Differential
Equations
& Applications
Volume 1, Number 4 (2009), 561–582

PSEUDO–ALMOST AUTOMORPHIC SOLUTIONS TO SOME CLASSES OF NONAUTONOMOUS PARTIAL EVOLUTION EQUATIONS

TOKA DIAGANA

(*Communicated by E. Hernandez*)

Abstract. In this paper we obtain the existence of pseudo-almost automorphic solutions to some classes of nonautonomous partial evolution equations. To illustrate our main result, we study the existence of a pseudo-almost automorphic solution to a nonautonomous heat equation.

1. Introduction

The impetus of this paper comes from one main source, that is, a recent paper by Xiao, Zhu, and Liang [44], in which the existence of pseudo-almost automorphic solutions to the non-autonomous differential equations given by

$$u'(t) = A(t)u(t) + h(t,u(t)), \tag{1.1}$$

where $A(t) : D(A(t)) \subset \mathbb{X} \mapsto \mathbb{X}$ is a family of densely-defined linear operators on a Banach space $\mathbb{X}$, and the function $h : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ is pseudo-almost automorphic, was established. For that, Xiao *et al.* [44] introduced a new concept called bi-almost automorphy and also assumed that the family of linear operators $A(t)$ satisfy the well-known Acquistapace-Terreni conditions [3], which in fact do guarantee the existence of an evolution family $\mathscr{T} = \{V(t,s)\}_{t \geqslant s}$ associated with the family of linear operators $A(t)$. The main result in [44] was then subsequently utilized to study the existence of pseudo-almost automorphic solutions to some functional differential equations with delay.

In this paper, we consider a more general setting, that is, we make extensive use of intermediate space techniques to study the existence of pseudo-almost automorphic solutions to the the class of abstract nonautonomous evolution equations

$$\frac{d}{dt}\Big[u(t) + f(t,Bu(t))\Big] = A(t)u(t) + g(t,Cu(t)), \ \ t \in \mathbb{R}, \tag{1.2}$$

where $A(t)$ for $t \in \mathbb{R}$ is a family of closed linear operators with domains $D(A(t))$ satisfying Acquistapace-Terreni conditions, and $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}^t_\beta$ $(0 < \alpha < \beta < 1)$ and

Mathematics subject classification (2000): 43A60, 34B05, 34C27, 42A75, 47D06, 35L90.

Keywords and phrases: sectorial operator, evolution family, exponential dichotomy, hyperbolic evolution family, Acquistapace and Terreni conditions, almost automorphic, pseudo-almost automorphic, hyperbolic evolution equation, nonautonomous equation, heat equation in divergence.

Paper DEA-01-30

$g : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ are pseudo-almost automorphic in $t \in \mathbb{R}$ uniformly in the second variable. It is well-known that in that case there exists an evolution family $\mathscr{U} = \{U(t,s)\}_{t \geqslant s}$ defined on $\mathbb{X}$ and associated with the family of closed linear operators $A(t)$. Assuming that the evolution family $\mathscr{U} = \{U(t,s)\}_{t \geqslant s}$ is exponentially dichotomic (hyperbolic) and under some additional assumptions, it will be shown that Eq. (1.2) has a unique pseudo-almost automorphic solution. It is worth mentioning that the main result of this paper (Theorem 3.1) generalizes, to some extent, most of known results on (pseudo) almost automorphic solutions to autonomous and nonautonomous differential equations, especially those in [8], [19], and [44].

Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ be a open bounded subset with C^2 boundary $\Gamma := \partial\Omega$ and let $\mathbb{X} = L^2(\Omega)$ be the space square integrable functions equipped with its natural topology. As an application to our abstract result, we consider and study the existence of pseudo-almost automorphic solutions to the N-dimensional heat equation given by

$$\begin{cases} \dfrac{\partial}{\partial t}\Big[\varphi + F(t, \widehat{\operatorname{div}}\,\varphi)\Big] = a(t,x)\Delta\varphi + G(t, \widehat{\operatorname{div}}\,\varphi), \text{ in } \mathbb{R} \times \Omega, \\ \\ \varphi = 0, \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \mathbb{R} \times \Gamma, \end{cases} \tag{1.3}$$

where $a : \mathbb{R} \times \Omega \mapsto \mathbb{R}$ is a function satisfying some additional conditions, the symbols $\widehat{\operatorname{div}}$ and Δ stand respectively for the first and second-order differential operators defined by

$$\widehat{\operatorname{div}} := \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \text{ and } \Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2},$$

and the coefficients $F, G : \mathbb{R} \times \mathbb{H}_0^1(\Omega) \mapsto L^2(\Omega)$ are pseudo-almost automorphic functions and satisfy some additional conditions.

The existence of almost periodic, almost automorphic, pseudo-almost periodic, and pseudo-almost automorphic constitutes one of the most attractive topics in qualitative theory of differential equations due essentially to their applications. Some contributions on pseudo-almost automorphic solutions to abstract differential and partial differential equations have recently been made, among them are [12], [14], [22], [23], [31], [32], [43], and [44]. However, the existence of pseudo-almost automorphic solutions to evolution equations of the form Eq. (1.2) in the non-autonomous case is an untreated original question, which in fact is the main motivation of the present paper.

The paper is organized as follows: Section 2 is devoted to preliminaries facts related to the existence of an evolution family. Some preliminary results on intermediate spaces are also stated there. In addition, basic definitions and results on the concept of pseudo-almost automorphic functions are given. In Section 3, we first state and prove a key technical lemma (Lemma 3.1) and next make use of it to prove the main result. In Section 4, we give an example to illustrate our main result.

2. Preliminaries

This section is devoted to some preliminary results needed in the sequel. We basically use the same setting as in [7] with slight adjustments. Throughout the rest of this paper, $(\mathbb{X}, \|\cdot\|)$ stands for a Banach space, $A(t)$ for $t \in \mathbb{R}$ is a family of closed linear operators on $D(A(t))$ satisfying the so-called Acquistapace-Terreni conditions (Hypothesis (H.1)). Moreover, the operators $A(t)$ are not necessarily densely defined. The linear operators B, C are (possibly unbounded) defined on $\mathbb{X}$ such that $A(t)+B+C$ is not trivial for each $t \in \mathbb{R}$. The functions, $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}_\beta^t$ $(0 < \alpha < \beta < 1)$, $g : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ are respectively jointly continuous satisfying some additional assumptions.

If L is a linear operator on the Banach space $\mathbb{X}$, then:

- $D(L)$ stands for its domain;
- $\rho(L)$ stands for its resolvent;
- $\sigma(L)$ stands for its spectrum;
- $N(L)$ stands for its null-space or kernel; and
- $R(L)$ stands for its range.

Moreover, one sets $R(\lambda, L) := (\lambda I - L)^{-1}$ for all $\lambda \in \rho(A)$. Furthermore, we set $Q = I - P$ for a projection P. The space $B(\mathbb{Y}, \mathbb{Z})$ denotes the collection of all bounded linear operators from $\mathbb{Y}$ into $\mathbb{Z}$ equipped with its natural topology. When $\mathbb{Y} = \mathbb{Z}$, then this is simply denoted by $B(\mathbb{Y})$.

Hypothesis (H.1). The family of closed linear operators $A(t)$ for $t \in \mathbb{R}$ on $\mathbb{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfy the so-called Acquistapace-Terreni conditions, that is, there exist constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $L > 0$ and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$ such that

$$\lambda \in \rho\big(A(t) - \omega\big), \qquad \big\|R\big(\lambda, A(t) - \omega\big)\big\| \leqslant \frac{K}{1 + |\lambda|}, \tag{2.1}$$

and

$$\big\|\big(A(t) - \omega\big) R\big(\lambda, A(t) - \omega\big) \big[R\big(\omega, A(t)\big) - R\big(\omega, A(s)\big)\big]\big\| \leqslant L|t-s|^\mu |\lambda|^{-\nu} \tag{2.2}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \Big\{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leqslant \theta\Big\}$.

Note that in the particular case when $A(t)$ has a constant domain $\mathbb{D} = D(A(t))$, it is well-known [4, 38] that Eq. (2.2) can be replaced with the following: There exist constants L and $0 < \mu \leqslant 1$ such that

$$\big\|\big(A(t) - A(s)\big) R\big(\omega, A(r)\big)\big\| \leqslant L|t-s|^\mu, \ \forall s, t, r \in \mathbb{R}.$$

It should mentioned that (H.1) was introduced in the literature by Acquistapace and Terreni in [2, 3] for $\omega = 0$. Among other things, it ensures that there exists a unique evolution family $\mathscr{U} = U(t,s)$ on $\mathbb{X}$ associated with $A(t)$ satisfying:

(a) $U(t,s)U(s,r) = U(t,r)$;

(b) $U(t,t) = I$ for $t \geqslant s \geqslant r$ in $\mathbb{R}$;

(c) $(t,s) \mapsto U(t,s) \in B(\mathbb{X})$ is continuous for $t > s$;

(d) $U(\cdot,s)\in C^1((s,\infty),B(\mathbb{X}))$, $\dfrac{\partial U}{\partial t}(t,s)=A(t)U(t,s)$ and

$$\|A(t)^k U(t,s)\| \leqslant K\,(t-s)^{-k}$$

for $0<t-s\leqslant 1$, $k=0,1$, $0\leqslant \alpha<\mu$, $x\in D((\omega-A(s))^\alpha)$, and a constant C depending only on the constants appearing in (H.1); and

(e) $\partial_s^+ U(t,s)x=-U(t,s)A(s)x$ for $t>s$ and $x\in D(A(s))$ with $A(s)x\in\overline{D(A(s))}$.

It should also be mentioned that the above-mentioned proprieties were mainly established in [1, Theorem 2.3] and [46, Theorem 2.1], see also [3, 45]. In this case we say that $A(\cdot)$ generates the evolution family $U(\cdot,\cdot)$.

One says that an evolution family $\mathscr{U}$ has an *exponential dichotomy* (or is *hyperbolic*) if there are projections $P(t)$ $(t\in\mathbb{R})$ that are uniformly bounded and strongly continuous in t and constants $\delta>0$ and $N\geqslant 1$ such that

(f) $U(t,s)P(s)=P(t)U(t,s)$,

(g) the restriction $U_Q(t,s): Q(s)\mathbb{X}\to Q(t)\mathbb{X}$ of $U(t,s)$ is invertible (we then set $\widetilde{U}_Q(s,t):=U_Q(t,s)^{-1}$), and

(h) $\big\|U(t,s)P(s)\big\|\leqslant Ne^{-\delta(t-s)}$ and $\big\|\widetilde{U}_Q(s,t)Q(t)\big\|\leqslant Ne^{-\delta(t-s)}$, $t\geqslant s$ and $t,s\in\mathbb{R}$.

According to [40], the following sufficient conditions are required for $A(t)$ to have exponential dichotomy.

(i) Let $(A(t),D(t))_{t\in\mathbb{R}}$ be generators of analytic semigroups on $\mathbb{X}$ of the same type. Suppose that $D(A(t))\equiv D(A(0))$, $A(t)$ is invertible,

$$\sup_{t,s\in\mathbb{R}}\big\|A(t)A(s)^{-1}\big\|$$

is finite, and

$$\big\|A(t)A(s)^{-1}-I\big\|\leqslant L_0|t-s|^\mu$$

for $t,s\in\mathbb{R}$ and constants $L_0\geqslant 0$ and $0<\mu\leqslant 1$.

(j) The semigroups $(e^{\tau A(t)})_{\tau\geqslant 0}$, $t\in\mathbb{R}$, are hyperbolic with projection P_t and constants $N,\delta>0$. Moreover, let

$$\big\|A(t)e^{\tau A(t)}P_t\big\|\leqslant\psi(\tau)$$

and

$$\big\|A(t)e^{\tau A_Q(t)}Q_t\big\|\leqslant\psi(-\tau)$$

for $\tau>0$ and a function ψ such that $\mathbb{R}\ni s\mapsto\varphi(s):=|s|^\mu\psi(s)$ is integrable with $L_0\|\varphi\|_{L^1(\mathbb{R})}<1$.

This setting requires some estimates related to $U(t,s)$. For that, we introduce the interpolation spaces for $A(t)$. We refer the reader to the following excellent books [4], [21], and [30] for proofs and further information on theses interpolation spaces.

Let A be a sectorial operator on $\mathbb{X}$ (in assumption (H.1), replace $A(t)$ with A) and let $\alpha \in (0,1)$. Define the real interpolation space

$$\mathbb{X}_\alpha^A := \left\{x \in \mathbb{X} : \left\|x\right\|_\alpha^A := \sup_{r>0} \left\|r^\alpha (A-\omega) R(r, A-\omega) x\right\| < \infty\right\},$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write

$$\mathbb{X}_0^A := \mathbb{X},\ \left\|x\right\|_0^A := \left\|x\right\|,\ \mathbb{X}_1^A := D(A),$$

and

$$\left\|x\right\|_1^A := \left\|(\omega - A)x\right\|.$$

Moreover, let $\hat{\mathbb{X}}^A := \overline{D(A)}$ of $\mathbb{X}$. In particular, we have the following continuous embedding

$$D(A) \hookrightarrow \mathbb{X}_\beta^A \hookrightarrow D((\omega - A)^\alpha) \hookrightarrow \mathbb{X}_\alpha^A \hookrightarrow \hat{\mathbb{X}}^A \hookrightarrow \mathbb{X}, \tag{2.3}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathbb{X}_\alpha^A$ and $\mathbb{X}$. However, we have the following continuous injection

$$\mathbb{X}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \tag{2.4}$$

for $0 < \alpha < \beta < 1$.

Given the family of linear operators $A(t)$ for $t \in \mathbb{R}$, satisfying (H.1), we set

$$\mathbb{X}_\alpha^t := \mathbb{X}_\alpha^{A(t)}, \quad \hat{\mathbb{X}}^t := \hat{\mathbb{X}}^{A(t)},$$

for $0 \leqslant \alpha \leqslant 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embedding in Eq. (2.3) holds with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathscr{J}_\alpha$ ([30, Definition 1.1.1]) and hence there is a constant $c(\alpha)$ such that

$$\left\|y\right\|_\alpha^t \leqslant c(\alpha) \left\|y\right\|^{1-\alpha} \left\|A(t)y\right\|^\alpha, \quad y \in D(A(t)). \tag{2.5}$$

We have the following fundamental estimates for the evolution family $\mathscr{U}$. Its proof was given in [7] though for the sake of clarity, we reproduce it here.

PROPOSITION 2.1. *For $x \in \mathbb{X}$, $0 \leqslant \alpha \leqslant 1$ and $t > s$, the following hold.*

(i) There is a constant $c(\alpha)$, such that

$$\left\|U(t,s)P(s)x\right\|_\alpha^t \leqslant c(\alpha) e^{-\frac{\delta}{2}(t-s)} (t-s)^{-\alpha} \|x\|. \tag{2.6}$$

(ii) There is a constant $m(\alpha)$, such that

$$\left\|\widetilde{U}_Q(s,t) Q(t) x\right\|_\alpha^s \leqslant m(\alpha) e^{-\delta(t-s)} \|x\|. \tag{2.7}$$

Proof. (i) Using (2.5) we obtain

$$
\begin{aligned}
\left\|U(t,s)P(s)x\right\|_\alpha^t &\leqslant c(\alpha)\left\|U(t,s)P(s)x\right\|^{1-\alpha}\left\|A(t)U(t,s)P(s)x\right\|^\alpha \\
&\leqslant c(\alpha)\left\|U(t,s)P(s)x\right\|^{1-\alpha}\left\|A(t)U(t,t-1)U(t-1,s)P(s)x\right\|^\alpha \\
&\leqslant c(\alpha)\left\|U(t,s)P(s)x\right\|^{1-\alpha}\left\|A(t)U(t,t-1)\right\|^\alpha\left\|U(t-1,s)P(s)x\right\|^\alpha \\
&\leqslant c(\alpha)N'\, e^{-\delta(t-s)(1-\alpha)}e^{-\delta(t-s-1)\alpha}\|x\| \\
&\leqslant c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}(t-s)^\alpha e^{-\frac{\delta}{2}(t-s)}\|x\|
\end{aligned}
$$

for $t-s\geqslant 1$ and $x\in\mathbb{X}$.

Since $(t-s)^\alpha e^{-\frac{\delta}{2}(t-s)}\to 0$ as $t\to\infty$ it easily follows that

$$\left\|U(t,s)P(s)x\right\|_\alpha^t\leqslant c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|x\|.$$

If $0<t-s\leqslant 1$, we have

$$
\begin{aligned}
\left\|U(t,s)P(s)x\right\|_\alpha^t &\leqslant c(\alpha)\left\|U(t,s)P(s)x\right\|^{1-\alpha}\left\|A(t)U(t,s)P(s)x\right\|^\alpha \\
&\leqslant c(\alpha)\left\|U(t,s)P(s)x\right\|^{1-\alpha}\left\|A(t)U(t,\frac{t+s}{2})U(\frac{t+s}{2},s)P(s)x\right\|^\alpha \\
&\leqslant c(\alpha)\left\|U(t,s)P(s)x\right\|^{1-\alpha}\left\|A(t)U(t,\frac{t+s}{2})\right\|^\alpha\left\|U(\frac{t+s}{2},s)P(s)x\right\|^\alpha \\
&\leqslant c(\alpha)Ne^{-\delta(t-s)(1-\alpha)}2^\alpha(t-s)^{-\alpha}e^{-\frac{\delta\alpha}{2}(t-s)}\|x\| \\
&\leqslant c(\alpha)Ne^{-\frac{\delta}{2}(t-s)(1-\alpha)}2^\alpha(t-s)^{-\alpha}e^{-\frac{\delta\alpha}{2}(t-s)}\|x\| \\
&\leqslant c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|,
\end{aligned}
$$

and hence

$$\left\|U(t,s)P(s)x\right\|_\alpha^t\leqslant c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|x\| \text{ for } t>s.$$

(ii)

$$
\begin{aligned}
\left\|\widetilde{U}_Q(s,t)Q(t)x\right\|_\alpha^s &\leqslant c(\alpha)\left\|\widetilde{U}_Q(s,t)Q(t)x\right\|^{1-\alpha}\left\|A(s)\widetilde{U}_Q(s,t)Q(t)x\right\|^\alpha \\
&\leqslant c(\alpha)\left\|\widetilde{U}_Q(s,t)Q(t)x\right\|^{1-\alpha}\left\|A(s)Q(s)\widetilde{U}_Q(s,t)Q(t)x\right\|^\alpha \\
&\leqslant c(\alpha)\left\|\widetilde{U}_Q(s,t)Q(t)x\right\|^{1-\alpha}\left\|A(s)Q(s)\right\|^\alpha\left\|\widetilde{U}_Q(s,t)Q(t)x\right\|^\alpha \\
&\leqslant c(\alpha)Ne^{-\delta(t-s)(1-\alpha)}\left\|A(s)Q(s)\right\|^\alpha e^{-\delta(t-s)\alpha}\|x\| \\
&\leqslant m(\alpha)e^{-\delta(t-s)}\|x\|.
\end{aligned}
$$

In the last inequality we have used that $\left\|A(s)Q(s)\right\|\leqslant c$ for some constant $c\geqslant 0$, see e.g. [42, Proposition 3.18].

In addition to above, we also need the following assumptions:

Hypothesis (H.2). The evolution family $\mathscr{U}$ generated by $A(\cdot)$ has an exponential dichotomy with constants $N,\delta>0$ and dichotomy projections $P(t)$ for $t\in\mathbb{R}$. Moreover, $0\in\rho(A(t))$ for each $t\in\mathbb{R}$ and the following holds

$$\sup_{t,s\in\mathbb{R}}\left\|A(s)A^{-1}(t)\right\|_{B(\mathbb{X}_\alpha,\mathbb{X})}<c_0. \tag{2.8}$$

REMARK 2.1. Note that Eq. (2.8) is satisfied in many cases in the literature. In particular, it holds when $A(t) = d(t)A$ where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is any closed linear operator such that $0 \in \rho(A)$ and $d : \mathbb{R} \mapsto \mathbb{R}$ with $\inf_{t\in\mathbb{R}} |d(t)| > 0$ and $\sup_{t\in\mathbb{R}} |d(t)| < \infty$.

Hypothesis (H.3). There exists $0 \leqslant \alpha < \beta < 1$ such that

$$\mathbb{X}_\alpha^t = \mathbb{X}_\alpha \text{ and } \mathbb{X}_\beta^t = \mathbb{X}_\beta$$

for all $t \in \mathbb{R}$, with uniform equivalent norms.

If $0 \leqslant \alpha < \beta < 1$, then we let $k(\alpha)$ and c' denote respectively the bounds of the embedding $\mathbb{X}_\beta \hookrightarrow \mathbb{X}_\alpha$ and $\mathbb{X}_\alpha \hookrightarrow \mathbb{X}$, that is,

$$\|u\|_\alpha \leqslant k(\alpha)\|u\|_\beta$$

for each $u \in \mathbb{X}_\beta$ and

$$\|u\| \leqslant c'\|u\|_\alpha$$

for each $u \in \mathbb{X}_\alpha$

2.1. Pseudo-Almost Automorphic Functions

Let $BC(\mathbb{R},\mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) denote the collection of all $\mathbb{X}$-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $F : \mathbb{R}\times\mathbb{Y} \mapsto \mathbb{X}$). The space $BC(\mathbb{R},\mathbb{X})$ equipped with its natural norm, that is, the sup norm defined by

$$\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|,$$

is a Banach space. Furthermore, $C(\mathbb{R},\mathbb{Y})$ (respectively, $C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ (respectively, the class of jointly continuous functions $F : \mathbb{R}\times\mathbb{Y} \mapsto \mathbb{X}$).

DEFINITION 2.1. A function $f \in C(\mathbb{R},\mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$.

If the convergence above is uniform in $t \in \mathbb{R}$, then f is almost periodic in the classical Bochner's sense. Denote by $AA(\mathbb{X})$ the collection of all almost automorphic functions $\mathbb{R} \mapsto \mathbb{X}$. Note that $AA(\mathbb{X})$ equipped with the sup-norm $\|\cdot\|_\infty$ turns out to be a Banach space.

Among other things, almost automorphic functions satisfy the following properties.

THEOREM 2.1. [35, 36] *If* $f, f_1, f_2 \in AA(\mathbb{X})$, *then:*

(i) $f_1 + f_2 \in AA(\mathbb{X})$,

(ii) $\lambda f \in AA(\mathbb{X})$ *for any scalar* λ,

(iii) $f_\alpha \in AA(\mathbb{X})$ *where* $f_\alpha : \mathbb{R} \to \mathbb{X}$ *is defined by* $f_\alpha(\cdot) = f(\cdot + \alpha)$,

(iv) *the range* $\mathscr{R}_f := \{f(t) : t \in \mathbb{R}\}$ *is relatively compact in* $\mathbb{X}$, *thus* f *is bounded in norm,*

(v) *if* $f_n \to f$ *uniformly on* $\mathbb{R}$ *where each* $f_n \in AA(\mathbb{X})$, *then* $f \in AA(\mathbb{X})$ *too.*

In addition to the above-mentioned properties, we have the the following property due to Bugajewski and Diagana [10]:

(vi) if $g \in L^1(\mathbb{R})$, then $f * g \in AA(\mathbb{R})$, where $f * g$ is the convolution of f with g on $\mathbb{R}$.

Let $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be another Banach space.

DEFINITION 2.2. A jointly continuous function $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ is said to be almost automorphic in $t \in \mathbb{R}$ if $t \mapsto F(t,x)$ is almost automorphic for all $x \in K$ ($K \subset \mathbb{Y}$ being any bounded subset). Equivalently, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$G(t,x) := \lim_{n\to\infty} F(t+s_n, x)$$

is well defined in $t \in \mathbb{R}$ and for each $x \in K$, and

$$\lim_{n\to\infty} G(t - s_n, x) = F(t,x)$$

for all $t \in \mathbb{R}$ and $x \in K$.

The collection of such functions will be denoted by $AA(\mathbb{Y}, \mathbb{X})$.

For more on almost automorphic functions and related issues, we refer the reader to the excellent book by N'Guérékata [35].

Define

$$PAP_0(\mathbb{R}, \mathbb{X}) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|f(s)\| ds = 0 \right\}.$$

Similarly, $PAP_0(\mathbb{Y}, \mathbb{X})$ will denote the collection of all bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ such that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|F(s,x)\| ds = 0$$

uniformly in $x \in K$, where $K \subset \mathbb{Y}$ is any bounded subset.

DEFINITION 2.3. (Liang *et al.* [31, 43]) A function $f \in BC(\mathbb{R},\mathbb{X})$ is called pseudo almost automorphic if it can be expressed as $f = g+\phi$, where $g \in AA(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X})$. The collection of such functions will be denoted by $PAA(\mathbb{X})$.

The functions g and ϕ appearing in Definition 2.3 are respectively called the *almost automorphic* and the *ergodic perturbation* components of f.

DEFINITION 2.4. A bounded continuous function $F : \mathbb{R}\times\mathbb{Y} \mapsto \mathbb{X}$ belongs to $AA(Y,\mathbb{X})$ whenever it can be expressed as $F = G+\Phi$, where $G \in AA(\mathbb{Y},\mathbb{X})$ and $\Phi \in PAP_0(\mathbb{Y},\mathbb{X})$. The collection of such functions will be denoted by $PAA(\mathbb{Y},\mathbb{X})$.

We now collect a few useful properties of pseudo almost automorphic functions.

PROPOSITION 2.2. *If $g \in L^1(\mathbb{R})$, $f \in PAA(\mathbb{R})$, then $f * g \in PAA(\mathbb{R})$, where $f*g$ is the convolution of f with g on $\mathbb{R}$.*

The proof of Proposition 2.2 is based upon Bugajewski and Diagana [10] and Bugajewski, Diagana, and [11].

A substantial result is the next theorem, which is due to Liang et al. [43].

THEOREM 2.2. [43] *The space $PAA(\mathbb{X})$ equipped with the sup norm $\|\cdot\|_\infty$ is a Banach space.*

The next composition result, that is Theorem 2.3, is a consequence of [32, Theorem 2.4] and is crucial for the proof of the main result of the paper.

THEOREM 2.3. *Suppose $f : \mathbb{R}\times\mathbb{Y} \mapsto \mathbb{X}$ belongs to $PAA(\mathbb{Y},\mathbb{X})$; $f = g+h$, with $x \mapsto g(t,x)$ being uniformly continuous on any bounded subset K of $\mathbb{Y}$ uniformly in $t \in \mathbb{R}$. Furthermore, we suppose that there exists $L > 0$ such that*

$$\|f(t,x)-f(t,y)\| \leqslant L\|x-y\|_{\mathbb{Y}}$$

for all $x,y \in \mathbb{Y}$ and $t \in \mathbb{R}$.

Then the function defined by $h(t) = f(t,\varphi(t))$ belongs to $PAA(\mathbb{X})$ provided $\varphi \in PAA(\mathbb{Y})$.

We also have:

THEOREM 2.4. [43] *If $f : \mathbb{R}\times\mathbb{Y} \mapsto \mathbb{X}$ belongs to $PAA(\mathbb{Y},\mathbb{X})$ and if $x \mapsto f(t,x)$ is uniformly continuous on any bounded subset K of $\mathbb{Y}$ for each $t \in \mathbb{R}$, then the function defined by $h(t) = f(t,\varphi(t))$ belongs to $PAA(\mathbb{X})$ provided $\varphi \in PAA(\mathbb{Y})$.*

3. Main results

To study the existence and uniqueness of pseudo-almost automorphic solutions to Eq. (1.2) we first introduce the notion of bounded solution to it.

DEFINITION 3.1. A function $u : \mathbb{R} \mapsto \mathbb{X}_\alpha$ is said to be a bounded solution to Eq. (1.2) provided that the function $s \to A(s)U(t,s)P(s)f(s,Bu(s))$ is integrable on $(-\infty,t)$, $s \to A(s)U(t,s)Q(s)f(s,Bu(s))$ is integrable on (t,∞) for each $t \in \mathbb{R}$, and

$$\begin{aligned} u(t) = -f(t,Bu(t)) - \int_{-\infty}^{t} A(s)U(t,s)P(s)f(s,Bu(s))ds \\ + \int_{t}^{\infty} A(s)U(t,s)Q(s)f(s,Bu(s))ds + \int_{-\infty}^{t} U(t,s)P(s)g(s,Cu(s))ds \\ - \int_{t}^{\infty} U(t,s)Q(s)g(s,Cu(s))ds \end{aligned}$$

for each $t \in \mathbb{R}$.

Throughout the rest of the paper we denote by $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4, the nonlinear integral operators defined by:

$$\begin{aligned} (\Gamma_1 u)(t) &:= \int_{-\infty}^{t} A(s)U(t,s)P(s)f(s,Bu(s))ds, \\ (\Gamma_2 u)(t) &:= \int_{t}^{\infty} A(s)U(t,s)Q(s)f(s,Bu(s))ds, \\ (\Gamma_3 u)(t) &:= \int_{-\infty}^{t} U(t,s)P(s)g(s,Cu(s))ds, \\ (\Gamma_4 u)(t) &:= \int_{t}^{\infty} U(t,s)Q(s)g(s,Cu(s))ds. \end{aligned}$$

Moreover, we suppose that the linear operators $B, C : \mathbb{X}_\alpha \mapsto \mathbb{X}$ are bounded and hence set

$$\varpi := \max\Big(\|B\|_{B(\mathbb{X}_\alpha,\mathbb{X})}, \|C\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\Big).$$

To study Eq. (1.2), in addition to the previous assumptions, we require the following additional assumptions.

(H.4) $R(\omega, A(\cdot))u \in AA(\mathbb{X}_\alpha)$ for all $u \in \mathbb{X}$. For any sequence of real numbers $(\tau'_n)_{n\in\mathbb{N}}$ there exist a subsequence $(\tau_n)_{n\in\mathbb{N}}$ and a well-defined function G such that for each $\varepsilon > 0$, one can find $N_0, N_1 \in \mathbb{N}$ such that

$$\big\|G(t,s)P(s)u - A(s+\tau_n)U(t+\tau_n, s+\tau_n)P(s+\tau_n)u\big\|_\alpha \leqslant \varepsilon H_0(t-s)$$

whenever $n > N_0$ for $t, s \in \mathbb{R}$, $t > s$, and

$$\big\|A(s)U(t,s)P(s)u - G(t-\tau_n, s-\tau_n)P(s-\tau_n)u\big\|_\alpha \leqslant \varepsilon H_1(t-s)$$

whenever $n > N_1$ for all $t,s \in \mathbb{R}$, $t > s$, for all $u \in \mathbb{X}_\alpha$, where $H_0, H_1 : [0,\infty) \mapsto [0,\infty)$ with $H_0, H_1 \in L^1[0,\infty)$.

(H.5) Let $0 < \alpha \leqslant \beta < 1$, and $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}_\beta$ belongs to $PAA(\mathbb{X}, \mathbb{X}_\beta)$ while $g : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ belongs to $PAA(\mathbb{X}, \mathbb{X})$. If $f = f_1 + f_2$ and $g = g_1 + g_2$ where $f_1 \in AA(\mathbb{X}, \mathbb{X}_\beta)$, $g_1 \in AA(\mathbb{X}, \mathbb{X})$, $f_2 \in PAP_0(\mathbb{X}, \mathbb{X}_\beta)$, and $g_2 \in PAP_0(\mathbb{X}, \mathbb{X})$, we suppose that $x \mapsto f_1(t,x), g_1(t,x)$ are uniformly continuous on bounded subsets uniformly in $t \in \mathbb{R}$. Moreover, the functions f,g are uniformly Lipschitz with respect to the second argument in the following sense: there exists $K > 0$ such that

$$\|f(t,u) - f(t,v)\|_\beta \leqslant K\|u - v\|,$$

and

$$\|g(t,u) - g(t,v)\| \leqslant K\|u - v\|$$

for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$.

The proof of our main result requires the following technical Lemma.

LEMMA 3.1. *Under* (H.1)-(H.3), *then there exist constant* $m(\alpha,\beta), n(\alpha) > 0$ *such that*

$$\left\|A(s)\widetilde{U}_Q(t,s)Q(s)x\right\|_\alpha \leqslant m(\alpha,\beta)e^{\delta(s-t)}\|x\|_\beta \qquad \text{for } t \leqslant s, \tag{3.1}$$

$$\left\|A(s)U(t,s)P(s)x\right\|_\alpha \leqslant n(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|x\|_\beta, \qquad \text{for } t > s. \tag{3.2}$$

Proof. Let $x \in \mathbb{X}_\beta$. Since the restriction of $A(s)$ to $R(Q(s))$ is a bounded linear operator it follows that

$$\begin{aligned}
\left\|A(s)\widetilde{U}_Q(t,s)Q(s)x\right\|_\alpha &\leqslant ck(\alpha)\left\|\widetilde{U}_Q(t,s)Q(s)x\right\|_\beta \\
&\leqslant ck(\alpha)m(\beta)e^{\delta(s-t)}\|x\| \\
&\leqslant m(\alpha,\beta)e^{\delta(s-t)}\|x\|_\beta
\end{aligned}$$

for $t \leqslant s$ by using Eq. (2.7).

Similarly, for each $x \in \mathbb{X}_\beta$, using Eq. (2.8), we obtain

$$\begin{aligned}
\left\|A(s)U(t,s)P(s)x\right\|_\alpha &= \left\|A(s)A(t)^{-1}A(t)U(t,s)P(s)x\right\|_\alpha \\
&\leqslant \left\|A(s)A(t)^{-1}\right\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\left\|A(t)U(t,s)P(s)x\right\|_\alpha \\
&\leqslant c_0\left\|A(t)U(t,s)P(s)x\right\|_\alpha
\end{aligned}$$

for $t \geqslant s$.

Note that $\left\|A(t)U(t,s)\right\| \leqslant K(t-s)^{-1}$ for all t,s such that $0 < t-s \leqslant 1$.

Now, let $t-s \geqslant 1$. Then, using Eq. (2.6), we obtain

$$
\begin{aligned}
\left\|A(t)U(t,s)P(s)x\right\|_\alpha &= \left\|A(t)U(t,t-1)U(t-1,s)P(s)x\right\|_\alpha \\
&\leqslant \left\|A(t)U(t,t-1)\right\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\left\|U(t-1,s)P(s)x\right\|_\alpha \\
&\leqslant Kc(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\| \\
&\leqslant KK'c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|_\alpha \\
&\leqslant KK'k(\alpha)c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|_\beta \\
&\leqslant n'(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|_\beta.
\end{aligned}
$$

Now, let $0 < t-s \leqslant 1$. Again, using Eq. (2.6), we obtain

$$
\begin{aligned}
\left\|A(t)U(t,s)P(s)x\right\|_\alpha &= \left\|A(t)U(t,\frac{t+s}{2})U(\frac{t+s}{2},s)P(s)x\right\|_\alpha \\
&\leqslant \left\|A(t)U(t,\frac{t+s}{2})\right\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\left\|U(\frac{t+s}{2},s)P(s)x\right\|_\alpha \\
&\leqslant Kc(\alpha)e^{-\frac{\delta}{4}(t-s)}2^\alpha(t-s)^{-\alpha}\|x\| \\
&\leqslant KK'c(\alpha)e^{-\frac{\delta}{4}(t-s)}2^\alpha(t-s)^{-\alpha}\|x\|_\alpha \\
&\leqslant KK'k(\alpha)c(\alpha)e^{-\frac{\delta}{4}(t-s)}2^\alpha(t-s)^{-\alpha}\|x\|_\beta \\
&\leqslant n''(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|_\beta.
\end{aligned}
$$

Therefore,

$$
\left\|A(t)U(t,s)P(s)x\right\|_\alpha \leqslant n(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|_\beta
$$

for all $t,s \in \mathbb{R}$ with $t \geqslant s$.

LEMMA 3.2. *Under assumptions* (H.1)-(H.5)*, the integral operators* Γ_3 *and* Γ_4 *defined above map* $PAA(\mathbb{X}_\alpha)$ *into itself.*

Proof. Let $u \in PAA(\mathbb{X}_\alpha)$. Now since $C \in B(\mathbb{X}_\alpha,\mathbb{X})$ it follows that $Cu \in PAA(\mathbb{X})$. Setting $h(t) = g(t,Cu(t))$ and using the theorem of composition of pseudo almost automorphic functions (Theorem 2.3) it follows that $h \in PAA(\mathbb{X})$. Now write $h = \phi + \zeta$ where $\phi \in AA(\mathbb{X})$ and $\zeta \in PAP_0(\mathbb{X})$. Thus $\Gamma_3 u$ can be rewritten as

$$
(\Gamma_3 u)(t) = \int_{-\infty}^t U(t,s)P(s)\phi(s)ds + \int_{-\infty}^t U(t,s)P(s)\zeta(s)ds.
$$

Set

$$
\Phi(t) = \int_{-\infty}^t U(t,s)P(s)\phi(s)ds, \text{ and } \Psi(t) = \int_{-\infty}^t U(t,s)P(s)\zeta(s)ds
$$

for each $t \in \mathbb{R}$.

The next step consists of showing that $\Phi \in AA(\mathbb{X}_\alpha)$ and $\Psi \in PAP_0(\mathbb{X}_\alpha)$. Obviously, $\Phi \in AA(\mathbb{X}_\alpha)$. Indeed, since $\phi \in AA(\mathbb{X})$, for every sequence of real numbers $(\tau_n')_{n\in\mathbb{N}}$ there exists a subsequence $(\tau_n)_{n\in\mathbb{N}}$ such that

$$\psi(t) := \lim_{n\to\infty} \phi(t+\tau_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} \psi(t-\tau_n) = \phi(t)$$

for each $t \in \mathbb{R}$.

Set $\Phi_1(t) = \int_{-\infty}^t U(t,s)P(s)\psi(s)ds$ for all $t \in \mathbb{R}$.

Now

$$\begin{aligned}
\Phi(t+\tau_n) - \Phi_1(t) &= \int_{-\infty}^{t+\tau_n} U(t+\tau_n,s)P(s)\phi(s)ds - \int_{-\infty}^t U(t,s)P(s)\psi(s)ds \\
&= \int_{-\infty}^t U(t+\tau,s+\tau_n)P(s+\tau_n)\phi(s+\tau_n)ds \\
&\quad - \int_{-\infty}^t U(t,s)P(s)\psi(s)ds \\
&= \int_{-\infty}^t U(t+\tau_n,s+\tau_n)P(s+\tau_n)\phi(s+\tau_n)ds \\
&\quad - \int_{-\infty}^t U(t+\tau_n,s+\tau_n)P(s+\tau_n)\psi(s)ds \\
&\quad + \int_{-\infty}^t U(t+\tau_n,s+\tau_n)P(s+\tau_n)\psi(s)ds \\
&\quad - \int_{-\infty}^t U(t,s)P(s)\psi(s)ds \\
&= \int_{-\infty}^t U(t+\tau_n,s+\tau_n)P(s+\tau_n)\big(\phi(s+\tau_n)-\psi(s)\big)ds \\
&\quad + \int_{-\infty}^t \big(U(t+\tau_n,s+\tau_n)P(s+\tau_n) - U(t,s)P(s)\big)\psi(s)ds.
\end{aligned}$$

Using Lebesgue Dominated Convergence Theorem, one can easily see that

$$\Big\| \int_{-\infty}^t U(t+\tau_n,s+\tau_n)P(s+\tau_n)\big(\phi(s+\tau_n)-\psi(s)\big)ds\Big\|_\alpha \to 0 \text{ as } n\to\infty,\ t\in\mathbb{R}.$$

Similarly, using [8, Proposition 3.3] it follows that

$$\Big\| \int_{-\infty}^t \big(U(t+\tau_n,s+\tau_n)P(s+\tau_n) - U(t,s)P(s)\big)\psi(s)ds\Big\|_\alpha \to 0 \text{ as } n\to\infty,\ t\in\mathbb{R}.$$

Thus

$$\Phi_1(t) = \lim_{n\to\infty} \Phi(t+\tau_n),\ \ t\in\mathbb{R}.$$

Similarly, one can easily see that

$$\Phi(t) = \lim_{n\to\infty} \Phi_1(t-\tau_n), \ \ t \in \mathbb{R}.$$

Therefore, $\Phi \in AA(\mathbb{X}_\alpha)$.

To complete the proof for Γ_3, we have to show that $\Psi \in PAP_0(\mathbb{X}_\alpha)$. First, note that $s \mapsto \Psi(s)$ is a bounded continuous function. It remains to show that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left\|\Psi(t)\right\|_\alpha dt = 0.$$

Again using Eq. (2.6) it follows that

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left\|\Psi(t)\right\|_\alpha dt &\leqslant \lim_{T\to\infty} \frac{c(\alpha)}{2T} \int_{-T}^{T}\int_0^{+\infty} s^{-\alpha} e^{-\frac{\delta}{2}s} \left\|\zeta(t-s)\right\|_\alpha ds dt \\ &\leqslant \lim_{T\to\infty} c(\alpha) \int_0^{+\infty} s^{-\alpha} e^{-\frac{\delta}{2}s} \frac{1}{2T} \int_{-T}^{T} \left\|\zeta(t-s)\right\|_\alpha dt ds.\end{aligned}$$

Let $\Gamma_s(T) = \dfrac{1}{2T}\displaystyle\int_{-T}^{T} \left\|\zeta(t-s)\right\|_\alpha dt$. Since $PAP_0(\mathbb{X}_\alpha)$ is translation invariant it follows that $t \mapsto \zeta(t-s)$ belongs to $PAP_0(\mathbb{X}_\alpha)$ for each $s \in \mathbb{R}$, and hence

$$\lim_{T\mapsto\infty} \frac{1}{2T} \int_{-T}^{T} \left\|\zeta(t-s)\right\|_\alpha dt = 0$$

for each $s \in \mathbb{R}$.

One completes the proof by using the well-known Lebesgue Dominated Convergence Theorem and the fact $\Gamma_s(T) \mapsto 0$ as $T \to \infty$ for each $s \in \mathbb{R}$.

The proof for $\Gamma_4 u(\cdot)$ is similar to that of $\Gamma_3 u(\cdot)$. However one makes use of Eq. (2.7) rather than Eq. (2.6).

LEMMA 3.3. *Under assumptions* (H.1)-(H.5)*, the integral operators* Γ_1 *and* Γ_2 *defined above map* $PAA(\mathbb{X}_\alpha)$ *into itself.*

Proof. Let $u \in PAA(\mathbb{X}_\alpha)$. Since $B \in B(\mathbb{X}_\alpha, \mathbb{X})$ it follows that the function $t \mapsto Bu(t)$ belongs to $PAA(\mathbb{X})$. Again, using the composition of pseudo almost automorphic functions (Theorem 2.3) it follows that $\psi(\cdot) = f(\cdot, Bu(\cdot))$ is in $PAA(\mathbb{X}_\beta)$ whenever $u \in PAA(\mathbb{X}_\alpha)$. In particular,

$$\left\|\psi\right\|_{\infty,\beta} = \sup_{t\in\mathbb{R}} \left\|f(t, Bu(t))\right\|_\beta < \infty.$$

Now write $\psi = w + z$, where $w \in AA(\mathbb{X}_\beta)$ and $z \in PAP_0(\mathbb{X}_\beta)$, that is, $\Gamma_1\phi = \Xi(w) + \Xi(z)$ where

$$\Xi w(t) := \int_{-\infty}^{t} A(s)U(t,s)P(s)w(s)ds, \ \text{ and } \ \Xi z(t) := \int_{-\infty}^{t} A(s)U(t,s)P(s)z(s)ds.$$

Clearly, $\Xi(w) \in AA(\mathbb{X}_\alpha)$. Indeed, since $w \in AA(\mathbb{X}_\alpha)$, for every sequence of real numbers $(\tau_n')_{n\in\mathbb{N}}$ there exists a subsequence $(\tau_n)_{n\in\mathbb{N}}$ such that

$$v(t) := \lim_{n\to\infty} w(t+\tau_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} v(t-\tau_n) = w(t)$$

for each $t \in \mathbb{R}$. And there exists a well-defined function G such that for each $\varepsilon > 0$, one can find $N_0, N_1 \in \mathbb{N}$ such that

$$\left\|G(t,s)u - A(s+\tau_n)U(t+\tau_n, s+\tau_n)u\right\|_\alpha \leqslant \varepsilon H_0(t-s)$$

whenever $n > N_0$ for $t,s \in \mathbb{R}$, $t > s$, and

$$\left\|A(s)U(t,s)u - G(t-\tau_n, s-\tau_n)u\right\|_\alpha \leqslant \varepsilon H_1(t-s)$$

whenever $n > N_1$ for all $t,s \in \mathbb{R}$, $t > s$, for all $u \in \mathbb{X}_\alpha$, where $H_0, H_1 : [0,\infty) \mapsto [0,\infty)$ with $H_0, H_1 \in L^1[0,\infty)$.

Set $\Xi_1(t) = \int_{-\infty}^t G(t,s)P(s)v(s)ds$ for all $t \in \mathbb{R}$.

Now

$$\begin{aligned}
\Xi w(t+\tau_n) - \Xi_1(t) &= \int_{-\infty}^{t+\tau_n} A(s)U(t+\tau_n,s)P(s)w(s)ds - \int_{-\infty}^t G(t,s)P(s)v(s)ds \\
&= \int_{-\infty}^t A(s+\tau_n)U(t+\tau_n,s+\tau_n)P(s+\tau_n)w(s+\tau_n)ds \\
&\quad - \int_{-\infty}^t G(t,s)P(s)v(s)ds \\
&= \int_{-\infty}^t A(s+\tau_n)U(t+\tau_n,s+\tau_n)P(s+\tau_n)w(s+\tau_n)ds \\
&\quad - \int_{-\infty}^t A(s+\tau_n)U(t+\tau_n,s+\tau_n)P(s+\tau_n)v(s)ds \\
&\quad + \int_{-\infty}^t A(s+\tau_n)U(t+\tau_n,s+\tau_n)P(s+\tau_n)v(s)ds \\
&\quad - \int_{-\infty}^t G(t,s)P(s)v(s)ds \\
&= \int_{-\infty}^t A(s+\tau_n)U(t+\tau_n,s+\tau_n)P(s+\tau_n)\big(w(s+\tau_n)-v(s)\big)ds \\
&+ \int_{-\infty}^t \big(A(s+\tau_n)U(t+\tau_n,s+\tau_n)P(s+\tau_n) - G(t,s)P(s)\big)v(s)ds.
\end{aligned}$$

Using Eq. (3.2) and the Lebesgue Dominated Convergence Theorem it follows that

$$\left\|\int_{-\infty}^t A(s+\tau_n)U(t+\tau_n,s+\tau_n)P(s+\tau_n)\big(w(s+\tau_n)-v(s)\big)ds\right\|_\alpha \to 0 \text{ as } n\to\infty,\ t\in\mathbb{R}.$$

Similarly, in view of the above (assumption (H. 4)), we can prove that

$$\Big\| \int_{-\infty}^{t} \big(A(s+\tau_n)U(t+\tau_n, s+\tau_n)P(s+\tau_n) - G(t,s)P(s)\big)v(s)ds \Big\|_\alpha \to 0 \text{ as } n \to \infty$$

for $t \in \mathbb{R}$.

Therefore,

$$\Xi_1(t) = \lim_{n\to\infty} \Xi(w)(t+\tau_n)$$

is well-defined for all $t \in \mathbb{R}$.

Similarly, one can easily see that

$$\Xi(w)(t) = \lim_{n\to\infty} \Xi_1(t-\tau_n)$$

for all $t \in \mathbb{R}$, and hence $\Xi(w) \in AA(\mathbb{X}_\alpha)$.

Now, let $T > 0$. Again from Eq. (3.2), we have

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T} \big\|(\Xi z)(t)\big\|_\alpha dt &\leqslant \frac{1}{2T}\int_{-T}^{T}\int_0^{+\infty} \big\|A(s)U(t,s)P(s)z(t-s)\big\|_\alpha ds dt \\
&\leqslant \frac{n(\alpha)}{2T}\int_{-T}^{T}\int_0^{+\infty} s^{-\alpha}e^{-\frac{\delta}{2}s}\big\|z(t-s)\big\|_\beta ds dt \\
&\leqslant n(\alpha)\int_0^{+\infty} s^{-\alpha}e^{-\frac{\delta}{2}s}\left(\frac{1}{2T}\int_{-T}^{T}\big\|z(t-s)\big\|_\beta dt\right) ds.
\end{aligned}$$

Now

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\big\|z(t-s)\big\|_\beta dt = 0,$$

as $t \mapsto z(t-s) \in PAP_0(\mathbb{X}_\beta)$ for every $s \in \mathbb{R}$. One completes the proof by using Lebesgue Dominated Convergence Theorem.

The proof for $\Gamma_2 u(\cdot)$ is similar to that of $\Gamma_1 u(\cdot)$ except that one makes use of Eq. (3.1) instead of Eq. (3.2).

THEOREM 3.1. *Under assumptions* (H.1)-(H.5)*, the evolution equation* (1.2) *has a unique pseudo-almost automorphic mild solution whenever K is small enough, that is, $K\Theta < 1$ where*

$$\Theta = \varpi\Big[k(\alpha) + \frac{\big(m(\alpha,\beta)+m(\alpha)\big)}{\delta} + \big(n(\alpha)+c(\alpha)\big)\frac{2^{1-\alpha}\,\Gamma(1-\alpha)}{\delta^{1-\alpha}}\Big].$$

Proof. Consider the nonlinear operator $\mathbb{M}$ defined on $PAP(\mathbb{X}_\alpha)$ by

$$\begin{aligned}
\mathbb{M}u(t) = &-f(t,Bu(t)) - \int_{-\infty}^{t} A(s)U(t,s)P(s)f(s,Bu(s))ds \\
&+ \int_t^{\infty} A(s)U(t,s)Q(s)f(s,Bu(s))ds + \int_{-\infty}^{t} U(t,s)P(s)g(s,Cu(s))ds \\
&- \int_t^{\infty} U(t,s)Q(s)g(s,Cu(s))ds
\end{aligned}$$

for each $t \in \mathbb{R}$.

As we have previously seen, for every $u \in PAA(\mathbb{X}_\alpha)$, $f(\cdot, Bu(\cdot)) \in PAA(\mathbb{X}_\beta) \subset PAA(\mathbb{X}_\alpha)$. In view of Lemma 3.2 and Lemma 3.3, it follows that $\mathbb{M}$ maps $PAA(\mathbb{X}_\alpha)$ into itself. To complete the proof one has to show that $\mathbb{M}$ has a unique fixed-point.

Let $v, w \in PAA(\mathbb{X}_\alpha)$

$$\begin{aligned}
\left\|\Gamma_1(v)(t) - \Gamma_1(w)(t)\right\|_\alpha &\leqslant \int_{-\infty}^{t} \left\|A(s)U(t,s)P(s)\left[f(s,Bv(s)) - f(s,Bw(s))\right]\right\|_\alpha ds \\
&\leqslant n(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\left\|f(s,Bv(s)) - f(s,Bw(s))\right\|_\beta ds \\
&\leqslant n(\alpha)K\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\left\|Bv(s) - Bw(s)\right\| ds \\
&\leqslant n(\alpha)K\varpi\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\left\|v(s) - w(s)\right\|_\alpha ds \\
&\leqslant n(\alpha)K\varpi\left\|v - w\right\|_{\infty,\alpha}\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)} ds \\
&= n(\alpha)\frac{2^{1-\alpha}\Gamma(1-\alpha)}{\delta^{1-\alpha}}K\varpi\left\|v - w\right\|_{\infty,\alpha}.
\end{aligned}$$

Now

$$\begin{aligned}
\left\|\Gamma_2(v)(t) - \Gamma_2(w)(t)\right\|_\alpha &\leqslant \int_{t}^{\infty} \left\|A(s)U(t,s)Q(s)\left[f(s,Bv(s)) - f(s,Bw(s))\right]\right\|_\alpha ds \\
&\leqslant m(\alpha,\beta)K\int_{t}^{\infty}\left\|f(s,Bv(s)) - f(s,Bw(s))\right\|_\beta ds \\
&\leqslant m(\alpha,\beta)K\int_{t}^{\infty}\left\|Bv(s) - Bw(s)\right\| ds \\
&\leqslant m(\alpha,\beta)K\varpi\int_{t}^{+\infty}e^{\delta(t-s)}\left\|v(s) - w(s)\right\|_\alpha ds \\
&\leqslant m(\alpha,\beta)K\varpi\left\|v - w\right\|_{\infty,\alpha}\int_{t}^{+\infty}e^{\delta(t-s)} ds \\
&= \frac{m(\alpha,\beta)K\varpi}{\delta}\left\|v - w\right\|_{\infty,\alpha}.
\end{aligned}$$

Now for Γ_3 and Γ_4, we have the following approximations

$$\begin{aligned}
\left\|\Gamma_3(v)(t) - \Gamma_3(w)(t)\right\|_\alpha &\leqslant \int_{-\infty}^{t} \left\|U(t,s)P(s)\left[g(s,Cv(s)) - g(s,Cw(s))\right]\right\|_\alpha ds \\
&\leqslant \int_{-\infty}^{t} c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\left\|g(s,Cv(s)) - g(s,Cw(s))\right\| ds \\
&\leqslant Kc(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\left\|Cv(s) - Cw(s)\right\| ds \\
&\leqslant \varpi Kc(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\left\|v(s) - w(s)\right\|_\alpha ds \\
&\leqslant \frac{K\varpi c(\alpha)2^{1-\alpha}\Gamma(1-\alpha)}{\delta^{1-\alpha}}\left\|v - w\right\|_{\infty,\alpha},
\end{aligned}$$

and

$$
\begin{aligned}
\big\|\Gamma_4(v)(t)-\Gamma_4(w)(t)\big\|_\alpha &\leqslant \int_t^\infty \big\|U(t,s)Q(s)\left[g(s,Cv(s))-g(s,Cw(s))\right]\big\|_\alpha ds\\
&\leqslant \int_t^\infty m(\alpha)e^{\delta(t-s)}\big\|g(s,Cv(s))-g(s,Cw(s))\big\|ds\\
&\leqslant \int_t^\infty m(\alpha)Ke^{\delta(t-s)}\big\|Cv(s)-Cw(s)\big\|ds\\
&\leqslant \varpi m(\alpha)K\int_t^\infty e^{\delta(t-s)}\big\|v(s)-w(s)\big\|_\alpha ds\\
&\leqslant Km(\alpha)\varpi\big\|v-w\big\|_{\infty,\alpha}\int_t^{+\infty} e^{\delta(t-s)}ds\\
&= \frac{K\varpi m(\alpha)}{\delta}\big\|v-w\big\|_{\infty,\alpha}.
\end{aligned}
$$

Combining previous approximations it follows that

$$\big\|\mathbb{M}v-\mathbb{M}w\big\|_{\infty,\alpha}\leqslant \Theta\,\|v-w\|_{\infty,\alpha},$$

and hence if K is small enough, that is, $K\Theta<1$, then Eq. (1.2) has a unique solution, which obviously is its only pseudo-almost automorphic solution.

EXAMPLE 3.1. Let $\Omega\subset\mathbb{R}^N$ $(N\geqslant 1)$ be a open bounded subset with C^2 boundary $\Gamma=\partial\Omega$ and let $\mathbb{X}=L^2(\Omega)$ equipped with its natural topology $\|\cdot\|_{L^2(\Omega)}$.

Define the linear operator appearing in Eq. (1.3) as follows:

$$A(t)u=a(t,x)\Delta u \;\text{ for all }\; u\in D(A(t))=\mathbb{H}_0^1(\Omega)\cap\mathbb{H}^2(\Omega),$$

where $a:\mathbb{R}\times\Omega\mapsto\mathbb{R}$ is a jointly continuous, almost automorphic and satisfying the following assumptions:

(H.6) $\inf\limits_{t\in\mathbb{R},x\in\Omega} a(t,x)=m_0>0$, and

(H.7) there exists $L>0$ and $0<\mu\leqslant 1$ such that

$$|a(t,x)-a(s,x)|\leqslant L|s-t|^\mu$$

for all $t,s\in\mathbb{R}$ uniformly in $x\in\Omega$.

First of all, note that in view of the above, $\sup\limits_{t\in\mathbb{R},x\in\Omega} a(t,x)<\infty$. Also, a classical example of a function a satisfying the above-mentioned assumptions is for instance

$$a_\gamma(t,x)=3+\sin|x|t+\sin\gamma|x|t,$$

where $|x|=(x_1^2+\ldots+x_N^2)^{1/2}$ for each $x=(x_1,x_2,\ldots,x_N)\in\Omega$ and $\gamma\in\mathbb{R}\setminus\mathbb{Q}$.

Under previous assumptions, it is clear that the operators $A(t)$ defined above are invertible and satisfy Acquistapace-Terreni conditions. Moreover, it can be easily shown that

$$R\big(\omega, a(\cdot,x)\Delta\big)\varphi = \frac{1}{a(\cdot,x)} R\Big(\frac{\omega}{a(\cdot,x)}, \Delta\Big)\varphi \in AA(\mathbb{H}_0^1(\Omega))$$

for each $\varphi \in L^2(\Omega)$ with

$$\big\|R\big(\omega, a\Delta\big)\big\|_{B(L^2(\Omega))} \leqslant \frac{const.}{|\omega|}.$$

Furthermore, assumptions (H.1)-(H.4) are fulfilled.

For each $\mu \in (0,1)$, we take $\mathbb{X}_\mu = D((-\Delta)^\mu)$ equipped with its μ-norm $\|\cdot\|_\mu$. Moreover, we let $\alpha = \frac{1}{2}$ and suppose that $\frac{1}{2} < \beta < 1$. Letting $Bu = Cu = \widehat{\mathrm{div}}\, u$ for all $u \in \mathbb{X}_{\frac{1}{2}} = D((-\Delta)^{\frac{1}{2}}) = \mathbb{H}_0^1(\Omega)$, one easily see that both operators are bounded from $\mathbb{H}_0^1(\Omega)$ in $L^2(\Omega)$ with $\varpi = 1$.

We require the following assumption:

(H.8) Let $\frac{1}{2} < \beta < 1$, and $F \in PAA(\mathbb{H}_0^1(\Omega), \mathbb{X}_\beta)$ and $G \in PAA(\mathbb{H}_0^1(\Omega), L^2(\Omega))$. If $F = F_1 + F_2$ and $G = G_1 + G_2$ where $F_1 \in AA(L^2(\Omega), \mathbb{X}_\beta)$, $G_1 \in AA(L^2(\Omega), L^2(\Omega))$, $F_2 \in PAP_0(L^2(\Omega), \mathbb{X}_\beta)$, and $G_2 \in PAP_0(L^2(\Omega), L^2(\Omega))$, we suppose that $u \mapsto F_1(t,u), G_1(t,u)$ are uniformly continuous on bounded subsets uniformly in $t \in \mathbb{R}$. Moreover, the functions F, G are uniformly Lipschitz with respect to the second argument in the following sense: there exists $K' > 0$ such that

$$\big\|F(t,u) - F(t,v)\big\|_\beta \leqslant K'\big\|u - v\big\|_{L^2(\Omega)},$$

and

$$\big\|G(t,u) - G(t,v)\big\|_{L^2(\Omega)} \leqslant K'\big\|u - v\big\|_{L^2(\Omega)}$$

for all $u, v \in L^2(\Omega)$ and $t \in \mathbb{R}$.

We have

THEOREM 3.2. *Under assumption* (H.6)-(H.8)*, then the N-dimensional heat equation Eq.* (1.3) *has a unique pseudo-almost automorphic solution* $\varphi \in \mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega)$ *whenever K' is small enough.*

Classical examples of the above-mentioned functions $F, G : \mathbb{R} \times \mathbb{H}_0^1(\Omega) \mapsto L^2(\Omega)$ are given as follows:

$$F(t, \widehat{\mathrm{div}}\, u) = \frac{Ke(t)}{1 + |\widehat{\mathrm{div}}\, u|} \quad \text{and} \quad G(t, \widehat{\mathrm{div}}\, u) = \frac{Km(t)}{1 + |\widehat{\mathrm{div}}\, u|}$$

where the functions $e, m : \mathbb{R} \mapsto \mathbb{R}$ are pseudo-almost automorphic.

In this particular case, the corresponding heat equation, that is,

$$\begin{cases} \dfrac{\partial}{\partial t}\left[\varphi + \dfrac{Ke(t)}{1+|\widehat{\operatorname{div}\varphi}|}\right] = a_\gamma(t,x)\Delta\varphi + \dfrac{Km(t)}{1+|\widehat{\operatorname{div}\varphi}|}, \; t \in \mathbb{R}, \;\; x \in \Omega, \\ \\ \varphi = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \; \Gamma, \end{cases}$$

has a unique pseudo-almost automorphic solution $\varphi \in \mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega)$ whenever K is small enough.

REFERENCES

[1] P. ACQUISTAPACE, *Evolution operators and strong solutions of abstract linear parabolic equations,* Differential Integral Equations, **1** (1988), 433–457.

[2] P. ACQUISTAPACE, F. FLANDOLI, B. TERRENI, *Initial boundary value problems and optimal control for nonautonomous parabolic systems*, SIAM J. Control Optim., **29** (1991), 89–118.

[3] P. ACQUISTAPACE, B. TERRENI, *A unified approach to abstract linear nonautonomous parabolic equations*, Rend. Sem. Mat. Univ. Padova, **78** (1987), 47–107.

[4] H. AMANN, *Linear and Quasilinear Parabolic Problems*, Birkhäuser, Berlin 1995.

[5] B. AMIR AND L. MANIAR, *Existence and some asymptotic behaviors of solutions to semilinear Cauchy problems with non dense domain via extrapolation spaces*, Rend. Circ. Mat. Palermo II, **49** (2000), 481–496.

[6] B. AMIR AND L. MANIAR, *Composition of pseudo-almost periodic functions and cauchy problems with perator of nondense domain*, Ann. Math. Blaise Pascal, **6**, 1 (1999), 1–11.

[7] M. BAROUN, S. BOULITE, T. DIAGANA, AND L. MANIAR, *Almost periodic solutions to some semilinear non-autonomous thermoelastic plate equations*, J. Math. Anal. Appl., **349**, 1 (2009), 74–84.

[8] M. BAROUN, S. BOULITE, G. M. N'GUÉRÉKATA, AND L. MANIAR, *Almost automorphy of semilinear parabolic equations*, Electron. J. Differential Equations, **2008**, 60 (2008), 1–9.

[9] S. BOULITE, L. MANIAR, AND G. M. N'GUÉREKATA, *Almost automorphic solutions for hyperbolic semilniear evolution equations*, Semigroup Forum, **71** (2005), 231–240.

[10] D. BUGAJEWSKI AND T. DIAGANA, *Almost automorphy of the convolution operator and applications to differential and functional-differential equations*, Nonlinear Stud., **13**, 2 (2006), 129–140.

[11] D. BUGAJEWSKI, T. DIAGANA, C. M. MAHOP, *Asymptotic and pseudo almost periodicity of the convolution operator and applications to differential and integral equations*, Z. Anal. Anwend., **25**, 3 (2006), 327–340.

[12] P. CIEUTAT AND K. EZZINBI, *Existence, uniqueness and attractiveness of a pseudo almost automorphic solutions for some dissipative differential equations in Banach spaces*, J. Math. Anal. Appl., **354**, 2 (2009), 494–506.

[13] G. DA PRATO AND P. GRISVARD, *Equations d'évolution abstraites non linéaires de type parabolique*, Ann. Mat. Pura Appl., **120**, 4 (1979), 329–396.

[14] T. DIAGANA, *Existence of pseudo-almost automorphic solutions to some abstract differential equations with $\mathbb{S}^p$-pseudo-almost automorphic coefficients*, Nonlinear Anal., **70**, 11 (2009), 3781–3790.

[15] T. DIAGANA, *Pseudo almost periodic functions in Banach spaces*, Nova Science Publishers, Inc., New York, 2007.

[16] T. DIAGANA AND E. HERNÀNDEZ, *Existence and uniqueness of pseudo almost periodic solutions to some abstract partial neutral functional-differential equations and applications*, J. Math. Anal. Appl., **327**, 2 (2007), 776–791.

[17] T. DIAGANA, *Existence and uniqueness of pseudo almost periodic solutions to some classes of partial evolution equations*, Nonlinear Anal., **66**, 2 (2007), 384–395.

[18] T. DIAGANA AND G. M. N'GUÉRÉKATA, *Pseudo almost periodic mild solutions to hyperbolic evolution equationa in abstract intermediate Banach spaces*, Applicable Anal., **85**, 6-7 (2006), 769–780.

[19] T. DIAGANA, N. HENRÍQUEZ, AND E. HERNÀNDEZ, *Almost automorphic mild solutions to some partial neutral functional-differential Equations and Applications*, Nonlinear Anal., **69**, 5 (2008), 1485–1493.

[20] T. DIAGANA AND G. M. N'GUÉRÉKATA, *Almost automorphic solutions to some classes of partial evolution equations*, Appl. Math. Lett., **20**, 4 (2007), 462–466.

[21] K. J. ENGEL AND R. NAGEL, *One parameter semigroups for linear evolution equations*, Graduate texts in Mathematics, Springer Verlag 1999.

[22] K. EZZINBI, S. FATAJOU AND G. M. NGUÉRÉKATA, *Pseudo almost automorphic solutions to some neutral partial functional differential equations in Banach space*, Nonlinear Anal., **70**, 4 (2009), 1641–1647.

[23] K. EZZINBI, S. FATAJOU AND G. M. NGUÉRÉKATA, *Pseudo almost automorphic solutions for dissipative differential equations in Banach spaces*, J. Math. Anal. Appl., **351**, 2 (2009), 765–772.

[24] A. M. FINK, *Almost Periodic Differential Equations,* Lecture Notes in Mathematics, **377**, Springer-Verlag, New York-Berlin, 1974.

[25] E. HERNÁNDEZ AND H. R. HENRÍQUEZ, *Existence of periodic solutions of partial neutral functional differential equations with unbounded delay*, J. Math. Anal. Appl., **221**, 2 (1998), 499–522.

[26] E. HERNÁNDEZ, *Existence Results for Partial Neutral Integro-differential Equations with Unbounded Delay*, J. Math. Anal. Appl., **292**, 1 (2004), 194–210.

[27] E. HERNÁNDEZ M., M. L. PELICER, AND J. P. C. DOS SANTOS, *Asymptotically almost periodic and almost periodic solutions for a class of evolution equations*, Electron. J. Diff. Eqns., **2004**, 61 (2004), 1–15.

[28] Y. HINO, T. NAITO, N. V. MINH, AND J. S. SHIN, *Almost Periodic Solutions of Differential Equations in Banach Spaces*, Stability and Control: Theory, Methods and Applications, **15**, Taylor and Francis, London, 2002.

[29] J.-L. LIONS AND J. PEETRE, *Sur une classe d'espaces d'interpolation*, Inst. Hautes tudes Sci. Publ. Math., **19** (1964), 5–68.

[30] A. LUNARDI, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, PNLDE, **16**, Birkhäauser Verlag, Basel, 1995.

[31] J. LIANG, J. ZHANG, AND T-J. XIAO, *Composition of pseudo almost automorphic and asymptotically almost automorphic functions*, J. Math. Anal. Appl., **340** (2008), 1493–1499.

[32] J. LIANG, G. M. N'GUÉRÉKATA, T-J. XIAO, AND J. ZHANG, *Some properties of pseudo almost automorphic functions and applications to abstract differential equations*, Nonlinear Anal., **70**, 7 (2009), 2731–2735.

[33] L. MANIAR AND R. SCHNAUBELT, *Almost periodicity of inhomogeneous parabolic evolution equations*, Lecture Notes in Pure and Appl. Math., **234**, Dekker, New York (2003), 299–318.

[34] M. G. NASO AND A. BENABDALLAH, *Thermoelastic plate with thermal interior control*, Mathematical models and methods for smart materials (Cortona, 2001), 247–250, Ser. Adv. Math. Appl. Sci., **62**, World Sci. Publ., River Edge, NJ, 2002.

[35] G. M. N'GUÉRÉKATA, *Almost automorphic functions and almost periodic functions in abstract spaces,* Kluwer Academic / Plenum Publishers, New York-London-Moscow, 2001.

[36] G. M. N'GUÉRÉKATA, *Topics in almost automorphy*, Springer, New york, Boston, Dordrecht, Lodon, Moscow 2005.

[37] G. M. N'GUÉRÉKATA AND A. PANKOV, *Stepanov-like almost automorphic functions and monotone evolution equations*, Nonlinear Anal., **68**, 9 (2008), 2658–2667.

[38] A. PAZY, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.

[39] J. PRÜSS, *Evolutionary Integral Equations and Applications*, Birkhäuser, 1993.

[40] R. SCHNAUBELT, *Sufficient conditions for exponential stability and dichotomy of evolution equations*, Forum Math., **11** (1999), 543–566.

[41] R. SCHNAUBELT, *Asymptotically autonomous parabolic evolution equations*, J. Evol. Equ., **1** (2001), 19–37.

[42] R. SCHNAUBELT, *Asymptotic behavior of parabolic nonautonomous evolution equations*, in: M. Iannelli, R. Nagel, S. Piazzera (Eds.), Functional Analytic Methods for Evolution Equations, in: Lecture Notes in Math., **1855**, Springer-Verlag, Berlin, 2004, 401–472.

[43] T-J. XIAO, J. LIANG, J. ZHANG, *Pseudo almost automorphic solutions to semilinear differential equations in Banach spaces*, Semigroup Forum, **76**, 3 (2008), 518–524.

[44] TI-J. XIAO, X-X. ZHU, J. LIANG, *Pseudo-almost automorphic mild solutions to nonautonomous differential equations and applications*, Nonlinear Anal., **70**, 11 (2009), 4079–4085.

[45] A. YAGI, *Parabolic equations in which the coefficients are generators of infinitely differentiable semi-groups II*, Funkcial. Ekvac., **33** (1990), 139–150.

[46] A. YAGI, *Abstract quasilinear evolution equations of parabolic type in Banach spaces*, Boll. Un. Mat. Ital. B (7), **5** (1991), 341–368.

[47] S. ZAIDMAN, *Topics in Abstract Differential Equations,* Pitman Research Notes in Mathematics Ser. II John Wiley and Sons, New York, 1994/1995.

(Received August 21, 2009)

Toka Diagana
Department of Mathematics
Howard University
2441 6th Street N.W., Washington, D.C. 20059
USA
e-mail: tdiagana@howard.edu