or Bartsch-Weth [4]). Bartsch-Weth [4] proved that equation $(E_{p,q})$ in a bounded domain Ω that contains a large ball has three nodal solutions in which two are 2-nodal solutions. Huang-Wu [17] proved that equation $(E_{p,q})$ in a finite strip with hole has at least four 2-nodal solutions. Wu [26], proved that equation $(E_{p,q})$ in an infinite upper strip with $m-$ holes has at least m^2 2-nodal solutions.

Motivated by the results of [4, 17, 26], we are interested in the relation between the "holes" of domain and the number of 2-nodal solutions of equation $(E_{p,q})$. Before stating our main results, we need the following notations. Denote the infinite strip $\mathbf{A}$, the upper half strip $\mathbf{A}^+$ and the finite strip $\mathbf{A}(s,l)$ as follows:

$$\mathbf{A} = \left\{(x',x_N) \in \Theta \times \mathbb{R} : \Theta \text{ is a bounded domain in } \mathbb{R}^{N-1}\right\};$$
$$\mathbf{A}^+ = \{(x',x_N) \in \mathbf{A} : x_N > 0\};$$
$$\mathbf{A}(s,l) = \{(x',x_N) \in \mathbf{A} : s < x_N < l\}.$$

Then we have the following result.

THEOREM 1.1. *There exists a positive number l_0 such that for $l > l_0$ and the bounded domain $\Omega(l)$ satisfy $\mathbf{A}(-l,l) \subset \Omega(l) \subset \mathbf{A}$, equation $(E_{p,q})$ in $\Omega(l)$ has at least two 2-nodal solutions.*

Let $\Omega(l)$ be a bounded domain as in Theorem 1.1 and let $\omega \subset \mathbf{A}$ be a nonempty open set such that $\mathbf{A}\left(-\widetilde{l},\widetilde{l}\right)\backslash\overline{\omega}$ is a domain in $\mathbb{R}^N$ for some $\widetilde{l} > 0$. Then [17] proved that equation $(E_{p,q})$ in $\Omega(l)\backslash\overline{\omega}$ has at least four 2-nodal solutions if l sufficiently large. Here we will use the generalized barycenter map to improve the result of [17]. Our result is the following theorem.

THEOREM 1.2. *There is $\overline{l}_0 > \widetilde{l}$ such that for $l > \overline{l}_0$ equation $(E_{p,q})$ in $\Omega(l)\backslash\overline{\omega}$ has at least six 2-nodal solutions.*

Next, we consider the upper infinite strip with m-holes

$$D_m(l) = \mathbf{A}^+ \backslash \left[\cup_{i=1}^m \left(\overline{\omega} + (0,il)\right)\right].$$

By Wu [26], we know that equation $(E_{p,q})$ in $D_m(l)$ has at least m^2 2-nodal solutions if l sufficiently large. Here we can show that existence of more than m^2 2-nodal solutions. Our result is the following theorem.

THEOREM 1.3. *There exists $\widetilde{l}_0 > \widetilde{l}$ such that for $l > \widetilde{l}_0$, equation $(E_{p,q})$ in $D_m(l)$ has at least $m \times (m+1)$ 2-nodal solutions.*

This paper is organized as follows. In Section 2, we set up preliminaries. In Sections 3-5, we complete the proofs of our Theorems 1.1-1.3.

2. Preliminaries

In this section, we recall several known results will be used in later section. First, we define the Palais-Smale (denoted by (PS)) sequences in $H_0^1(\Omega)$ for J as follows.

DEFINITION 2.1. For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J if $J(u_n) = \beta + o(1)$ and $J'(u_n) = o(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$.

Now, we consider the minimization problems

$$\alpha^{\pm}(\Omega) = \inf_{u \in \mathbf{M}^{\pm}(\Omega)} J(u);\ \theta(\Omega) = \inf_{u \in \mathbf{N}(\Omega)} J(u),$$

where

$$\mathbf{M}^{\pm}(\Omega) = \left\{ u \in H_0^1(\Omega) \setminus \{0\} \mid \left\langle J'(u), u \right\rangle = 0, \pm u \geqslant 0 \right\}$$

and

$$\mathbf{N}(\Omega) = \left\{ u \in H_0^1(\Omega) \mid u^+ \in \mathbf{M}^+(\Omega), u^- \in \mathbf{M}^-(\Omega) \right\}.$$

Clearly, $\alpha^+(\Omega) + \alpha^-(\Omega) \leqslant \theta(\Omega)$. We need the following definition.

DEFINITION 2.2. (i) The domain Ω is called a large domain in $\mathbf{A}$ if $\Omega \subset \mathbf{A}$ and for any $n > 0$ there exist $s < l$ such that $l - s = n$ and $\mathbf{A}(s,l) \subset \Omega$;

(ii) The domain Ω is called a strictly large domain in $\mathbf{A}$ if Ω is a large domain in $\mathbf{A}$ and $\Omega \neq \mathbf{A}$.

Note that the infinite strip $\mathbf{A}$ is a large domain in itself and the upper half strip with m-holes $D_m(l)$ is a strictly large domain in $\mathbf{A}$ for all $l > 0$. Furthermore, by Lien-Tzeng-Wang [19, Lemma 2.5] we have the following result.

LEMMA 2.3. *If Ω is a large domain in $\mathbf{A}$, then $\alpha^{\pm}(\Omega) = \alpha^{\pm}(\mathbf{A})$. Furthermore, if Ω is a strictly large domain in $\mathbf{A}$, then equation $(E_{p,q})$ in Ω does not admits any least energy one sign solution.*

LEMMA 2.4. *If u is a nodal solution of the equation $(E_{p,q})$ in Ω and $J(u) < \alpha^+(\Omega) + \alpha^-(\Omega) + \min\{\alpha^+(\Omega), \alpha^-(\Omega)\}$, then u is a 2-nodal solution of equation $(E_{p,q})$ in Ω.*

Proof. The proof is similar to that of Proposition 3.1 in Furtado [15] (or see Bartsch-Weth [4]. □

Now, we recall the generalized barycenter map (cf. Bartsch-Weth [4, Theorem 2.1] and Cerami-Passaseo [10]) given by $\Phi : L^p(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ a continuous map satisfying

$$\Phi(u(z - \zeta)) = \zeta + \Phi(u) \text{ and } \Phi\left(u \circ A^{-1}\right) = A\Phi(u)$$

for every $\zeta \in \mathbb{R}^N$, every orthogonal $N \times N$ matrix A and every $u \in L^P(\mathbb{R}^N) \setminus \{0\}$. Since $L^p(\mathbf{A}) \subset L^p(\mathbb{R}^N)$ and the infinite strip $\mathbf{A}$ is only translation invariant and symmetric on x_N-axis. Therefore, we may redefine a new generalized barycenter map $h : L^p(\mathbf{A}) \setminus \{0\} \to \mathbb{R}$ such that for $\xi \in \mathbb{R}$ and $u \in L^p(\mathbf{A}) \setminus \{0\}$, we have

$$h\left(u\left(x', x_N - \xi\right)\right) = \xi + h\left(u\left(x', x_N\right)\right) \text{ and } h\left(u\left(x', -x_N\right)\right) = -h\left(u\left(x', x_N\right)\right). \quad (1)$$

Then we have the following result.

LEMMA 2.5. *Let Ω be a large domain in $\mathbf{A}$. Then for each positive number L there exists a positive number $\delta(L)$ such that for $u \in \mathbf{N}(\Omega)$ with $J(u) \leqslant \theta(\mathbf{A}) + \delta(L)$ we have either $h(u^+) - h(u^-) > L$ or $h(u^-) - h(u^+) > L$. Furthermore, if Ω is a strictly large domain in $\mathbf{A}$, then:*

(i) the function u^+ satisfies either $h(u^+) > L$ or $h(u^+) < -L$;
(ii) the function u^- satisfies either $h(u^-) > L$ or $h(u^-) < -L$.

Proof. Suppose otherwise, then there exist $L_0 > 0$ and a sequence $\{u_n\} \subset \mathbf{N}(\Omega)$ such that $J(u_n) = \theta(\mathbf{A}) + o(1)$,

$$h(u_n^+) - h(u_n^-) \leqslant L_0 \text{ and } h(u_n^-) - h(u_n^+) \leqslant L_0. \tag{2}$$

By Lemma 2.3 and Wu [26, Theorem 1.2]

$$\lim_{n\to\infty} J(u_n) = \theta(\mathbf{A}) = \alpha^+(\mathbf{A}) + \alpha^-(\mathbf{A})$$

and

$$J(u_n) = J\left(u_n^+\right) + J\left(u_n^-\right) \geqslant \alpha^+(\mathbf{A}) + \alpha^-(\mathbf{A}).$$

This implies $\lim_{n\to\infty} J(u_n^\pm) = \alpha^\pm(\mathbf{A})$. Since $u_n^\pm \in \mathbf{M}^\pm(\Omega) \subset \mathbf{M}^\pm(\mathbf{A})$, by Wang-Wu [22, Lemma 7], $\{u_n^\pm\}$ are $(\mathrm{PS})_{\alpha^\pm(\mathbf{A})}$-sequences in $H_0^1(\mathbf{A})$ for J. Clearly, $\{u_n^\pm\}$ are bounded sets in $H_0^1(\mathbf{A})$. Then by using a similar argument as in Lien-Tzeng-Wang [19, Theorem 4.1], there exist $R, C_0 > 0$ and $\{y_n^\pm\} \subset \mathbb{R}$ with $|y_n^\pm| \to \infty$ such that

$$\int_{\mathbf{A}(-R,R)-\left(0,y_n^\pm\right)} \left(u_n^\pm\right)^2 dx \geqslant C_0 \text{ for all } n \in \mathbb{N}.$$

Moreover, by Lien-Tzeng-Wang [19, Theorem 4.1] and Chen-Chen-Wang [9], equation $(E_{p,q})$ in $\mathbf{A}$ has a positive solution u_0^+ and a negative solution u_0^- such that $u_0^\pm$ are axially symmetric in x_N-axis and

$$\left\|u_n^\pm - u_0^\pm\left(x', x_N - y_n^\pm\right)\right\|_{H^1} \to 0 \text{ as } n \to \infty. \tag{3}$$

Now we will show that $|y_n^+ - y_n^-| \to \infty$ as $n \to \infty$. Suppose otherwise, then we can assume that $y_n^+ - y_n^- \to y_0$ for some $y_0 \in \mathbb{R}$. Then by (3)

$$\begin{aligned}
0 &= \int_{\mathbf{A}} \left|u_n^+\right|^r \left|u_n^-\right|^s dx = \int_{\mathbf{A}} \left|u_0^+\left(x', x_N - y_n^+\right)\right|^r \left|u_0^-\left(x', x_N - y_n^-\right)\right|^s dx + o(1) \\
&= \int_{\mathbf{A}} \left|u_0^+\left(x', x_N\right)\right|^r \left|u_0^-\left(x', x_N - y_n^- + y_n^+\right)\right|^s dx + o(1) \\
&= \int_{\mathbf{A}} \left|u_0^-\left(x', x_N\right)\right|^r \left|u_0^-\left(x', x_N + y_0\right)\right|^s dx + o(1),
\end{aligned}$$

which is a contradiction, where $\frac{r}{p} + \frac{s}{q} = 1$. Thus, $|y_n^+ - y_n^-| \to \infty$ as $n \to \infty$. Moreover, by (3) we can conclude

$$h(u_n^+) = h(u_0^+\left(x', x_N - y_n^+\right)) + o(1) = h(u_0^+) + y_n^+ + o(1) \tag{4}$$

and

$$h(u_n^-) = h(u_0^-\left(x', x_N - y_n^-\right)) + o(1) = h(u_0^-) + y_n^- + o(1). \tag{5}$$

This implies

$$\left|h(u_n^+) - h(u_n^-)\right| \to \infty \text{ as } n \to \infty$$

this contradicts (2). Next, for Ω is a strictly large domain in $\mathbf{A}$, then by (4) and (5) we only need to prove that

$$\left|y_n^+\right| \to \infty \text{ and } \left|y_n^-\right| \to \infty \text{ as } n \to \infty.$$

The proofs of two cases are similar argument. Therefore, we only need to prove the case $|y_n^+| \to \infty$ as $n \to \infty$. Suppose otherwise, then $\{y_n^+\}$ is a bounded sequence in $\mathbb{R}$. Without loss of generality, we may assume that $y_n^+ \to y_0$ for some $y_0 \in \mathbb{R}$. Since $u_n^+ \in H_0^1(\Omega)$. Then

$$u_n^+ \left(x', x_N + y_n^+\right) \in H_0^1\left(\Omega - \left(0, y_n^+\right)\right).$$

By (3) and $\lim_{n\to\infty}[\Omega - (0, y_n^+)] \to [\Omega - (0, y_0)]$ we obtain $[\Omega - (0, y_0)] \neq \mathbf{A}$ and $u_0^+ \in H_0^1(\Omega - (0, y_0))$ which contradicts the fact that u_0^+ is a positive solution of equation $(E_{p.q})$ in $\mathbf{A}$. □

3. Proof of Theorem 1.1

For positive numbers L, δ and the domain $\Omega \subset \mathbf{A}$, we denote

$$\begin{aligned} \mathbf{N}(\delta, \Omega) &= \{u \in \mathbf{N}(\Omega) : J(u) \leqslant \theta(\mathbf{A}) + \delta\}, \\ \mathbf{N}_i(L, \delta, \Omega) &= \left\{u \in \mathbf{N}(\delta, \Omega) : (-1)^i g(u) > L\right\}, \end{aligned}$$

where $g(u) = h(u^+) - h(u^-)$. Then we have the following results.

LEMMA 3.1. *Let L and $\delta(L)$ be positive numbers as in Lemma 2.5. Then there exists a positive number l_0 such that for $l > l_0$ and the bounded domain $\Omega(l)$ satisfy $\mathbf{A}(-l, l) \subset \Omega(l) \subset \mathbf{A}$, we have:*

(i) $\mathbf{N}_i(L, \delta(L), \Omega(l)) \neq \emptyset$ *for all* $i = 1, 2$;
(ii) $\mathbf{N}(\delta(L), \Omega(l)) = \mathbf{N}_1(L, \delta(L), \Omega(l)) \cup \mathbf{N}_2(L, \delta(L), \Omega(l))$;
(iii) $\mathbf{N}_1(L, \delta(L), \Omega(l)) \cap \mathbf{N}_2(L, \delta(L), \Omega(l)) = \emptyset$.

Proof. By $\mathbf{N}(\Omega(l)) \subset \mathbf{N}(\mathbf{A})$ for all $l > 0$ and Lemma 2.5, we only need to prove that there exists $l_0 > 0$ such that $\mathbf{N}_i(L, \delta(L), \Omega(l)) \neq \emptyset$ for all $l > l_0$ and $i = 1, 2$. By Lien-Tzeng-Wang [19, Lemma 2.2],

$$\alpha^{\pm}\left(\mathbf{A}\left(-\frac{l}{2}, \frac{l}{2}\right)\right) \searrow \alpha^{\pm}(\mathbf{A}) \text{ as } l \nearrow \infty.$$

Then there exists $l_0 > L$ such that

$$\alpha^{\pm}\left(\mathbf{A}\left(-\frac{l}{2}, \frac{l}{2}\right)\right) < \alpha^{\pm}(\mathbf{A}) + \frac{\delta(L)}{2} \text{ for all } l > l_0.$$

Moreover, by Ambrosetti-Rabinowitz [1] and Chen-Chen-Wang [9], equation $(E_{p,q})$ in $\mathbf{A}(-l/2, l/2)$ has a positive solution $v^+ \in \mathbf{M}^+(\mathbf{A}(-l/2, l/2))$ and a negative solution $v^- \in \mathbf{M}^-(\mathbf{A}(-l/2, l/2))$ such that $J(v^{\pm}) = \alpha^{\pm}(\mathbf{A}(-l/2, l/2))$ and $v^{\pm}(x, -y) = v^{\pm}(x, y)$. Clearly, $h(v^{\pm}) = 0$.

Setting

$$v_i^+\left(x', x_N\right) = v^+\left(x', x_N - (-1)^i \frac{l}{2}\right), \quad v_i^-\left(x', x_N\right) = v^-\left(x', x_N + (-1)^i \frac{l}{2}\right) \text{ for } i = 1, 2.$$

From the translation invariance of the functional in x_N-axis we get that $v_i^{\pm} \in \mathbf{M}^{\pm}(\Omega(l))$ and $h(v_i^{\pm}) = \pm\frac{(-1)^i l}{2}$.

Setting $v_i = v_i^+ + v_i^-$, we obtain $v_i \in \mathbf{N}(\Omega(l)), (-1)^i g(v_i) = l > L$ and

$$J(v_i) < \alpha^+(\mathbf{A}) + \alpha^-(\mathbf{A}) + \delta(L) = \theta(\mathbf{A}) + \delta(L) \text{ for all } l > l_0.$$

This implies $\mathbf{N}_i(L, \delta(L), \Omega(l)) \neq \emptyset$ for all $l > l_0$. □

Furthermore, we have the following results.

LEMMA 3.2. *Let $l_0 > 0$ as in Lemma 3.1. Then for each $l > l_0$, we have that $\mathbf{N}_i(L, \delta(L), \Omega(l))$ are closed sets.*

Proof. The proofs of cases "$i = 1$" and "$i = 2$" are similar argument. Therefore, we only need to prove the case "$i = 1$". Suppose that u_0 is a limits point of $\mathbf{N}_1(L, \delta(L), \Omega(l))$, then $-g(u_0) \geqslant L$ and $J(u_0) \leqslant \theta(\mathbf{A}) + \delta(L)$. This implies $u_0 \in \mathbf{N}(\delta(L), \Omega(l))$. If $-g(u_0) = L$, then by Lemma 3.1 $u_0 \in \mathbf{N}_2(L, \delta(L), \Omega(l))$. We obtain

$$-L = g(u_0) > L$$

which is a contradiction. Thus, $g(u_0) < L$, and so $u_0 \in \mathbf{N}_1(L, \delta(L), \Omega(l))$. Therefore, $\mathbf{N}_i(L, \delta(L), \Omega(l))$ are closed sets. □

Now we consider the minimization problem in $\mathbf{N}_i(L, \delta(L), \Omega(l))$ for J

$$\theta_i(\Omega(l)) = \inf_{u \in \mathbf{N}_i(L, \delta(L), \Omega(l))} J(u) \text{ for all } i = 1, 2.$$

Then we have the following result.

LEMMA 3.3. *For each $v_0 \in \mathbf{N}_i(L, \delta(L), \Omega(l))$ there is a map $\Phi : H_0^1(\Omega(l)) \to \mathbb{R}^2$ such that:*

(i) $\Phi\left(t_1 v_0^+ + t_2 v_0^-\right) = (t_1, t_2)$ *for* $t_1, t_2 \geqslant 0$;
(ii) $\Phi(u) = (1, 1)$ *if and only if* $u \in \mathbf{N}(\Omega(l))$.

Proof. Similarly to the method used in Clapp-Weth [11, Lemma 13]. □

The next result is a variant of Proposition 14 in Clapp-Weth [11], and its proof follows from the arguments of applying the Leray-Schauder continuation principle. Let

$$b = \theta(\mathbf{A}) + \min\left\{\alpha^+(\mathbf{A}), \alpha^-(\mathbf{A}), \delta(L)\right\}$$

and $\operatorname{dist}_H(u, \mathbf{D}) = \inf\left\{\|u - v\| : v \in \mathbf{D} \subset H_0^1(\Omega)\right\}$. Then we have the following results.

PROPOSITION 3.4. *Let $\lambda_0 = b - \theta_i(\Omega(l))$. Then for each $\lambda \in (0, \lambda_0)$ and $\mu > 0$ there exists $u_0 \in H_0^1(\Omega(l))$ such that:*

(i) $\operatorname{dist}_H(u_0, \mathbf{N}_i(L, \delta(L), \Omega(l))) \leqslant \mu$;
(ii) $J(u_0) \in [\theta_i(\Omega(l)), \theta_i(\Omega(l)) + \lambda)$;
(iii) $\|\nabla J(u_0)\| \leqslant \max\left\{\sqrt{\lambda}, \frac{\lambda}{\mu}\right\}$.

Proof. The proofs of cases "$i=1$" and "$i=2$" are similar argument. Therefore, we only need to prove the case "$i=1$". Fix $v_0 \in \mathbf{N}_1(L,\delta(L),\Omega(l))$ such that $J(v_0) < \theta_1(\Omega(l))+\lambda$, and fix $d_0 > 1$ such that $J\left(d_0 v_0^{\pm}\right) \leqslant 0$. Let $\Phi : H_0^1(\Omega(l)) \to \mathbb{R}^2$ be as in Lemma 3.3. We put $K = [0,d_0]\times[0,d_0]$ and define

$$\eta : K \to H_0^1(\Omega(l)),\ \eta(s_1,s_2) = s_1 v_0^+ + s_2 v_0^-.$$

Then $\Phi\circ\eta = id : K \to K$, in particular

$$\deg(\Phi\circ\eta, K, (1,1)) = 1. \tag{6}$$

Notice also that

$$J(\eta(s_1,s_2)) \leqslant J(v_0) < \theta_1(\Omega(l))+\lambda \text{ for all } (s_1,s_2)\in K. \tag{7}$$

We now choose a Lipschitz continuous function $\chi:\mathbb{R}\to\mathbb{R}$ such that $0\leqslant\chi\leqslant 1, \chi(s) = 1$ for $s\geqslant 0$ and $\chi(s) = 0$ for $s\leqslant -1$. Then since $J\in C^2\left(H_0^1(\Omega(l)),\mathbb{R}\right)$, there is a semiflow $\varphi : [0,\infty)\times H_0^1(\Omega(l)) \to H_0^1(\Omega(l))$ satisfying

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t,u) = -\chi(J(\varphi(t,u)))\nabla J(\varphi(t,u)), \\ \varphi(0,u) = u. \end{cases}$$

We will frequently write φ^t in place of $\varphi(t,\cdot)$. Since

$$J\left(v_0^{\pm}\right) < \alpha^+(\mathbf{A})+\alpha^-(\mathbf{A}) \text{ and } J\left(d_0 v_0^{\pm}\right)\leqslant 0,$$

it follows that

$$\sup J(\eta(\partial K)) < \alpha^+(\mathbf{A})+\alpha^-(\mathbf{A}) = \theta(\mathbf{A}).$$

Hence

$$\left(\varphi^t\circ\eta\right)(\partial K)\cap\mathbf{N}(\Omega(l)) = \emptyset \text{ for all } t\geqslant 0$$

and, by Lemma 3.3, this implies

$$\left(\Phi\circ\varphi^t\circ\eta\right)(y)\neq(1,1) \text{ for all } y\in\partial K, t\geqslant 0.$$

Equality (6) and the global continuation principle of Leray-Schauder (see e.g. Zeidler [27, p.629]) imply that there exists a connected subset $Z\subset K\times[0,1]$ such that

$$\begin{array}{l} (1,1,0)\in Z; \\ \varphi^t(\eta(s_1,s_2))\in\mathbf{N}(\Omega(l)) \text{ for all } (s_1,s_2,t)\in Z; \\ Z\cap(K\times\{1\})\neq\emptyset. \end{array}$$

We put

$$\widetilde{Z} = \left\{\varphi^t(\eta(s_1,s_2))\in\mathbf{N}(\Omega(l)) : (s_1,s_2,t)\in Z\right\}.$$

By inequality (7) and Lemma 3.1 (ii), we obtain

$$\widetilde{Z}\subset\mathbf{N}_1(L,\delta(L),\Omega(l))\cup\mathbf{N}_2(L,\delta(L),\Omega(l)).$$

Since Z is connected, we obtain that $\widetilde{Z} \subset \mathbf{N}_1(L, \delta(L), \Omega(l))$. We now pick $(\bar{s}_1, \bar{s}_2, 1) \in Z \cap (K \times \{1\})$ and write

$$v_1 := \eta(\bar{s}_1, \bar{s}_2), v_2 := \varphi^1(v_1).$$

Then $v_2 \in \widetilde{Z} \subset \mathbf{N}_1(L, \delta(L), \Omega(l))$. We distinguish two case.

Case 1. $\|\varphi^t(v_1) - v_2\| \leqslant \mu$ for all $t \in [0,1]$. We choose $t_0 \in [0,1]$ with

$$\left\|\nabla J\left(\varphi^{t_0}(v_1)\right)\right\| = \min_{0 \leqslant t \leqslant 1} \left\|\nabla J\left(\varphi^t(v_1)\right)\right\|$$

and put $u_0 = \varphi^{t_0}(v_1)$. Then

$$\begin{aligned} \lambda &\geqslant J(v_1) - J(v_2) = -\int_0^1 \frac{\partial}{\partial t} J\left(\varphi^t(v_1)\right) ds \\ &= \int_0^1 \left\|\nabla J\left(\varphi^t(v_1)\right)\right\|^2 dt \geqslant \|\nabla J(u_0)\|^2. \end{aligned}$$

Hence u_0 has the desired properties.

Case 2. There exists $\bar{t} \in [0,1]$ such that $\left\|\varphi^{\bar{t}}(v_1) - v_2\right\| > \mu$. Then let

$$t_1 = \sup\left\{t \geqslant \bar{t} : \left\|\varphi^t(v_1) - v_2\right\| > \mu\right\}.$$

We choose $t_0 \in [t_1, 1]$ with

$$\left\|\nabla J\left(\varphi^{t_0}(v_1)\right)\right\| = \min_{t_1 \leqslant t \leqslant 1} \left\|\nabla J\left(\varphi^t(v_1)\right)\right\|$$

and put $u_0 = \varphi^{t_0}(v_1)$. Then

$$\mu \leqslant \int_{t_1}^1 \left\|\frac{\partial}{\partial t}\varphi^t(v_1)\right\| dt \leqslant \int_{t_1}^1 \left\|\nabla J\left(\varphi^t(v_1)\right)\right\| dt$$

and

$$\begin{aligned} \lambda &\geqslant J\left(\varphi^{t_1}(v_1)\right) - J(v_2) = \int_{t_1}^1 \left\|\nabla J\left(\varphi^t(v_1)\right)\right\|^2 dt \\ &\geqslant \|\nabla J(u_0)\| \int_{t_1}^1 \left\|\nabla J\left(\varphi^t(v_1)\right)\right\| dt. \end{aligned}$$

We conclude that $\|\nabla J(u_0)\| \leqslant \frac{\lambda}{\mu}$. Thus, u_0 has the desired properties. □

COROLLARY 3.5. *For each $l > l_0$ there exists a sequence $\left\{u_n^{(i)}\right\} \subset H_0^1(\Omega(l))$ such that:*

(i) $\operatorname{dist}_H\left(u_n^{(i)}, \mathbf{N}_i(K, \delta(K), \Omega(l))\right) \to 0$;

(ii) $J\left(u_n^{(i)}\right) \to \theta_i(\Omega(l)) < \theta(\mathbf{A}) + \min\{\alpha^+(\mathbf{A}), \alpha^-(\mathbf{A}), \delta(K)\}$;

(iii) $J'(u_n^{(i)}) = o(1)$ *strongly in* $H^{-1}(\Omega(l))$.

We begin to show the proof of Theorem 1.1 for $l > l_0$. Then by Corollary 3.5, there exist sequences $\{u_n^{(i)}\} \subset H_0^1(\Omega(l))$ such that $(i)-(iii)$ in Corollary 3.5 are hold. Then by the Rellich compactness theorem and Lemma 3.2 there exist subsequences $\{u_n^{(i)}\}$ and a nodal solution $u_0^{(i)} \in \mathbf{N}_i(K, \delta(K), \Omega(l))$ such that $u_n^{(i)} \to u_0^{(i)}$ strongly in $H_0^1(\Omega(l))$ and $J(u_0^{(i)}) = \theta_i(\Omega(l))$. Thus, by Lemmas 3.1 $u_0^{(1)}$ and $u_0^{(2)}$ are different. Since

$$\theta_i(\Omega(l)) < \theta(\mathbf{A}) + \min\{\alpha^+(\mathbf{A}), \alpha^-(\mathbf{A}), \delta(K)\},$$

by Lemma 2.4 $u_0^{(1)}$ and $u_0^{(2)}$ are 2-nodal solutions of equation $(E_{p,q})$ in $\Omega(l)$.

4. Proof of Theorem 1.2

Throughout this paper, we let $\omega \subset \mathbf{A}$ be a nonempty open set such that the set $\mathbf{A}(-\tilde{l}, \tilde{l}) \backslash \overline{\omega}$ is a domain in $\mathbb{R}^N$ for some $\tilde{l} > 0$. Then $\mathbf{A} \backslash \overline{\omega}$ is a strictly large domain in $\mathbf{A}$. Furthermore, by Lemma 2.5 we have the following result.

LEMMA 4.1. *For each positive number L there exists a positive number $\delta(L)$ such that for $u \in \mathbf{N}(\mathbf{A} \backslash \overline{\omega})$ with $J(u) \leqslant \theta(\mathbf{A}) + \delta(L)$ we have:*
(i) either $h(u^+) - h(u^-) > L$ or $h(u^-) - h(u^+) > L$;
(ii) the function u^+ satisfies either $h(u^+) > L$ or $h(u^+) < -L$;
(iii) the function u^- satisfies either $h(u^-) > L$ or $h(u^-) < -L$.

For positive numbers L, δ, we denote:

$$\begin{aligned}
\widetilde{\mathbf{N}}_1(L,l) &= \{u \in \mathbf{N}(\delta, D(l)) : h(u^+) > L \text{ and } h(u^-) < -L\},\\
\widetilde{\mathbf{N}}_2(L,l) &= \{u \in \mathbf{N}(\delta, D(l)) : h(u^+) < -L \text{ and } h(u^-) > L\},\\
\widetilde{\mathbf{N}}_3(L,l) &= \{u \in \mathbf{N}(\delta, D(l)) : h(u^+) > L, h(u^-) > L \text{ and } g(u) > L\},\\
\widetilde{\mathbf{N}}_4(L,l) &= \{u \in \mathbf{N}(\delta, D(l)) : h(u^+)) > L, h(u^-) > L \text{ and } -g(u) > L\},\\
\widetilde{\mathbf{N}}_5(L,l) &= \{u \in \mathbf{N}(\delta, D(l)) : h(u^+) < -L, h(u^-) < -L \text{ and } g(u) > L\},\\
\widetilde{\mathbf{N}}_6(L,l) &= \{u \in \mathbf{N}(\delta, D(l)) : h(u^+) < -L, h(u^-) < -L \text{ and } -g(u) > L\},
\end{aligned}$$

where $g(u) = h(u^+) - h(u^-)$. Then we have the following result.

LEMMA 4.2. *For each positive number L, there exist positive numbers $\delta(L)$ and $\bar{l}_0$ such that for $l > \bar{l}_0$:*
(i) $\widetilde{\mathbf{N}}_i(L,l) \neq \emptyset$ for all $i = 1, 2, \ldots, 6$;
(ii) $\widetilde{\mathbf{N}}_i(L,l) \cap \widetilde{\mathbf{N}}_j(L,l) = \emptyset$ for all $i \neq j$;
(iii) $\mathbf{N}(\delta, D(l)) = \cup_{i=1}^6 \widetilde{\mathbf{N}}_i(L,l)$.

Proof. Since $\mathbf{N}(D(l)) \subset \mathbf{N}(\mathbf{A} \backslash \overline{\omega})$ for all $l > \tilde{l}$. By Lemma 4.1, we only need to prove that there exists $\bar{l}_0 > \tilde{l}$ such that $\widetilde{\mathbf{N}}_i(L,l) \neq \emptyset$ for all $l > \bar{l}_0$ and $i = 1, 2, \ldots, 6$.

Moreover, the proofs of all cases are similar argument. Thus, we only need to prove the case "$\widetilde{\mathbf{N}}_1(L,l) \neq \emptyset$". By Lien-Tzeng-Wang [19, Lemma 2.2], we have

$$\alpha^{\pm}\left(\mathbf{A}\left(-\frac{l}{2}+\widetilde{l},\frac{l}{2}-\widetilde{l}\right)\right) \searrow \alpha^{\pm}(\mathbf{A}) \text{ as } l \nearrow \infty.$$

Thus, there exists $\bar{l}_0 > \max\{2L,\widetilde{l}\}$ such that

$$\alpha^{\pm}\left(\mathbf{A}\left(-\frac{l}{2}+\widetilde{l},\frac{l}{2}-\widetilde{l}\right)\right) < \alpha^{\pm}(\mathbf{A}) + \frac{\delta(L)}{2} \text{ for all } l > \bar{l}_0.$$

Moreover, by Ambrosetti-Rabinowitz [1] and Chen-Chen-Wang [9], equation $(E_{p,q})$ in $\mathbf{A}(-l/2+\widetilde{l},l/2-\widetilde{l})$ has a positive solution $v^+ \in \mathbf{M}^+(\mathbf{A}(-l/2+\widetilde{l},l/2-\widetilde{l}))$ and a negative solution $v^- \in \mathbf{M}^-(\mathbf{A}(-l/2+\widetilde{l},l/2-\widetilde{l}))$ such that

$$J(v^{\pm}) = \alpha^{\pm}\left(\mathbf{A}\left(-\frac{l}{2}+\widetilde{l},\frac{l}{2}-\widetilde{l}\right)\right)$$

and $v^{\pm}(x,-y) = v^{\pm}(x,y)$. Clearly, $h(v^{\pm}) = 0$. Setting $u^+(x,y) = v^+(x,y-l/2)$ and $u^-(x,y) = v^-(x,y+l/2)$. From the translation invariance of the functional in y-axis we get that $u^{\pm} \in \mathbf{M}^{\pm}(D(l))$ and $h(u^{\pm}) = \pm l/2$. Setting $u = u^+ + u^-$, we obtain $u \in \mathbf{N}(D(l)), h_+(u^+) > L$ and $h_-(u^-) < -L$. Moreover

$$J(u) < \alpha^+(\mathbf{A}) + \alpha^-(\mathbf{A}) + \delta(L) = \theta(\mathbf{A}) + \delta(L) \text{ for all } l > \bar{l}_0.$$

This implies $\widetilde{\mathbf{N}}_1(L,l) \neq \emptyset$ for all $l > \bar{l}_0$. □

Similar to the argument in Lemma 3.2 we have the following result.

LEMMA 4.3. *Let $\bar{l}_0 > 0$ as in Lemma 3.1. Then for each $l > \bar{l}_0$, we have that $\widetilde{\mathbf{N}}_i(L,l)$ is closed set for all $i = 1,2\ldots,6$.*

Now we consider the minimization problem in $\widetilde{\mathbf{N}}_i(L,l)$ for J

$$\widetilde{\theta}_i(D(l)) = \inf_{u\in\widetilde{\mathbf{N}}_i(L,l)} J(u) \text{ for all } i = 1,2\ldots,6.$$

Similar to the method used in the proof of Proposition 3.4, we can get the following result.

PROPOSITION 4.4. *For each $l > \bar{l}_0$ there exist sequences $\{u_n^{(i)}\} \subset H_0^1(D(l))$ such that:*
(i) $\text{dist}_H\left(u_n^{(i)},\widetilde{\mathbf{N}}_i(L,l)\right) \to 0$;
(ii) $J\left(u_n^{(i)}\right) \to \widetilde{\theta}_i(D(l)) < \theta(\mathbf{A}) + \min\{\alpha^+(\mathbf{A}),\alpha^-(\mathbf{A}),\delta(L)\}$;
(iii) $J'(u_n^{(i)}) = o(1)$ *strongly in* $H^{-1}(D(l))$.

We begin to show the proof of Theorem 1.2 for $l > \bar{l}_0$. By Proposition 4.4, there exist sequence $\{u_n^{(i)}\} \subset H_0^1(D(l))$ such that $(i)-(iii)$ in Proposition 4.4 are hold. Then by the Rellich compactness theorem and Lemma 4.3 there exist subsequences $\{u_n^{(i)}\}$ and nodal solutions $u_0^{(i)} \in \widetilde{\mathbf{N}}_i(L,l)$ such that $u_n^{(i)} \to u_0^{(i)}$ strongly in $H_0^1(D(l))$ and $J\big(u_0^{(i)}\big) = \widetilde{\theta}_i(D(l))$ for all $i = 1,2\ldots,6$. Thus, by Lemmas 4.2 $u_0^{(1)}, u_0^{(2)}, \ldots, u_0^{(6)}$ are different. Since

$$\widetilde{\theta}_i(D(l)) < \theta(\mathbf{A}) + \min\left\{\alpha^+(\mathbf{A}), \alpha^-(\mathbf{A}), \delta(K)\right\},$$

by Lemma 2.4 $u_0^{(1)}, u_0^{(2)}, \ldots, u_0^{(6)}$ are 2-nodal solutions of equation $(E_{p,q})$ in $D(l)$.

5. Proof of Theorem 1.3

For each $i \in \{1,2,\ldots,m\}$ and $l > 2\widetilde{l}$ we denote the set $\mathbf{B}(i,l)$ as follows:

$$\mathbf{B}(i,l) = \{y \in \mathbb{R} : y = (i-1)l \text{ or } y = il\}.$$

Furthermore, we denote:

$$\begin{aligned}
\mathbf{M}_i^+(l) &= \left\{u \in \mathbf{M}^+(D_m(l)) : (i-1)l < h(u) < il\right\}, \\
\partial\mathbf{M}_i^+(l) &= \left\{u \in \mathbf{M}^+(D_m(l)) : h(u) \in \mathbf{B}(i,l)\right\}, \\
\mathbf{M}_i^-(l) &= \left\{u \in \mathbf{M}^-(D_m(l)) : (i-1)l < h(u) < il\right\}, \\
\partial\mathbf{M}_i^-(l) &= \left\{u \in \mathbf{M}^-(D_m(l)) : h(u) \in \mathbf{B}(i,l)\right\}, \\
\mathbf{N}_{i,j}(l) &= \left\{u \in H_0^1(D_m(l)) : u^+ \in \mathbf{M}_i^+(\varepsilon,t) \text{ and } u^- \in \mathbf{M}_j^-(\varepsilon,t)\right\}, \\
\partial\mathbf{N}_{i,j}(l) &= \Big\{u \in H_0^1(D_m(l)) : u^+ \in \overline{\mathbf{M}_i^+(l)}, u^- \in \overline{\mathbf{M}_j^-(l)} \text{ and} \\
&\qquad \text{either } u^+ \in \partial\mathbf{M}_i^+(l) \text{ or } u^- \in \partial\mathbf{M}_j^-(l)\Big\},
\end{aligned}$$

where $\overline{\mathbf{M}_i^{\pm}(l)} = \mathbf{M}_i^{\pm}(l) \cup \partial\mathbf{M}_i^{\pm}(l)$. Then we have the following result.

LEMMA 5.1. *For each $l > 2\widetilde{l}$, we have that $\mathbf{N}_{i,j}(l)$ are mutually disjoint.*

Proof. The proofs of all cases are similar argument. Thus, we only need to prove the case "1,1" and "1,2". Suppose otherwise, then there exists $v_0 \in H_0^1(D_m(l))$ such that $v_0 \in N_{1,1}(l) \cap N_{1,2}(l)$. Then

$$0 < h(v_0^-) < l \text{ and } l < h(v_0^-) < 2l,$$

this contradicts $\mathbf{A}(0,l) \cap \mathbf{A}(l,2l) = \emptyset$ for all $l > 2\widetilde{l}$. □

Define the minimization problems in $\mathbf{N}_{i,j}(l)$ and $\partial\mathbf{N}_{i,j}(l)$ for J,

$$\gamma_{i,j}(l) = \inf_{v \in \mathbf{N}_{i,j}(l)} J(v) \text{ and } \widetilde{\gamma}_{i,j}(l) = \inf_{v \in \partial\mathbf{N}_{i,j}(l)} J(v).$$

Similar to the method used the proof of Lemma 3.1, we can get the following result.

LEMMA 5.2. *For each positive number* $\sigma \leqslant \min\{\alpha^+(\mathbf{A}), \alpha^-(\mathbf{A}), \delta(L)\}$ *there exists* $l_1 > 2\widetilde{l}$ *such that* $\mathbf{N}_{i,j}(l) \neq \emptyset$ *and* $\gamma_{i,j}(l) < \theta(\mathbf{A}) + \sigma$ *for all* $i,j = 1,2,\ldots,m$ *and* $l > l_1$.

Furthermore, we have the following result.

LEMMA 5.3. *There exist positive numbers* δ *and* $l_2 > 2\widetilde{l}$ *such that for each* $i,j \in \{1,2,\ldots,m\}$ *we have*

$$\widetilde{\gamma}_{i,j}(l) > \theta(\mathbf{A}) + \delta \text{ for all } l \geqslant l_2.$$

Proof. Fix $i,j \in \{1,2,\ldots,m\}$. Suppose otherwise, then there exist $l_n \to \infty$ as $n \to \infty$ and $\{u_n\} \subset \partial\mathbf{N}_{i,j}(l)$ such that:

$$J(u_n) \to \theta(\mathbf{A}), \tag{8}$$

$$\int_{D_m(l_n)} \left[\left|\nabla u_n^+\right|^2 + \left(u_n^+\right)^2\right] dx = \int_{D_m(l_n)} \left|u_n^+\right|^p dx, \tag{9}$$

$$\int_{D_m(l_n)} \left[\left|\nabla u_n^-\right|^2 + \left(u_n^-\right)^2\right] dx = \int_{D_m(l_n)} \left|u_n^-\right|^q dx \tag{10}$$

and either $u_n^+ \in \partial\mathbf{M}_i^+(l_n)$ or $u_n^- \in \partial\mathbf{M}_j^-(l_n)$. Since $u_n^\pm \in \mathbf{M}^\pm(D_m(l_n)) \subset \mathbf{M}^\pm(\mathbf{A}^+) \subset \mathbf{M}^\pm(\mathbf{A}), \theta(\mathbf{A}) = \alpha^+(\mathbf{A}) + \alpha^-(\mathbf{A})$ and $J(u_n) = J(u_n^+) + J(u_n^-)$. By (8) and Wang-Wu [22, Lemma 7] $\{u_n^\pm\}$ are $(\mathrm{PS})_{\alpha^\pm(\mathbf{A})}$-sequences in $H_0^1(\mathbf{A})$ for J. Moreover, by (10) and the Sobolev imbedding theorem, there exists $c > 0$ such that

$$\int_{\mathbf{A}^+} \left[\left|\nabla u_n^\pm\right|^2 + \left(u_n^\pm\right)^2\right] dx > c \text{ for all } n.$$

From the concentration compactness principle of Lions [20] (or see [19, Theorem 4.1]), there exist $R > 0, d > 0$ and $\{(0,y_n^\pm)\} \subset \mathbb{R}^{N-1} \times \mathbb{R}^+$ such that

$$\int_{\mathbf{A}(-R,R)+\left(0,y_n^+\right)} \left|u_n^+\right|^p dx \geqslant d \text{ and } \int_{\mathbf{A}(-R,R)+\left(0,y_n^-\right)} \left|u_n^-\right|^q dx \geqslant d \text{ for all } n.$$

Without loss of generality, we may assume that $u_n^+ \in \partial\mathbf{M}_i^+(l_n)$ that is, $h_+(u_n^+) \in \mathbf{B}(i,l_n)$. Set $\bar{u}_n(x',x_N) = u_n^+(x',x_N + y_n^+)$. From the translation invariance of the functional in x_N-axis, we get that $\{\bar{u}_n\}$ satisfies

$$\bar{u}_n \in \mathbf{M}^+\left(\mathbf{A}^+ - \left(0,y_n^+\right)\right) \subset \mathbf{M}^+(\mathbf{A}) \tag{11}$$

and is $(\mathrm{PS})_{\alpha^+(\mathbf{A})}$-sequences in $H_0^1(\mathbf{A})$ for J. Then there exist a subsequence $\{\bar{u}_n\}$ and a nonnegative function $u_0 \in H_0^1(\mathbf{A})$ such that

$$\bar{u}_n \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\mathbf{A}) \text{ as } n \to \infty,$$
$$\bar{u}_n \to u_0 \quad \text{a.e. in } \mathbf{A} \text{ as } n \to \infty,$$

and

$$\int_{\mathbf{A}(-R,R)} |\bar{u}_n|^p dx \to \int_{\mathbf{A}(-R,R)} |u_0|^p dx \geqslant d \quad \text{as } n \to \infty. \tag{12}$$

This implies u_0 is a nonzero nonnegative solution of equation $(E_{p,q})$ in $\mathbf{A}$. By the Fatou lemma

$$\begin{aligned}\alpha^+(\mathbf{A}) \leqslant J(u_0) &= \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbf{A}} |u_0|^p\,dx \\ &\leqslant \liminf \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbf{A}} |\overline{u}_n|^p\,dx = \alpha^+(\mathbf{A}),\end{aligned}$$

and so $J(u_0)=\alpha^+(\mathbf{A})$. Moreover, by the maximum principle and Chen-Chen-Wang [9], u_0 is a positive solution of equation $(E_{p,q})$ in $\mathbf{A}$ and is axially symmetric in y-axis. Set $w_n=\overline{u}_n-u_0$, Since $\{\overline{u}_n\}$ is uniformly bounded, by Brézis-Lieb lemma [7] we obtain

$$\int_{\mathbf{A}} |w_n|^p\,dx = \int_{\mathbf{A}} |\overline{u}_n|^p\,dx - \int_{\mathbf{A}} |u_0|^p\,dx + o(1). \tag{13}$$

Moreover, $\overline{u}_n \rightharpoonup u_0$ weakly in $H_0^1(\mathbf{A})$ we have

$$\|w_n\|^2 = \|\overline{u}_n\|^2 - \|u_0\|^2 + o(1). \tag{14}$$

Then

$$\|w_n\|^2 = \int_{\mathbf{A}} |w_n|^p\,dx + o(1)$$

and so

$$\left(\frac{1}{2}-\frac{1}{p}\right)\|w_n\|^2 = J(w_n) = J(\overline{u}_n) - J(u_0) + o(1) = o(1).$$

This implies $\bar{u}_n \to u_0$ strongly in $H_0^1(\mathbf{A})$ as $n\to\infty$. We will show that $y_n^+ \to \infty$ as $n\to\infty$. Suppose otherwise, then $\{y_n^+\}$ is a bounded sequence in $\mathbb{R}$ or there exists a subsequence $\{y_n^+\}$ such that $y_n^+ \to -\infty$ as $n\to\infty$. If $\{y_n^+\}$ is a bounded sequence in $\mathbb{R}$. Without loss of generality, we may assume that $y_n^+ \to y_0$. Since $u_n^+ \in \mathbf{M}^+(D_m(l_n)) \subset \mathbf{M}^+(\mathbf{A}^+)$ and $\bar{u}_n(x,y) = u_n^+(x', x_N + y_n^+)$, by (11) and

$$\lim_{n\to\infty}\left[\mathbf{A}^+ - \left(0, y_n^+\right)\right] \to \mathbf{A}_{y_0}^+ = \left\{\left(x', x_N\right) \in \mathbf{A} : x_N > -y_0\right\}$$

we have $u_0 \in \mathbf{M}^+\left(\mathbf{A}_{y_0}^+\right)$ which contradicts the fact that u_0 is a positive solution of equation $(E_{p,q})$ in $\mathbf{A}$. If $y_n^+ \to -\infty$ as $n\to\infty$, then there exists $n_0\in\mathbb{N}$ such that

$$\left[\mathbf{A}^+ - \left(0, y_n^+\right)\right] \cap \mathbf{A}(-R,R) = \emptyset \text{ for all } n \geqslant n_0,$$

this implies $\bar{u}_n \equiv 0$ on $\mathbf{A}(-R,R)$ for all $n \geqslant n_0$ which contradicts (12). Thus $y_n^+ \to \infty$ as $n\to\infty$. Moreover, $\bar{u}_n \to u_0$ strongly in $H_0^1(\mathbf{A})$ as $n\to\infty$, we have

$$h(\overline{u}_n) = h(u_0) + o(1) \tag{15}$$

and so

$$h(u_n^+) = h\left(\overline{u}_n\left(x', x_N + y_n^+\right)\right) = h(u_0) - y_n^+ + o(1). \tag{16}$$

This implies $\operatorname{dist}(y_n^+, \mathbf{B}(i, l_n)) \to 0$ as $n\to\infty$. By passing to a subsequence, we may assume that one of the following cases occurs:

(a) $|il_n - y_n^+| \to 0$ as $n \to \infty$ for a subsequence;
(b) $|(i-1)l_n - y_n^+| \to 0$ as $n \to \infty$ for a subsequence.

In case (a) since $y_n^+ \to \infty$ as $n \to \infty$ and $l_n \to \infty$ as $n \to \infty$. Then $(i-1)l_n - y_n^+ \to -\infty$ as $n \to \infty$ and $(i+1)l_n - y_n^+ \to \infty$ as $n \to \infty$, this implies

$$\lim_{n\to\infty}\left[D_m(l_n) - \left(0, y_n^+\right)\right] \to \mathbf{A}\backslash\overline{\varpi}.$$

Moreover, $\bar{u}_n \to u_0$ strongly in $H_0^1(\mathbf{A})$ as $n \to \infty$ and

$$\bar{u}_n \in H_0^1\left(D_m(l_n) - \left(0, y_n^+\right)\right).$$

Thus, $u_0 \in \mathbf{M}^+(\mathbf{A}\backslash\overline{\varpi})$ which contradicts to the fact that u_0 is a positive solution of equation $(E_{p,q})$ in $\mathbf{A}$.

In case (b) : since $\operatorname{dist}(y_n^+, \mathbf{B}(i,l)) \to 0$ as $n \to \infty$ and $l_n \to \infty$ as $n \to \infty$, we may assume $il_n - y_n^+ \to \infty$ as $n \to \infty$. If $i = 1$, then $(i-1)l_n - y_n^+ \to -\infty$ as $n \to \infty$, this is a contradiction. For $i \geqslant 2$. Then $(i-2)l_n - y_n^+ \to -\infty$ as $n \to \infty$, this implies

$$\lim_{n\to\infty}\left[D_m(l_n) - \left(0, y_n^+\right)\right] \to \mathbf{A}\backslash\overline{\varpi}.$$

Moreover, $\bar{u}_n \to u_0$ strongly in $H_0^1(\mathbf{A})$ as $n \to \infty$ and

$$\bar{u}_n \in H_0^1\left(D_m(l_n) - \left(0, y_n^+\right)\right).$$

Thus, $u_0 \in \mathbf{M}^+(\mathbf{A}\backslash\overline{\varpi})$ which contradicts to the fact that u_0 is a positive solution of equation $(E_{p,q})$ in $\mathbf{A}$. Therefore, we have completed our proof. □

By Lemmas 5.2, 5.3, there exists $\widetilde{l}_0 > 2\widetilde{l}$ such that for $l > \widetilde{l}_0$

$$\gamma_{i,j}(l) < \min\left\{\theta(\mathbf{A}) + \min\left\{\alpha^+(\mathbf{A}), \alpha^-(\mathbf{A}), \delta(L)\right\}, \widetilde{\gamma}_{i,j}(l)\right\} \tag{17}$$

for all $i, j \in \{1, 2, \ldots, m\}$. Similar to the method used in the proof of Lemma 3.1, we can get the following result.

LEMMA 5.4. *Let L and $\delta(L)$ be positive numbers as in Lemma 2.5. Then for each $i \in \{1, 2, \ldots, m\}$ and $l > \widetilde{l}_0$, we have:*

(i) $\mathbf{N}_{i,i}^{(k)}(l) \neq \emptyset$ *for all* $k = 1, 2$;
(ii) $\mathbf{N}_{i,i}(l) = \mathbf{N}_{i,i}^{(1)}(l) \cup \mathbf{N}_{i,i}^{(2)}(l)$;
(iii) $\mathbf{N}_{i,i}^{(1)}(l) \cap \mathbf{N}_{i,i}^{(2)}(l) = \emptyset$, *where* $\mathbf{N}_{i,i}^{(k)}(l) = \left\{u \in \mathbf{N}_{i,i}(l) : (-1)^k g(u) > L\right\}$.

Similar to the method used in the proof of Proposition 3.4 (or see Wu [26, Proposition 3.6]), we can get the following result.

PROPOSITION 5.5. *For each $l > \widetilde{l}_0$ there exist sequences $\left\{u_n^{(i,j)}\right\} \subset H_0^1(D_m(l))$ such that:*

(i) $\operatorname{dist}_H(u_n^{(i,j)}, \mathbf{N}_{i,j}(l)) \to 0$;

(ii) $J(u_n^{(i,j)}) \to \gamma_{i,j}(l)$;
(iii) $J'(u_n^{(i,j)}) = o(1)$ *strongly in* $H^{-1}(D_m(l))$;
(iv) $\mathrm{dist}(h((u_n^{(i,j)})^+), ((i-1)l, il)) \to 0$ *and* $\mathrm{dist}(h((u_n^{(i,j)})^-), ((j-1)l, jl)) \to 0$.

Then we have the following result.

THEOREM 5.6. *For each* $L > 0$, *there exists* $\widetilde{l_0} > 0$ *such that for* $l > \widetilde{l_0}$, *equation* $(E_{p,q})$ *in* $D_m(l)$ *has* m^2 *2-nodal solutions* $\{u_0^{(i,j)}$
big$\}_{i,j\in\{1,2,\dots,m\}}$ *with* $u_0^{(i,j)} \in \mathbf{N}_{i,j}(l)$ *for all* $i,j \in \{1,2,\dots,m\}$.

Proof. It follows from Proposition 5.5 that there exists $\widetilde{l_0} > 2\widetilde{l}$ such that for each $l > \widetilde{l_0}$ and $i,j \in \{1,2,\dots,m\}$ we can find a sequence $\{u_n^{(i,j)}\} \subset H_0^1(D_m(l))$ such that $(i)-(iv)$ in Proposition 5.5 hold. Since $\{u_n^{(i,j)}\}$ is bounded in $H_0^1(D_m(l))$, we have $\{(u_n^{(i,j)})^+\}$ and $\{(u_n^{(i,j)})^-\}$ are also bounded in $H_0^1(D_m(l))$ and

$$\left\|\left(u_n^{(i,j)}\right)^+\right\|^2 = \int_{D_m(l)} \left|\left(u_n^{(i,j)}\right)^+\right|^p dx + o(1),$$

and

$$\left\|\left(u_n^{(i,j)}\right)^-\right\|^2 = \int_{D_m(l)} \left|\left(u_n^{(i,j)}\right)^-\right|^q dx + o(1).$$

Thus, there exist a subsequence $\{u_n^{(i,j)}\}$ and $u_0^{(i,j)}$ in $H_0^1(D_m(l))$ such that

$$u_n^{(i,j)} \rightharpoonup u_0^{i,j}; \left(u_n^{(i,j)}\right)^{\pm} \rightharpoonup \left(u_0^{(i,j)}\right)^{\pm} \text{ weakly in } H_0^1(D_m(l))$$

and

$$u_n^{(i,j)} \to u_0^{(i,j)}; \left(u_n^{(i,j)}\right)^{\pm} \to \left(u_0^{(i,j)}\right)^{\pm} \text{ a.e. in } D_m(l).$$

Moreover, $u_0^{(i,j)}$ is a solution of equation $(E_{p,q})$ in $D_m(l)$. We will show that $\left(u_0^{(i,j)}\right)^{\pm} \not\equiv 0$. If not, then we may assume that $\left(u_0^{(i,j)}\right)^+ \equiv 0$. Since $\mathrm{dist}\left(u_n^{(i,j)}, \mathbf{N}_{i,j}(l)\right) \to 0$ as $n \to \infty$ and $\gamma_{i,j}(l) > 0$, we deduce from the Sobolev imbedding theorem that

$$\left\|\left(u_n^{(i,j)}\right)^+\right\| > \nu > 0 \text{ for some constant } \nu \text{ and for all } n.$$

Applying the concentration-compactness principle of P. L. Lions [20], there are positive constants R, c_0 and a unbounded sequence $\{y_n^+\} \subset \mathbb{R}$ such that

$$\int_{\mathbf{A}(-R,R)} \left|\left(u_n^{(i,j)}\right)^+\left(x', x_N + y_n^+\right)\right|^p dx \geqslant c_0 \text{ for } n \text{ sufficiently large.} \tag{18}$$

Set $\widetilde{u}_n^{(i,j)}(x', x_N) = \left(u_n^{(i,j)}\right)^+(x, y + y_n^+)$. Since $\{\widetilde{u}_n^{(i,j)}\}$ is bounded in $H_0^1(\mathbf{A})$, we may assume that there exists $\widetilde{u}_0^{(i,j)} \in H_0^1(\mathbf{A})$ such that $\widetilde{u}_n^{(i,j)} \rightharpoonup \widetilde{u}_0^{(i,j)}$ weakly in $H_0^1(\mathbf{A})$. From (18) we have $\widetilde{u}_0^{(i,j)} \geqslant 0$ and $\widetilde{u}_0^{(i,j)} \not\equiv 0$ in $\mathbf{A}$. Set $v_n = \widetilde{u}_n^{(i,j)} - \widetilde{u}_0^{(i,j)}$. We distinguish between two cases:

Case I: $\|v_n\| \to 0$ as $n \to \infty$;

Case II: $\|v_n\| \geqslant c$ for large n and for some constant $c > 0$.

Assume Case I, we employ the argument in Lemma 5.3 to obtain that there exists $c_0 > 0$ such that

$$\operatorname{dist}\Big(h\big((u_n^{(i,j)})^+\big), ((i-1)l, il)\Big) > c_0 \text{ for large } n,$$

which is a contradiction.

In Case II, we notice first that $J'(u_n^{(i,j)}) \to 0$ strongly in $H^{-1}(\mathbf{A})$ and $\operatorname{dist}\big(u_n^{(i,j)}, \mathbf{N}_{i,j}(l)\big) \to 0$ as $n \to \infty$ implies

$$\int_{\mathbf{A}} \Big[\big|\nabla \widetilde{u}_0^{(i,j)}\big|^2 + \big(\widetilde{u}_0^{(i,j)}\big)^2\Big] dx - \int_{\mathbf{A}} \big|\widetilde{u}_0^{(i,j)}\big|^p dx = o(1) \tag{19}$$

and

$$\int_{\mathbf{A}} \Big[\big|\nabla \widetilde{u}_n^{(i,j)}\big|^2 + \big(\widetilde{u}_n^{(i,j)}\big)^2\Big] dx - \int_{\mathbf{A}} \big|\widetilde{u}_n^{(i,j)}\big|^p dx = o(1). \tag{20}$$

By $(19),(20)$ and Brezis-Lieb lemma [7] we obtain

$$\int_{\mathbf{A}} \Big[|\nabla v_n|^2 + v_n^2\Big] dx + \int_{\mathbf{A}} |v_n|^p dx = o(1).$$

Since $\|v_n\| \geqslant c$ for large n, is is easy to find a sequence $\{s_n\} \subset \mathbb{R}^+$ with $s_n \to 1$ as $n \to \infty$ such that $s_n v_n \in \mathbf{M}^+(\mathbf{A})$, and so

$$\frac{1}{2}\int_{\mathbf{A}} \Big[|\nabla v_n|^2 + v_n^2\Big] dx - \frac{1}{p}\int_{\mathbf{A}} |v_n|^p dx \geqslant \alpha^+(\mathbf{A}) + o(1).$$

Similarly

$$\frac{1}{2}\int_{\mathbf{A}} \Big[\big|\nabla \widetilde{u}_0^{(i,j)}\big|^2 + \big(\widetilde{u}_0^{(i,j)}\big)^2\Big] dx - \frac{1}{p}\int_{\mathbf{A}} \big|\widetilde{u}_0^{(i,j)}\big|^p dx \geqslant \alpha^+(\mathbf{A}) + o(1)$$

and

$$\frac{1}{2}\int_{\mathbf{A}} \Big[\big|\nabla \big(u_n^{(i,j)}\big)^-\big|^2 + \big(\big(u_n^{(i,j)}\big)^-\big)^2\Big] dx - \frac{1}{q}\int_{\mathbf{A}} \big|\big(u_n^{(i,j)}\big)^-\big|^q dx \geqslant \alpha^-(\mathbf{A}) + o(1).$$

Thus by Brezis-Lieb lemma [7] we have

$$\begin{aligned}
J\big(u_n^{(i,j)}\big) &= \frac{1}{2}\int_{\mathbf{A}} \Big[\big|\nabla \big(\widetilde{u}_n^{(i,j)}\big)^+\big|^2 + \big(\big(\widetilde{u}_n^{(i,j)}\big)^+\big)^2\Big] dx - \frac{1}{p}\int_{\mathbf{A}} \big|\big(\widetilde{u}_n^{(i,j)}\big)^-\big|^p dx \\
&\quad + \frac{1}{2}\int_{\mathbf{A}} \Big[\big|\nabla \big(u_n^{(i,j)}\big)^-\big|^2 + \big(\big(u_n^{(i,j)}\big)^-\big)^2\Big] dx - \frac{1}{q}\int_{\mathbf{A}} \big|\big(u_n^{(i,j)}\big)^-\big|^q dx \\
&= \frac{1}{2}\int_{\mathbf{A}} \Big[|\nabla v_n|^2 + v_n^2\Big] dx - \frac{1}{p}\int_{\mathbf{A}} |v_n|^p dx \\
&\quad + \frac{1}{2}\int_{\mathbf{A}} \Big[\big|\nabla \widetilde{u}_0^{(i,j)}\big|^2 + \big(\widetilde{u}_0^{(i,j)}\big)^2\Big] dx - \frac{1}{p}\int_{\mathbf{A}} \big|\widetilde{u}_0^{(i,j)}\big|^p dx \\
&\quad + \frac{1}{2}\int_{\mathbf{A}} \Big[\big|\nabla \big(u_n^{(i,j)}\big)^-\big|^2 + \big(\big(u_n^{(i,j)}\big)^-\big)^2\Big] dx - \frac{1}{q}\int_{\mathbf{A}} \big|\big(u_n^{(i,j)}\big)^-\big|^q dx + o(1) \\
&\geqslant 2\alpha^+(\mathbf{A}) + \alpha^-(\mathbf{A}) + o(1)
\end{aligned}$$

and so

$$\lim_{n\to\infty} J\left(u_n^{(i,j)}\right) = \gamma_{i,j}(l) \geqslant \theta(\mathbf{A}) + \alpha^+(\mathbf{A}), \tag{21}$$

this contradicts (17). Next we will show that $u_n^{(i,j)} \to u_0^{(i,j)}$ strongly in $H_0^1(D_m(l))$. Using Case II we can prove that we result, otherwise, we may use a similar argument as above to reach the contradiction (21). Finally, we will show that $u_0^{(i,j)} \in \mathbf{N}_{i,j}(l)$. Since

$$\operatorname{dist}\Big(h\big(\big(u_n^{(i,j)}\big)^+\big), ((i-1)l, il)\Big) \to 0 \quad \text{and} \quad \operatorname{dist}\Big(h\big(\big(u_n^{(i,j)}\big)^-\big), ((j-1)l, jl)\Big) \to 0,$$

we have $u_0^{(i,j)} \in \overline{\mathbf{N}_{i,j}(l)}$. Moreover, $J\big(u_0^{(i,j)}\big) = \gamma_{i,j}(l) < \widetilde{\gamma}_{i,j}(l)$ and so $u_0^{(i,j)} \notin \partial\mathbf{N}_{i,j}(l)$. Thus, $u_0^{(i,j)} \in \mathbf{N}_{i,j}(l)$ and so $\{u_0^{(i,j)}\}_{i,j\in\{1,2,\ldots,m\}}$ are different. This completes the proof. □

THEOREM 5.7. *For each* $L>0$, *there exists* $\widetilde{l}_0 > 0$ *such that for* $l > \widetilde{l}_0$ *and* $i \in \{1,2,\ldots,m\}$, *equation* $(E_{p,q})$ *in* $D_m(l)$ *has two 2-nodal solutions* $u_0^{(i,i,1)}$ *and* $u_0^{(i,i,2)}$ *such that* $u_0^{(i,i,k)} \in \mathbf{N}_{i,i}(l)$ *for all* $k=1,2$.

Proof. Similar to the method used in the proof of Proposition 3.4 (or see Wu [26, Proposition 3.6]), there exists a sequence $\{u_n^{(i,i,k)}\} \subset H_0^1(D_m(l))$ such that:

(i) $\operatorname{dist}_H\left(u_n^{(i,i,k)}, \mathbf{N}_{i,i}^{(k)}(l)\right) \to 0$;

(ii) $J\left(u_n^{(i,i,k)}\right) \to \gamma_{i,i,k}(l) = \inf_{u\in\mathbf{N}_{i,i}^{(k)}(l)} J(u)$;

(iii) $J'(u_n^{(i,i,k)}) = o(1)$ strongly in $H^{-1}(D_m(l))$;

(iv) $\operatorname{dist}\Big(h\big(\big(u_n^{(i,i,k)}\big)^{\pm}\big), ((i-1)l, il)\Big) \to 0$ and $(-1)^k g\big(u_n^{(i,i,k)}\big) > L$.

Then we may use a similar argument as in Theorem 5.6, there exist subsequences $\{u_n^{(i,i,k)}\}$ and $u_0^{(i,i,k)} \in \mathbf{N}_{i,i}^{(k)}(l)$ such that $u_n^{(i,i,k)} \to u_0^{(i,i,k)}$ strongly in $H_0^1(D_m(l))$. Furthermore, $u_0^{(i,i,1)}$ and $u_0^{(i,i,2)}$ are 2-nodal solutions of equation $(E_{p,q})$ in $D_m(l)$. By Lemma 5.4, $u_0^{(i,i,1)}$ and $u_0^{(i,i,2)}$ are different. □

The proof of Theorem 1.3 follows by a combination of the results of Theorems 5.6 and 5.7 and so, we have equation $(E_{p,q})$ in $D_m(l)$ has at least $m\times(m+1)$ 2-nodal solutions.

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