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VARIATIONAL PROBLEMS WITH POINTWISE CONSTRAINTS AND DEGENERATION IN VARIABLE DOMAINS

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Abstract. In this article we deal with a sequence of functionals defined on weighted Sobolev spaces. The spaces are associated with a sequence of domains Ω_s contained in a bounded domain Ω of $\mathbb{R}^n$. The main structural components of the functionals are integral functionals whose integrands satisfy a growth and coercivity condition with a weight and additional terms $\psi_s \in L^1(\Omega_s)$. For the given functionals we consider variational problems with sets of constraints for functions v of the kind $h(x,v(x)) \leqslant 0$ a.e. in Ω_s, where $h : \Omega \times \mathbb{R} \to \mathbb{R}$. We establish conditions on h and ψ_s and on the given domains, weighted spaces and functionals under which solutions of the variational problems under consideration converge in a certain sense to a solution of a limit variational problem with the set of constraints defined by the same function h.

1. Introduction

In this article we deal with a sequence of functionals $I_s : \widetilde{W}_0^{1,p}(\nu,\Omega_s) \to \mathbb{R}$, $s \in \mathbb{N}$, where Ω_s is a domain contained in a bounded domain Ω of $\mathbb{R}^n (n \geqslant 2)$, $p > 1$, $\nu : \Omega \to \mathbb{R}$, and $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ denotes a Sobolev space associated with the domain Ω_s, the exponent p and the weighted function ν. The precise description of the spaces $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ as well as a limit space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ is given in the beginning of Section 2. We suppose that for every $s \in \mathbb{N}$ the functional I_s has the following structure: $I_s = J_s + G_s$, where J_s is an integral functional whose value on every element $u \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ depends on the gradient of u, and G_s is a weakly continuous functional on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$.

For a given function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ we consider the sets

$$V = \{v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) : h(x,v(x)) \leqslant 0 \text{ for a. e. } x \in \Omega\}$$

and

$$V_s = \{v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) : h(x,v(x)) \leqslant 0 \text{ for a. e. } x \in \Omega_s\}, \quad s \in \mathbb{N}.$$

Our aim is to find out conditions on the function h and the given domains, spaces and functionals under which any sequence of minimizers u_s of the functionals I_s on the sets V_s converges in a certain sense to a minimizer of a functional $I : \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \to \mathbb{R}$ on the set V, and the minimum values of the functionals I_s on the sets V_s also tend to the minimum of the functional I on the set V.

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We assume that the integrands $f_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ of the functionals J_s satisfy the following condition: for every $s \in \mathbb{N}$, almost every $x \in \Omega_s$ and every $\xi \in \mathbb{R}^n$,

$$c_1 \nu(x)|\xi|^p - \psi_s(x) \leqslant f_s(x,\xi) \leqslant c_2 \nu(x)|\xi|^p + \psi_s(x),$$

where $c_1, c_2 > 0$ and $\psi_s \in L^1(\Omega_s)$, $\psi_s \geqslant 0$ in Ω_s. The function ν may characterize the degeneration or singularity of the integrands with respect to the spatial variable, and the functions ψ_s may describe an additional strong oscillation of the integrands. It is not supposed that the functions ψ_s have a pointwise majorant, and in general the presence of the sequence ψ_s makes the problem more difficult and requires some reasonable restrictions on the behaviour of this sequence (see (3.1) and condition ($*_4$) of Theorem 3.1).

We note that among the main conditions under which we establish a weak convergence of the minimizers and the convergence of the minimum values of the functionals I_s on the sets V_s are the Γ-convergence of the sequence $\{J_s\}$ to a functional $J : \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \to \mathbb{R}$ and the strong connectedness of the sequence of spaces $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ with the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ (see Theorems 3.1 and 3.12).

The role of the Γ-convergence of functionals in the study of the convergence of their minimizers and minimum values is well known (see for instance [9], [15], [22], [24], [37] and [38]). We only remark here that the notion of the Γ-convergence of functionals defined on the spaces $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ to a functional defined on the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ was introduced in [26] and the corresponding theorems on Γ-compactness for integral functionals were given in [26], [31] and [32].

In the study of the homogenization of variational problems in variable domains (particularly, in strongly perforated sets) along with the Γ-convergence of functionals a certain connection of the domains with a limit domain or more precisely, a connection of the corresponding spaces is important as well (see for instance [20], [22], [24], [28] and [38]). The notion of the strong connectedness of the sequence of the spaces $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ with the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ used in the present work was introduced and studied in [27].

As far as conditions on the function h are concerned we consider the following two cases:
(i) $h(x,\eta)$ has a special behaviour with respect to the variable η (in particular, $h(x,\eta)$ may be nonincreasing with respect to η for almost every $x \in \Omega$);
(ii) the value $h(x,\eta)$ does not depend on x.

In case (i) the main result on the convergence of minimizers of the functionals I_s on the sets V_s is given in Theorem 3.1. We note that the statement of the theorem contains the next "exhaustion" condition on the domains Ω_s:

$$\text{for every increasing sequence } \{m_j\} \subset \mathbb{N},\ \operatorname{meas}(\Omega \setminus \bigcup_j \Omega_{m_j}) = 0,$$

and generally speaking this condition cannot be omitted. We justify this fact in Example 4.13.

Observe that the same "exhaustion" condition has already been used in [21] for the investigation of both a convergence of sets of variable Sobolev spaces and the coercivity of the Γ-limit of functionals defined on these spaces.

We also show that in case (i) the sets V and V_s have the following representations:

$$V = \{v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) : v \geqslant \bar{h} \text{ a.e. in } \Omega\}$$

and

$$V_s = \{v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) : v \geqslant \bar{h} \text{ a.e. in } \Omega_s\}, \ \ s \in \mathbb{N},$$

where $\bar{h} : \Omega \to \bar{\mathbb{R}}$ is a function defined by the function h. These representations are not utilized in the proof of Theorem 3.1. However, if $\bar{h} = z$ a.e. in Ω, where $z \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, with the use of the given representations we demonstrate that the above-mentioned "exhaustion" condition on the domains Ω_s in Theorem 3.1 is unnecessary (see Remark 3.4).

Moreover, we give an application of Theorem 3.1 to the study of the convergence of minimizers of the functionals I_s on the sets defined by varying unilateral constraints (see Theorem 3.6).

In case (ii) we establish that the sets V and V_s have the following representations:

$$V = \{v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) : \alpha_- \leqslant v \leqslant \alpha_+ \text{ a.e. in } \Omega\}$$

and

$$V_s = \{v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) : \alpha_- \leqslant v \leqslant \alpha_+ \text{ a.e. in } \Omega_s\}, \ \ s \in \mathbb{N},$$

where $\alpha_- \in [-\infty, 0]$ and $\alpha_+ \in [0, +\infty]$ (see Lemma 3.11). The main result on the convergence of minimizers of the functionals I_s on the sets V_s in this case is given in Theorem 3.12.

The consideration of cases where the behaviour of the function h is different from that prescribed by cases (i) and (ii) is also possible with the use of techniques similar to those given in the article, although for this additional constructions are required too. For instance in the case where $h(x,\eta) = (\eta - \varphi(x))(\eta - \psi(x))$ with φ, $\psi \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and $\varphi \leqslant \psi$ a.e. in Ω (this corresponds to the variational problems with bilateral obstacles of the kind $\varphi \leqslant v \leqslant \psi$ a.e. in Ω) a delicate moment is the behaviour of the difference $\psi - \varphi$. Cases like this will be considered in the further publications of the authors.

On the whole the present article is organized as follows. In Section 2 we describe functional spaces and give definitions used in the work. In Section 3 we state the main results of the paper. Finally, Section 4 is devoted to comments and various examples concerning the realization of conditions under which the main results of the article are established.

Now let us mention some other works related to the topic. The convergence of solutions of variational problems with unilateral and bilateral obstacles in general variable domains for Γ-convergent integral functionals with the same nondegenerate integrand was established in [22]. At the same time it was assumed that the obstacles are regular, i.e. they belong to the Sobolev spaces on which the functionals are defined. Close results for solutions of variational inequalities with G-convergent nondegenerate nonlinear elliptic operators and strongly convergent regular unilateral and bilateral obstacles in perforated domains were obtained in [23].

The convergence of solutions of nondegenerate elliptic variational inequalities with obstacles was also studied in [2], [8], [29], [33] and [34].

With the use of techniques of the Γ-convergence theory the convergence of minimum points and minimum values in variational problems with general varying unilateral obstacles in a fixed domain for integral functionals with nondegenerate integrands satisfying a uniform growth and coercivity condition was studied in [7]. Analogous questions concerning variational problems with general varying bilateral obstacles for a quadratic integral functional were investigated in [6]. Results close to those of [6] and [7] were also obtained in [1].

The homogenization of variational problems with pointwise gradient constraints was studied for instance in [3]. A bibliography on this and close questions one can find in [4].

Γ-convergence of quadratic integral functionals having periodic quickly oscillating coefficients and defined on a weighted Sobolev space was established in [11]. The convergence of solutions of the Dirichlet problem for integral functionals or elliptic equations with degenerations in a fixed domain was studied in [10], [12], [13] and [16].

Finally, we remark that in connection with the study of the Dirichlet problems in variable domains the Γ-convergence of integral functionals defined on weighted spaces with a weight in a Muckenhoupt class was proved in [14], and the convergence of solutions of the Dirichlet problems for degenerate nonlinear elliptic second-order equations in domains with fine-grained boundary was studied for instance in [35].

2. Preliminaries

Let Ω be a bounded domain of $\mathbb{R}^n$ $(n \geqslant 2)$, $p > 1$, and let ν be a nonnegative function on Ω with the properties: $\nu > 0$ almost everywhere in Ω and

$$\nu \in L^1_{\text{loc}}(\Omega), \quad \left(\frac{1}{\nu}\right)^{1/(p-1)} \in L^1_{\text{loc}}(\Omega). \tag{2.1}$$

We denote by $L^p(\nu,\Omega)$ the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that the function $\nu|u|^p$ is summable in Ω. $L^p(\nu,\Omega)$ is a Banach space with the norm

$$\|u\|_{L^p(\nu,\Omega)} = \left(\int_\Omega \nu|u|^p\,dx\right)^{1/p}.$$

We note that by virtue of Young's inequality and the second inclusion of (2.1) we have $L^p(\nu,\Omega) \subset L^1_{\text{loc}}(\Omega)$.

We denote by $W^{1,p}(\nu,\Omega)$ the set of all functions $u \in L^p(\nu,\Omega)$ such that for every $i \in \{1,\dots,n\}$ there exists the weak derivative D_iu, $D_iu \in L^p(\nu,\Omega)$. $W^{1,p}(\nu,\Omega)$ is a reflexive Banach space with the norm

$$\|u\|_{1,p,\nu} = \left(\int_\Omega \nu|u|^p\,dx + \sum_{i=1}^n \int_\Omega \nu|D_iu|^p\,dx\right)^{1/p}.$$

Due to the first inclusion of the assumption (2.1) we have $C_0^\infty(\Omega) \subset W^{1,p}(\nu,\Omega)$. We denote by $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in $W^{1,p}(\nu,\Omega)$.

Next, let $\{\Omega_s\}$ be a sequence of domains of $\mathbb{R}^n$ which are contained in Ω.

By analogy with the spaces introduced above we define the functional spaces corresponding to the domains Ω_s.

Let $s \in \mathbb{N}$. We denote by $L^p(\nu,\Omega_s)$ the set of all measurable functions $u : \Omega_s \to \mathbb{R}$ such that the function $\nu|u|^p$ is summable in Ω_s. $L^p(\nu,\Omega_s)$ is a Banach space with the norm

$$\|u\|_{L^p(\nu,\Omega_s)} = \left(\int_{\Omega_s} \nu|u|^p\,dx\right)^{1/p}.$$

By virtue of the second inclusion of (2.1) we have $L^p(\nu,\Omega_s) \subset L^1_{\mathrm{loc}}(\Omega_s)$. We denote by $W^{1,p}(\nu,\Omega_s)$ the set of all functions $u \in L^p(\nu,\Omega_s)$ such that for every $i \in \{1,\dots,n\}$ there exists the weak derivative D_iu, $D_iu \in L^p(\nu,\Omega_s)$. $W^{1,p}(\nu,\Omega_s)$ is a Banach space with the norm

$$\|u\|_{1,p,\nu,s} = \left(\int_{\Omega_s} \nu|u|^p\,dx + \sum_{i=1}^n \int_{\Omega_s} \nu|D_iu|^p\,dx\right)^{1/p}.$$

We denote by $\widetilde{C}_0^\infty(\Omega_s)$ the set of all restrictions on Ω_s of functions from $C_0^\infty(\Omega)$. Due to the first inclusion of (2.1) we have $\widetilde{C}_0^\infty(\Omega_s) \subset W^{1,p}(\nu,\Omega_s)$. We denote by $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ the closure of the set $\widetilde{C}_0^\infty(\Omega_s)$ in $W^{1,p}(\nu,\Omega_s)$.

We observe that if $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and $s \in \mathbb{N}$, then $u|_{\Omega_s} \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$.

DEFINITION 2.1. If $s \in \mathbb{N}$, q_s is the mapping from $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ into $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that for every function $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, $\quad q_su = u|_{\Omega_s}$.

DEFINITION 2.2. We say that the sequence of the spaces $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ is strongly connected with the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ if there exists a sequence of linear continuous operators $l_s : \widetilde{W}_0^{1,p}(\nu,\Omega_s) \to \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that:
(i) the sequence of the norms $\|l_s\|$ is bounded;
(ii) for every $s \in \mathbb{N}$ and $u \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ we have $\quad q_s(l_su) = u\;$ a.e. in Ω_s.

PROPOSITION 2.3. *Suppose that the embedding of* $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ *into* $L^p(\nu,\Omega)$ *is compact, and the sequence of the spaces* $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ *is strongly connected with the space* $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. *Let for every* $s \in \mathbb{N}$, $u_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$, *and let the sequence of the norms* $\|u_s\|_{1,p,\nu,s}$ *be bounded. Then there exist an increasing sequence* $\{s_j\} \subset \mathbb{N}$ *and a function* $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ *such that* $\lim\limits_{j\to\infty} \|u_{s_j} - q_{s_j}u\|_{L^p(\nu,\Omega_{s_j})} = 0$.

The proof of this result is simple (see [27]).

DEFINITION 2.4. Let for every $s \in \mathbb{N}$, I_s be a functional on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$, and let I be a functional on $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. We say that the sequence $\{I_s\}$ Γ-converges to the functional I if the following conditions are satisfied:
(i) for every function $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ there exists a sequence $w_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that $\lim\limits_{s\to\infty} \|w_s - q_s u\|_{L^p(\nu,\Omega_s)} = 0$ and $\lim\limits_{s\to\infty} I_s(w_s) = I(u)$;
(ii) for every function $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and every sequence $u_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that $\lim\limits_{s\to\infty} \|u_s - q_s u\|_{L^p(\nu,\Omega_s)} = 0$ we have $\liminf\limits_{s\to\infty} I_s(u_s) \geqslant I(u)$.

THEOREM 2.5. *Suppose that the embedding of $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ into $L^p(\nu,\Omega)$ is compact, and the sequence of the spaces $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ is strongly connected with the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Let for every $s \in \mathbb{N}$, I_s be a functional on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$, let I be a functional on $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, and let the sequence $\{I_s\}$ Γ-converge to the functional I. Let for every $s \in \mathbb{N}$ the function u_s minimize the functional I_s on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$, and let the sequence of the norms $\|u_s\|_{1,p,\nu,s}$ be bounded. Then there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that the function u minimizes the functional I on $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, $\lim\limits_{j\to\infty} \|u_{s_j} - q_{s_j} u\|_{L^p(\nu,\Omega_{s_j})} = 0$ and $\lim\limits_{j\to\infty} I_{s_j}(u_{s_j}) = I(u)$.*

The proof of the theorem is simple: first Proposition 2.3 is applied and then Γ-convergence of the sequence $\{I_s\}$ is standardly used [32].

We note that in the nonweighted case results similar to Theorem 2.5 were established for different kinds of the domains Ω_s in [21], [22] and [24]. These results along with Theorem 2.5 are analogs of the variational property of Γ-convergence of functionals with the same domain of definition (see [9] and [15]).

Our aim in this article is to obtain assertions analogous to those of Theorem 2.5 for minimizers of some functionals $I_s : \widetilde{W}_0^{1,p}(\nu,\Omega_s) \to \mathbb{R}$ on sets with certain pointwise constraints. The corresponding results we give in the next section.

3. Main results

Let c_1, $c_2 > 0$, and let for every $s \in \mathbb{N}$, $\psi_s \in L^1(\Omega_s)$ and $\psi_s \geqslant 0$ in Ω_s. We shall assume that

$$\text{the sequence of the norms } \|\psi_s\|_{L^1(\Omega_s)} \text{ is bounded.} \tag{3.1}$$

Let $f_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$, $s \in \mathbb{N}$, be a sequence of functions such that:

$$\text{for every } s \in \mathbb{N} \text{ and } \xi \in \mathbb{R}^n \text{ the function } f_s(\cdot,\xi) \text{ is measurable in } \Omega_s; \tag{3.2}$$

$$\text{for every } s \in \mathbb{N} \text{ and almost every } x \in \Omega_s \text{ the function } f_s(x,\cdot) \text{ is convex in } \mathbb{R}^n; \tag{3.3}$$

$$\begin{cases} \text{for every } s \in \mathbb{N}, \text{ almost every } x \in \Omega_s \text{ and every } \xi \in \mathbb{R}^n, \\ c_1\nu(x)|\xi|^p - \psi_s(x) \leqslant f_s(x,\xi) \leqslant c_2\nu(x)|\xi|^p + \psi_s(x). \end{cases} \tag{3.4}$$

From (3.2)-(3.4) it follows that for every $s \in \mathbb{N}$, f_s is a Carathéodory function, and if $s \in \mathbb{N}$ and $u \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$, the function $f_s(x,\nabla u)$ is summable in Ω_s.

For every $s \in \mathbb{N}$ we define the functional $J_s : \widetilde{W}_0^{1,p}(\nu,\Omega_s) \to \mathbb{R}$ by

$$J_s(u) = \int_{\Omega_s} f_s(x,\nabla u)\,dx, \qquad u \in \widetilde{W}_0^{1,p}(\nu,\Omega_s).$$

We observe that due to (3.3) and (3.4) for every $s \in \mathbb{N}$ the functional J_s is weakly lower semicontinuous on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$.

Next, let c_3, $c_4 > 0$, and let for every $s \in \mathbb{N}$, G_s be weakly continuous functional on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$. We shall suppose that the following conditions are satisfied:

$$\begin{cases} \text{for every sequence } u_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) \text{ such that the sequence of the} \\ \text{norms } \|u_s\|_{1,p,\nu,s} \text{ is bounded, the sequence } \{G_s(u_s)\} \text{ is bounded,} \end{cases} \tag{3.5}$$

$$\text{for every } s \in \mathbb{N} \text{ and } u \in \widetilde{W}_0^{1,p}(\nu,\Omega_s),\ G_s(u) \geqslant c_3\|u\|^p_{L^p(\nu,\Omega_s)} - c_4. \tag{3.6}$$

Obviously, for every $s \in \mathbb{N}$ the functional $J_s + G_s$ is weakly lower semicontinuous on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$. Moreover, owing to (3.4) and (3.6) for every $s \in \mathbb{N}$ we have $(J_s + G_s)(u) \to +\infty$ if $\|u\|_{1,p,\nu,s} \to +\infty$. In view of known results on the existence of the minimizers of functionals (see for instance [36]), these properties of the functionals $J_s + G_s$ imply that the next assertion holds true:

$$\begin{cases} \text{if } s \in \mathbb{N} \text{ and } U \text{ is a nonempty sequentially weakly closed set in } \widetilde{W}_0^{1,p}(\nu,\Omega_s), \\ \text{there exists a function } u \in U \text{ minimizing the functional } J_s + G_s \text{ on } U. \end{cases} \tag{3.7}$$

In connection with Theorem 2.5 we note that if for every $s \in \mathbb{N}$ the function u_s minimizes the functional $J_s + G_s$ on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$, the sequence of the norms $\|u_s\|_{1,p,\nu,s}$ is bounded. This fact follows from (3.1) and (3.4)-(3.6).

Further, let $h : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that

$$\text{for almost every } x \in \Omega \text{ the function } h(x,\cdot) \text{ is continuous in } \mathbb{R}. \tag{3.8}$$

We set

$$V = \{v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) : h(x,v(x)) \leqslant 0 \text{ for a. e. } x \in \Omega\} \tag{3.9}$$

and suppose that $V \neq \emptyset$.

For every $s \in \mathbb{N}$ we define

$$V_s = \{v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) : h(x,v(x)) \leqslant 0 \text{ for a. e. } x \in \Omega_s\}. \tag{3.10}$$

If $v \in V$ and $s \in \mathbb{N}$, we have $q_s v \in V_s$. Therefore, for every $s \in \mathbb{N}$ the set V_s is nonempty.

We observe that by virtue of the second inclusion of (2.1) and (3.8) the set V is sequentially weakly closed in $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and for every $s \in \mathbb{N}$ the set V_s is sequentially weakly closed in $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$. The latter fact and (3.7) imply that for every $s \in \mathbb{N}$ there exists a function $u_s \in V_s$ minimizing the functional $J_s + G_s$ on V_s.

THEOREM 3.1. *Suppose that the following conditions are satisfied:*

$(*_1)$ *the embedding of* $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ *into* $L^p(\nu,\Omega)$ *is compact;*

$(*_2)$ *the sequence of the spaces* $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ *is strongly connected with* $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$*;*

$(*_3)$ *for every increasing sequence* $\{m_j\}\subset\mathbb{N}$, $\operatorname{meas}(\Omega\setminus\bigcup_j\Omega_{m_j})=0$*;*

$(*_4)$ *if* $\varepsilon>0$, *there exists* $\delta>0$ *such that for every measurable set* $E\subset\Omega$, $\operatorname{meas}E\leqslant\delta$, *we have* $\limsup\limits_{s\to\infty}\int_{E\cap\Omega_s}\psi_s\,dx\leqslant\varepsilon$*;*

$(*_5)$ *the sequence* $\{J_s\}$ Γ*-converges to a functional* $J:\overset{\circ}{W}{}^{1,p}(\nu,\Omega)\to\mathbb{R}$*;*

$(*_6)$ *there exists a functional* $G:\overset{\circ}{W}{}^{1,p}(\nu,\Omega)\to\mathbb{R}$ *such that for every* $v\in\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ *and every sequence* $v_s\in\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ *with the property* $\|v_s-q_sv\|_{L^p(\nu,\Omega_s)}\to 0$ *we have* $G_s(v_s)\to G(v)$*;*

$(*_7)$ *for almost every* $x\in\Omega$ *from* $\eta\in\mathbb{R}$ *and* $h(x,\eta)\leqslant 0$ *it follows that for every* $\eta'\geqslant\eta$, $h(x,\eta')\leqslant 0$.

Moreover, assume that for every $s\in\mathbb{N}$, u_s *is function in* V_s *minimizing the functional* J_s+G_s *on* V_s. *Finally, let* $\{\bar{s}_k\}\subset\mathbb{N}$ *be an increasing sequence.*

Then there exist an increasing sequence $\{s_j\}\subset\{\bar{s}_k\}$ *and a function* $u\in V$ *such that the following assertions hold true:*

$$\text{the function } u \text{ minimizes the functional } J+G \text{ on } V, \tag{3.11}$$

$$\lim_{j\to\infty}\|u_{s_j}-q_{s_j}u\|_{L^p(\nu,\Omega_{s_j})}=0, \tag{3.12}$$

$$\lim_{j\to\infty}(J_{s_j}+G_{s_j})(u_{s_j})=(J+G)(u). \tag{3.13}$$

Proof. We fix $w\in V$. Clearly, for every $s\in\mathbb{N}$, $q_sw\in V_s$. Then for every $s\in\mathbb{N}$ we have $(J_s+G_s)(u_s)\leqslant(J_s+G_s)(q_sw)$. Hence using conditions (3.1), (3.4), (3.6) and $(*_6)$, we establish that

$$\text{the sequence of the norms } \|u_s\|_{1,p,\nu,s} \text{ is bounded.} \tag{3.14}$$

Next, by virtue of condition $(*_2)$ there exists a sequence of linear continuous operators $l_s:\widetilde{W}_0^{1,p}(\nu,\Omega_s)\to\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that

$$\text{the sequence of the norms } \|l_s\| \text{ is bounded,} \tag{3.15}$$

$$\forall s\in\mathbb{N},\quad q_s(l_su_s)=u_s \text{ a.e. in } \Omega_s. \tag{3.16}$$

From (3.14) and (3.15) it follows that the sequence $\{l_su_s\}$ is bounded in $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Due to this fact and condition $(*_1)$ there exist an increasing sequence $\{s_j\}\subset\{\bar{s}_k\}$ and a function $u\in\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that

$$l_{s_j}u_{s_j}\to u \text{ strongly in } L^p(\nu,\Omega), \tag{3.17}$$

$$l_{s_j}u_{s_j}\to u \text{ a.e. in } \Omega. \tag{3.18}$$

Let us show that $u \in V$. First, we observe that owing to (3.8) and (3.18) there exists a set $E' \subset \Omega$ with measure zero such that

$$\forall x \in \Omega \setminus E', \qquad h(x,(l_{s_j}u_{s_j})(x)) \to h(x,u(x)). \tag{3.19}$$

Moreover, in view of (3.16) and the inclusions $u_s \in V_s$, $s \in \mathbb{N}$, there exists a set $E'' \subset \Omega$ with measure zero such that

$$\forall j \in \mathbb{N} \text{ and } x \in \Omega_{s_j} \setminus E'' \text{ we have } (l_{s_j}u_{s_j})(x) = u_{s_j}(x) \text{ and } h(x,u_{s_j}(x)) \leqslant 0. \tag{3.20}$$

For every $r \in \mathbb{N}$ we set $E^{(r)} = \Omega \setminus \bigcup\limits_{j=r}^{\infty} \Omega_{s_j}$. By virtue of condition ($*_3$) for every $r \in \mathbb{N}$, $\operatorname{meas} E^{(r)} = 0$. We define $E = \bigcup\limits_{r=1}^{\infty} E^{(r)}$. Clearly, $\operatorname{meas} E = 0$. Let $x \in \Omega \setminus (E \cup E' \cup E'')$ and $\varepsilon > 0$. Due to (3.19) there exists $j_0 \in \mathbb{N}$ such that for every $j \in \mathbb{N}$, $j \geqslant j_0$,

$$h(x,u(x)) \leqslant h(x,(l_{s_j}u_{s_j})(x)) + \varepsilon. \tag{3.21}$$

Obviously, $x \notin E^{(j_0)}$. Therefore, there exists $j \in \mathbb{N}$, $j \geqslant j_0$, such that $x \in \Omega_{s_j}$. Then from (3.20) and (3.21) we get $h(x,u(x)) \leqslant \varepsilon$. Hence because of the arbitrariness of ε we obtain $h(x,u(x)) \leqslant 0$. Consequently, $u \in V$.

We note that by virtue of (3.16) and (3.17) equality (3.12) holds true.

Now we define the sequence $\{\bar{u}_s\}$ by

$$\bar{u}_s = \begin{cases} u_s & \text{if } s = s_j \text{ for some } j \in \mathbb{N}, \\ q_s u & \text{if } s \neq s_j \text{ for every } j \in \mathbb{N}. \end{cases}$$

It is evident that for every $s \in \mathbb{N}$, $\bar{u}_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$. Owing to (3.12) we have

$$\lim_{s\to\infty} \|\bar{u}_s - q_s u\|_{L^p(\nu,\Omega_s)} = 0.$$

Then by virtue of conditions $(*_5)$ and $(*_6)$,

$$\liminf_{s\to\infty} (J_s + G_s)(\bar{u}_s) \geqslant (J+G)(u).$$

This implies that

$$\liminf_{j\to\infty} (J_{s_j} + G_{s_j})(u_{s_j}) \geqslant (J+G)(u). \tag{3.22}$$

Further, we fix $v \in V$. Let us show that

$$\limsup_{s\to\infty} (J_s + G_s)(u_s) \leqslant (J+G)(v). \tag{3.23}$$

In view of condition ($*_5$) there exists a sequence $v_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that

$$\lim_{s\to\infty} \|v_s - q_s v\|_{L^p(\nu,\Omega_s)} = 0, \tag{3.24}$$

$$\lim_{s\to\infty} J_s(v_s) = J(v). \tag{3.25}$$

We observe that owing to (3.1), (3.4) and (3.25) there exists $c \geqslant 1$ such that for every $s \in \mathbb{N}$,

$$\int_{\Omega_s} (\nu|\nabla v_s|^p + \psi_s)\,dx \leqslant c. \tag{3.26}$$

For every $k \in \mathbb{N}$ we set $H_k = \{x \in \Omega : d(x, \partial\Omega) \geqslant 1/k\}$. Clearly,

$$\lim_{k\to\infty} \operatorname{meas}(\Omega \setminus H_k) = 0. \tag{3.27}$$

Now we fix an arbitrary $t \in \mathbb{N}$. Obviously, there exists $\delta_1 > 0$ such that

$$\begin{cases} \text{for every measurable set } H \subset \Omega,\ \operatorname{meas} H \leqslant \delta_1, \text{ we have} \\ \int_H \nu|\nabla v|^p\,dx \leqslant 1/t. \end{cases} \tag{3.28}$$

Moreover, by condition $(*_4)$ there exists $\delta_2 > 0$ such that

$$\begin{cases} \text{for every measurable set } H \subset \Omega,\ \operatorname{meas} H \leqslant \delta_2, \text{ we have} \\ \limsup\limits_{s\to\infty} \int_{H\cap\Omega_s} \psi_s\,dx \leqslant 1/t. \end{cases} \tag{3.29}$$

We set $\delta = \min(\delta_1, \delta_2)$. By virtue of (3.27) there exists $k \in \mathbb{N}$ such that $\operatorname{int} H_k \neq \emptyset$ and

$$\operatorname{meas}(\Omega \setminus H_k) < \delta. \tag{3.30}$$

Let $\varphi \in C_0^\infty(\Omega)$ be a function with the properties: $0 \leqslant \varphi \leqslant 1$ in Ω, $\varphi = 1$ in H_k, $\varphi = 0$ in $\Omega \setminus H_{2k}$ and $|\nabla\varphi| \leqslant c_0 k$ in Ω, where $c_0 > 0$ depends only on n.

For every $s \in \mathbb{N}$ we set

$$\mu_s = \left\{ \int_{H_k} \left(\frac{1}{\nu}\right)^{1/(p-1)} dx \right\}^{(p-1)/2p} \left\{ \|v_s - q_s v\|_{L^p(\nu,\Omega_s)} + s^{-1} \right\}^{1/2}.$$

From (3.24) it follows that

$$\lim_{s\to\infty} \mu_s = 0. \tag{3.31}$$

For every $s \in \mathbb{N}$ we define

$$w_s = \max\{v_s + \mu_s q_s \varphi,\ q_s v\} \ \text{ and } \ E_s = \{v_s + \mu_s q_s \varphi \leqslant q_s v\}.$$

For every $s \in \mathbb{N}$ we have $w_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$. Moreover, it is useful to note that

$$\text{if } s \in \mathbb{N} \text{ and } \operatorname{meas}(\Omega_s \setminus E_s) > 0,\ \nabla w_s = \nabla v_s + \mu_s \nabla(q_s\varphi) \text{ a.e. in } \Omega_s \setminus E_s, \tag{3.32}$$

$$\text{if } s \in \mathbb{N} \text{ and } \operatorname{meas} E_s > 0,\ \nabla w_s = \nabla(q_s v) \text{ a.e. in } E_s. \tag{3.33}$$

These facts are established by analogy with the standard chain rule for the functions in nonweighted Sobolev spaces (see for instance [17, Chapter 7]).

If $s \in \mathbb{N}$, by virtue of the definition of the function w_s we have $w_s \geqslant q_s v$ in Ω_s. This along with condition $(*_7)$ and the inclusion $v \in V$ implies that for every $s \in \mathbb{N}$, $w_s \in V_s$. Hence taking into account that for every $s \in \mathbb{N}$ the function u_s minimizes the functional $J_s + G_s$ on V_s, we get

$$\forall s \in \mathbb{N}, \quad (J_s + G_s)(u_s) \leqslant (J_s + G_s)(w_s). \tag{3.34}$$

Moreover, if $s \in \mathbb{N}$, we have $\|w_s - q_s v\|_{L^p(\nu,\Omega_s)} \leqslant \|v_s - q_s v\|_{L^p(\nu,\Omega_s)} + \mu_s \|\varphi\|_{L^p(\nu,\Omega)}$. From this and (3.24) and (3.31) we deduce that $\lim\limits_{s\to\infty} \|w_s - q_s v\|_{L^p(\nu,\Omega_s)} = 0$. This equality and condition $(*_6)$ imply that

$$\lim_{s\to\infty} G_s(w_s) = G(v). \tag{3.35}$$

In what follows we shall estimate from above $J_s(w_s)$ for sufficiently large s.

First we observe that

$$\forall s \in \mathbb{N}, \qquad \operatorname{meas}(H_k \cap E_s) \leqslant \mu_s. \tag{3.36}$$

In fact, let $s \in \mathbb{N}$. Suppose that $H_k \cap E_s \neq \emptyset$. Taking into account that $\varphi = 1$ in H_k, from the definition of the set E_s we derive that for every $x \in H_k \cap E_s$, $\mu_s \leqslant |v_s - q_s v|(x)$. Then

$$\mu_s \operatorname{meas}(H_k \cap E_s) \leqslant \int_{H_k \cap E_s} |v_s - q_s v|\, dx. \tag{3.37}$$

Using Hölder's inequality, we obtain

$$\int_{H_k \cap E_s} |v_s - q_s v|\, dx \leqslant \left\{ \int_{H_k} \left(\frac{1}{\nu}\right)^{1/(p-1)} dx \right\}^{(p-1)/p} \|v_s - q_s v\|_{L^p(\nu,\Omega_s)} \leqslant \mu_s^2.$$

From this and (3.37) it follows that $\operatorname{meas}(H_k \cap E_s) \leqslant \mu_s$. Obviously, this inequality also holds true if $H_k \cap E_s = \emptyset$. Thus, assertion (3.36) is proved.

Next, for every $s \in \mathbb{N}$ we have

$$\operatorname{meas} E_s \leqslant \operatorname{meas}(H_k \cap E_s) + \operatorname{meas}(\Omega \setminus H_k), \tag{3.38}$$

$$\int_{E_s} \psi_s\, dx \leqslant \int_{H_k \cap E_s} \psi_s\, dx + \int_{(\Omega \setminus H_k) \cap \Omega_s} \psi_s\, dx. \tag{3.39}$$

Owing to (3.30), (3.31), (3.36), (3.38) and (3.28) there exists $s' \in \mathbb{N}$ such that

$$\forall s \in \mathbb{N},\, s \geqslant s', \qquad \int_{E_s} \nu |\nabla v|^p\, dx \leqslant 1/t. \tag{3.40}$$

We note that

$$\int_{H_k \cap E_s} \psi_s\, dx \to 0 \text{ as } s \to \infty. \tag{3.41}$$

Indeed, suppose that assertion (3.41) is not valid. Then there exist $\varepsilon_1 > 0$ and an increasing sequence $\{n_j\} \subset \mathbb{N}$ such that

$$\forall j \in \mathbb{N}, \qquad \int_{H_k \cap E_{n_j}} \psi_{n_j}\, dx > \varepsilon_1. \tag{3.42}$$

By virtue of condition $(*_4)$ there exists $\delta' > 0$ such that

$$\begin{cases} \text{for every measurable set } H \subset \Omega, \text{ meas} H \leqslant \delta', \text{ we have} \\ \limsup\limits_{s\to\infty} \int_{H\cap\Omega_s} \psi_s\,dx < \varepsilon_1/2. \end{cases} \tag{3.43}$$

From (3.31) and (3.36) it follows that there exists an increasing sequence $\{r_i\} \subset \{n_j\}$ such that

$$\forall i \in \mathbb{N}, \qquad \text{meas}(H_k \cap E_{r_i}) \leqslant \delta'/2^i. \tag{3.44}$$

We set $H' = \bigcup\limits_{i=1}^{\infty} (H_k \cap E_{r_i})$. Due to (3.44) we have $\text{meas} H' \leqslant \delta'$. Then by (3.43)

$$\limsup_{s\to\infty} \int_{H'\cap\Omega_s} \psi_s\,dx < \varepsilon_1/2.$$

Therefore, there exists $N \in \mathbb{N}$ such that

$$\int_{H'\cap\Omega_{r_N}} \psi_{r_N}\,dx \leqslant \varepsilon_1. \tag{3.45}$$

Clearly, $H_k \cap E_{r_N} \subset H' \cap \Omega_{r_N}$. This and (3.45) imply that

$$\int_{H_k\cap E_{r_N}} \psi_{r_N}\,dx \leqslant \varepsilon_1.$$

Evidently, this inequality contradicts (3.42). The contradiction obtained proves that assertion (3.41) is valid.

Owing to (3.29), (3.30), (3.41) and (3.39) there exists $s'' \in \mathbb{N}$ such that

$$\forall s \in \mathbb{N},\, s \geqslant s'', \qquad \int_{E_s} \psi_s\,dx \leqslant 2/t. \tag{3.46}$$

Finally, because of (3.31) there exists $s''' \in \mathbb{N}$ such that

$$\forall s \in \mathbb{N},\, s \geqslant s''', \qquad \mu_s \Big\{ 1 + (c_0 k)^p \int_{H_{2k}} \nu\,dx \Big\} \leqslant 1/t. \tag{3.47}$$

We set $\tilde{s} = \max(s', s'', s''')$ and fix $s \in \mathbb{N}$, $s \geqslant \tilde{s}$. Obviously,

$$J_s(w_s) = \int_{\Omega_s\setminus E_s} f_s(x, \nabla w_s)\,dx + \int_{E_s} f_s(x, \nabla w_s)\,dx. \tag{3.48}$$

Taking into account that by (3.47) $\mu_s \leqslant 1$ and using (3.32), (3.3) and (3.4), we establish that

$$\begin{aligned} \int_{\Omega_s\setminus E_s} f_s(x, \nabla w_s)\,dx \leqslant \int_{\Omega_s\setminus E_s} f_s(x, \nabla v_s)\,dx + 2\mu_s \int_{\Omega_s} \psi_s\,dx \\ + 2^{p-1} c_2\, \mu_s \Big\{ \int_{\Omega_s} \nu |\nabla v_s|^p\,dx + \int_{\Omega} \nu |\nabla \varphi|^p\,dx \Big\}. \end{aligned} \tag{3.49}$$

Moreover, (3.4) and (3.46) imply that

$$\int_{\Omega_s \setminus E_s} f_s(x, \nabla v_s)\, dx \leqslant J_s(v_s) + 2/t. \tag{3.50}$$

Taking into account the properties of the function φ, from (3.49), (3.50), (3.26) and (3.47) we obtain

$$\int_{\Omega_s \setminus E_s} f_s(x, \nabla w_s)\, dx \leqslant J_s(v_s) + [2 + 2^p c\,(c_2 + 1)]/t. \tag{3.51}$$

Besides, using (3.4), (3.33), (3.40) and (3.46), we find that

$$\int_{E_s} f_s(x, \nabla w_s)\, dx \leqslant (c_2 + 2)/t. \tag{3.52}$$

From (3.48), (3.51) and (3.52) we deduce that

$$J_s(w_s) \leqslant J_s(v_s) + (2^p + 1)(c_2 + 4)c/t.$$

This and (3.34) imply that

$$(J_s + G_s)(u_s) \leqslant J_s(v_s) + G_s(w_s) + (2^p + 1)(c_2 + 4)c/t.$$

Then taking into account (3.25) and (3.35), we get

$$\limsup_{s \to \infty} (J_s + G_s)(u_s) \leqslant (J + G)(v) + (2^p + 1)(c_2 + 4)c/t.$$

Hence due to the arbitrariness of t we obtain inequality (3.23).

From (3.22) and (3.23) we infer that assertion (3.11) holds true. Finally, from (3.23) with $v = u$ and (3.22) we derive equality (3.13). □

REMARK 3.2. The assertions of Theorem 3.1 remain true if in the statement of the theorem instead of condition $(*_7)$ we use the next one: for almost every $x \in \Omega$ from $\eta \in \mathbb{R}$ and $h(x, \eta) \leqslant 0$ it follows that for every $\eta' \leqslant \eta$, $h(x, \eta') \leqslant 0$. In this case in the above-given proof instead of the functions w_s and sets E_s one should use the functions and sets defined by $\bar{w}_s = \min\{v_s - \mu_s q_s \varphi, q_s v\}$, $\bar{E}_s = \{v_s - \mu_s q_s \varphi \geqslant q_s v\}$.

Further, let us show that under condition $(*_7)$ of Theorem 3.1 V and V_s are actually the sets with unilateral constraints, and the values of the function which determines the constraints lie in $\bar{\mathbb{R}}$.

For every $x \in \Omega$ we set

$$M_h(x) = \{\eta \in \mathbb{R} : h(x, \eta) \leqslant 0\}.$$

Define the function $\bar{h} : \Omega \to \bar{\mathbb{R}}$ by

$$\bar{h}(x) = \begin{cases} \inf\limits_{\eta \in M_h(x)} \eta & \text{if} \quad M_h(x) \neq \emptyset, \\ +\infty & \text{if} \quad M_h(x) = \emptyset. \end{cases}$$

Since $V \neq \emptyset$, we have $\bar{h}(x) \neq +\infty$ for a. e. $x \in \Omega$.

PROPOSITION 3.3. *Suppose that for almost every $x \in \Omega$ from $\eta \in \mathbb{R}$ and $h(x,\eta) \leqslant 0$ it follows that for every $\eta' \geqslant \eta$, $h(x,\eta') \leqslant 0$. Then*

$$V = \{v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) : v \geqslant \bar{h} \text{ a.e. in } \Omega\}$$

and for every $s \in \mathbb{N}$,

$$V_s = \{v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) : v \geqslant \bar{h} \text{ a.e. in } \Omega_s\}.$$

We omit the proof of the proposition because of its simplicity.

REMARK 3.4. We observe that in the case $\bar{h} = z$ a.e. in Ω, where $z \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, condition $(*_3)$ in Theorem 3.1 is unnecessary. In fact, first of all we recall that in the proof of Theorem 3.1 condition $(*_3)$ is used only in order to establish that $u \in V$. Suppose that all the conditions of Theorem 3.1 are satisfied except for condition $(*_3)$. If $\bar{h} = z$ a.e. in Ω, where $z \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, instead of the consideration given in the proof of Theorem 3.1 below (3.16) up to the conclusions that $u \in V$ and equality (3.12) holds true we argue as follows. Taking into account Proposition 3.3 and (3.16) and setting for every $s \in \mathbb{N}$, $z_s = \max\{l_s u_s, z\}$, we obtain that for every $s \in \mathbb{N}$, $z_s \in V$ and $q_s z_s = u_s$ a.e. in Ω_s. Since by (3.14) and (3.15) the sequence $\{l_s u_s\}$ is bounded in $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, the sequence $\{z_s\}$ also is bounded in $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Therefore, there exist an increasing sequence $\{s_j\} \subset \{\bar{s}_k\}$ and a function $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that $z_{s_j} \to u$ weakly in $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. From this, taking into account the above-mentioned properties of the sequence $\{z_s\}$, the sequential weak closedness of the set V in $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and condition $(*_1)$ of Theorem 3.1, we obtain that $u \in V$ and equality (3.12) holds true.

However, in general case condition $(*_3)$ of Theorem 3.1 cannot be omitted. The corresponding example is given in Section 4.

REMARK 3.5. If $z : \Omega \to \overline{\mathbb{R}}$, $U = \{v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) : v \geqslant z$ a.e. in $\Omega\}$ and for every $s \in \mathbb{N}$, $U_s = \{v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) : v \geqslant z$ a.e. in $\Omega_s\}$, defining the function $\sigma : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\sigma(x,\eta) = \begin{cases} -\eta + z(x) & \text{if } z(x) \in \mathbb{R}, \\ 0 & \text{if } z(x) = -\infty, \\ 1 & \text{if } z(x) = +\infty, \end{cases}$$

we easily get $\bar{\sigma} = z$, $U = \{v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) : \sigma(x,v(x)) \leqslant 0$ for a.e. $x \in \Omega\}$ and for every $s \in \mathbb{N}$, $U_s = \{v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) : \sigma(x,v(x)) \leqslant 0$ for a.e. $x \in \Omega_s\}$.

Besides, if $z \neq +\infty$ a.e. in Ω, the function σ has the same property as that of the function h assumed in Theorem 3.1 and Proposition 3.3.

Now we give an application of Theorem 3.1 to variational problems with varying obstacles.

For every $y \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ we set

$$W^{(y)} = \{v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) : v \geqslant y \text{ a.e. in } \Omega\}.$$

Let for every $s \in \mathbb{N}$, $y_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$.
For every $s \in \mathbb{N}$ we set

$$W_s = \{v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) : \ v \geqslant y_s \ \text{ a.e. in } \Omega_s\}.$$

Clearly, if $s \in \mathbb{N}$, the set W_s is nonempty and sequentially weakly closed in $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$. This fact and (3.7) imply that for every $s \in \mathbb{N}$ there exists a function $u_s \in W_s$ minimizing the functional $J_s + G_s$ on W_s.

THEOREM 3.6. *Suppose that conditions $(*_1)$, $(*_2)$ and $(*_4)$-$(*_6)$ of Theorem 3.1 are satisfied. Moreover, assume that the following conditions are satisfied:*
$(*_1')$ *the sequence of the norms $\|y_s\|_{1,p,\nu,s}$ is bounded;*
$(*_2')$ *if $\varepsilon > 0$, there exists $\delta > 0$ such that for every measurable set $E \subset \Omega$, $\operatorname{meas} E \leqslant \delta$, we have $\limsup_{s\to\infty} \int_{E\cap\Omega_s} \nu|\nabla y_s|^p\,dx \leqslant \varepsilon$.*

Finally, let for every $s \in \mathbb{N}$, u_s be a function in W_s minimizing the functional $J_s + G_s$ on W_s.

Then there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and functions $y \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and $u \in W^{(y)}$ such that the following assertions hold true:

$$\lim_{j\to\infty} \|y_{s_j} - q_{s_j} y\|_{L^p(\nu,\Omega_{s_j})} = 0, \tag{3.53}$$

$$\textit{the function } u \textit{ minimizes the functional } J+G \textit{ on } W^{(y)}, \tag{3.54}$$

$$\lim_{j\to\infty} \|u_{s_j} - q_{s_j} u\|_{L^p(\nu,\Omega_{s_j})} = 0, \tag{3.55}$$

$$\lim_{j\to\infty} (J_{s_j} + G_{s_j})(u_{s_j}) = (J+G)(u). \tag{3.56}$$

Proof. Since conditions $(*_1)$ and $(*_2)$ of Theorem 3.1 and condition $(*_1')$ are satisfied, by virtue of Proposition 2.3 there exist an increasing sequence $\{\bar{s}_k\} \subset \mathbb{N}$ and a function $y \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that

$$\lim_{k\to\infty} \|y_{\bar{s}_k} - q_{\bar{s}_k} y\|_{L^p(\nu,\Omega_{\bar{s}_k})} = 0. \tag{3.57}$$

We define the sequence $\{\bar{y}_s\}$ by

$$\bar{y}_s = \begin{cases} y_s & \text{if } \ s = \bar{s}_k \ \text{ for some } k \in \mathbb{N}, \\ q_s y & \text{if } \ s \neq \bar{s}_k \ \text{ for every } k \in \mathbb{N}. \end{cases}$$

Clearly, for every $s \in \mathbb{N}$, $\bar{y}_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$. Moreover, in view of (3.57) we have

$$\lim_{s\to\infty} \|\bar{y}_s - q_s y\|_{L^p(\nu,\Omega_s)} = 0. \tag{3.58}$$

Next, we set $\bar{c}_1 = 2^{1-p}c_1$, $\bar{c}_2 = 2^{p-1}c_2$ and for every $s \in \mathbb{N}$ we define

$$\bar{\psi}_s = \psi_s + \bar{c}_2 \nu |\nabla \bar{y}_s|^p.$$

Obviously, for every $s \in \mathbb{N}$, $\bar{\psi}_s \in L^1(\Omega_s)$ and $\bar{\psi}_s \geqslant 0$ in Ω_s. Furthermore, (3.1) and condition $(*_1')$ imply that the sequence of the norms $\|\bar{\psi}_s\|_{L^1(\Omega_s)}$ is bounded. Finally, by virtue of condition $(*_4)$ of Theorem 3.1 and condition $(*_2')$ the following assertion holds true: if $\varepsilon > 0$, there exists $\delta > 0$ such that for every measurable set $E \subset \Omega$, $\operatorname{meas} E \leqslant \delta$, we have

$$\limsup_{s\to\infty} \int_{E\cap\Omega_s} \bar{\psi}_s \, dx \leqslant \varepsilon.$$

For every $s \in \mathbb{N}$ we define the function $\bar{f}_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ by

$$\bar{f}_s(x,\xi) = f_s(x, \xi + \nabla \bar{y}_s(x)), \quad (x,\xi) \in \Omega_s \times \mathbb{R}^n.$$

Owing to (3.2)-(3.4) the following assertions hold true: for every $s \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$ the function $\bar{f}_s(\cdot,\xi)$ is measurable in Ω_s; for every $s \in \mathbb{N}$ and almost every $x \in \Omega_s$ the function $\bar{f}_s(x,\cdot)$ is convex in $\mathbb{R}^n$; for every $s \in \mathbb{N}$, almost every $x \in \Omega_s$ and every $\xi \in \mathbb{R}^n$,

$$\bar{c}_1 \nu(x)|\xi|^p - \bar{\psi}_s(x) \leqslant \bar{f}_s(x,\xi) \leqslant \bar{c}_2 \nu(x)|\xi|^p + \bar{\psi}_s(x).$$

Thus, the described properties of the functions $\bar{\psi}_s$ and $\bar{f}_s$ are the same as the properties of the functions ψ_s and f_s stated in the beginning of the section. Besides, the functions $\bar{\psi}_s$ satisfy the condition similar to condition $(*_4)$.

For every $s \in \mathbb{N}$ we define the functional $\bar{J}_s : \widetilde{W}_0^{1,p}(\nu,\Omega_s) \to \mathbb{R}$ by

$$\bar{J}_s(v) = \int_{\Omega_s} \bar{f}_s(x, \nabla v)\, dx, \qquad v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s).$$

If $s \in \mathbb{N}$ and $v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$, we have $\bar{J}_s(v) = J_s(v + \bar{y}_s)$.

We define the functional $\bar{J} : \mathring{W}^{1,p}(\nu,\Omega) \to \mathbb{R}$ by $\bar{J}(v) = J(v+y)$, $v \in \mathring{W}^{1,p}(\nu,\Omega)$.

From condition $(*_5)$ of Theorem 3.1 and (3.58) it follows that the sequence $\{\bar{J}_s\}$ Γ-converges to the functional $\bar{J}$.

Further, in view of condition $(*_1')$ there exists $c_5 > 0$ such that

$$\forall s \in \mathbb{N}, \qquad \|\bar{y}_s\|^p_{L^p(\nu,\Omega_s)} \leqslant c_5. \tag{3.59}$$

We set $\bar{c}_3 = 2^{1-p}c_3$, $\bar{c}_4 = c_4 + c_3 c_5$ and for every $s \in \mathbb{N}$ define the functional $\bar{G}_s : \widetilde{W}_0^{1,p}(\nu,\Omega_s) \to \mathbb{R}$ by

$$\bar{G}_s(v) = G_s(v + \bar{y}_s), \qquad v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s).$$

Clearly, if $s \in \mathbb{N}$, the functional $\bar{G}_s$ is weakly continuous on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$. Moreover, owing to conditions $(*_1')$ and (3.5) the following assertion holds true: for every sequence $v_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that the sequence of the norms $\|v_s\|_{1,p,\nu,s}$ is bounded, the sequence $\{\bar{G}_s(v_s)\}$ is bounded. Finally, using (3.6) and (3.59), we establish that for every $s \in \mathbb{N}$ and $v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$,

$$\bar{G}_s(v) \geqslant \bar{c}_3 \|v\|^p_{L^p(\nu,\Omega_s)} - \bar{c}_4.$$

We define the functional $\bar{G} : \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \to \mathbb{R}$ by $\bar{G}(v) = G(v+y)$, $v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$.

By virtue of condition $(*_6)$ of Theorem 3.1 and (3.58) the following assertion holds true: for every $v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and every sequence $v_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ with the property $\|v_s - q_s v\|_{L^p(\nu,\Omega_s)} \to 0$ we have $\bar{G}_s(v_s) \to \bar{G}(v)$.

Thus, $\bar{J}_s$ and $\bar{G}_s$ are functionals of the same kind as the functionals J_s and G_s, and they satisfy conditions analogous to conditions $(*_5)$ and $(*_6)$ of Theorem 3.1.

Next, let $\hat{h} : \Omega \times \mathbb{R} \to \mathbb{R}$ be the function such that for every $(x,\eta) \in \Omega \times \mathbb{R}$, $\hat{h}(x,\eta) = -\eta$. Obviously, for every $x \in \Omega$ the function $\hat{h}(x,\cdot)$ is continuous and nonincreasing in $\mathbb{R}$. We set

$$\hat{V} = \{v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) : \hat{h}(x,v(x)) \leqslant 0 \text{ for a.e. } x \in \Omega\}$$

and for every $s \in \mathbb{N}$ define

$$\hat{V}_s = \{v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) : \hat{h}(x,v(x)) \leqslant 0 \text{ for a.e. } x \in \Omega_s\}.$$

Evidently, the function $\hat{h}$ satisfies conditions analogous to condition (3.8) and condition $(*_7)$ of Theorem 3.1, and $\hat{V}$ and $\hat{V}_s$ are the sets of the same kind as the sets V and V_s.

For every $s \in \mathbb{N}$ we define

$$\hat{W}_s = \{v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) : v \geqslant \bar{y}_s \text{ a.e. in } \Omega_s\}.$$

It is easy to see that for every $s \in \mathbb{N}$ there exists a function $\hat{u}_s \in \hat{W}_s$ minimizing the functional $J_s + G_s$ on $\hat{W}_s$. We define the sequence $\{\bar{u}_s\}$ by

$$\bar{u}_s = \begin{cases} u_s & \text{if } s = \bar{s}_k \text{ for some } k \in \mathbb{N}, \\ \hat{u}_s & \text{if } s \neq \bar{s}_k \text{ for every } k \in \mathbb{N}. \end{cases}$$

It is not difficult to verify that for every $s \in \mathbb{N}$, $\bar{u}_s - \bar{y}_s$ is a function in $\hat{V}_s$ minimizing the functional $\bar{J}_s + \bar{G}_s$ on $\hat{V}_s$.

Now taking into account Remark 3.4 and applying Theorem 3.1, we conclude that there exist an increasing sequence $\{s_j\} \subset \{\bar{s}_k\}$ and a function $\bar{u} \in \hat{V}$ such that the following assertions hold true:

$$\text{the function } \bar{u} \text{ minimizes the functional } \bar{J} + \bar{G} \text{ on } \hat{V}, \tag{3.60}$$

$$\lim_{j\to\infty} \|\bar{u}_{s_j} - \bar{y}_{s_j} - q_{s_j}\bar{u}\|_{L^p(\nu,\Omega_{s_j})} = 0, \tag{3.61}$$

$$\lim_{j\to\infty} (\bar{J}_{s_j} + \bar{G}_{s_j})(\bar{u}_{s_j} - \bar{y}_{s_j}) = (\bar{J} + \bar{G})(\bar{u}). \tag{3.62}$$

We set $u=\bar{u}+y$. Since $\bar{u}\in\hat{V}$, we have $u\in W^{(y)}$. Observe that by virtue of (3.57) assertion (3.53) holds true. Moreover, owing to (3.60) assertion (3.54) holds true. Using (3.58) and (3.61), we obtain that assertion (3.55) holds true, and finally from (3.62) we deduce that assertion (3.56) holds true. □

In what follows we give a result on the behaviour of a sequence of minimizers of the functionals J_s+G_s on the sets V_s without the assumption that for almost every $x\in\Omega$ from $\eta\in\mathbb{R}$ and $h(x,\eta)\leqslant 0$ it follows that for every $\eta'\geqslant\eta$, $h(x,\eta')\leqslant 0$, but under the condition that the values $h(x,\eta)$ of the function h do not depend on the variable x.

Before to make this we prove several useful lemmas.

LEMMA 3.7. *Let* $v\in\overset{\circ}{W}{}^{1,1}(\Omega)$*. Then the following assertions hold true:*
(i) for every $\lambda>0$, $\operatorname{meas}\{|v|<\lambda\}>0$;
(ii) if $0\leqslant\lambda_1<\lambda_2\leqslant\lambda$ *and* $\operatorname{meas}\{v>\lambda\}>0$, *we have* $\operatorname{meas}\{\lambda_1<v<\lambda_2\}>0$;
(iii) if $\lambda>0$ *and* $\varepsilon>0$, *there exists a measurable set* $H\subset\Omega$ *such that* $\operatorname{meas}H>0$ *and for every* $x\in H$, $|v(x)|<\lambda$ *and* $d(x,\partial\Omega)<\varepsilon$.

Proof. As is known (see for instance [17, Chapter 7]) there exists $C>0$ such that for every $z\in\overset{\circ}{W}{}^{1,1}(\Omega)$,

$$\int_\Omega |z|dx\leqslant C\int_\Omega |\nabla z|dx. \tag{3.63}$$

Let φ be a function in $C^1(\mathbb{R})$ such that φ is nondecreasing in $\mathbb{R}$, $\varphi=0$ in $(-\infty,0]$ and $\varphi=1$ in $[1,+\infty)$.

Let $\lambda>0$. Define the function $\varphi_\lambda:\mathbb{R}\to\mathbb{R}$ by $\varphi_\lambda(\eta)=\varphi(\eta/\lambda)$, $\eta\in\mathbb{R}$, and set $w=|v|$. Since $v\in\overset{\circ}{W}{}^{1,1}(\Omega)$, we have $w\in\overset{\circ}{W}{}^{1,1}(\Omega)$. This and the properties of the function φ imply that $\varphi_\lambda(w)\in\overset{\circ}{W}{}^{1,1}(\Omega)$ and $|\nabla\varphi_\lambda(w)|=\varphi'_\lambda(w)|\nabla w|$ a. e. in Ω. Therefore, using (3.63), we get

$$\int_\Omega \varphi_\lambda(w)dx\leqslant C\int_\Omega \varphi'_\lambda(w)|\nabla w|dx. \tag{3.64}$$

Suppose that $\operatorname{meas}\{|v|<\lambda\}=0$. Then taking into account that $\varphi_\lambda=1$ in $[\lambda,+\infty)$, from (3.64) we obtain $\operatorname{meas}\{|v|\geqslant\lambda\}=0$. This contradicts $\operatorname{meas}\Omega>0$. Therefore, $\operatorname{meas}\{|v|<\lambda\}>0$. Thus, we conclude that assertion (i) holds true.

Next, let $0\leqslant\lambda_1<\lambda_2\leqslant\lambda$ and $\operatorname{meas}\{v>\lambda\}>0$. We define the function $\bar{\varphi}:\mathbb{R}\to\mathbb{R}$ by

$$\bar{\varphi}(\eta)=\varphi\left(\frac{\eta-\lambda_1}{\lambda_2-\lambda_1}\right),\quad \eta\in\mathbb{R}.$$

We have $\bar{\varphi}(v)\in\overset{\circ}{W}{}^{1,1}(\Omega)$ and $|\nabla\bar{\varphi}(v)|=\bar{\varphi}'(v)|\nabla v|$ a. e. in Ω. Then by virtue of (3.63)

$$\int_\Omega \bar{\varphi}(v)dx\leqslant C\int_\Omega \bar{\varphi}'(v)|\nabla v|dx. \tag{3.65}$$

Suppose that $\operatorname{meas}\{\lambda_1 < v < \lambda_2\} = 0$. Then taking into account that $\bar\varphi = 0$ in $(-\infty, \lambda_1]$ and $\bar\varphi = 1$ in $[\lambda_2, +\infty)$, from (3.65) we get $\operatorname{meas}\{v \geqslant \lambda_2\} = 0$. This contradicts $\operatorname{meas}\{v > \lambda\} > 0$. Therefore, $\operatorname{meas}\{\lambda_1 < v < \lambda_2\} > 0$. Thus, we conclude that assertion (ii) holds true.

Now let $\lambda > 0$ and $\varepsilon > 0$. We fix a point $x_0 \in \Omega$ and ε_1 such that $0 < \varepsilon_1 < \min\{\varepsilon, d(x_0, \partial\Omega)\}$. Define

$$H' = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon_1\}, \qquad H'' = \{x \in \Omega : d(x, \partial\Omega) \leqslant \varepsilon_1/2\},$$
$$H = (\Omega \setminus H') \cap \{|v| < \lambda\}.$$

Let ρ be a function in $C_0^\infty(\Omega)$ with the properties: $0 \leqslant \rho \leqslant 1$ in Ω, $\rho = 1$ in H' and $\rho = 0$ in H''. We set $\bar w = |v|(1-\rho) + \lambda\rho$. It is easy to see that $\bar w \in \mathring{W}^{1,1}(\Omega)$. Then by assertion (i) we have $\operatorname{meas}\{|\bar w| < \lambda\} > 0$. Suppose that $\operatorname{meas} H = 0$. Let $x \in \{|\bar w| < \lambda\} \setminus H$. Hence $|\bar w(x)| < \lambda$ and $x \notin H$. If $x \in H'$, we have $\rho(x) = 1$ and therefore, $\bar w(x) = \lambda$. This contradicts $|\bar w(x)| < \lambda$. Hence $x \in \Omega \setminus H'$. Therefore, since $x \notin H$, we get $|v(x)| \geqslant \lambda$. Then $\bar w(x) \geqslant \lambda$. This also contradicts $|\bar w(x)| < \lambda$. Hence we infer that $\operatorname{meas} H > 0$. Obviously, for every $x \in H$, $|v(x)| < \lambda$ and $d(x, \partial\Omega) < \varepsilon$. Thus, assertion (iii) holds true. □

LEMMA 3.8. *Let*

$$\left(\frac{1}{\nu}\right)^{1/(p-1)} \in L^1(\Omega). \tag{3.66}$$

Then

$$\mathring{W}^{1,p}(\nu, \Omega) \subset \mathring{W}^{1,1}(\Omega). \tag{3.67}$$

Proof. We set

$$K = (n+1)\left\{\int_\Omega \left(\frac{1}{\nu}\right)^{1/(p-1)} dx\right\}^{(p-1)/p}.$$

Let $v \in W^{1,p}(\nu, \Omega)$. Clearly,

$$|v| = \nu^{1/p}|v| \cdot \left(\frac{1}{\nu}\right)^{1/p} \quad \text{a.e. in } \Omega. \tag{3.68}$$

Using this fact and Young's inequality, we obtain

$$|v| \leqslant \nu|v|^p + \left(\frac{1}{\nu}\right)^{1/(p-1)} \quad \text{a.e. in } \Omega.$$

This along with the inclusion $\nu|v|^p \in L^1(\Omega)$ and (3.66) implies that $v \in L^1(\Omega)$. Analogously, for every $i \in \{1, \ldots, n\}$ we have $D_i v \in L^1(\Omega)$. Thus, $v \in W^{1,1}(\Omega)$. Moreover, using (3.68) and Hölder's inequality, we get

$$\int_\Omega |v| dx \leqslant \frac{K}{n+1}\left(\int_\Omega \nu|v|^p dx\right)^{1/p}.$$

This and analogous inequalities for the derivatives $D_i v$ imply that

$$\|v\|_{W^{1,1}(\Omega)} \leqslant K\|v\|_{1,p,\nu}\,. \tag{3.69}$$

Now let $v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Evidently, $v \in W^{1,1}(\Omega)$. Taking a sequence $\{v_j\} \subset C_0^\infty(\Omega)$ such that $\|v_j - v\|_{1,p,\nu} \to 0$, from (3.69) we obtain $v_j \to v$ strongly in $W^{1,1}(\Omega)$. Hence $v \in \overset{\circ}{W}{}^{1,1}(\Omega)$. □

REMARK 3.9. Inclusion (3.66) is essential for (3.67). In Section 4 we give an example which shows that without assumption (3.66) inclusion (3.67) may not be valid.

LEMMA 3.10. *Suppose that $(1/\nu)^{1/(p-1)} \in L^1(\Omega)$ and the embedding of the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ into $L^p(\nu,\Omega)$ is compact. Let $s \in \mathbb{N}$ and let the set $\Omega \setminus \Omega_s$ be closed. Assume that the following condition is satisfied:*

$$\begin{cases} \textit{for every bounded sequence } \{v_j\} \subset \widetilde{W}{}_0^{1,p}(\nu,\Omega_s) \textit{ there exists a bounded} \\ \textit{sequence } \{\bar v_j\} \subset \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \textit{ such that } \forall j \in \mathbb{N},\ \bar v_j = v_j \textit{ a. e. in } \Omega_s. \end{cases} \tag{3.70}$$

Then the next assertion holds true: if $v \in \widetilde{W}{}_0^{1,p}(\nu,\Omega_s)$, $0 \leqslant \lambda_1 < \lambda_2 \leqslant \lambda$ and $\operatorname{meas}\{v > \lambda\} > 0$, *we have* $\operatorname{meas}\{\lambda_1 < v < \lambda_2\} > 0$.

Proof. First of all let us show that the following assertion holds true:

$$\begin{cases} \text{there exists } K_s > 0 \text{ such that for every } w \in \widetilde{W}{}_0^{1,p}(\nu,\Omega_s),\ \|w\|_{L^p(\nu,\Omega_s)} = 1, \\ \text{we have } K_s \int_{\Omega_s} \nu |\nabla w|^p dx \geqslant 1. \end{cases} \tag{3.71}$$

Suppose that this assertion is not valid. Then there exists a sequence $\{w_j\} \subset \widetilde{W}{}_0^{1,p}(\nu,\Omega_s)$ such that for every $j \in \mathbb{N}$,

$$\|w_j\|_{L^p(\nu,\Omega_s)} = 1, \tag{3.72}$$

$$\int_{\Omega_s} \nu |\nabla w_j|^p dx < 1/j. \tag{3.73}$$

Clearly, the sequence $\{w_j\}$ is bounded in $\widetilde{W}{}_0^{1,p}(\nu,\Omega_s)$. Therefore, by condition (3.70) there exists a bounded sequence $\{\bar w_j\} \subset \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that

$$\forall j \in \mathbb{N}, \qquad \bar w_j = w_j \ \text{ a. e. in } \ \Omega_s\,. \tag{3.74}$$

Obviously, there exist an increasing sequence $\{j_k\} \subset \mathbb{N}$ and a function $z \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that

$$\bar w_{j_k} \to z \ \text{ weakly in } \ \overset{\circ}{W}{}^{1,p}(\nu,\Omega). \tag{3.75}$$

Owing to (3.74) and (3.75) we have $w_{j_k} \to q_s z$ weakly in $\widetilde{W}{}_0^{1,p}(\nu,\Omega_s)$. Hence

$$\liminf_{k\to\infty} \int_{\Omega_s} \nu |\nabla w_{j_k}|^p dx \geqslant \int_{\Omega_s} \nu |\nabla (q_s z)|^p dx.$$

This and (3.73) imply that

$$\nabla(q_s z) = 0 \ \text{a.e. in } \Omega_s. \tag{3.76}$$

Since $(1/\nu)^{1/(p-1)} \in L^1(\Omega)$, we have $q_s z \in W^{1,1}(\Omega_s)$. By virtue of this inclusion and (3.76) and the connectedness of Ω_s there exists $\tau_s \in \mathbb{R}$ such that

$$q_s z = \tau_s \ \text{a.e. in } \Omega_s. \tag{3.77}$$

Let us demonstrate that $\tau_s = 0$. Since the set $\Omega \setminus \Omega_s$ is closed, we have $d(\Omega \setminus \Omega_s, \partial\Omega) > 0$. We fix ε such that $0 < \varepsilon < d(\Omega \setminus \Omega_s, \partial\Omega)$ and define $H^{(\varepsilon)} = \{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}$. It is easy to see that

$$H^{(\varepsilon)} \subset \Omega_s. \tag{3.78}$$

Suppose that $\tau_s \neq 0$. Due to Lemma 3.8 we have $z \in \overset{\circ}{W}{}^{1,1}(\Omega)$. Then by virtue of assertion (iii) of Lemma 3.7 and (3.78) there exists a measurable set $H \subset \Omega_s$ such that $\operatorname{meas} H > 0$ and $|z| < |\tau_s|$ in H. However, this contradicts (3.77). The contradiction obtained proves that

$$\tau_s = 0. \tag{3.79}$$

Since the embedding of $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ into $L^p(\nu, \Omega)$ is compact, from (3.74), (3.75), (3.77) and (3.79) we deduce that $\lim\limits_{k\to\infty} \|w_{j_k}\|_{L^p(\nu,\Omega_s)} = 0$. This contradicts (3.72). Hence we conclude that assertion (3.71) is valid. Therefore, for every $w \in \widetilde{W}{}_0^{1,p}(\nu, \Omega_s)$,

$$\int_{\Omega_s} \nu |w|^p dx \leqslant K_s \int_{\Omega_s} \nu |\nabla w|^p dx. \tag{3.80}$$

Now let $v \in \widetilde{W}{}_0^{1,p}(\nu, \Omega_s)$, $0 \leqslant \lambda_1 < \lambda_2 \leqslant \lambda$ and $\operatorname{meas}\{v > \lambda\} > 0$. Taking a function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi = 0$ in $(-\infty, \lambda_1]$, $\varphi = 1$ in $[\lambda_2, +\infty)$ and $\varphi' \geqslant 0$ in $\mathbb{R}$, we have $\varphi(v) \in \widetilde{W}{}_0^{1,p}(\nu, \Omega_s)$ and $|\nabla\varphi(v)| = \varphi'(v)|\nabla v|$ a.e. in Ω_s. Then applying (3.80), we get

$$\int_{\Omega_s} \nu (\varphi(v))^p dx \leqslant K_s \int_{\Omega_s} \nu (\varphi'(v))^p |\nabla v|^p dx. \tag{3.81}$$

Suppose that $\operatorname{meas}\{\lambda_1 < v < \lambda_2\} = 0$. Then taking into account the properties of the function φ, from (3.81) we obtain that $\varphi(v) = 0$ a.e. in $\{v > \lambda\}$. However, this contradicts the fact that $\varphi = 1$ in $[\lambda_2, +\infty)$. Thus, we conclude that $\operatorname{meas}\{\lambda_1 < v < \lambda_2\} > 0$. $\square$

LEMMA 3.11. *Suppose that the following conditions are satisfied:*

$$\left(\frac{1}{\nu}\right)^{1/(p-1)} \in L^1(\Omega), \tag{3.82}$$

$$\textit{the embedding of } \overset{\circ}{W}{}^{1,p}(\nu, \Omega) \textit{ into } L^p(\nu, \Omega) \textit{ is compact}, \tag{3.83}$$

$$\textit{the sequence } \{\widetilde{W}{}_0^{1,p}(\nu, \Omega_s)\} \textit{ is strongly connected with } \overset{\circ}{W}{}^{1,p}(\nu, \Omega), \tag{3.84}$$

$$\textit{for every } s \in \mathbb{N} \textit{ the set } \Omega \setminus \Omega_s \textit{ is closed}, \tag{3.85}$$

$$\textit{for every } x', x'' \in \Omega \textit{ we have } h(x', \cdot) = h(x'', \cdot). \tag{3.86}$$

Then there exist $\alpha_- \in [-\infty, 0]$ *and* $\alpha_+ \in [0, +\infty]$ *such that*

$$V = \{v \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega) : \alpha_- \leqslant v \leqslant \alpha_+ \ \textit{a.e. in} \ \Omega\} \tag{3.87}$$

and for every $s \in \mathbb{N}$,

$$V_s = \{v \in \widetilde{W}{}^{1,p}_0(\nu, \Omega_s) : \alpha_- \leqslant v \leqslant \alpha_+ \ \textit{a.e. in} \ \Omega_s\}. \tag{3.88}$$

Proof. We fix $x_0 \in \Omega$ and set $\gamma = h(x_0, \cdot)$. From (3.8) and (3.86) it follows that $\gamma \in C(\mathbb{R})$. Observe that $\gamma(0) \leqslant 0$. In fact, since $V \neq \emptyset$, by virtue of (3.9) there exists a function $z \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ such that $h(x, z(x)) \leqslant 0$ for a. e. $x \in \Omega$. Then owing to (3.86) and the definition of γ there exists a set $E \subset \Omega$ with measure zero such that

$$\forall x \in \Omega \setminus E, \qquad \gamma(z(x)) \leqslant 0. \tag{3.89}$$

Let $\varepsilon > 0$. Since $\gamma \in C(\mathbb{R})$, there exists $\delta > 0$ such that

$$\forall \eta \in \mathbb{R},\ |\eta| < \delta, \qquad |\gamma(\eta) - \gamma(0)| < \varepsilon. \tag{3.90}$$

Due to (3.82) and Lemma 3.8 $z \in \overset{\circ}{W}{}^{1,1}(\Omega)$. Then by assertion (i) of Lemma 3.7 we have $\operatorname{meas}\{|z| < \delta\} > 0$. Let $x \in \{|z| < \delta\} \setminus E$. Therefore, $|z(x)| < \delta$. This along with (3.89) and (3.90) implies that $\gamma(0) < \varepsilon$. Hence because of the arbitrariness of ε we get $\gamma(0) \leqslant 0$.

We set

$$\alpha_- = \inf\{\eta \in (-\infty, 0] : \forall \lambda \in [\eta, 0],\ \gamma(\lambda) \leqslant 0\},$$
$$\alpha_+ = \sup\{\eta \in [0, +\infty) : \forall \lambda \in [0, \eta],\ \gamma(\lambda) \leqslant 0\}.$$

Obviously, $\alpha_- \in [-\infty, 0]$ and $\alpha_+ \in [0, +\infty]$. Moreover, observe that owing to the definition of α_- and α_+ and the continuity of the function γ the following assertion holds true:

$$\text{if } \eta \in \mathbb{R} \text{ and } \alpha_- \leqslant \eta \leqslant \alpha_+, \text{ we have } \gamma(\eta) \leqslant 0. \tag{3.91}$$

Using this assertion, the definition of γ and (3.86) and (3.9), we establish that the next assertion holds true:

$$\text{if } v \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega) \text{ and } \alpha_- \leqslant v \leqslant \alpha_+ \text{ a.e in } \Omega, \text{ we have } v \in V. \tag{3.92}$$

Now let $v \in V$. Clearly, $v \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ and there exists a set $E' \subset \Omega$ with measure zero such that

$$\forall x \in \Omega \setminus E', \qquad \gamma(v(x)) \leqslant 0. \tag{3.93}$$

We set $A_+ = \{x \in \Omega : \gamma(v(x)) \leqslant 0,\ v(x) > \alpha_+\}$. Let us show that

$$\operatorname{meas} A_+ = 0. \tag{3.94}$$

If $\alpha_+ = +\infty$, we have $A_+ = \emptyset$ and therefore, equality (3.94) holds true. Suppose that $\alpha_+ \neq +\infty$. For every $m \in \mathbb{N}$ we set $A_+^{(m)} = \{x \in \Omega : \gamma(v(x)) \leqslant 0,\ v(x) > \alpha_+ + 1/m\}$. Obviously,

$$A_+ = \bigcup_{m=1}^{\infty} A_+^{(m)} \quad \text{and} \quad \forall m \in \mathbb{N},\quad A_+^{(m)} \subset A_+^{(m+1)}.$$

Hence

$$\lim_{m\to\infty} \operatorname{meas} A_+^{(m)} = \operatorname{meas} A_+. \tag{3.95}$$

Let $m \in \mathbb{N}$. Using the definition of α_+ and the continuity of the function γ, we find that there exist $\eta', \eta'' \in \mathbb{R}$ such that

$$\alpha_+ \leqslant \eta' < \eta'' \leqslant \alpha_+ + 1/m, \tag{3.96}$$

$$\forall \eta \in (\eta', \eta''),\ \ \gamma(\eta) > 0. \tag{3.97}$$

Assume that $\operatorname{meas} A_+^{(m)} > 0$. Then

$$\operatorname{meas}\{v > \alpha_+ + 1/m\} > 0. \tag{3.98}$$

Due to (3.82) and Lemma 3.8 $v \in \overset{\circ}{W}{}^{1,1}(\Omega)$. Therefore, using assertion (ii) of Lemma 3.7 and (3.96) and (3.98), we get $\operatorname{meas}\{\eta' < v < \eta''\} > 0$. Let $x \in \{\eta' < v < \eta''\} \setminus E'$. Clearly, $v(x) \in (\eta', \eta'')$ and then by (3.97) $\gamma(v(x)) > 0$. On the other hand, by (3.93) $\gamma(v(x)) \leqslant 0$. The contradiction obtained proves that for every $m \in \mathbb{N}$, $\operatorname{meas} A_+^{(m)} = 0$. This and (3.95) imply that equality (3.94) is valid.

From (3.93) and (3.94) we obtain that

$$v \leqslant \alpha_+ \ \text{ a.e. in } \ \Omega. \tag{3.99}$$

Next, we set $A_- = \{x \in \Omega : \gamma(v(x)) \leqslant 0,\ v(x) < \alpha_-\}$. If $\alpha_- = -\infty$, we have $A_- = \emptyset$ and therefore, $\operatorname{meas} A_- = 0$. Suppose that $\alpha_- \neq -\infty$. For every $m \in \mathbb{N}$ we set $A_-^{(m)} = \{x \in \Omega : \gamma(v(x)) \leqslant 0,\ v(x) < \alpha_- - 1/m\}$. Evidently,

$$\lim_{m\to\infty} \operatorname{meas} A_-^{(m)} = \operatorname{meas} A_-. \tag{3.100}$$

Let $m \in \mathbb{N}$. Using the definition of α_- and the continuity of the function γ, we establish that there exist $\eta_1, \eta_2 \in \mathbb{R}$ such that

$$\alpha_- - 1/m \leqslant \eta_1 < \eta_2 \leqslant \alpha_-, \tag{3.101}$$

$$\forall \eta \in (\eta_1, \eta_2),\ \ \gamma(\eta) > 0. \tag{3.102}$$

Assume that $\operatorname{meas} A_-^{(m)} > 0$. Then setting $w = -v$, we have $\operatorname{meas}\{w > -\alpha_- + 1/m\} > 0$. From this and (3.101) and assertion (ii) of Lemma 3.7 we deduce that $\operatorname{meas}\{-\eta_2 < w < -\eta_1\} > 0$. Hence $\operatorname{meas}\{\eta_1 < v < \eta_2\} > 0$. Let $x \in \{\eta_1 < v < \eta_2\} \setminus E'$. By (3.102) $\gamma(v(x)) > 0$ and by (3.93) $\gamma(v(x)) \leqslant 0$. The contradiction obtained proves that for every $m \in \mathbb{N}$, $\operatorname{meas} A_-^{(m)} = 0$. This and (3.100) imply that $\operatorname{meas} A_- = 0$. Owing to this equality and (3.93) we obtain that $v \geqslant \alpha_-$ a. e. in Ω. Then taking into account (3.99), we get $\alpha_- \leqslant v \leqslant \alpha_+$ a. e. in Ω.

Thus, the following assertion holds true: if $v \in V$, we have $v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and $\alpha_- \leqslant v \leqslant \alpha_+$ a. e. in Ω. From this assertion and (3.92) we infer (3.87).

Further, let $s \in \mathbb{N}$. Using (3.91), the definition of γ and (3.86) and (3.10), we establish that the next assertion holds true:

$$\text{if } v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) \text{ and } \alpha_- \leqslant v \leqslant \alpha_+ \text{ a. e. in } \Omega_s, \text{ we have } v \in V_s. \tag{3.103}$$

Now let $v \in V_s$. Clearly, $v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ and there exists a set $E_s \subset \Omega_s$ with measure zero such that

$$\forall x \in \Omega_s \setminus E_s, \quad \gamma(v(x)) \leqslant 0. \tag{3.104}$$

Moreover, taking into account conditions (3.82)-(3.85), from Lemma 3.10 we deduce that the following assertions hold true:

$$\text{if } 0 \leqslant \lambda_1 < \lambda_2 \leqslant \lambda \text{ and } \operatorname{meas}\{v > \lambda\} > 0, \text{ then } \operatorname{meas}\{\lambda_1 < v < \lambda_2\} > 0, \tag{3.105}$$

$$\text{if } \lambda \leqslant \lambda_1 < \lambda_2 \leqslant 0 \text{ and } \operatorname{meas}\{v < \lambda\} > 0, \text{ then } \operatorname{meas}\{\lambda_1 < v < \lambda_2\} > 0. \tag{3.106}$$

Using (3.104)-(3.106), by analogy with the above consideration for the set V we establish that $\alpha_- \leqslant v \leqslant \alpha_+$ a. e. in Ω_s. Thus, if $v \in V_s$, we have $v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ and $\alpha_- \leqslant v \leqslant \alpha_+$ a.e. in Ω_s. This and assertion (3.103) imply (3.88). □

THEOREM 3.12. *Suppose that* $(1/\nu)^{1/(p-1)} \in L^1(\Omega)$, *conditions* $(*_1)$, $(*_2)$ *and* $(*_4)$-$(*_6)$ *of Theorem 3.1 are satisfied and for every* $s \in \mathbb{N}$ *the set* $\Omega \setminus \Omega_s$ *is closed. Moreover, assume that the following conditions are satisfied:*

$$\textit{for every } x', x'' \in \Omega \textit{ we have } h(x',\cdot) = h(x'',\cdot), \tag{3.107}$$

$$\textit{there exists a function } w \in V \textit{ such that } \operatorname{meas}\{w \neq 0\} > 0. \tag{3.108}$$

Finally, suppose that for every $s \in \mathbb{N}$, u_s *is a function in* V_s *minimizing the functional* $J_s + G_s$ *on* V_s. *Then there exist an increasing sequence* $\{s_j\} \subset \mathbb{N}$ *and a function* $u \in V$ *such that assertions* (3.11)-(3.13) *hold true.*

Proof. By virtue of Lemma 3.11 there exist $\alpha_- \in [-\infty, 0]$ and $\alpha_+ \in [0, +\infty]$ such that

$$V = \{v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) : \alpha_- \leqslant v \leqslant \alpha_+ \text{ a. e. in } \Omega\} \tag{3.109}$$

and for every $s \in \mathbb{N}$,

$$V_s = \{v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) : \alpha_- \leqslant v \leqslant \alpha_+ \text{ a. e. in } \Omega_s\}. \tag{3.110}$$

Observe that owing to (3.108) and (3.109), $\alpha_- \neq \alpha_+$. Thus, we have $\alpha_- < \alpha_+$.

Clearly, the following cases are admissible: (i) $\alpha_- \neq -\infty$ and $\alpha_+ \neq +\infty$; (ii) $\alpha_- \neq -\infty$ and $\alpha_+ = +\infty$; (iii) $\alpha_- = -\infty$ and $\alpha_+ \neq +\infty$; (iv) $\alpha_- = -\infty$ and $\alpha_+ = +\infty$.

First we assume that $\alpha_- \neq -\infty$ and $\alpha_+ \neq +\infty$. Using (3.1), (3.4)-(3.6) and the fact that for every $s \in \mathbb{N}$ the function u_s minimizes the functional $J_s + G_s$ on V_s, we establish that the sequence of the norms $\|u_s\|_{1,p,\nu,s}$ is bounded. This and condition $(*_2)$ of Theorem 3.1 imply that there exists a bounded sequence $\{\tilde{u}_s\}$ in $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that

$$\forall s \in \mathbb{N}, \qquad \tilde{u}_s = u_s \quad \text{a.e. in} \quad \Omega_s. \tag{3.111}$$

For every $s \in \mathbb{N}$ we set $u_s^{(1)} = \max\{\tilde{u}_s, \alpha_-\}$. Clearly, $\{u_s^{(1)}\}$ is a bounded sequence in $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Using the inclusions $u_s \in V_s$, (3.110) and (3.111) and the definition of the functions $u_s^{(1)}$, we find that

$$\forall s \in \mathbb{N}, \qquad u_s^{(1)} \geqslant \alpha_- \ \text{ in } \ \Omega \ \text{ and } \ u_s^{(1)} = u_s \ \text{ a.e. in } \ \Omega_s. \tag{3.112}$$

Now for every $s \in \mathbb{N}$ we set $u_s^{(2)} = \min\{u_s^{(1)}, \alpha_+\}$. Obviously, $\{u_s^{(2)}\}$ is a bounded sequence in $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, and from the definition of the functions $u_s^{(2)}$, the inclusions $u_s \in V_s$ and (3.109), (3.110) and (3.112) it follows that $\{u_s^{(2)}\} \subset V$ and for every $s \in \mathbb{N}$, $u_s^{(2)} = u_s$ a.e. in Ω_s. These facts and condition $(*_1)$ of Theorem 3.1 imply that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $u \in V$ such that equality (3.12) holds true.

Using (3.12) and conditions $(*_5)$ and $(*_6)$ of Theorem 3.1, we get

$$\liminf_{j\to\infty} (J_{s_j} + G_{s_j})(u_{s_j}) \geqslant (J+G)(u). \tag{3.113}$$

Further, we fix $v \in V$. Let us show that

$$\limsup_{s\to\infty} (J_s + G_s)(u_s) \leqslant (J+G)(v). \tag{3.114}$$

We fix an arbitrary $\varepsilon > 0$. Since the function $\nu|\nabla v|^p$ is summable in Ω, there exists $\delta_1 > 0$ such that

$$\begin{cases} \text{for every measurable set } H \subset \Omega, \text{ meas} H \leqslant \delta_1, \text{ we have} \\ \int_H \nu|\nabla v|^p dx \leqslant \varepsilon. \end{cases} \tag{3.115}$$

Moreover, by virtue of condition $(*_4)$ of Theorem 3.1 there exists $\delta_2 > 0$ such that

$$\begin{cases} \text{for every measurable set } H \subset \Omega, \text{ meas} H \leqslant \delta_2, \text{ we have} \\ \limsup\limits_{s\to\infty} \int_{H\cap\Omega_s} \psi_s\, dx \leqslant \varepsilon. \end{cases} \tag{3.116}$$

We set $\delta = \min(\delta_1, \delta_2)$, fix a nonempty closed set $\tilde{H}$ in $\mathbb{R}^n$ such that $\tilde{H} \subset \Omega$ and

$$\text{meas}(\Omega \setminus \tilde{H}) < \delta \tag{3.117}$$

and take a function $\varphi \in C_0^\infty(\Omega)$ such that $0 \leqslant \varphi \leqslant 1$ in Ω and $\varphi = 1$ in $\tilde{H}$.

In view of condition $(*_5)$ of Theorem 3.1 there exists a sequence $v_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that

$$\lim_{s\to\infty} \|v_s - q_s v\|_{L^p(\nu,\Omega_s)} = 0, \tag{3.118}$$

$$\lim_{s\to\infty} J_s(v_s) = J(v). \tag{3.119}$$

For every $s \in \mathbb{N}$ we set

$$\mu_s = \left\{ \int_\Omega \left(\frac{1}{\nu}\right)^{1/(p-1)} dx \right\}^{(p-1)/2p} \{\|v_s - q_s v\|_{L^p(\nu,\Omega_s)} + s^{-1}\}^{1/2}.$$

Due to (3.118) we have

$$\lim_{s\to\infty} \mu_s = 0. \tag{3.120}$$

For every $s \in \mathbb{N}$ we set

$$v_s^{(1)} = \max\{v_s, q_s v - \mu_s q_s \varphi\} \quad \text{and} \quad E_s^{(1)} = \{v_s \leqslant q_s v - \mu_s q_s \varphi\}.$$

If $s \in \mathbb{N}$, we have $v_s^{(1)} \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ and

$$|v_s^{(1)} - q_s v| \leqslant |v_s - q_s v| \quad \text{in } \Omega_s. \tag{3.121}$$

Moreover, using (3.4), we establish that for every $s \in \mathbb{N}$,

$$J_s(v_s^{(1)}) \leqslant J_s(v_s) + 2\int_{E_s^{(1)}} \psi_s\, dx \\ + 2^{p-1} c_2 \int_{E_s^{(1)}} \nu |\nabla v|^p dx + 2^{p-1} \mu_s^p c_2 \int_\Omega \nu |\nabla \varphi|^p dx. \tag{3.122}$$

Next, for every $s \in \mathbb{N}$ we set

$$v_s^{(2)} = \min\{v_s^{(1)}, q_s v + \mu_s q_s \varphi\} \ \text{ and } \ E_s^{(2)} = \{v_s^{(1)} \geqslant q_s v + \mu_s q_s \varphi\}.$$

If $s \in \mathbb{N}$, we have $v_s^{(2)} \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ and

$$|v_s^{(2)} - q_s v| \leqslant |v_s^{(1)} - q_s v| \quad \text{in } \Omega_s. \tag{3.123}$$

Moreover, using (3.4), we obtain that for every $s \in \mathbb{N}$,

$$J_s(v_s^{(2)}) \leqslant J_s(v_s^{(1)}) + 2\int_{E_s^{(2)}} \psi_s\, dx \\ + 2^{p-1} c_2 \int_{E_s^{(2)}} \nu |\nabla v|^p dx + 2^{p-1} \mu_s^p c_2 \int_\Omega \nu |\nabla \varphi|^p dx. \tag{3.124}$$

Observe that owing to (3.118), (3.121) and (3.123)

$$\lim_{s\to\infty} \|v_s^{(2)} - q_s v\|_{L^p(\nu,\Omega_s)} = 0. \tag{3.125}$$

Besides, by virtue of the definition of the functions $v_s^{(1)}$ and $v_s^{(2)}$, the inclusion $v \in V$ and (3.109) for every $s \in \mathbb{N}$,

$$\alpha_- - \mu_s q_s \varphi \leqslant v_s^{(2)} \leqslant \alpha_+ + \mu_s q_s \varphi \quad \text{a.e. in } \Omega_s. \tag{3.126}$$

For every $s \in \mathbb{N}$ we set $E_s = E_s^{(1)} \cup E_s^{(2)}$. Clearly, for every $s \in \mathbb{N}$,

$$\operatorname{meas} E_s \leqslant \operatorname{meas}(\tilde{H} \cap E_s) + \operatorname{meas}(\Omega \setminus \tilde{H}), \tag{3.127}$$

$$\int_{E_s} \psi_s \, dx \leqslant \int_{\tilde{H} \cap E_s} \psi_s \, dx + \int_{(\Omega \setminus \tilde{H}) \cap \Omega_s} \psi_s \, dx. \tag{3.128}$$

Moreover, taking into account that $\varphi = 1$ in $\tilde{H}$, from the definition of the sets $E_s^{(1)}$ and $E_s^{(2)}$ and (3.121) we deduce that for every $s \in \mathbb{N}$,

$$\operatorname{meas}(\tilde{H} \cap E_s) \leqslant \mu_s. \tag{3.129}$$

This along with (3.120) and condition $(*_4)$ of Theorem 3.1 implies that

$$\lim_{s \to \infty} \int_{\tilde{H} \cap E_s} \psi_s \, dx = 0. \tag{3.130}$$

Using (3.116), (3.117), (3.128) and (3.130), we get

$$\limsup_{s \to \infty} \int_{E_s} \psi_s \, dx \leqslant \varepsilon. \tag{3.131}$$

Furthermore, by virtue of (3.115), (3.117), (3.120), (3.127) and (3.129) we have

$$\limsup_{s \to \infty} \int_{E_s} v |\nabla v|^p dx \leqslant \varepsilon. \tag{3.132}$$

From (3.119), (3.120), (3.122), (3.124), (3.131) and (3.132) we infer that

$$\limsup_{s \to \infty} J_s(v_s^{(2)}) \leqslant J(v) + (4 + 2^p c_2)\varepsilon. \tag{3.133}$$

Now we set

$$\tau_0 = \max\{\alpha_+, -\alpha_-\} \text{ and } \tau = 1 - 2\operatorname{sign}(\tau_0 - \alpha_+).$$

Since $\alpha_- < \alpha_+$, we have $\tau_0 > 0$. Besides, observe that $\tau = 1$ if $\alpha_+ \geqslant -\alpha_-$ and $\tau = -1$ if $\alpha_+ < -\alpha_-$.

For every $s \in \mathbb{N}$ we define

$$\beta_s = \frac{\tau_0 - \mu_s}{\tau_0 + \mu_s}.$$

For every $s \in \mathbb{N}$ we have $\beta_s < 1$ and

$$\frac{\mu_s}{1 - \beta_s} = \frac{1}{2}(\tau_0 + \mu_s). \tag{3.134}$$

Moreover, (3.120) implies that

$$\lim_{s\to\infty} \beta_s = 1. \tag{3.135}$$

For every $s \in \mathbb{N}$ we set

$$w_s = \beta_s v_s^{(2)} + \tau\mu_s q_s \varphi.$$

Obviously, if $s \in \mathbb{N}$, we have $w_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$. From (3.120), (3.125) and (3.135) it follows that

$$\lim_{s\to\infty} \|w_s - q_s v\|_{L^p(\nu,\Omega_s)} = 0. \tag{3.136}$$

Next, by virtue of (3.135) there exists $s' \in \mathbb{N}$ such that

$$\forall s \in \mathbb{N},\, s \geqslant s', \qquad \beta_s \in (0,1). \tag{3.137}$$

Using (3.3) and (3.137), we obtain that for every $s \in \mathbb{N},\ s \geqslant s'$,

$$J_s(w_s) \leqslant \beta_s J_s(v_s^{(2)}) + (1-\beta_s) J_s\left(\frac{\tau\mu_s}{1-\beta_s} q_s \varphi\right). \tag{3.138}$$

Furthermore, in view of (3.1), (3.4), (3.120), (3.134) and (3.135)

$$\limsup_{s\to\infty} (1-\beta_s) J_s\left(\frac{\tau\mu_s}{1-\beta_s} q_s \varphi\right) \leqslant 0.$$

This along with (3.133), (3.135), (3.137) and (3.138) implies that

$$\limsup_{s\to\infty} J_s(w_s) \leqslant J(v) + (4 + 2^p c_2)\varepsilon. \tag{3.139}$$

Besides, by virtue of (3.136) and condition $(*_6)$ of Theorem 3.1 we have

$$\lim_{s\to\infty} G_s(w_s) = G(v). \tag{3.140}$$

Now observe that owing to (3.126), (3.137) and (3.110) for every $s \in \mathbb{N},\ s \geqslant s'$, we have $w_s \in V_s$. Then taking into account that for every $s \in \mathbb{N}$ the function u_s minimizes the functional $J_s + G_s$ on V_s, for every $s \in \mathbb{N},\ s \geqslant s'$, we get $(J_s + G_s)(u_s) \leqslant (J_s + G_s)(w_s)$. From this along with relations (3.139) and (3.140) and the arbitrariness of ε we conclude that inequality (3.114) holds true. This inequality and (3.113) imply that assertions (3.11) and (3.13) hold true.

Thus, in the case where $\alpha_- \neq -\infty$ and $\alpha_+ \neq +\infty$ the conclusion of the theorem is valid.

In cases (ii) $\alpha_- \neq -\infty$ and $\alpha_+ = +\infty$ and (iii) $\alpha_- = -\infty$ and $\alpha_+ \neq +\infty$ the conclusion of the theorem is valid as well. Proving this fact, we obtain the inclusion $u \in V$ and (3.12) arguing by analogy with the corresponding part of the above consideration for case (i) $\alpha_- \neq -\infty$ and $\alpha_+ \neq +\infty$, and after that we establish the validity of assertions (3.11) and (3.13) by analogy with the corresponding part of the proof of Theorem 3.1.

Finally, consider case (iv) $\alpha_- = -\infty$ and $\alpha_+ = +\infty$. In this case due to (3.109) $V = \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and due to (3.110) for every $s \in \mathbb{N}$, $V_s = \widetilde{W}_0^{1,p}(\nu,\Omega_s)$. Moreover, observe that by virtue of conditions $(*_5)$ and $(*_6)$ of Theorem 3.1 the sequence $\{J_s + G_s\}$ Γ-converges to the functional $J+G$. Besides, in view of the fact that for every $s \in \mathbb{N}$ the function u_s minimizes the functional $J_s + G_s$ on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ and (3.1), (3.4)-(3.6) we have: the sequence of the norms $\|u_s\|_{1,p,\nu,s}$ is bounded. Therefore, taking into account conditions $(*_1)$ and $(*_2)$ of Theorem 3.1 and applying Theorem 2.5, we establish that the conclusion of the given theorem is valid. □

REMARK 3.13. If condition (3.108) is not satisfied, and all other conditions of Theorem 3.12 are satisfied, for every $s \in \mathbb{N}$ we have $u_s = 0$ a. e in Ω_s, and for the function $u : \Omega \to \mathbb{R}$ such that $u = 0$ in Ω we have $u \in V$, the function u minimizes the functional $J+G$ on V and $\|u_s - q_s u\|_{L^p(\nu,\Omega_s)} \to 0$. However, generally speaking there is no any increasing sequence $\{s_j\} \subset \mathbb{N}$ such that $(J_{s_j} + G_{s_j})(u_{s_j}) \to (J+G)(u)$. The corresponding example will be considered in the next section.

4. Comments and examples

In this section we make comments and give examples concerning the realization of conditions under which the main results of Section 3 were obtained.

As far as condition $(*_1)$ of Theorem 3.1 is concerned the following propositions hold true.

PROPOSITION 4.1. *Let $p < n$, $t \geqslant 1/(p-1)$, $t > n/p$, $t_1 > nt/(tp-n)$, and let $1/\nu \in L^t(\Omega)$ and $\nu \in L^{t_1}(\Omega)$. Then the embedding of $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ into $L^p(\nu,\Omega)$ is compact.*

PROPOSITION 4.2. *Let the function ν be the restriction on Ω of a function from the Muckenhoupt class A_p. Then the embedding of $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ into $L^p(\nu,\Omega)$ is compact.*

The detailed proofs of these propositions one can find for instance in [27].

Observe that under conditions on the weighted function of such a kind as in Proposition 4.1 the embeddings of weighted Sobolev spaces into nonweighted and weighted Lebesgue spaces were considered for instance in [5], [18], [25] and [30].

Concerning the definition of the Muckenhoupt class A_p see [19]. For example functions of the form $w(x) = |x|^\beta$, $x \in \mathbb{R}^n \backslash \{0\}$, where $\beta \in (-n, n(p-1))$, belong to this class.

Condition $(*_2)$ of Theorem 3.1 is satisfied for instance in the case of special strongly perforated structure of domains Ω_s and certain behaviour of the function ν in neighbourhoods of holes (see details in [27]; we only note that a power weight is admissible if the distance between some neighbourhoods of the holes and the point of the degeneration or singularity of the weight may go to zero sufficiently slowly).

Next, let us state a proposition concerning condition $(*_3)$ of Theorem 3.1.

PROPOSITION 4.3. *Suppose that* $c > 0$ *and for every open set* H *in* $\mathbb{R}^n$ *such that* $H \subset \Omega$ *we have*

$$\liminf_{s\to\infty} \operatorname{meas}(H \cap \Omega_s) \geqslant c \operatorname{meas} H. \tag{4.1}$$

Then for every increasing sequence $\{m_j\} \subset \mathbb{N}$*,*

$$\operatorname{meas}(\Omega \setminus \bigcup_j \Omega_{m_j}) = 0. \tag{4.2}$$

Proof. First of all we observe that by virtue of the condition of the proposition inequality (4.1) holds true for every measurable set H in $\mathbb{R}^n$ such that $H \subset \Omega$. We also note that (4.1) implies that $c \leqslant 1$.

For every $s \in \mathbb{N}$ we set $\Phi_s = \Omega \setminus \Omega_s$.

Now let $\{m_j\} \subset \mathbb{N}$ be an arbitrary increasing sequence. We fix $\varepsilon > 0$. In view of the condition of the proposition there exists $s^{(1)} \in \mathbb{N}$ such that for every $s \in \mathbb{N}$, $s \geqslant s^{(1)}$, $\operatorname{meas}\Omega_s \geqslant c \operatorname{meas}\Omega - \varepsilon/2$. Then fixing $j_1 \in \mathbb{N}$ such that $m_{j_1} \geqslant s^{(1)}$, we get

$$\operatorname{meas}\Phi_{m_{j_1}} \leqslant (1-c)\operatorname{meas}\Omega + \frac{\varepsilon}{2}. \tag{4.3}$$

Applying (4.1) for the set $\Phi_{m_{j_1}}$, we establish that there exists $s^{(2)} \in \mathbb{N}$ such that for every $s \in \mathbb{N}$, $s \geqslant s^{(2)}$,

$$\operatorname{meas}(\Phi_{m_{j_1}} \cap \Omega_s) \geqslant c \operatorname{meas}\Phi_{m_{j_1}} - \frac{\varepsilon}{4}. \tag{4.4}$$

We fix $j_2 \in \mathbb{N}$ such that $j_2 > j_1$ and $m_{j_2} \geqslant s^{(2)}$. Clearly,

$$\operatorname{meas}(\Phi_{m_{j_1}} \cap \Phi_{m_{j_2}}) = \operatorname{meas}\Phi_{m_{j_1}} - \operatorname{meas}(\Phi_{m_{j_1}} \setminus \Phi_{m_{j_2}}),$$
$$\Phi_{m_{j_1}} \setminus \Phi_{m_{j_2}} = \Phi_{m_{j_1}} \cap \Omega_{m_{j_2}}.$$

These equalities along with (4.3) and (4.4) imply that

$$\operatorname{meas}(\Phi_{m_{j_1}} \cap \Phi_{m_{j_2}}) \leqslant (1-c)^2 \operatorname{meas}\Omega + \left(\frac{1}{2} + \frac{1}{4}\right)\varepsilon. \tag{4.5}$$

After that applying (4.1) for the set $\Phi_{m_{j_1}} \cap \Phi_{m_{j_2}}$ and using (4.5), we find $j_3 \in \mathbb{N}$, $j_3 > j_2$, such that

$$\operatorname{meas}(\Phi_{m_{j_1}} \cap \Phi_{m_{j_2}} \cap \Phi_{m_{j_3}}) \leqslant (1-c)^3 \operatorname{meas}\Omega + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)\varepsilon.$$

Proceeding the selection of numbers $j_r \in \mathbb{N}$, $r = 4, 5, \ldots$, by the described way and taking into account that $1 - c \in [0, 1)$, for some $k \in \mathbb{N}$ we obtain $j_1 < j_2 < \ldots < j_k$,

$$\operatorname{meas}\left(\bigcap_{r=1}^{k} \Phi_{m_{j_r}}\right) \leqslant (1-c)^k \operatorname{meas}\Omega + \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^k}\right)\varepsilon$$

and $(1-c)^k \operatorname{meas}\Omega \leqslant \varepsilon$. Therefore, $\operatorname{meas}\big(\bigcap_j \Phi_{m_j}\big) = 0$. This means that equality (4.2) holds true. $\square$

We note that in the case of perforated structure of the domains Ω_s which was considered in [27] the condition of Proposition 4.3 is satisfied and consequently, condition $(*_3)$ of Theorem 3.1 is satisfied.

Further, let us give some results concerning condition $(*_4)$ of Theorem 3.1.

PROPOSITION 4.4. *Let $\psi \in L^1(\Omega)$ and $\|\psi_s - \psi\|_{L^1(\Omega_s)} \to 0$. Then condition $(*_4)$ of Theorem 3.1 is satisfied.*

PROPOSITION 4.5. *Let $r > 1$ and $M > 0$, and let for every $s \in \mathbb{N}$, $\psi_s \in L^r(\Omega_s)$ and $\|\psi_s\|_{L^r(\Omega_s)} \leqslant M$. Then condition $(*_4)$ of Theorem 3.1 is satisfied.*

The proofs of these propositions are simple.

Observe that the latter proposition is a particular case of the following result.

PROPOSITION 4.6. *Let $F : [0,+\infty) \to \mathbb{R}$ be a nonnegative and nondecreasing function such that $F(\eta) \to +\infty$ as $\eta \to +\infty$. Let $M > 0$. Suppose that for every $s \in \mathbb{N}$, $F(\psi_s)\psi_s \in L^1(\Omega_s)$ and $\|F(\psi_s)\psi_s\|_{L^1(\Omega_s)} \leqslant M$. Then condition $(*_4)$ of Theorem 3.1 is satisfied.*

Proof. Let $\varepsilon > 0$. We fix $\lambda > 0$ such that

$$F(\lambda) \geqslant 2M/\varepsilon \tag{4.6}$$

and set

$$\delta = \varepsilon/(2\lambda). \tag{4.7}$$

Let $E \subset \Omega$ be an arbitrary measurable set such that

$$\operatorname{meas} E \leqslant \delta. \tag{4.8}$$

Now we fix $s \in \mathbb{N}$ and set $H'_s = E \cap \{\psi_s \leqslant \lambda\}$, $H''_s = E \cap \{\psi_s > \lambda\}$. Clearly,

$$\int_{E \bigcap \Omega_s} \psi_s \, dx = \int_{H'_s} \psi_s \, dx + \int_{H''_s} \psi_s \, dx, \tag{4.9}$$

$$\int_{H'_s} \psi_s \, dx \leqslant \lambda \operatorname{meas} E. \tag{4.10}$$

Assume that $H''_s \neq \emptyset$. Since the function F is nondecreasing, we have

$$\psi_s \leqslant \frac{1}{F(\lambda)} F(\psi_s)\psi_s \quad \text{in } H''_s.$$

Therefore,

$$\int_{H''_s} \psi_s \, dx \leqslant \frac{1}{F(\lambda)} \int_{\Omega_s} F(\psi_s)\psi_s \, dx. \tag{4.11}$$

Obviously, this inequality also holds true if $H_s'' = \emptyset$. By virtue of (4.9)-(4.11)

$$\int_{E\bigcap\Omega_s} \psi_s\,dx \leqslant \lambda\,\text{meas}\,E + \frac{1}{F(\lambda)}\|F(\psi_s)\psi_s\|_{L^1(\Omega_s)}.$$

Hence taking into account that $\|F(\psi_s)\psi_s\|_{L^1(\Omega_s)} \leqslant M$ and using (4.6)-(4.8), we get

$$\int_{E\bigcap\Omega_s} \psi_s\,dx \leqslant \varepsilon.$$

Thus, we conclude that condition $(*_4)$ of Theorem 3.1 is satisfied. □

COROLLARY 4.7. *Let $M > 0$, and let for every $s \in \mathbb{N}$, $\psi_s \ln(1+\psi_s) \in L^1(\Omega_s)$ and $\|\psi_s \ln(1+\psi_s)\|_{L^1(\Omega_s)} \leqslant M$. Then condition $(*_4)$ of Theorem 3.1 is satisfied.*

For the proof of this result it suffices to apply Proposition 4.6 to the function $F\colon [0,+\infty) \to \mathbb{R}$ defined by $F(\eta) = \ln(1+\eta)$, $\eta \in [0,+\infty)$.

Now we state a useful criterion for condition $(*_4)$ of Theorem 3.1.

Let $\bar{\psi} : [0,+\infty) \to \mathbb{R}$ be the function such that for every $\eta \in [0,+\infty)$,

$$\bar{\psi}(\eta) = \limsup_{s\to\infty} \int_{\{\psi_s>\eta\}} \psi_s\,dx.$$

PROPOSITION 4.8. *Condition $(*_4)$ of Theorem 3.1 is satisfied if and only if*

$$\bar{\psi}(\eta) \to 0 \quad as \quad \eta \to +\infty. \tag{4.12}$$

Proof. By virtue of condition (3.1) there exists $M > 0$ such that for every $s \in \mathbb{N}$, $\|\psi_s\|_{L^1(\Omega_s)} \leqslant M$. Using this fact, we obtain that for every $\eta \in (0,+\infty)$ and $s \in \mathbb{N}$,

$$\text{meas}\,\{\psi_s > \eta\} \leqslant M/\eta\,. \tag{4.13}$$

Let condition $(*_4)$ of Theorem 3.1 be satisfied. Suppose that assertion (4.12) does not hold true. Then there exist $\varepsilon > 0$ and a sequence $\{\eta_k\} \subset (0,+\infty)$ such that for every $k \in \mathbb{N}$, $\eta_k \geqslant k$ and $\bar{\psi}(\eta_k) \geqslant \varepsilon$. Therefore, there exists an increasing sequence $\{s_k\} \subset \mathbb{N}$ such that for every $k \in \mathbb{N}$,

$$\int_{\{\psi_{s_k}>\eta_k\}} \psi_{s_k}\,dx \geqslant \frac{\varepsilon}{2}\,. \tag{4.14}$$

Since by assumption condition $(*_4)$ of Theorem 3.1 is satisfied, there exists $\delta > 0$ such that

$$\begin{cases} \text{for every measurable set } E \subset \Omega,\ \text{meas}\,E \leqslant \delta, \text{ we have} \\ \limsup_{s\to\infty} \int_{E\bigcap\Omega_s} \psi_s\,dx \leqslant \varepsilon/8\,. \end{cases} \tag{4.15}$$

Moreover, since $\eta_k \to +\infty$, there exists an increasing sequence $\{k_i\} \subset \mathbb{N}$ such that

$$\forall i \in \mathbb{N},\ \eta_{k_i} \geqslant 2^i M/\delta\,. \tag{4.16}$$

For every $i \in \mathbb{N}$ we set $H_i = \{\psi_{s_{k_i}} > \eta_{k_i}\}$. From (4.13) and (4.16) it follows that

$$\forall i \in \mathbb{N},\ \operatorname{meas} H_i \leqslant \delta/2^i. \tag{4.17}$$

We set $H = \bigcup_{i=1}^{\infty} H_i$. By virtue of (4.17) we have $\operatorname{meas} H \leqslant \delta$. Due to this and (4.15) there exists $s' \in \mathbb{N}$ such that

$$\forall s \in \mathbb{N},\ s \geqslant s',\ \int_{H\cap\Omega_s} \psi_s\, dx \leqslant \frac{\varepsilon}{4}. \tag{4.18}$$

Fixing $i \in \mathbb{N}$ such that $s_{k_i} \geqslant s'$, from (4.18) we get

$$\int_{H_i} \psi_{s_{k_i}}\, dx \leqslant \frac{\varepsilon}{4}.$$

However, this inequality contradicts (4.14). The contradiction obtained proves that assertion (4.12) holds true.

Conversely, let assertion (4.12) hold true. Then taking an arbitrary $\varepsilon > 0$, we fix $\eta > 0$ such that

$$\bar{\psi}(\eta) \leqslant \varepsilon/4 \tag{4.19}$$

and set

$$\delta = \varepsilon/(2\eta). \tag{4.20}$$

In view of (4.19) and the definition of the function $\bar{\psi}$ there exists $s'' \in \mathbb{N}$ such that

$$\forall s \in \mathbb{N},\ s \geqslant s'',\ \int_{\{\psi_s > \eta\}} \psi_s\, dx \leqslant \frac{\varepsilon}{2}. \tag{4.21}$$

Let $E \subset \Omega$ be an arbitrary measurable set such that $\operatorname{meas} E \leqslant \delta$. Then fixing $s \in \mathbb{N}$, $s \geqslant s''$, with the use of (4.20) and (4.21) we obtain

$$\int_{E\cap\Omega_s} \psi_s\, dx = \int_{E\cap\{\psi_s \leqslant \eta\}} \psi_s\, dx + \int_{E\cap\{\psi_s > \eta\}} \psi_s\, dx \leqslant \eta \operatorname{meas} E + \int_{\{\psi_s > \eta\}} \psi_s\, dx \leqslant \varepsilon.$$

Hence

$$\limsup_{s\to\infty} \int_{E\cap\Omega_s} \psi_s\, dx \leqslant \varepsilon.$$

Thus, we conclude that condition $(*_4)$ of Theorem 3.1 is satisfied. □

We utilize Proposition 4.8 for the justification of the immediate example. In this connection we introduce the following notation: for every $s \in \mathbb{N}$,

$$Z_s = \{z \in \mathbb{R}^n : sz_i \in \mathbb{Z},\ \ i = 1, \dots, n\};$$

for every $z \in \mathbb{R}^n$ and $s \in \mathbb{N}$,

$$Q_s(z) = \{x \in \mathbb{R}^n : |x_i - z_i| < \frac{1}{2s},\ i = 1, \dots, n\}.$$

Observe that

$$\forall s \in \mathbb{N}, \ \ \mathbb{R}^n = \bigcup_{z \in Z_s} \overline{Q_s(z)}, \tag{4.22}$$

$$\text{if } \ s \in \mathbb{N}, z, z' \in Z_s \ \text{ and } \ z \neq z', \ \text{ we have } \ Q_s(z) \cap Q_s(z') = \emptyset. \tag{4.23}$$

EXAMPLE 4.9. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative 1-periodic function such that $\psi|_{Q_1(0)} \in L^1(Q_1(0))$. Suppose that for every $s \in \mathbb{N}$ the function ψ_s is defined on Ω_s by

$$\psi_s(x) = \psi(sx), \quad x \in \Omega_s.$$

Then condition (3.1) and condition $(*_4)$ of Theorem 3.1 are satisfied.

In fact, let R be a positive number such that for every $x \in \Omega$, $|x| \leqslant R$. We denote by $\hat{c}$ the measure of the open ball with center at zero and radius $R+n$. Finally, we set $\hat{\psi} = \psi|_{Q_1(0)}$. Now we fix $s \in \mathbb{N}$ and set

$$Z_{0,s} = \{z \in Z_s : Q_s(z) \cap \Omega \neq \emptyset\}, \qquad Q^{(s)} = \bigcup_{z \in Z_{0,s}} \overline{Q_s(z)}\,.$$

Clearly,

$$\operatorname{meas} Q^{(s)} \leqslant \hat{c}. \tag{4.24}$$

Let $\tilde{\psi}_s : Q^{(s)} \to \mathbb{R}$ be the function such that for every $x \in Q^{(s)}$, $\tilde{\psi}_s(x) = \psi(sx)$. Taking into account the nonnegativity and periodicity of ψ and the summability of $\hat{\psi}$ and using (4.22)-(4.24), we establish that $\psi_s \in L^1(\Omega_s)$ and

$$\|\psi_s\|_{L^1(\Omega_s)} \leqslant \int_{Q^{(s)}} \tilde{\psi}_s \, dx \leqslant \hat{c} \int_{Q_1(0)} \hat{\psi} \, dx. \tag{4.25}$$

Next, let $\eta \in [0, +\infty)$. Due to the facts which we just mentioned

$$\int_{\{\psi_s > \eta\}} \psi_s \, dx \leqslant \int_{\{\tilde{\psi}_s > \eta\}} \tilde{\psi}_s \, dx = \sum_{z \in Z_{0,s}} \int_{Q_s(z) \cap \{\tilde{\psi}_s > \eta\}} \tilde{\psi}_s \, dx \leqslant \hat{c} \int_{\{\hat{\psi} > \eta\}} \hat{\psi} \, dx. \tag{4.26}$$

In view of (4.25) condition (3.1) is satisfied and by virtue of (4.26) $\bar{\psi}(\eta) \to 0$ as $\eta \to +\infty$. The latter assertion and Proposition 4.8 imply that condition $(*_4)$ of Theorem 3.1 is satisfied.

As far as condition $(*_5)$ of Theorem 3.1 is concerned we note the following. The Γ-convergence of the sequence $\{J_s\}$ to a functional $J\colon \overset{\circ}{W}{}^{1,p}(\nu, \Omega) \to \mathbb{R}$ holds true for instance in the case of certain periodicity of both the integrands $f_s(x, \xi)$ with respect to the spatial variable x and the perforated structure of the domains Ω_s. We add that in this case J is an integral functional, and for its integrand there is an effective representation. The corresponding results will be given in a forthcoming publication of the authors. Moreover, we remark that in the general case theorems on the selection from the sequence $\{J_s\}$ of a subsequence Γ-convergent to an integral functional defined on $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ are given in [26], [31] and [32].

Next, consider an example where all the assumptions made in Section 3 on the functionals G_s are realized and condition $(*_6)$ of Theorem 3.1 is satisfied.

EXAMPLE 4.10. Let $c', c'' > 0$, $\psi \in L^1(\Omega)$, $\psi \geqslant 0$ in Ω, and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheódory function such that for almost every $x \in \Omega$ and every $\eta \in \mathbb{R}$,

$$c'\nu(x)|\eta|^p - \psi(x) \leqslant g(x,\eta) \leqslant c''\nu(x)|\eta|^p + \psi(x). \tag{4.27}$$

Suppose that for every $s \in \mathbb{N}$ the functional G_s is defined on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ as follows:

$$G_s(u) = \int_{\Omega_s} g(x,u)dx, \quad u \in \widetilde{W}_0^{1,p}(\nu,\Omega_s).$$

Using (4.27), it is easy to verify that condition (3.5) is satisfied and condition (3.6) is fulfilled with $c_3 = c'$ and $c_4 = \|\psi\|_{L^1(\Omega)} + 1$. Furthermore, if the embedding of $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ into $L^p(\nu,\Omega)$ is compact and the sequence of the spaces $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ is strongly connected with the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, for every $s \in \mathbb{N}$, G_s is a weakly continuous functional on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$.

Using (4.27) and Egoroff's theorem, we establish the following fact:

$$\begin{cases} \text{for evary } v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \text{ and every sequence } v_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) \text{ such that} \\ \|v_s - q_s v\|_{L^p(\nu,\Omega_s)} \to 0 \text{ we have } G_s(v_s) - G_s(q_s v) \to 0. \end{cases} \tag{4.28}$$

Now suppose that the next condition is satisfied:

$$\begin{cases} \text{there exists a bounded measurable function } b \text{ on } \Omega \text{ such that} \\ \text{for every open cube } Q \subset \Omega,\ \lim_{s\to\infty} \operatorname{meas}(Q \cap \Omega_s) = \int_Q b\,dx. \end{cases} \tag{4.29}$$

We define the functional $G : \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \to \mathbb{R}$ by

$$G(v) = \int_\Omega b\,g(x,v)dx, \quad v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega).$$

It is not difficult to see that in view of condition (4.29) for every function $v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, $G_s(q_s v) \to G(v)$. This and (4.28) imply that condition $(*_6)$ of Theorem 3.1 is satisfied.

It remains to note that condition (4.29) is satisfied for instance in the case where the domains Ω_s have a periodic perforated structure.

Further, we pass to the consideration of condition $(*_7)$ of Theorem 3.1. An example of the realization of this condition is actually given in Remark 3.5. Evidently, condition $(*_7)$ of Theorem 3.1 is satisfied if for almost every $x \in \Omega$ the function $h(x,\cdot)$ is nonincreasing in $\mathbb{R}$.

EXAMPLE 4.11. Let $w \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, H be a measurable set in $\mathbb{R}^n$ such that $H \subset \Omega$ and $\operatorname{meas} H > 0$, and let the function h be defined on $\Omega \times \mathbb{R}$ by

$$h(x,\eta) = \begin{cases} -\eta + w(x) & \text{if } \ x \in H, \\ 0 & \text{if } \ x \in \Omega \setminus H. \end{cases}$$

Clearly, condition (3.8) and condition $(*_7)$ of Theorem 3.1 are satisfied, and the set V has the following representation: $V = \{v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) : v \geqslant w \text{ a.e. in } H\}$. Moreover, if $s \in \mathbb{N}$ and $\operatorname{meas}(H \cap \Omega_s) > 0$, we have $V_s = \{v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s) : v \geqslant q_s w \text{ a.e. in } H \cap \Omega_s\}$.

EXAMPLE 4.12. Let $\mu : \Omega \to \mathbb{R}$ be a function such that $\sqrt{2/e} \leqslant \mu \leqslant 1$ in Ω, and suppose that the function h satisfies the next conditions: if $(x,\eta) \in \Omega \times \mathbb{R}$ and $\eta < -3\pi/2$, $h(x,\eta) > 0$; if $(x,\eta) \in \Omega \times \mathbb{R}$ and $\eta \geqslant -3\pi/2$, $h(x,\eta) = \mu(x)\sin\eta - e^{\eta}$. It is easy to verify that for every $x \in \Omega$ the function $h(x,\cdot)$ is not nonincreasing in $\mathbb{R}$. However, condition $(*_7)$ of Theorem 3.1 is satisfied.

Now we give an example demonstrating the significance of condition $(*_3)$ of Theorem 3.1.

EXAMPLE 4.13. Let us suppose that $\nu = 1$ in Ω. Then $L^p(\nu,\Omega) = L^p(\Omega)$ and $\overset{\circ}{W}{}^{1,p}(\nu,\Omega) = \overset{\circ}{W}{}^{1,p}(\Omega)$. Therefore, condition $(*_1)$ of Theorem 3.1 is satisfied.

Let B be an open ball of $\mathbb{R}^n$ such that $\overline{B} \subset \Omega$. Owing to known extension results for Sobolev spaces (see for instance [17, Th.7.25]) there exists a linear continuous operator $l : W^{1,p}(\Omega \setminus \overline{B}) \to W^{1,p}(\Omega)$ such that for every $u \in W^{1,p}(\Omega \setminus \overline{B})$, $lu = u$ in $\Omega \setminus \overline{B}$.

Suppose that for every $s \in \mathbb{N}$, $\Omega_s = \Omega \setminus \overline{B}$. Obviously, for every $s \in \mathbb{N}$,

$$\widetilde{W}_0^{1,p}(\nu,\Omega_s) \subset W^{1,p}(\Omega \setminus \overline{B}).$$

Moreover, if $s \in \mathbb{N}$ and $v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$, we have $lv \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. For every $s \in \mathbb{N}$ we define the operator $l_s : \widetilde{W}_0^{1,p}(\nu,\Omega_s) \to \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ by $l_s v = lv$, $v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$. It is easy to see that for every $s \in \mathbb{N}$ the operator l_s is linear and continuous. Besides, the sequence of the norms $\|l_s\|$ is bounded. Finally, for every $s \in \mathbb{N}$ and $u \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ we have $q_s(l_s u) = u$.

Thus, we conclude that the sequence of the spaces $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ is strongly connected with the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. This means that condition $(*_2)$ of Theorem 3.1 is satisfied.

Evidently, $B \subset \Omega \setminus \bigcup_s \Omega_s$. This implies that $\operatorname{meas}(\Omega \setminus \bigcup_s \Omega_s) > 0$. Therefore, condition $(*_3)$ of Theorem 3.1 is not satisfied.

Next, let $\varphi \in C_0^\infty(\Omega)$, $-1 \leqslant \varphi \leqslant 0$ in Ω and $\varphi = -1$ in B. Suppose that $c_1 = 2^{1-p}$, $c_2 = 2^{p-1}$ and for every $s \in \mathbb{N}$ the function ψ_s be defined on Ω_s by $\psi_s(x) = 2^{p-1}|\nabla\varphi(x)|^p$, $x \in \Omega_s$. Clearly, condition (3.1) and condition $(*_4)$ of Theorem 3.1 are satisfied. Suppose that for every $s \in \mathbb{N}$ the function f_s is defined on $\Omega_s \times \mathbb{R}^n$ by

$$f_s(x,\xi) = |\xi - \nabla\varphi(x)|^p, \ \ (x,\xi) \in \Omega_s \times \mathbb{R}^n.$$

Obviously, conditions (3.2)-(3.4) are satisfied.

Let $\chi : \Omega \to \mathbb{R}$ be the characteristic function of $\Omega \setminus \overline{B}$, and let $J : \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \to \mathbb{R}$ be the functional such that for every $v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$,

$$J(v) = \int_\Omega \chi |\nabla v - \nabla \varphi|^p dx.$$

Observe that

$$\text{for every } v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \text{ and } s \in \mathbb{N},\ J_s(q_s v) = J(v). \tag{4.30}$$

Using this fact, we establish that

$$\text{for every } s \in \mathbb{N} \text{ and } v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s),\ J_s(v) = J(lv). \tag{4.31}$$

Let us show that the sequence $\{J_s\}$ Γ-converges to the functional J. We fix a function $v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and a sequence $v_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that

$$\lim_{s\to\infty} \|v_s - q_s v\|_{L^p(\nu,\Omega_s)} = 0. \tag{4.32}$$

Define $a = \liminf_{s\to\infty} J_s(v_s)$. Clearly, $a \in [0,+\infty]$ and there exists an increasing sequence $\{s_k\} \subset \mathbb{N}$ such that $J_{s_k}(v_{s_k}) \to a$. This and (4.31) imply that

$$J(lv_{s_k}) \to a. \tag{4.33}$$

Suppose that $a \neq +\infty$. Then in view of (4.32), (4.33) and the properties of the operator l the sequence $\{lv_{s_k}\}$ is bounded in $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Therefore, there exist an increasing sequence $\{m_j\} \subset \{s_k\}$ and a function $w \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that

$$lv_{m_j} \to w \quad \text{weakly in } \overset{\circ}{W}{}^{1,p}(\nu,\Omega). \tag{4.34}$$

Hence $\liminf_{j\to\infty} J(lv_{m_j}) \geqslant J(w)$. From this and (4.33) we get

$$a \geqslant J(w). \tag{4.35}$$

Note that

$$J(w) = J(v). \tag{4.36}$$

In fact, since the embedding of $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ into $L^p(\nu,\Omega)$ is compact, from (4.34) we obtain that $lv_{m_j} \to w$ strongly in $L^p(\Omega)$. Then $v_{m_j} \to w|_{\Omega\setminus\overline{B}}$ strongly in $L^p(\Omega\setminus\overline{B})$. This and (4.32) imply that $w = v$ a.e. in $\Omega\setminus\overline{B}$. Hence $\nabla w = \nabla v$ a.e. in $\Omega\setminus\overline{B}$. Therefore, equality (4.36) holds true. Then due to (4.35) $a \geqslant J(v)$. Obviously, this inequality also holds true in the case $a = +\infty$. Thus, we get $\liminf_{s\to\infty} J_s(v_s) \geqslant J(v)$. The result obtained and assertion (4.30) allow us to conclude that the sequence $\{J_s\}$ Γ-converges to the functional J. This means that condition $(*_5)$ of Theorem 3.1 is satisfied.

Next, we observe that for every open cube $Q \subset \Omega$ and $s \in \mathbb{N}$,

$$\operatorname{meas}(Q \cap \Omega_s) = \int_Q \chi\, dx. \tag{4.37}$$

Suppose that for every $s \in \mathbb{N}$ the functional G_s is defined on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ by

$$G_s(v) = \int_{\Omega\setminus\overline{B}} |v-\varphi|^p dx, \qquad v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s),$$

and let $G : \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \to \mathbb{R}$ be the functional such that for every $v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$,

$$G(v) = \int_\Omega \chi |v-\varphi|^p dx.$$

From the considerations given in Example 4.10 and (4.37) we deduce that for every $s \in \mathbb{N}$, G_s is a weakly continuous functional on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$, and conditions (3.5), (3.6) and condition $(*_6)$ of Theorem 3.1 are satisfied.

Next, suppose that the function h is defined on $\Omega\times\mathbb{R}$ by

$$h(x,\eta) = -\eta + \chi(x)\varphi(x), \quad (x,\eta) \in \Omega\times\mathbb{R}.$$

Clearly, condition (3.8) and condition $(*_7)$ of Theorem 3.1 are satisfied. Moreover, $V \neq \emptyset$. For instance $-\varphi \in V$.

For every $s \in \mathbb{N}$ we set $u_s = q_s\varphi$. It is easy to see that for every $s \in \mathbb{N}$, $u_s \in V_s$ and the function u_s minimizes the functional $J_s + G_s$ on the set V_s .

Now let us prove the following assertion:

$$\text{if } u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \text{ and } \|u_s - q_s u\|_{L^p(\nu,\Omega_s)} \to 0, \text{ we have } u \notin V. \tag{4.38}$$

In fact, let $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and $\|u_s - q_s u\|_{L^p(\nu,\Omega_s)} \to 0$. Hence

$$u = \varphi \quad \text{a.e. in} \quad \Omega\setminus\overline{B}. \tag{4.39}$$

Let σ be a function in $C^1(\mathbb{R})$ such that σ is nondecreasing in $\mathbb{R}$, $\sigma = 0$ in $(-\infty,0]$ and $\sigma = 1$ in $[1/2,+\infty)$. Since $u-\varphi \in \overset{\circ}{W}{}^{1,1}(\Omega)$, we have $\sigma(u-\varphi) \in \overset{\circ}{W}{}^{1,1}(\Omega)$ and $\nabla\sigma(u-\varphi) = \sigma'(u-\varphi)\nabla(u-\varphi)$ a.e. in Ω. This and (3.63) imply that

$$\int_\Omega \sigma(u-\varphi)dx \leqslant C\int_\Omega \sigma'(u-\varphi)|\nabla(u-\varphi)|dx. \tag{4.40}$$

We set $B' = \{x \in B : u(x)-\varphi(x) < 1/2\}$ and suppose that $\operatorname{meas} B' = 0$. Then using (4.39) and taking into account that $\sigma' = 0$ in $[1/2,+\infty)$, we get that the integral in the right-hand side of (4.40) is equal to zero. Therefore, $\sigma(u-\varphi) = 0$ a.e. in Ω. On the other hand, since $\sigma = 1$ in $[1/2,+\infty)$ we have $\sigma(u-\varphi) = 1$ in $B\setminus B'$. The contradiction obtained shows that $\operatorname{meas} B' > 0$. Suppose that $u \in V$. In view of (3.9) there exists a set $E \subset \Omega$ with measure zero such that for every $x \in \Omega\setminus E$, $h(x,u(x)) \leqslant 0$.

Let $x \in B' \setminus E$. We have $0 \geqslant h(x,u(x)) = -u(x) + \chi(x)\varphi(x) = -u(x)$. Hence $u(x) \geqslant 0$. However, taking into account the definition of the set B' and the fact that $\varphi = -1$ in B, we obtain $u(x) < -1/2$. The contradiction obtained proves that $u \notin V$.

Thus, assertion (4.38) holds true. By virtue of this assertion there are no any increasing sequence $\{s_j\} \subset \mathbb{N}$ and any function $u \in V$ such that assertion (3.12) holds true.

Now we conclude that all the conditions of Theorem 3.1 are satisfied except for condition $(*_3)$, and the conclusion of the theorem does not hold true.

Thus, condition $(*_3)$ in Theorem 3.1 is significant.

At the same time in connection with Remark 3.4 we observe that for every $x \in \Omega$, $\bar{h}(x) = (\chi\varphi)(x)$, and the function $\chi\varphi$ does not belong to $W^{1,1}(\Omega)$. Consequently, there is no any function $z \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that $\bar{h} = z$ a.e. in Ω.

Further, we give an example where conditions $(*_1')$ and $(*_2')$ of Theorem 3.6 are satisfied.

EXAMPLE 4.14. For every $s \in \mathbb{N}$ we set $Z_s' = \{z \in Z_s : Q_s(z) \subset \Omega\}$. Obviously, there exists $s' \in \mathbb{N}$ such that for every $s \in \mathbb{N}$, $s > s'$, the set Z_s' is nonempty. Let for every $s \in \mathbb{N}$, $s > s'$, and $z \in Z_s'$, $w_{s,z}$ be a function in $C_0^\infty(Q_1(0))$. We assume that there exists a nonnegative function $a \in L^1(Q_1(0))$ such that for every $s \in \mathbb{N}$, $s > s'$, and $z \in Z_s'$,

$$|\nabla w_{s,z}|^p \leqslant a \quad \text{in } Q_1(0).$$

For every $s \in \mathbb{N}$, $s > s'$, we define the function $w_s : \Omega \to \mathbb{R}$ by

$$w_s(x) = \begin{cases} \dfrac{1}{s} w_{s,z}(s(x-z)) & \text{if} \quad x \in Q_s(z) \text{ and } z \in Z_s', \\ 0 & \text{if} \quad x \in \Omega \setminus \bigcup\limits_{z \in Z_s'} Q_s(z). \end{cases}$$

For every $s \in \mathbb{N}$, $s \leqslant s'$, we set $w_s = w_{s'+1}$. Clearly, $\{w_s\} \subset C_0^\infty(\Omega)$. Let for every $s \in \mathbb{N}$, $y_s = w_s|_{\Omega_s}$. Evidently, for every $s \in \mathbb{N}$ we have $y_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$. Supposing that the function ν is bounded in Ω, we establish that the sequence of the functions y_s satisfies conditions $(*_1')$ and $(*_2')$ of Theorem 3.6.

Next, consider an example which shows that inclusion (3.66) is essential for (3.67).

EXAMPLE 4.15. For every $m \in \mathbb{N}$ we set $B^{(m)} = \{x \in \mathbb{R}^n : |x| \leqslant 1 - 1/2^m\}$. Suppose that $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ and the function ν is defined on Ω by

$$\nu(x) = \begin{cases} 1 & \text{if} \quad x \in B^{(1)}, \\ 2^{-(m+1)p} & \text{if} \quad x \in B^{(m+1)} \setminus B^{(m)} \text{ and } m \in \mathbb{N}. \end{cases}$$

Obviously, $\nu > 0$ in Ω, $\nu \in L^1(\Omega)$ and $(1/\nu)^{1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega)$. Moreover, for every $m \in \mathbb{N}$ we have

$$\int_{B^{(m+1)} \setminus B^{(m)}} \left(\frac{1}{\nu}\right)^{1/(p-1)} dx > 2^{(m+p)/(p-1)-n} \operatorname{meas} \Omega.$$

This implies that $(1/\nu)^{1/(p-1)} \notin L^1(\Omega)$.

Now let $\varphi : \Omega \to \mathbb{R}$ be the function such that for every $x \in \Omega$, $\varphi(x) = 1$. Clearly,

$$\varphi \notin \overset{\circ}{W}{}^{1,1}(\Omega). \tag{4.41}$$

However,

$$\varphi \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega). \tag{4.42}$$

In fact, it is easy to see that $\varphi \in W^{1,p}(\nu,\Omega)$. For every $m \in \mathbb{N}$ we fix a function $\varphi_m \in C_0^\infty(\Omega)$ such that $0 \leqslant \varphi_m \leqslant 1$ in Ω, $\varphi_m = 1$ in $B^{(m)}$, $\varphi_m = 0$ in $\Omega \setminus B^{(m+1)}$ and $|\nabla \varphi_m| \leqslant 2^m c_0$ in Ω, where $c_0 > 0$ depends only on n. Then for every $m \in \mathbb{N}$ we obtain that $\|\varphi_m - \varphi\|^p_{1,p,\nu} \leqslant 2^{-m} n^2 (c_0^p + 1) \operatorname{meas} \Omega$. Hence $\varphi_m \to \varphi$ strongly in $W^{1,p}(\nu,\Omega)$. Therefore, inclusion (4.42) is valid.

From (4.41) and (4.42) it follows that the inclusion $\overset{\circ}{W}{}^{1,p}(\nu,\Omega) \subset \overset{\circ}{W}{}^{1,1}(\Omega)$ does not hold true.

In conclusion of the section we consider an example where all the conditions of Theorem 3.12 are satisfied except for condition (3.108), and there is no any increasing sequence $\{s_j\} \subset \mathbb{N}$ such that $(J_{s_j} + G_{s_j})(u_{s_j}) \to (J+G)(u)$, where u minimizes the functional $J+G$ on V.

EXAMPLE 4.16. Let $\varphi \in C_0^\infty(Q_1(0))$ be a function such that $\varphi = 1$ in $Q_2(0)$. We set

$$\alpha = \int_{Q_1(0)} |\nabla \varphi|^p \, dx \quad \text{and} \quad \beta = \max_{Q_1(0)} |\nabla \varphi|.$$

Obviously, $\alpha > 0$.

Let for every $s \in \mathbb{N}$ the set Z'_s be defined as in Example 4.14. Clearly, there exists $s' \in \mathbb{N}$ such that for every $s \in \mathbb{N}$, $s > s'$, the set Z'_s is nonempty. For every $s \in \mathbb{N}$, $s > s'$, we define the function $\varphi_s : \Omega \to \mathbb{R}$ by

$$\varphi_s(x) = \begin{cases} \dfrac{1}{s}\, \varphi(s(x-z)) & \text{if} \quad x \in Q_s(z) \text{ and } z \in Z'_s, \\ 0 & \text{if} \quad x \in \Omega \setminus \bigcup\limits_{z \in Z'_s} Q_s(z). \end{cases}$$

For every $s \in \mathbb{N}$, $s \leqslant s'$, we set $\varphi_s = \varphi_{s'+1}$. Evidently, $\{\varphi_s\} \subset C_0^\infty(\Omega)$. Moreover, we have

$$\lim_{s \to \infty} \|\varphi_s\|_{L^p(\Omega)} = 0, \tag{4.43}$$

$$\lim_{s \to \infty} \int_\Omega |\nabla \varphi_s|^p \, dx = \alpha \operatorname{meas} \Omega, \tag{4.44}$$

$$\forall s \in \mathbb{N}, \quad |\nabla \varphi_s| \leqslant \beta \quad \text{in } \Omega. \tag{4.45}$$

Let $u : \Omega \to \mathbb{R}$ be the function such that for every $x \in \Omega$, $u(x) = 0$. From (4.43) and (4.44) it follows that

$$\varphi_s \to u \quad \text{weakly in } \overset{\circ}{W}{}^{1,p}(\Omega). \tag{4.46}$$

Suppose that $\nu = 1$ in Ω. Obviously, $(1/\nu)^{1/(p-1)} \in L^1(\Omega)$, $L^p(\nu,\Omega) = L^p(\Omega)$ and $\overset{\circ}{W}{}^{1,p}(\nu,\Omega) = \overset{\circ}{W}{}^{1,p}(\Omega)$. Therefore, condition $(*_1)$ of Theorem 3.1 is satisfied.

Assume that for every $s \in \mathbb{N}$, $\Omega_s = \Omega$. Clearly, if $s \in \mathbb{N}$, the set $\Omega \setminus \Omega_s$ is closed and $\widetilde{W}_0^{1,p}(\nu,\Omega_s) = \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Furthermore, condition $(*_2)$ of Theorem 3.1 is satisfied.

Next, suppose that $c_1 = 2^{1-p}$, $c_2 = 2^{p-1}$ and for every $s \in \mathbb{N}$ the function ψ_s is defined on Ω_s by $\psi_s(x) = 2^{p-1}|\nabla\varphi_s(x)|^p$, $x \in \Omega_s$. We observe that due to (4.44) condition (3.1) is satisfied, and owing to (4.45) condition $(*_4)$ of Theorem 3.1 is satisfied.

Now assume that for every $s \in \mathbb{N}$ the function f_s is defined on $\Omega_s \times \mathbb{R}^n$ by

$$f_s(x,\xi) = |\xi + \nabla\varphi_s(x)|^p, \qquad (x,\xi) \in \Omega_s \times \mathbb{R}^n.$$

Evidently, conditions (3.2)-(3.4) are satisfied.

Let $J : \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \to \mathbb{R}$ be the functional such that for every $v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$,

$$J(v) = \int_\Omega |\nabla v|^p\,dx.$$

It is easy to see that for every $s \in \mathbb{N}$ and $v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$,

$$J_s(v) = J(v + \varphi_s). \tag{4.47}$$

Using (4.46) and (4.47), we establish that the sequence $\{J_s\}$ Γ-converges to the functional J. Thus, condition $(*_5)$ of Theorem 3.1 is satisfied.

Next, let $G : \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \to \mathbb{R}$ be the functional such that for every $v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$,

$$G(v) = \int_\Omega |v|^p\,dx.$$

We suppose that $c_3 = c_4 = 1$ and for every $s \in \mathbb{N}$ the functional G_s is defined on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ by

$$G_s(v) = G(v), \qquad v \in \widetilde{W}_0^{1,p}(\nu,\Omega_s). \tag{4.48}$$

Obviously, for every $s \in \mathbb{N}$ the functional G_s is weakly continuous on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$. Moreover, conditions (3.5) and (3.6) and condition $(*_6)$ of Theorem 3.1 are satisfied.

Assume that the function h is defined on $\Omega \times \mathbb{R}$ by $h(x,\eta) = |\eta|$, $(x,\eta) \in \Omega \times \mathbb{R}$. Clearly, condition (3.8) is satisfied, and $V \neq \emptyset$, since $u \in V$. Furthermore, it is easy to see that condition (3.107) is satisfied, and condition (3.108) is not satisfied.

Finally, suppose that for every $s \in \mathbb{N}$, u_s is a function in V_s minimizing the functional $J_s + G_s$ on V_s.

Thus, all the conditions of Theorem 3.12 are satisfied except for condition (3.108). At the same time the function u minimizes the functional $J + G$ on V and for every $s \in \mathbb{N}$, $u_s = 0$ a.e. in Ω_s. The latter fact along with (4.47) and (4.48) implies that for every $s \in \mathbb{N}$, $\|u_s - q_s u\|_{L^p(\nu,\Omega_s)} = 0$ and

$$(J_s + G_s)(u_s) = \int_\Omega |\nabla\varphi_s|^p\,dx. \tag{4.49}$$

From (4.44) and (4.49) it follows that $(J_s+G_s)(u_s) \to \alpha\,\mathrm{meas}\,\Omega$. Then taking into account that $\alpha \neq 0$ and $(J+G)(u)=0$, we conclude that there is no any increasing sequence $\{s_j\} \subset \mathbb{N}$ such that $(J_{s_j}+G_{s_j})(u_{s_j}) \to (J+G)(u)$.

Thus, in the case under consideration the conclusion of Theorem 3.12 is not valid.

REFERENCES

[1] H. ATTOUCH AND C. PICARD, *Variational inequalities with varying obstacles: the general form of the limit problem*, J. Funct. Anal., **50**, 3 (1983), 329–386.

[2] L. BOCCARDO AND F. MURAT, *Homogenization of nonlinear unilateral problems, Composite Media and Homogenization Theory*, Progr. Nonlinear Differential Equations Appl., **5**, Birkhäuser, Boston, (1991), 81–105.

[3] L. CARBONE AND R. DE ARCANGELIS, *Unbounded functionals: applications to the homogenization of gradient constrained problems*, Ricerche Mat., **48**, 1 (1999), 139–182.

[4] L. CARBONE AND R. DE ARCANGELIS, *Unbounded Functionals in the Calculus of Variations*, Chapman & Hall/CRC, Boca Raton, 2002.

[5] G. R. CIRMI AND M. M. PORZIO, *L^∞-solutions for some nonlinear degenerate elliptic and parabolic equations*, Ann. Mat. Pura Appl. (4), **169** (1995), 67–86.

[6] G. DAL MASO, *Asymptotic behaviour of minimum problems with bilateral obstacles*, Ann. Mat. Pura Appl. (4), **129** (1981), 327–366.

[7] G. DAL MASO, *Limits of minimum problems for general integral functionals with unilateral obstacles*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), **74**, 2 (1983), 55–61.

[8] G. DAL MASO AND A. DEFRANCESCHI, *Convergence of unilateral problems for monotone operators*, J. Analyse Math., **53** (1989), 269–289.

[9] G. DAL MASO, *An Introduction to Γ-convergence*, Birkhäuser, Boston, 1993.

[10] R. DE ARCANGELIS, *Compactness and convergence of minimum points for a class of nonlinear nonequicoercive functionals*, Nonlinear Anal., Theory Methods Appl., **15**, 4 (1990), 363–380.

[11] R. DE ARCANGELIS AND P. DONATO, *Homogenization in weighted Sobolev spaces*, Ricerche Mat., **34**, 2 (1985), 289–308.

[12] R. DE ARCANGELIS AND F. SERRA CASSANO, *On the homogenization of degenerate elliptic equations in divergence form*, J. Math. Pures Appl. (9), **71**, 2 (1992), 119–138.

[13] R. DE ARCANGELIS AND F. SERRA CASSANO, *On the convergence of solutions of degenerate elliptic equations in divergence form*, Ann. Mat. Pura Appl. (4), **167** (1994), 1–23.

[14] V. DE CICCO, *Limits of nonlinear weighted Dirichlet problems in varying domains*, Nonlinear Anal. Ser. A: Theory Methods, **48**, 2 (2002), 274–270.

[15] E. DE GIORGI AND T. FRANZONI, *Su un tipo di convergenza variazionale*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), **58**, 6 (1975), 842–850.

[16] A. GAUDIELLO, *Sulla convergenza dei punti di minimo di funzionali degeneri non equicoercitivi*, Rend. Accad. Sci. Fis. Mat. Napoli (4), **57** (1990), 125– 168.

[17] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.

[18] F. GUGLIELMINO AND F. NICOLOSI, *W-solutions of boundary value problems for degenerate elliptic operators*, Ricerche Mat., **36** (1987), 59–72.

[19] J. HEINONEN, T. KILPELÄINEN AND O. MARTIO, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, The Clarendon Press, Oxford University Press, New York, 1993.

[20] E. YA. KHRUSLOV, *Asymptotic behavior of the solutions of the second boundary value problem in the case of the refinement of the boundary of the domain*, Mat. Sb., **106**, 4 (1978), 604–621.

[21] A. A. KOVALEVSKII, *On the connectedness of subsets of Sobolev spaces and the Γ-convergence of functionals with varying domain of definition*, Nelinejnye granichnye zadachi, **1** (1989), 48–54.

[22] A. A. KOVALEVSKII, *Some problems connected with the problem of averaging variational problems for functionals with a variable domain*, Current Analysis and its Applications, Naukova Dumka, Kiev, (1989), 62–70.

[23] A. A. KOVALEVSKII, *G-convergence and homogenization of nonlinear elliptic operators in divergence form with variable domain*, Russian Acad. Sci. Izv. Math., **44**, 3 (1995), 431–460.

[24] A. A. KOVALEVSKII, *On the Γ-convergence of integral functionals defined on Sobolev weakly connected spaces*, Ukrainian Math. J., **48**, 5 (1996), 683–698.
[25] A. KOVALEVSKY AND F. NICOLOSI, *On the convergence of solutions of degenerate nonlinear elliptic high order equations*, Nonlinear Anal. Ser. A: Theory Methods, **49**, 3 (2002), 335–360.
[26] A. A. KOVALEVSKII AND O. A. RUDAKOVA, *On the Γ-compactness of integral functionals with degenerate integrands*, Nelinejnye granichnye zadachi, **15** (2005), 149–153.
[27] A. A. KOVALEVSKII AND O. A. RUDAKOVA, *On the strong connectedness of weighted Sobolev spaces and the compactness of sequences of their elements*, Tr. Inst. Prikl. Mat. Mekh. Nats. Akad. Nauk Ukrainy, **12** (2006), 85–99.
[28] V. A. MARCHENKO AND E. YA. KHRUSLOV, *Homogenization of Partial Differential Equations*, Birkhäuser, Boston, 2006.
[29] F. MURAT, *Sur l'homogeneisation d'inequations elliptiques du 2ème ordre, relatives au convexe $K(\psi_1,\psi_2)=\{v\in H_0^1(\Omega)\,|\,\psi_1\leqslant v\leqslant\psi_2$ p. p. dans $\Omega\}$,* Publ. Laboratoire d'Analyse Numérique, no. 76013, Univ. Paris VI, 1976.
[30] M. K. V. MURTHY AND G. STAMPACCHIA, *Boundary value problems for some degenerate-elliptic operators*, Ann. Mat. Pura Appl. (4), **80** (1968), 1–122.
[31] O. A. RUDAKOVA, *On the coercivity of the integrand of the Γ-limit functional of a sequence of integral functionals defined on different weighted Sobolev spaces*, Tr. Inst. Prikl. Mat. Mekh. Nats. Akad. Nauk Ukrainy, **15** (2007), 171–180.
[32] O. A. RUDAKOVA, *On the Γ-convergence of integral functionals defined on different weighted Sobolev spaces*, Ukrain. Mat. Zh., **61**, 1 (2009), 99–115.
[33] G. V. SANDRAKOV, *On the homogenization of variational inequalities for problems of nonlinear diffusion in perforated domains*, Izv. Math., **69**, 5 (2005), 1035–1059.
[34] G. V. SANDRAKOV, *Homogenization of variational inequalities and of equations defined by a pseudomonotone operator*, Sb. Math., **199**, 1–2 (2008), 67–98.
[35] I. V. SKRYPNIK AND D. V. LARIN, *Principle of additivity in averaging of degenerate nonlinear Dirichlet problems*, Ukrainian Math. J., **50**, 1 (1998), 136–154.
[36] M. M. VAINBERG, *Variational Method and Method of Monotone Operators in the Theory of Nonlinear Equations*, Halsted Press, New York-Toronto, 1973.
[37] V. V. ZHIKOV, *Questions of convergence, duality and averaging for functionals of the calculus of variations*, Izv. Akad. Nauk SSSR Ser. Mat., **47**, 5 (1983), 961–998.
[38] V. V. ZHIKOV, *Passage to the limit in nonlinear variational problems*, Mat. Sb., **183**, 8 (1992), 47–84.

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