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REGULARITY FOR THE NAVIER–STOKES–FOURIER SYSTEM

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Abstract. We prove the existence of strong 2-dimensional solutions for two Cauchy-Dirichlet problems to the Navier-Stokes-Fourier system which characterizes the Newtonian fluids under heat-conducting effects. The nonstationary Navier-Stokes system for an incompressible homogeneous fluid with temperature dependent viscosity is completed by the equation of balance of energy which includes the term of dissipative heating. The regularity of solutions to the problems under study is proved through compactness methods and fixed point arguments, instead assuming the existence of weak solutions to the problems.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain sufficiently regular and $T > 0$. Let us consider the Cauchy-Dirichlet problem in the following form:

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}(\mu(\theta)D\mathbf{u}) = \mathbf{f} - \nabla p \quad \text{in } Q_T := \Omega\times]0,T[, \\ \operatorname{div}\mathbf{u} = 0 \quad \text{in } Q_T; \end{aligned} \tag{1}$$

$$\partial_t \theta + \mathbf{u}\cdot\nabla\theta - \operatorname{div}(k(\theta)\nabla\theta) = \mu(\theta)|D\mathbf{u}|^2 + g \quad \text{in } Q_T, \tag{2}$$

$$\mathbf{u}\big|_{t=0} = \mathbf{u}_0,\ \theta\big|_{t=0} = \theta_0,\ \text{ in } \Omega, \tag{3}$$

$$\mathbf{u} = \bar{\mathbf{u}},\ \theta = \bar{\theta},\ \text{on } \partial\Omega\times]0,T[, \tag{4}$$

where $\mathbf{u}$ denotes the velocity of the fluid and $D\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$, θ the temperature, p the pressure, μ the viscosity, k the thermal conductivity, $\mathbf{f}$ denotes the given external body forces and g the heat source. In the present work the product of two tensors is given by $D : \tau = D_{ij}\tau_{ij}$, under the Einstein convention, and the norm by $|D|^2 = D : D$.

The Navier-Stokes-Fourier system arises from fluid thermomechanics. In fact, it is constituted by momentum and energy equations when the constitutive relations for the Cauchy stress and heat flux are assumed linear. The density is constant and assumed

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equal to one. The initial conditions are given in (3) and we assume Dirichlet boundary conditions in (4). For the sake of clarity we found convenient that the boundary conditions are taken to be homogeneous,

$$\bar{\mathbf{u}} \equiv \mathbf{0} \quad \text{and} \quad \bar{\theta} \equiv 0. \tag{5}$$

The abstract mathematical viscosity can be illustrated with Arrhenius law:

$$\mu(\theta) = \mu_0 e^{E_0(1/\theta - 1/\theta_r)/R},$$

where μ_0, E_0, θ_r are constants of reference and R is the gas constant.

In the seventies, Lauder and Spalding proposed the $k-\varepsilon$ model (it consists of two equations for the turbulence kinetic energy k and the rate of dissipation ε of the turbulent energy) to describe the mean of a turbulence flow. Unfortunately, the turbulence is essentially a three dimensional phenomenon and it is not clear that this model produces physically relevant results (a positive energy, for example). Despite the fact that the validity of $k-\varepsilon$ model is not universal, it presents a good compromise between simplicity and generality (see [18]). In this context, the equations (1) represent the averaged Navier-Stokes equation in which u, π and f are the mean values of velocity, pressure and external forces, respectively, the viscosity is the eddy viscosity, and the equation (2) represents the $k-\varepsilon$ model, that is, θ denotes the mean turbulent kinetic energy and $g = -\theta|\theta|^{1/2}$ denoting the Navier-Stokes turbulence. More physical motivation can be found in [1, 3] for instance.

Several authors proved existence of solutions to similar mathematical problems in fluid thermomechanics (see for example [4, 8, 13, 16, 19, 23] and the references therein). The existence of at least a weak solution is given in [6] for different constitutive relations in the Cauchy stress and in the Fourier heat flux. We refer to [7] for the existence of strong and classical solutions to the stationary coupled system under general constitutive relations.

Although the continuity of the coefficients, to prove the regularity of solution to the coupled system, additional terms appear which invalidate the direct application of known regularity results ([9, 11, 12, 15, 17, 25] between others). Notice that if the velocity $\mathbf{u}$ is a weak solution to (1) in the sense of $\mathbf{u} \in L^2(0,T;\mathbf{H}^1_0(\Omega))$, the Joule effect term $\mu(\theta)|D\mathbf{u}|^2$ belongs to $L^1(Q_T)$ and the existence of a solution of the energy equation (2) requires L^1-theory (see [5, 6] and the references therein). We wish to emphasize that at the present work we do not show regularity for every weak solution, we prove existence results under smallness restrictions only on the ratio between the derivatives of the viscosity and the thermal conductivity functions and their lower bounds (cf. (12)). Indeed, here we prove that the equation (which is satisfied by the solution) is valid almost everywhere in Q_T, which means that the strong solutions coincide by the uniqueness result with the weak solutions in a smaller space.

The outline of the paper is as follows. In next section we present the appropriate functional framework and we state two main existence results and the corresponding uniqueness without any additional assumption on the data. First existence result is established under a given heat-production profile $g = g(x,t)$ and the second one is given for $g = -\theta|\theta|^{1/2}$. In Section 3, we recall and prove some technical results for any dimension $n \geqslant 2$. In Section 4, we deal with the apriori estimates. Sections 5 and 6 are devoted to the proofs of the solvability of the problems under study. The uniqueness results are proved in Section 7.

2. Assumptions and main results

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with sufficiently smooth boundary $\partial\Omega$. In the framework of Lebesgue and Sobolev spaces, we introduce for $q > 1$, see [11],

$$\mathbf{J}^{1,q}(\Omega) = \{\mathbf{u} \in \mathbf{W}^{1,q}(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega\},$$

where the vector spaces of vector-valued or tensor-valued functions are denoted by bold. It is known that $\mathbf{J}_0^{1,q}(\Omega) = \mathbf{J}^{1,q}(\Omega) \cap \mathbf{W}_0^{1,q}(\Omega)$ with norm

$$\|\cdot\|_{(1),q,\Omega} = \|\nabla \cdot\|_{q,\Omega}.$$

We will use the following Banach spaces, for $1 \leqslant q, r \leqslant \infty$, see [12],

$$\begin{aligned} L^{q,r}(Q_T) &= L^r(0,T;L^q(\Omega)) \\ W_q^{1,0}(Q_T) &= L^q(0,T;W^{1,q}(\Omega)) \\ W_q^{1,1}(Q_T) &= L^q(0,T;W^{1,q}(\Omega)) \cap W^{1,q}(0,T;L^q(\Omega)) \\ W_q^{2,1}(Q_T) &= L^q(0,T;W^{2,q}(\Omega)) \cap W^{1,q}(0,T;L^q(\Omega)). \end{aligned}$$

We recall that the following continuous inclusion $W_q^{2,1}(Q_T) \hookrightarrow C^{k,\alpha}(\bar{Q}_T)$ only occurs if $q > 4/(2-k)$ and $0 \leqslant \alpha < 2-k-4/q$. This means for $k=0$ that $q > 2$, i.e., the Banach space $W_2^{2,1}(Q_T)$ is not embedded in the Banach space of Hölder continuous functions with exponent α in the x-variables and $\alpha/2$ in the t-variable. Note that $u(0)$ makes sense for all $u \in W_q^{1,1}(Q_T)$ since $W_q^{1,1}(Q_T) \hookrightarrow C([0,T];L^q(\Omega))$.

The following assertions on data are assumed as well as the following assumptions on the physical parameters appearing in the equations are established:

- $\mathbf{f} : Q_T \to \mathbb{R}^2$ is given such that $\mathbf{f} \in \mathbf{L}^2(Q_T)$ and

$$\partial_t \mathbf{f} \in \mathbf{L}^2(Q_T); \tag{6}$$

- $\mu, k : \mathbb{R} \to \mathbb{R}$ are functions of class C^1 such that

$$0 < \mu_0 \leqslant \mu(s) \leqslant \mu_1,\ |\mu'(s)| \leqslant \mu_2,\ \forall s \in \mathbb{R}, \tag{7}$$

$$0 < k_0 \leqslant k(s) \leqslant k_1,\ |k'(s)| \leqslant k_2,\ \forall s \in \mathbb{R}; \tag{8}$$

- $\mathbf{u}_0 \in \mathbf{J}_0^{1,2}(\Omega) \cap \mathbf{H}^2(\Omega)$ and $\theta_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfy the following compatibility conditions

$$\nabla \mathbf{u}_0 \cdot \mathbf{n} = \mathbf{0}, \quad \nabla \theta_0 \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega, \tag{9}$$

where $\mathbf{n}$ denotes the unit outward normal to the boundary $\partial\Omega$.

The system (1)-(5) has the variational formulation

$$\begin{cases} \int_{Q_T} \partial_t \mathbf{u}\cdot\mathbf{v}\,dxdt + \int_{Q_T}\Big(\mu(\theta)D\mathbf{u}:D\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\Big)dxdt = \int_{Q_T}\mathbf{f}\cdot\mathbf{v}\,dxdt, \\ \qquad \forall \mathbf{v}\in L^2(0,T;\mathbf{J}_0^{1,2}(\Omega)),\ \mathbf{u}\big|_{t=0} = \mathbf{u}_0 \text{ in } \Omega; \\ \int_{Q_T}(\partial_t\theta)\eta\,dxdt + \int_{Q_T}\Big(k(\theta)\nabla\theta\cdot\nabla\eta + \mathbf{u}\cdot\nabla\theta\eta\Big)dxdt = \int_{Q_T}\Big(\mu(\theta)|D\mathbf{u}|^2+g\Big)\eta\,dxdt, \\ \qquad \forall\eta\in L^2(0,T;H_0^1(\Omega)),\ \theta\big|_{t=0} = \theta_0 \text{ in } \Omega. \end{cases} \tag{10}$$

REMARK 2.1. For all $\mathbf{u},\mathbf{v}\in\mathbf{J}_0^{1,2}(\Omega)$, the convective term verifies

$$\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,dx = \int_\Omega \nabla\mathbf{u}:\mathbf{v}\otimes\mathbf{u}\,dx = \int_\Omega(\nabla\mathbf{u})^T:\mathbf{u}\otimes\mathbf{v}\,dx = 2\int_\Omega D\mathbf{u}:\mathbf{u}\otimes\mathbf{v}\,dx.$$

THEOREM 2.1. *Suppose that* (6)-(9) *be fulfilled. Let* $g: Q_T\to\mathbb{R}$ *be such that* $g\in L^2(Q_T)$ *and*

$$\partial_t g\in L^2(Q_T). \tag{11}$$

Under the assumption that

$$\frac{\mu_2^2}{\mu_0},\ \frac{\mu_2^4}{\mu_0^3},\ \frac{k_2^4}{k_0^2},\ \frac{k_2^4}{k_0^3},\quad \textit{and}\quad \frac{\mu_2^4}{k_0}\ \textit{are sufficiently small,} \tag{12}$$

then the problem (10) *admits, at least, one solution*

$$(\mathbf{u},\theta)\in L^2(0,T;\mathbf{J}_0^{1,2}(\Omega))\times L^2(0,T;H_0^1(\Omega)),$$

which is strong, i.e.

$$(\mathbf{u},\theta)\in\mathbf{W}_2^{2,1}(Q_T)\times W_2^{2,1}(Q_T).$$

Moreover, such solution is Hölder continuous, $(\mathbf{u},\theta)\in\mathbf{C}^{0,\alpha}(\bar{Q}_T)\times C^{0,\alpha}(\bar{Q}_T)$*, for some* $\alpha>0$.

REMARK 2.2. The smallness of the data in (12) is not explicitly given, because there does not exist a unique expression. For instance, we can take

$$\frac{\mu_2^2}{\mu_0},\ \frac{\mu_2^4}{2\mu_0^3},\ \frac{2k_2^4}{k_0^2},\ \frac{2k_2^4}{k_0^3},\quad \text{and}\quad \frac{\mu_2^4}{2k_0}\leqslant\frac{1}{R},$$

with $R = \mathscr{H}(1,1)+\mathscr{Q}(1,1,1,1,1)$ (cf. (36)). Even more the estimative functions stated in Section 4 depend on the application of the Young's inequality $ab\leqslant a^r/r+b^s/s$, for $a,b>0$ and $r,s>1$ such that $1/r+1/s=1$. Notice that (12) is verified if the viscosity and thermal conductivity are constants, i.e. $\mu_2=k_2=0$.

It is known that the pressure is recovered as a distribution from the variational formulation thanks to the De Rham Theorem [14]. Using Theorem 2.1 we can rewrite (1) as

$$\nabla p = \mathbf{f}-\partial_t\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{u}+\mu'(\theta)\nabla\theta D\mathbf{u}+\mu(\theta)\Delta\mathbf{u}\in\mathbf{L}^2(Q_T).$$

THEOREM 2.2. *Suppose that* (6)-(9) *and* (12) *be fulfilled. Under* $g = -\theta|\theta|^{1/2}$, *the problem* (10) *admits, at least, one strong solution. Moreover, such solution is Hölder continuous, and* $\theta \in W_r^{2,1}(Q_T)$, *for all* $r < 3$.

Finally, let us state the uniqueness result.

THEOREM 2.3. *The solutions obtained in Theorem* 2.1 *and* 2.2 *are unique.*

Henceforth we denote by C every positive constant depending on the data, but not on the unknown functions $\mathbf{u}$, p or θ.

3. Technical results

Here we assume that $\Omega \subset \mathbb{R}^n$, for any dimension $n \geqslant 2$. Let us begin to recall two important results. Indeed their application in the present work will be only on the two-dimensional case.

LEMMA 3.1. (interpolative inequalities [21]) *The interpolative inequalities hold*

$$\forall v \in H^2(\Omega), \quad \|\nabla v\|_{q,\Omega} \leqslant \|v\|_{2,\Omega}^{\frac{(2-n)q+2n}{4q}} \|\nabla^2 v\|_{2,\Omega}^{\frac{(2+n)q-2n}{4q}}, \tag{13}$$

$$\forall v \in H_0^1(\Omega), \quad \|v\|_{q,\Omega} \leqslant \|v\|_{r,\Omega}^{\frac{r(2q+(2-q)n)}{q(2r+(2-r)n)}} \|\nabla v\|_{2,\Omega}^{\frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{r}-\frac{n-2}{2n}}}.$$

In particular, $L^{r,\infty}(Q_T) \cap L^2(0,T;H^1(\Omega)) \hookrightarrow L^{2(n+r)/n}(Q_T)$ *and*

$$\forall v \in H_0^1(\Omega), \quad \|v\|_{q,\Omega} \leqslant \|v\|_{2,\Omega}^{\frac{2q+(2-q)n}{2q}} \|\nabla v\|_{2,\Omega}^{\left(\frac{1}{2}-\frac{1}{q}\right)n}, \tag{14}$$

$$\forall v \in H^1(\Omega), \quad \|v\|_{2(n+2)/n,\Omega} \leqslant \|v\|_{2,\Omega}^{\frac{2}{n+2}} \|\nabla v\|_{2,\Omega}^{\frac{n}{n+2}}. \tag{15}$$

Taking $n = 2$ in Lemma 3.1 we obtain $L^{2,\infty}(Q_T) \cap L^2(0,T;H^1(\Omega)) \hookrightarrow L^4(Q_T)$, and if we take $n = 3$ we have $L^{2,\infty}(Q_T) \cap L^2(0,T;H^1(\Omega)) \hookrightarrow L^{10/3}(Q_T)$.

LEMMA 3.2. ([20, Lemma 4]) *Let* $\delta > 0$, $\alpha > 0$ *and* $q > 1$. *For any function* $v \in L^\infty(0,T;C^{0,\alpha}(\Omega))$ *verifying* $\partial_t v \in L^q(Q_T)$, *there exists a constant* $C > 0$ *such that*

$$|v(x,t_1) - v(x,t_2)| \leqslant C(\|v\|_{L^\infty(0,T;C^{0,\alpha}(\Omega))} + \|\partial_t v\|_{q,Q_T})|t_1 - t_2|^\beta,$$

for every $x \in B_\delta$ *and every* $t_1, t_2 \in]0,T[$, *where* $\beta = \alpha(q-1)/(\alpha q + n(q-1))$. *In particular,* v *is Hölder continuous in* Q_T.

Next let us prove a crucial embedding proposition.

PROPOSITION 3.1. *Assuming* $v \in L^{2(q-1)}(Q_T)$, *for any* $q \geqslant 2$, *and* $\partial_t v \in L^2(Q_T)$ *then* v *belongs to* $L^\infty(0,T;L^q(\Omega))$.

Proof. Let us argue as in [10] writing

$$|v(\cdot,t)|^q = \int_0^t \frac{d}{ds}|v(\cdot,s)|^q ds = q\int_0^t |v(\cdot,s)|^{q-2}v(\cdot,s)\partial_s v(\cdot,s)ds.$$

Integrating with respect to the space variable and using the Schwarz's inequality, it follows

$$\begin{aligned}\|v\|_{q,\Omega}^q(t) &\leqslant q\int_0^t \||v|^{q-1}\|_{2,\Omega}\|\partial_s v\|_{2,\Omega}\,ds \\ &\leqslant q\left(\int_0^t \|v\|_{2(q-1),\Omega}^{2(q-1)}ds\right)^{1/2}\left(\int_0^t \|\partial_s v\|_{2,\Omega}^2\,ds\right)^{1/2}.\end{aligned}$$

Then we can conclude

$$\sup_{0\leqslant t\leqslant T}\|v\|_{q,\Omega}^q(t) \leqslant q\|v\|_{2(q-1),Q_T}^{q-1}\|\partial_t v\|_{2,Q_T}.$$

Finally, let us prove the following regularity result.

PROPOSITION 3.2. *Assuming $v\in L^{2,\infty}(Q_T)\cap L^2(0,T;H^1(\Omega))$ and $\partial_t v\in L^2(Q_T)$ then v belongs to $L^q(Q_T)$, for any $q<2(n+1)/(n-1)$. Moreover, v belongs to $L^{r,\infty}(Q_T)$, for any $r<2n/(n-1)$.*

Proof. From Lemma 3.1, we have $v\in L^{2(n+2)/n}(Q_T)$ and we apply Proposition 3.1 with $2(q-1)=2(n+2)/n$, i.e., $q=1+(n+2)/n$. Next, using Lemma 3.1 with $v\in L^{1+(n+2)/n,\infty}(Q_T)\cap L^2(0,T;H^1(\Omega))$ we obtain $v\in L^{2(n+1+(n+2)/n)/n}(Q_T)$. Define

$$q_0 = \frac{n+2}{n}, \qquad q_1 = \frac{n+1+q_0}{n}$$

and arguing by iteration, we apply Proposition 3.1 with $2(q-1)=2q_k$, i.e., $q=1+q_k$. Now, using Lemma 3.1 with $v\in L^{1+q_k,\infty}(Q_T)\cap L^2(0,T;H^1(\Omega))$ we obtain $v\in L^{2\frac{n+1+q_k}{n}}(Q_T)$.

Thus defining by recurrence

$$q_{k+1} = \frac{n+1+q_k}{n}, \quad k\in\mathbb{N},$$

this sequence is monotone increasing, bounded onto $]0,(n+1)/(n-1)[$ and its limit is $q=(n+1)/(n-1)$, which concludes the first statement of the proof of Proposition 3.2.

Again applying Proposition 3.1 with $2(r-1)=q<2(n+1)/(n-1)$, i.e., $r=q/2+1<2n/(n-1)$, we get $v\in L^{q/2+1,\infty}(Q_T)$ if $q\geqslant 2$.

4. Apriori estimates

The main result theorems 2.1 and 2.2 are proved using the following fixed point argument. We fix $\xi \in W_4^{1,1}(Q_T)$ and we consider the following auxiliary problems:

$$\langle \partial_t \mathbf{u}, \mathbf{v}\rangle_{(\mathbf{J}_0^{1,2}(\Omega))'\times \mathbf{J}_0^{1,2}(\Omega)} + \int_\Omega \Big(\mu(\xi) D\mathbf{u} : D\mathbf{v} + \mathbf{v}\otimes\mathbf{u} : \nabla\mathbf{u}\Big)\, dx$$
$$= \int_\Omega \mathbf{f}\cdot\mathbf{v}\, dx, \quad \text{a.e. } t\in[0,T], \quad \forall \mathbf{v}\in \mathbf{J}_0^{1,2}(\Omega),\ \mathbf{u}\big|_{t=0} = \mathbf{u}_0 \text{ in } \Omega; \tag{16}$$

$$\langle \partial_t \theta, \eta\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} + \int_\Omega \Big(k(\xi)\nabla\theta\cdot\nabla\eta + \mathbf{u}\cdot\nabla\theta\,\eta\Big)\, dx$$
$$= \int_\Omega \Big(\mu(\xi)|D\mathbf{u}|^2 + g\Big)\eta\, dx, \quad \text{a.e. } t\in[0,T],\ \forall \eta\in H_0^1(\Omega),\ \theta\big|_{t=0} = \theta_0 \text{ in } \Omega. \tag{17}$$

In this section, we assume the existence of solutions to (16)-(17) and we prove some apriori estimates. In order to emphasize the key ideas, in the sequel the *apriori estimates* technique is taken care of (see Remark 4.1).

PROPOSITION 4.1. *Under the assumptions* $\mathbf{f}\in \mathbf{L}^2(Q_T)$, (7) *and* $\mathbf{u}_0 \in \mathbf{J}_0^{1,2}(\Omega)$, *if* $\nabla\xi \in \mathbf{L}^4(Q_T)$ *then any possible solution* $\mathbf{u}$ *of* (16) *is such that* $\nabla\mathbf{u}$ *belongs to a bounded set of* $L^\infty(0,T;\mathbf{L}^2(\Omega))\cap L^2(0,T;\mathbf{J}_0^{1,2}(\Omega))$ *depending on* ξ *in the sense of* (19) *and* (20), *respectively. Moreover, it satisfies*

$$\|\nabla\mathbf{u}\|_{4,Q_T}^4 \leqslant \mathscr{F}\Big(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4\Big), \tag{18}$$

with $\mathscr{F}$ *the positive strictly increasing function on its argument defined by*

$$\mathscr{F}(d) = \frac{4}{\mu_0}\big(T\|\nabla\mathbf{u}_0\|_{2,\Omega}^2 + \frac{4}{\mu_0}\|\mathbf{f}\|_{2,Q_T}^2\big)^2 F(d);$$
$$F(d) = \exp[d](1 + d\exp[d]).$$

Proof. We choose $\mathbf{v} = \Delta\mathbf{u}$ as a test function in (16) (cf. Remark 4.1), then

$$\int_\Omega \partial_t(\nabla\mathbf{u})\cdot\nabla\mathbf{u}\, dx + \int_\Omega \{\mu'(\xi)\nabla\xi\otimes D\mathbf{u} + \mu(\xi)\nabla^2\mathbf{u}\} : \nabla^2\mathbf{u}\, dx$$
$$= -\int_\Omega \Delta\mathbf{u}\otimes\mathbf{u} : \nabla\mathbf{u}\, dx - \int_\Omega \mathbf{f}\cdot\Delta\mathbf{u}\, dx.$$

Here $\nabla^2\mathbf{u} = (\partial_k D_{ij})$ is a third order tensor, with $D_{ij} = ((\partial_j u_i + \partial_i u_j)/2)$, and $\nabla\xi\otimes D\mathbf{u} : \nabla^2\mathbf{u} = \partial_k\xi D_{ij}\partial_k D_{ij}$ under the Einstein convention.

Using the assumption (7) and the property of the convective term vanishes in the two dimensional space (cf. [17], for instance) we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_{2,\Omega}^2 + \mu_0\int_\Omega |\nabla^2\mathbf{u}|^2 dx \leqslant \mu_2\int_\Omega |\nabla\xi\otimes D\mathbf{u} : \nabla^2\mathbf{u}|\, dx + \int_\Omega |\mathbf{f}\cdot\Delta\mathbf{u}|\, dx.$$

Applying the Hölder's inequality and integrating in time for each $t \in]0,T[$, it follows

$$\frac{1}{2}\|\nabla \mathbf{u}\|_{2,\Omega}^2(t) + \mu_0 \int_0^t \int_\Omega |\nabla^2 \mathbf{u}|^2 dxds \leqslant \frac{1}{2}\|\nabla \mathbf{u}(0)\|_{2,\Omega}^2 + \mu_2 \int_0^t \|\nabla \xi\|_{4,\Omega}\|\nabla \mathbf{u}\|_{4,\Omega}\|\nabla^2 \mathbf{u}\|_{2,\Omega}\, ds + \int_0^t \|\mathbf{f}\|_{2,\Omega}\|\nabla^2 \mathbf{u}\|_{2,\Omega}\, ds.$$

Applying Lemma 3.1, (15) with $n=2$ and $v=\nabla \mathbf{u}$, to the second term on right hand side of the above inequality and successively using the Young's inequality, we obtain

$$\frac{1}{2}\|\nabla \mathbf{u}\|_{2,\Omega}^2(t) + \mu_0 \int_0^t \|\nabla^2 \mathbf{u}\|_{2,\Omega}^2 ds \leqslant \frac{1}{2}\|\nabla \mathbf{u}(0)\|_{2,\Omega}^2 ds + \frac{\mu_2^4}{4\mu_0^3}\int_0^t \|\nabla \xi\|_{4,\Omega}^4 \|\nabla \mathbf{u}\|_{2,\Omega}^2 ds + \frac{3\mu_0}{4}\int_0^t \|\nabla^2 \mathbf{u}\|_{2,\Omega}^2 ds + \frac{2}{\mu_0}\int_0^t \|\mathbf{f}\|_{2,\Omega}^2 ds + \frac{\mu_0}{8}\int_0^t \|\nabla^2 \mathbf{u}\|_{2,\Omega}^2 ds.$$

Thus, we deduce

$$\frac{1}{2}\|\nabla \mathbf{u}\|_{2,\Omega}^2(t) + \frac{\mu_0}{8}\int_0^t \|\nabla^2 \mathbf{u}\|_{2,\Omega}^2 ds \leqslant \frac{1}{2}\|\nabla \mathbf{u}_0\|_{2,\Omega}^2 + \frac{2}{\mu_0}\int_0^t \|\mathbf{f}\|_{2,\Omega}^2 ds + \frac{\mu_2^4}{4\mu_0^3}\int_0^t \|\nabla \xi\|_{4,\Omega}^4 \|\nabla \mathbf{u}\|_{2,\Omega}^2 ds.$$

Thus we conclude the estimate in $L^\infty(0,T;\mathbf{L}^2(\Omega))$ with help of the Gronwall's lemma

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\nabla \mathbf{u}\|_{2,\Omega}^2 \leqslant (T\|\nabla \mathbf{u}_0\|_{2,\Omega}^2 + \frac{4}{\mu_0}\|\mathbf{f}\|_{2,Q_T}^2)\exp[\mathscr{G}(\xi)] \tag{19}$$

where $\mathscr{G}(\xi) = \frac{\mu_2^4}{2\mu_0^3}\|\nabla \xi\|_{4,Q_T}^4$.

Next the estimate in $L^2(0,T;\mathbf{H}_0^1(\Omega))$ follows

$$\|\nabla^2 \mathbf{u}\|_{2,Q_T}^2 \leqslant \frac{4}{\mu_0}(T\|\nabla \mathbf{u}_0\|_{2,\Omega}^2 + \frac{4}{\mu_0}\|\mathbf{f}\|_{2,Q_T}^2)\,(1+\mathscr{G}(\xi)\exp[\mathscr{G}(\xi)]). \tag{20}$$

Finally applying Lemma 3.1, (15) with $n=2$ and $v=\nabla \mathbf{u}$, $\nabla \mathbf{u}$ belonging to $L^2(0,T;\mathbf{H}_0^1(\Omega)\cap \mathbf{L}^{2,\infty}(Q_T)$ implies (18).

REMARK 4.1. The correctness of the test function should be understood as local, after flattening the boundary, and by the well-known method of tangential differential quotients due to Nirenberg, i.e. making use of the operator

$$\tau_{i,h}v(x,t) = v(x+h\mathbf{e}_i,t),\ 1 \leqslant i \leqslant n-1,\ h\in\mathbb{R},$$

and $\mathbf{e}_i$ being the unit vector in the direction x_i then, for $h \neq 0$ small enough, we take

$$\left(\frac{\tau_{i,h}+\tau_{i,-h}-2}{h^2}\right)v.$$

Next applying discrete integration by parts, using the fact that the adjoint of the operator $\tau_{i,h}$ is $\tau_{i,-h}$, the square integrability to all second derivatives of $\mathbf{u}$ follows from the passage to the limit as h tends to zero.

LEMMA 4.1. *Under the assumptions of Proposition* 4.1, (8), $\theta_0 \in L^2(\Omega)$ *and* $g \in L^2(Q_T)$, *then any possible solution* $(\mathbf{u}, \theta)$ *of* (16)-(17) *satisfies*

$$\|\mathbf{u}\|_{4,Q_T}^4 \leqslant \frac{1}{\mu_0}\big(T\|\mathbf{u}_0\|_{2,\Omega}^2 + \frac{1}{\mu_0}\|\mathbf{f}\|_{2,Q_T}^2\big)^2, \tag{21}$$

$$\|\theta\|_{4,Q_T}^4 \leqslant \frac{1}{k_0}\left(T\|\theta_0\|_{2,\Omega}^2 + \frac{2}{k_0}\left(\mu_1^2\mathscr{F}(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4) + \|g\|_{2,Q_T}^2\right)\right)^2. \tag{22}$$

Proof. The energy inequalities hold

$$\|\mathbf{u}\|_{2,\Omega}^2(t) + \mu_0\int_0^t \|\nabla\mathbf{u}\|_{2,\Omega}^2 ds \leqslant \|\mathbf{u}_0\|_{2,\Omega}^2 + \frac{1}{\mu_0}\int_0^t \|\mathbf{f}\|_{2,\Omega}^2 ds; \tag{23}$$

$$\|\theta\|_{2,\Omega}^2(t) + k_0\int_0^t \|\nabla\theta\|_{2,\Omega}^2 ds \leqslant \|\theta_0\|_{2,\Omega}^2 + \frac{1}{k_0}\int_0^t \big(\mu_1\|\nabla\mathbf{u}\|_{4,\Omega}^2 + \|g\|_{2,\Omega}\big)^2 ds.$$

To prove the estimates in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ we do simultaneously as in standard manner. Using (15) and (18) we obtain (21)-(22).

REMARK 4.2. If $g = -\theta|\theta|^{1/2}$ the energy inequality on θ given in the proof of Lemma 4.1 can be simplified since

$$\int_\Omega g\theta dx = -\int_\Omega |\theta|^{1/2}\theta^2 dx \leqslant 0.$$

Consequently (22) reads

$$\|\theta\|_{4,Q_T}^4 \leqslant \frac{1}{k_0}\left(T\|\theta_0\|_{2,\Omega}^2 + \frac{\mu_1^2}{k_0}\mathscr{F}(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4)\right)^2. \tag{24}$$

PROPOSITION 4.2. *Under the assumptions of Lemma* 4.1, *and* $\theta_0 \in H_0^1(\Omega)$, *if* $\nabla\xi \in \mathbf{L}^4(Q_T)$ *then any possible solution* θ *of* (17) *is such that* $\nabla\theta$ *belongs to a bounded set of* $L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}_0^1(\Omega))$ *depending on* ξ *in the sense of* (26) *and* (27), *respectively. Moreover, it satisfies*

$$\|\nabla\theta\|_{4,Q_T}^4 \leqslant \mathscr{H}\left(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4, \frac{2k_2^4}{k_0^3}\|\nabla\xi\|_{4,Q_T}^4\right) \tag{25}$$

where $\mathscr{H}$ *is defined, for all* $d_1, d_2 \in \mathbb{R}$, *by*

$$\mathscr{H}(d_1,d_2) = \frac{4}{k_0}\left(T\|\nabla\theta_0\|_{2,\Omega}^2 + \frac{4}{k_0}(\mu_1^2\mathscr{F}(d_1) + \|g\|_{2,Q_T}^2)\right)^2$$
$$F(d_2 + \frac{2}{\mu_0 k_0^3}(T\|\mathbf{u}_0\|_{2,\Omega}^2 + \frac{1}{\mu_0}\|\mathbf{f}\|_{2,Q_T}^2)^2),$$

with $\mathscr{F}$ *and* F *given as in Proposition* 4.1.

Proof. Proceeding as in the proof of Proposition 4.1, we choose $\eta = \Delta\theta$ as a test function in (17), then we have

$$\frac{1}{2}\|\nabla\theta\|_{2,\Omega}^2(t) + k_0\int_0^t \|\nabla^2\theta\|_{2,\Omega}^2 ds \leqslant \frac{1}{2}\|\nabla\theta(0)\|_{2,\Omega}^2$$
$$+ k_2\int_0^t \|\nabla\xi\|_{4,\Omega}\|\nabla\theta\|_{4,\Omega}\|\nabla^2\theta\|_{2,\Omega} ds + \int_0^t \|\mathbf{u}\|_{4,\Omega}\|\nabla\theta\|_{4,\Omega}\|\nabla^2\theta\|_{2,\Omega} ds$$
$$+ \mu_1\int_0^t \|\nabla\mathbf{u}\|_{4,\Omega}^2\|\nabla^2\theta\|_{2,\Omega} ds + \int_0^t \|g\|_{2,\Omega}\|\nabla^2\theta\|_{2,\Omega} ds,$$

denoting by ∇^2 the tensor (∂_{ij}) of second order spatial derivatives.

Applying interpolation (15) and the Young's inequality, we deduce

$$\frac{1}{2}\|\nabla\theta\|_{2,\Omega}^2(t) + k_0\int_0^t \|\nabla^2\theta\|_{2,\Omega}^2 ds \leqslant \frac{1}{2}\|\nabla\theta_0\|_{2,\Omega}^2$$
$$+ \frac{1}{4k_0^3}\int_0^t (k_2\|\nabla\xi\|_{4,\Omega} + \|\mathbf{u}\|_{4,\Omega})^4 \|\nabla\theta\|_{2,\Omega}^2 ds$$
$$+ \frac{3k_0}{4}\int_0^t \|\nabla^2\theta\|_{2,\Omega}^2 ds + \frac{2}{k_0}\int_0^t \left(\mu_1^2\|\nabla\mathbf{u}\|_{4,\Omega}^4 ds + \|g\|_{2,\Omega}^2\right) ds + \frac{k_0}{8}\int_0^t \|\nabla^2\theta\|_{2,\Omega}^2 ds.$$

Using (18) and (21), from the Gronwall's lemma we conclude

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\nabla\theta\|_{2,\Omega}^2$$
$$\leqslant \left(T\|\nabla\theta_0\|_{2,\Omega}^2 + \frac{4}{k_0}(\mu_1^2\mathscr{F}(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4) + \|g\|_{2,Q_T}^2)\right)\exp[\mathscr{I}(\xi)] \tag{26}$$

where

$$\mathscr{I}(\xi) = \frac{2k_2^4}{k_0^3}\|\nabla\xi\|_{4,Q_T}^4 + \frac{2}{\mu_0 k_0^3}\left(T\|\mathbf{u}_0\|_{2,\Omega}^2 + \frac{1}{\mu_0}\|\mathbf{f}\|_{2,Q_T}^2\right)^2.$$

Then we get

$$\|\nabla^2\theta\|_{2,Q_T}^2 \leqslant \frac{4}{k_0}\left(T\|\nabla\theta_0\|_{2,\Omega}^2 + \frac{4}{k_0}(\mu_1^2\mathscr{F}(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4) + \|g\|_{2,Q_T}^2)\right)$$
$$(1 + \mathscr{I}(\xi)\exp[\mathscr{I}(\xi)]) \tag{27}$$

and consequently (25).

REMARK 4.3. For $g = -\theta|\theta|^{1/2}$, the proof of Proposition 4.2 is still valid if we take into account that

$$\int_\Omega g\Delta\theta dx \leqslant \|\theta\|_{4,\Omega}\|\theta\|_{2,\Omega}^{1/2}\|\Delta\theta\|_{2,\Omega}.$$

Consequently, using (24) we conclude (25) with $\|g\|_{2,Q_T}$ replaced by

$$C(|\Omega|)((T\|\theta_0\|_{2,\Omega}^2 + \frac{\mu_1^2}{k_0}\mathscr{F}(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4))/k_0)^{3/8}.$$

PROPOSITION 4.3. *Under the assumptions* (6)-(7) *and* (9), *if* $\xi \in W_4^{1,1}(Q_T)$ *such that* $\nabla\xi(0) = \nabla\theta_0$ *then any solution* $\mathbf{u}$ *of* (16) *is such that* $\partial_t\mathbf{u}$ *belongs to a bounded set of* $L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}_0^1(\Omega))$ *depending on* ξ *in the sense of* (30) *and* (31), *respectively. Moreover, it satisfies*

$$\|\partial_t\mathbf{u}\|_{4,Q_T}^4 \leqslant \mathscr{P}\left(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4, \frac{\mu_2^2}{\mu_0}\|\partial_t\xi\|_{4,Q_T}^4\right) \tag{28}$$

where $\mathscr{P}$ *is defined, for all* $d_1, d_2 \in \mathbb{R}$, *by*

$$\mathscr{P}(d_1,d_2) = \frac{1}{\mu_0}\left(TA^2 + \|\partial_t\mathbf{f}\|_{2,Q_T}^2 + d_2 + \frac{\mu_2^2}{\mu_0}\mathscr{F}(d_1)\right)^2 F(\mathscr{A}),$$
$$\mathscr{A} = T + \frac{2}{\mu_0^2}\left(T\|\mathbf{u}_0\|_{2,\Omega}^2 + \frac{1}{\mu_0}\|\mathbf{f}\|_{2,Q_T}^2\right),$$

with $\mathscr{F}$ *and* F *given as in Proposition* 4.1, *and* A *denoting some constant depending on* μ_1, μ_2, $\|\nabla\mathbf{u}_0\|_{(1),2,\Omega}$, $\|\nabla\theta_0\|_{4,\Omega}$ *and* $\|\mathbf{f}(0)\|_{2,\Omega}$.

Proof. Differentiating (16) with respect to time and choosing $\mathbf{v} = \partial_t\mathbf{u}$ as a test function (cf. Remark 4.4), and using the orthogonality property of the convective term (cf. [11, p. 128]), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_t\mathbf{u}\|_{2,\Omega}^2 &+ \int_\Omega [\mu'(\xi)\partial_t\xi D\mathbf{u} + \mu(\xi)\partial_t D\mathbf{u}] : D\partial_t\mathbf{u}\,dx \\ &= \int_\Omega \partial_t^2\mathbf{u}\partial_t\mathbf{u}\,dx + \int_\Omega \partial_t[\mu(\xi)D\mathbf{u}] : D\partial_t\mathbf{u}\,dx \\ &= -\int_\Omega \partial_t\mathbf{u}\otimes\partial_t\mathbf{u} : \nabla\mathbf{u}\,dx + \int_\Omega \partial_t\mathbf{f}\cdot\partial_t\mathbf{u}\,dx.\end{aligned}$$

Using the assumption (7), the Hölder's inequality and integrating in time for each $t \in]0,T[$, it follows

$$\begin{aligned}\frac{1}{2}\|\partial_t\mathbf{u}\|_{2,\Omega}^2 + \mu_0\int_0^t\int_\Omega |\nabla\partial_t\mathbf{u}|^2 dxds &\leqslant \frac{1}{2}\|\partial_t\mathbf{u}(0)\|_{2,\Omega}^2 + \int_0^t \|\partial_t\mathbf{f}\|_{2,\Omega}\|\partial_t\mathbf{u}\|_{2,\Omega}\,ds \\ &+\mu_2\int_0^t \|\partial_t\xi\|_{4,\Omega}\|\nabla\mathbf{u}\|_{4,\Omega}\|\nabla\partial_t\mathbf{u}\|_{2,\Omega}\,ds + \int_0^t \|\partial_t\mathbf{u}\|_{4,\Omega}^2\|\nabla\mathbf{u}\|_{2,\Omega}\,ds.\end{aligned} \tag{29}$$

Let us study separately each term of the right hand side of the above inequality. *First term.* To estimate $\|\partial_t\mathbf{u}(0)\|_{2,\Omega}^2$ we choose $\mathbf{v} = \partial_t\mathbf{u}(0)$ as a test function in (16) for the particular case $t = 0$. Thus, we observe that

$$\begin{aligned}\|\mathbf{u}'(0)\|_{2,\Omega}^2 + \int_\Omega \mathbf{u}'(0)\otimes\mathbf{u}_0 : \nabla\mathbf{u}_0\,dx - \int_\Omega \mathbf{f}(0)\cdot\mathbf{u}'(0)dx \\ = -\int_\Omega \mu(\xi(0))D\mathbf{u}_0 : D\mathbf{u}'(0)dx.\end{aligned}$$

Using the Green's formula and (9) it follows

$$\|\mathbf{u}'(0)\|_{2,\Omega}^2 + \int_\Omega \mathbf{u}'(0)\otimes\mathbf{u}_0 : \nabla\mathbf{u}_0\,dx - \int_\Omega \mathbf{f}(0)\cdot\mathbf{u}'(0)dx$$
$$= \int_\Omega \mu'(\xi(0))D\mathbf{u}_0 : \nabla\xi(0)\otimes\mathbf{u}'(0)dx + \int_\Omega \mu(\xi(0))\Delta\mathbf{u}_0\cdot\mathbf{u}'(0)dx.$$

Applying the Hölder's inequality, we get

$$\|\mathbf{u}'(0)\|_{2,\Omega} \leqslant \mu_2\|\nabla\theta_0\|_{4,\Omega}\|\nabla\mathbf{u}_0\|_{4,\Omega} + \mu_1\|\nabla\mathbf{u}_0\|_{(1),2,\Omega} + $$
$$+\|\mathbf{u}_0\|_{4,\Omega}\|\nabla\mathbf{u}_0\|_{4,\Omega} + \|\mathbf{f}(0)\|_{2,\Omega} := A.$$

Note that $\mathbf{f}\in\mathbf{L}^2(Q_T)$ and $\partial_t\mathbf{f}\in\mathbf{L}^2(Q_T)$ then $\mathbf{f}\in C([0,T];\mathbf{L}^2(\Omega))$.
Second term. It is sufficient the use of the Young's inequality.
Third term. Applying the Young's inequality, it follows:

$$\mu_2\|\partial_t\xi\|_{4,\Omega}\|\nabla\mathbf{u}\|_{4,\Omega}\|\nabla\partial_t\mathbf{u}\|_{2,\Omega} \leqslant \frac{\mu_2^2}{2\mu_0}\{\|\partial_t\xi\|_{4,\Omega}^4 + \|\nabla\mathbf{u}\|_{4,\Omega}^4\}$$
$$+\frac{\mu_0}{4}\|\nabla\partial_t\mathbf{u}\|_{2,\Omega}^2.$$

Forth term. Applying the interpolation inequality (15), with $n=2$ and $v=\partial_t\mathbf{u}$, and using the Young's inequality, it follows

$$\|\partial_t\mathbf{u}\|_{4,\Omega}^2\|\nabla\mathbf{u}\|_{2,\Omega} \leqslant \|\partial_t\mathbf{u}\|_{2,\Omega}\|\nabla\partial_t\mathbf{u}\|_{2,\Omega}\|\nabla\mathbf{u}\|_{2,\Omega}$$
$$\leqslant \frac{1}{\mu_0}\|\nabla\mathbf{u}\|_{2,\Omega}^2\|\partial_t\mathbf{u}\|_{2,\Omega}^2 + \frac{\mu_0}{4}\|\nabla\partial_t\mathbf{u}\|_{2,\Omega}^2.$$

Substituting each calculation in (29), we get

$$\|\partial_t\mathbf{u}\|_{2,\Omega}^2 + \mu_0\int_0^t \|\nabla\partial_t\mathbf{u}\|_{2,\Omega}^2 ds \leqslant A^2 + \int_0^t \|\partial_t\mathbf{f}\|_{2,\Omega}^2 ds$$
$$+\frac{\mu_2^2}{\mu_0}\int_0^t \left(\|\partial_t\xi\|_{4,\Omega}^4 + \|\nabla\mathbf{u}\|_{4,\Omega}^4\right)ds + \int_0^t \left(1+\frac{2}{\mu_0}\|\nabla\mathbf{u}\|_{2,\Omega}^2\right)\|\partial_t\mathbf{u}\|_{2,\Omega}^2 ds.$$

From the Gronwall's lemma and using (18) and (23), we conclude that

$$\operatorname*{ess\ sup}_{t\in[0,T]}\|\partial_t\mathbf{u}\|_{2,\Omega}^2 \leqslant \Big[TA^2 + \|\partial_t\mathbf{f}\|_{2,Q_T}^2$$
$$+\frac{\mu_2^2}{\mu_0}\Big(\|\partial_t\xi\|_{4,Q_T}^4 + \mathscr{F}(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4)\Big)\Big]\exp[\mathscr{A}]. \tag{30}$$

Subsequently we obtain that

$$\|\nabla\partial_t\mathbf{u}\|_{2,Q_T}^2 \leqslant \frac{1}{\mu_0}\Big[TA^2 + \|\partial_t\mathbf{f}\|_{2,Q_T}^2$$
$$+\frac{\mu_2^2}{\mu_0}\Big(\|\partial_t\xi\|_{4,Q_T}^4 + \mathscr{F}(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4)\Big)\Big](1+\mathscr{A}\exp[\mathscr{A}]) \tag{31}$$

and that $\partial_t\mathbf{u}$ verifies (28).

REMARK 4.4. To differentiate (16) with respect to the variable t and the choice of the test function have meaning in the following sense. We first form the difference ratio of the identity with respect to t, choose as a test function the difference ratio of the solution with respect to t, and then pass to the limit as $\triangle t \to 0$. Or equivalently, the argument of Remark 4.1 can be repeated.

PROPOSITION 4.4. *Under the assumptions* (6)-(9) *and* (11), *if* $\xi \in W_4^{1,1}(Q_T)$ *such that* $\nabla\xi(0) = \nabla\theta_0$ *any solution* θ *given at Proposition* 4.2 *is such that* $\partial_t\theta$ *belongs to* $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. *In particular, the following estimate holds*

$$\|\partial_t\theta\|_{4,Q_T}^4 \leqslant \mathscr{Q}\Big(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4, \frac{2k_2^4}{k_0^3}\|\nabla\xi\|_{4,Q_T}^4,$$
$$\frac{\mu_2^2}{\mu_0}\|\partial_t\xi\|_{4,Q_T}^4, \frac{2k_2^4}{k_0^2}\|\partial_t\xi\|_{4,Q_T}^4, \frac{\mu_2^4}{2k_0}\|\partial_t\xi\|_{4,Q_T}^4\Big), \tag{32}$$

with $\mathscr{Q}$ *the positive strictly increasing function on its arguments defined by*

$$\mathscr{Q}(d_1,d_2,d_3,d_4,d_5) = (TB^2 + d_4 + \mathscr{H}(d_1,d_2)$$
$$+ \frac{1}{\mu_0}\left(TA^2 + \|\partial_t\mathbf{f}\|_{2,Q_T}^2 + d_3 + \frac{\mu_2^2}{\mu_0}\mathscr{F}(d_1)\right)(1+\mathscr{A}\exp[\mathscr{A}])$$
$$+ \mathscr{P}(d_1,d_3) + \mathscr{F}(d_1) + \|\partial_t g\|_{2,Q_T}^2)F(T + \frac{1}{2k_0}\mathscr{F}(d_1) + d_5),$$

with correspondent estimative functions according to Propositions 4.1, 4.2, 4.3, *and* B *denoting some constant depending on* k_1, k_2, μ_1, $\|\nabla\mathbf{u}_0\|_{4,\Omega}$, $\|\nabla\theta_0\|_{(1),2,\Omega}$ *and* $\|g(0)\|_{2,\Omega}$. *Moreover,* $\nabla\theta$ *belongs to* $\mathbf{L}^q(Q_T)$ *for any* $q < 6$.

Proof. We recall the equation in the sense of distributions

$$\partial_t\theta + \mathbf{u}\cdot\nabla\theta - \nabla\cdot(k(\xi)\nabla\theta) = \mu(\xi)|D\mathbf{u}|^2 + g \text{ in } Q_T.$$

Differentiating the above equation with respect to time, we deduce

$$\partial_t^2\theta - \nabla\cdot(k(\xi)\partial_t\nabla\theta) = \nabla\cdot(k'(\xi)\partial_t\xi\nabla\theta) - \partial_t\mathbf{u}\cdot\nabla\theta$$
$$- \mathbf{u}\cdot\partial_t\nabla\theta + \mu'(\xi)\partial_t\xi|D\mathbf{u}|^2 + \mu(\xi)2D\mathbf{u} : \partial_t D\mathbf{u} + \partial_t g. \tag{33}$$

If multiply (33) by $\eta = \partial_t\theta$, using the orthogonality property of the convective term, after standard calculations it follows

$$\frac{1}{2}\int_0^t \frac{d}{dt}\|\partial_t\theta\|_{2,\Omega}^2 ds + k_0\int_0^t\int_\Omega |\nabla\partial_t\theta|^2 dxds$$
$$\leqslant \int_0^t\int_\Omega k'(\xi)\partial_t\xi\nabla\theta\cdot\nabla\partial_t\theta dxds + \int_0^t\int_\Omega \partial_t\mathbf{u}\cdot\nabla\theta\partial_t\theta dxds$$
$$+ \int_0^t\int_\Omega(\mu'(\xi)\partial_t\xi|D\mathbf{u}|^2 + 2\mu(\xi)\partial_t D\mathbf{u} : D\mathbf{u})\partial_t\theta dxds + \int_0^t\int_\Omega \partial_t g\partial_t\theta dxds.$$

Using the assumption (7) and the Hölder's inequality, we get

$$\begin{aligned}
\frac{1}{2}\|\partial_t\theta\|^2_{2,\Omega}(t)+k_0\int_0^t\|\nabla\partial_t\theta\|^2_{2,\Omega}ds\leqslant\frac{1}{2}\|\partial_t\theta(0)\|^2_{2,\Omega}\\
+k_2\int_0^t\|\partial_t\xi\|_{4,\Omega}\|\nabla\theta\|_{4,\Omega}\|\nabla\partial_t\theta\|_{2,\Omega}ds\\
+\int_0^t\|\partial_t\mathbf{u}\|_{4,\Omega}\|\nabla\theta\|_{4,\Omega}\|\partial_t\theta\|_{2,\Omega}ds+\mu_2\int_0^t\|\partial_t\xi\|_{4,\Omega}\|\nabla\mathbf{u}\|^2_{4,\Omega}\|\partial_t\theta\|_{4,\Omega}ds\\
+2\mu_1\int_0^t\|\nabla\partial_t\mathbf{u}\|_{2,\Omega}\|\nabla\mathbf{u}\|_{4,\Omega}\|\partial_t\theta\|_{4,\Omega}ds+\int_0^t\|\partial_t g\|_{2,\Omega}\|\partial_t\theta\|_{2,\Omega}ds. \qquad (34)
\end{aligned}$$

Let us examine separately each term of RHS of the above inequality.
First term. Choosing $\eta=\partial_t\theta(0)$ as a test function in (17) for the particular case $t=0$ and applying (9), we obtain

$$\begin{aligned}
\|\theta'(0)\|^2_{2,\Omega}+\int_\Omega\mathbf{u}_0\cdot\nabla\theta_0\theta'(0)dx-\int_\Omega g(0)\theta'(0)dx-\int_\Omega\mu(\xi(0))|D\mathbf{u}_0|^2\theta'(0)dx\\
=-\int_\Omega k(\xi(0))\nabla\theta_0\cdot\nabla\theta'(0)dx=\int_\Omega\left(k(\xi(0))\Delta\theta_0+k'(\xi(0))|\nabla\theta_0|^2\right)\theta'(0)dx.
\end{aligned}$$

Consequently, we find

$$\begin{aligned}
\|\theta'(0)\|_{2,\Omega}\leqslant k_1\|\nabla\theta_0\|_{(1),2,\Omega}+k_2\|\nabla\theta_0\|^2_{4,\Omega}+\|\mathbf{u}_0\|_{4,\Omega}\|\nabla\theta_0\|_{4,\Omega}\\
+\|g(0)\|_{2,\Omega}+\mu_1\|\nabla\mathbf{u}_0\|^2_{4,\Omega}:=B.
\end{aligned}$$

Note that $g\in L^2(Q_T)$ and $\partial_t g\in L^2(Q_T)$ then $g\in C([0,T];L^2(\Omega))$.
Second and third terms. Applying the Young's inequality, we have

$$\begin{aligned}
k_2\|\partial_t\xi\|_{4,\Omega}\|\nabla\theta\|_{4,\Omega}\|\nabla\partial_t\theta\|_{2,\Omega}\leqslant\frac{k_2^4}{k_0^2}\|\partial_t\xi\|^4_{4,\Omega}+\frac{1}{4}\|\nabla\theta\|^4_{4,\Omega}+\frac{k_0}{4}\|\nabla\partial_t\theta\|^2_{2,\Omega};\\
\|\partial_t\mathbf{u}\|_{4,\Omega}\|\nabla\theta\|_{4,\Omega}\|\partial_t\theta\|_{2,\Omega}\leqslant\frac{1}{4}\|\partial_t\mathbf{u}\|^4_{4,\Omega}+\frac{1}{4}\|\nabla\theta\|^4_{4,\Omega}+\frac{1}{2}\|\partial_t\theta\|^2_{2,\Omega}.
\end{aligned}$$

Forth term. Using interpolation inequality (15) and the Young's inequality we get

$$\begin{aligned}
\mu_2\|\partial_t\xi\|_{4,\Omega}\|\nabla\mathbf{u}\|^2_{4,\Omega}\|\partial_t\theta\|_{4,\Omega}\leqslant\mu_2\|\partial_t\xi\|_{4,\Omega}\|\nabla\mathbf{u}\|^2_{4,\Omega}\|\nabla\partial_t\theta\|^{1/2}_{2,\Omega}\|\partial_t\theta\|^{1/2}_{2,\Omega}\\
\leqslant\frac{1}{2}\|\nabla\mathbf{u}\|^4_{4,\Omega}+\frac{\mu_2^4}{4k_0}\|\partial_t\xi\|^4_{4,\Omega}\|\partial_t\theta\|^2_{2,\Omega}+\frac{k_0}{4}\|\nabla\partial_t\theta\|^2_{2,\Omega},
\end{aligned}$$

Fifth term. Using interpolation inequality (15) and the Young's inequality we get

$$\begin{aligned}
\mu_1\|\nabla\partial_t\mathbf{u}\|_{2,\Omega}\|\nabla\mathbf{u}\|_{4,\Omega}\|\partial_t\theta\|_{4,\Omega}\leqslant\mu_1\|\nabla\partial_t\mathbf{u}\|_{2,\Omega}\|\nabla\mathbf{u}\|_{4,\Omega}\|\nabla\partial_t\theta\|^{1/2}_{2,\Omega}\|\partial_t\theta\|^{1/2}_{2,\Omega}\\
\leqslant\frac{\mu_1^2}{2}\|\nabla\partial_t\mathbf{u}\|^2_{2,\Omega}+\frac{1}{4k_0}\|\nabla\mathbf{u}\|^4_{4,\Omega}\|\partial_t\theta\|^2_{2,\Omega}+\frac{k_0}{4}\|\nabla\partial_t\theta\|^2_{2,\Omega}.
\end{aligned}$$

Sixth term. It is sufficient the use of the Young's inequality.

Then, introducing all these terms in (34) we conclude

$$\|\partial_t\theta\|_{2,\Omega}^2(t)+\frac{k_0}{2}\int_0^t\|\nabla\partial_t\theta\|_{2,\Omega}^2ds\leqslant B^2+\frac{2k_2^4}{k_0^2}\int_0^t\|\partial_t\xi\|_{4,\Omega}^4ds+\int_0^t\|\nabla\theta\|_{4,\Omega}^4ds$$
$$+\mu_1^2\int_0^t\|\nabla\partial_t\mathbf{u}\|_{2,\Omega}^2ds+\int_0^t\big(1+\frac{1}{2k_0}\|\nabla\mathbf{u}\|_{4,\Omega}^4+\frac{\mu_2^4}{2k_0}\|\partial_t\xi\|_{4,\Omega}^4\big)\|\partial_t\theta\|_{2,\Omega}^2ds$$
$$+\frac{1}{2}\int_0^t\|\partial_t\mathbf{u}\|_{4,\Omega}^4ds+\int_0^t\|\nabla\mathbf{u}\|_{4,\Omega}^4ds+\int_0^t\|\partial_t g\|_{2,\Omega}^2ds.$$

Recalling Propositions 4.1, 4.2 and 4.3, we use the Gronwall's lemma to obtain

$$\operatorname*{ess\,sup}_{t\in[0,T]}\|\partial_t\theta\|_{2,\Omega}^2\leqslant\exp[\mathscr{B}(\xi)]\Big(TB^2+\frac{2k_2^4}{k_0^2}\|\partial_t\xi\|_{4,Q_T}^4$$
$$+\mathscr{H}\left(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4,\frac{2k_2^4}{k_0^3}\|\nabla\xi\|_{4,Q_T}^4\right)+\mu_1^2\|\nabla\partial_t\mathbf{u}\|_{2,Q_T}^4$$
$$+\mathscr{P}\left(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4,\frac{\mu_2^2}{\mu_0}\|\partial_t\xi\|_{4,Q_T}^4\right)+\mathscr{F}\left(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4\right)+\|\partial_t g\|_{2,Q_T}^2\Big),\quad(35)$$

with

$$\mathscr{B}(\xi)=T+\left(\mathscr{F}(\frac{\mu_2^4}{2\mu_0^3}\|\nabla\xi\|_{4,Q_T}^4)+\mu_2^4\|\partial_t\xi\|_{4,Q_T}^4\right)/(2k_0).$$

Applying (31) and this result in the last expression we conclude the desired result and consequently (32).

Finally applying Proposition 4.2, $\nabla\theta$ belongs to $L^2(0,T;\mathbf{H}^1(\Omega))\cap L^\infty(0,T;\mathbf{L}^2(\Omega))$. Since we also have $\partial_t\nabla\theta\in\mathbf{L}^2(Q_T)$ then Proposition 3.2 implies that $\nabla\theta$ belongs to $\mathbf{L}^q(Q_T)$ for any $q<6$.

REMARK 4.5. For $g=-\theta|\theta|^{1/2}$, the result of Proposition 4.4 is still valid if in its proof we take into account that

$$\int_\Omega\partial_t g\partial_t\theta dx=-\frac{3}{2}\int_\Omega|\theta|^{1/2}|\partial_t\theta|^2dx\leqslant 0.$$

PROPOSITION 4.5. *Under the assumptions* (7) *and* (11)*, if* θ *is any solution given at Proposition* 4.4*, then* $\partial_t^2\theta$ *belongs to* $L^2(0,T;H^{-1}(\Omega))$.

Proof. Considering (33) and since we already proved that

$$\partial_t\mathbf{u}\text{ and }\nabla\theta\in\mathbf{L}^4(Q_T),$$
$$\mathbf{u}\in L^\infty(0,T;\mathbf{L}^2(\Omega))\text{ and }\partial_t\nabla\theta\in\mathbf{L}^2(Q_T),$$
$$\nabla\mathbf{u}\in L^\infty(0,T;\mathbf{L}^2(\Omega))\text{ and }\nabla\partial_t\mathbf{u}\in\mathbf{L}^2(Q_T),$$
$$|\nabla\mathbf{u}|^2\in L^4(0,T;L^{4/3}(\Omega)),$$

we have

$$\partial_t^2\theta - \nabla\cdot(k'(\xi)\partial_t\xi\nabla\theta + k(\xi)\nabla\partial_t\theta) \in L^2(0,T;L^1(\Omega)).$$

Taking the operator div: $\mathbf{L}^2(\Omega)\to H^{-1}(\Omega)$, it permits to get

$$\nabla\cdot(k'(\xi)\partial_t\xi\nabla\theta + k(\xi)\nabla\partial_t\theta) \in L^2(0,T;H^{-1}(\Omega)),$$

then it results that $\partial_t^2\theta \in L^2(0,T;H^{-1}(\Omega))$.

PROPOSITION 4.6. *Under the assumptions of Propositions* 4.2, 4.3 *and* 4.4*, any solution* θ *of* (17) *is such that* $\partial_t\theta$ *belongs to* $L^\infty(0,T;L^{2+\delta}(\Omega))$ *for some* $\delta>0$*, and consequently to* $L^{4+\delta}(Q_T)$.

Proof. In consequence of Proposition 4.5, $\partial_t^2\theta$ does not belong in $L^2(Q_T)$ and we cannot apply the argument used in Proposition 4.4. Let us argue as in [2], multiplying (33) by $\eta=\partial_t\theta|\partial_t\theta|^\delta$ and integrating over the space variable, we obtain

$$\int_\Omega \partial_t^2\theta\partial_t\theta|\partial_t\theta|^\delta dx + (1+\delta)\int_\Omega (k(\xi)\partial_t\nabla\theta + k'(\xi)\partial_t\xi\nabla\theta)\cdot\nabla(\partial_t\theta|\partial_t\theta|^\delta)dx$$
$$+\int_\Omega \partial_t(\mathbf{u}\cdot\nabla\theta)\partial_t\theta|\partial_t\theta|^\delta dx = \int_\Omega \partial_t(\mu(\xi)|D\mathbf{u}|^2)\partial_t\theta|\partial_t\theta|^\delta dx + \int_\Omega \partial_t g\partial_t\theta|\partial_t\theta|^\delta dx.$$

Applying the Hölder's inequality under the relations for the exponents

$$\frac{1}{4}+\frac{1-2\delta}{4}+\frac{1+\delta}{2}=1,\quad \frac{1}{2}+\frac{1}{5}+\frac{3-5\delta}{10}+\frac{\delta}{2}=1,\quad \delta<\frac{1}{2}\left(<\frac{3}{5}\right),$$

it follows

$$\frac{1}{2+\delta}\int_0^t \frac{d}{dt}\|\partial_t\theta\|_{2+\delta,\Omega}^{2+\delta}ds + k_0(1+\delta)\int_0^t\int_\Omega |\nabla\partial_t\theta|^2|\partial_t\theta|^\delta dxds$$
$$\leqslant I_1+I_2+I_3+I_4+I_5+\int_0^t\int_\Omega \partial_t g\partial_t\theta dxds,$$

with

$$I_1 := k_2(1+\delta)\int_0^t \|\partial_t\xi\|_{4,\Omega}\|\nabla\theta\|_{4/(1-2\delta),\Omega}\|\nabla\partial_t\theta\|_{2,\Omega}\|\partial_t\theta\|_{2,\Omega}^\delta ds,$$
$$I_2 := \int_0^t \|\partial_t\mathbf{u}\|_{4,\Omega}\|\nabla\theta\|_{4/(1-2\delta),\Omega}\|\partial_t\theta\|_{2,\Omega}^{1+\delta} ds,$$
$$I_3 := \int_0^t \|\mathbf{u}\|_{5,\Omega}\|\nabla\partial_t\theta\|_{2,\Omega}\|\partial_t\theta\|_{10/(3-5\delta),\Omega}\|\partial_t\theta\|_{2,\Omega}^\delta ds,$$
$$I_4 := \mu_2\int_0^t \|\partial_t\xi\|_{4,\Omega}\|D\mathbf{u}\|_{8/(1-2\delta),\Omega}^2\|\partial_t\theta\|_{2,\Omega}^{1+\delta} ds,$$
$$I_5 := 2\mu_1\int_0^t \|\partial_t D\mathbf{u}\|_{2,\Omega}\|D\mathbf{u}\|_{5,\Omega}\|\partial_t\theta\|_{10/(3-5\delta),\Omega}\|\partial_t\theta\|_{2,\Omega}^\delta ds.$$

From Proposition 4.4, $\partial_t\theta$ belongs to $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ and $\nabla\theta$ belongs to $\mathbf{L}^q(Q_T)\hookrightarrow\mathbf{L}^{4/(1-2\delta),4}(Q_T)\hookrightarrow\mathbf{L}^{4/(1-2\delta),4/3}(Q_T)$ for any $q<6$. Consequently, choosing $\delta<1/6$ we obtain

$$I_1 \leqslant C\|\partial_t\xi\|_{4,Q_T}\|\nabla\theta\|_{4/(1-2\delta),4,Q_T}\|\nabla\partial_t\theta\|_{2,Q_T}\operatorname*{ess\,sup}_{t\in[0,T]}\|\partial_t\theta\|_{2,\Omega}^{\delta},$$
$$I_2 \leqslant \|\partial_t\mathbf{u}\|_{4,Q_T}\|\nabla\theta\|_{4/(1-2\delta),4/3,Q_T}\operatorname*{ess\,sup}_{t\in[0,T]}\|\partial_t\theta\|_{2,\Omega}^{1+\delta},$$

observing that from Proposition 4.3 and Lemma 3.1, $\partial_t\mathbf{u}$ belongs to $L^2(0,T;\mathbf{H}^1(\Omega))\cap L^\infty(0,T;\mathbf{L}^2(\Omega))\hookrightarrow\mathbf{L}^4(Q_T)$.

Thanks to Proposition 4.1, we get $\mathbf{u}\in L^\infty(0,T;\mathbf{H}^1(\Omega))\hookrightarrow L^\infty(0,T;\mathbf{L}^5(\Omega))$, and thanks to Proposition 4.4, we get $\partial_t\theta$ belongs to $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))\hookrightarrow L^{10/(3-5\delta),2}(Q_T)$ for any $\delta<3/5$. Then, we obtain

$$I_3 \leqslant \|\mathbf{u}\|_{5,\infty,Q_T}\|\nabla\partial_t\theta\|_{2,Q_T}\|\partial_t\theta\|_{10/(3-5\delta),2,Q_T}\operatorname*{ess\,sup}_{t\in[0,T]}\|\partial_t\theta\|_{2,\Omega}^{\delta}.$$

Thanks to Propositions 4.1 and 3.2, $\nabla\mathbf{u}$ belongs to

$$L^2(0,T;\mathbf{H}^1(\Omega))\cap L^\infty(0,T;\mathbf{L}^3(\Omega))\hookrightarrow\mathbf{L}^{8/(1-2\delta),8/3}(Q_T)$$

for any $\delta<1/6$. Then, it follows

$$I_4 \leqslant C\|\partial_t\xi\|_{4,Q_T}\|\nabla\mathbf{u}\|^2_{8/(1-2\delta),8/3,Q_T}\operatorname*{ess\,sup}_{t\in[0,T]}\|\partial_t\theta\|_{2,\Omega}^{1+\delta}.$$

From Proposition 4.1, $\nabla\mathbf{u}$ belongs to $L^2(0,T;\mathbf{H}^1(\Omega))\cap L^\infty(0,T;\mathbf{L}^2(\Omega))$. From Proposition 4.3, $\partial_t\mathbf{u}$ belongs to $L^2(0,T;\mathbf{H}^1(\Omega))\cap L^\infty(0,T;\mathbf{L}^2(\Omega))$. In particular, applying Proposition 3.2 we have $\nabla\mathbf{u}\in\mathbf{L}^5(Q_T)$. From Proposition 4.4, $\partial_t\theta$ belongs to $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))\hookrightarrow L^{10/(3-5\delta),10/3}(Q_T)$ for any $\delta<1/5$. Then, we obtain

$$I_5 \leqslant C\|\nabla\mathbf{u}\|_{5,Q_T}\|\partial_t\nabla\mathbf{u}\|_{2,Q_T}\|\partial_t\theta\|_{10/(3-5\delta),10/3,Q_T}\operatorname*{ess\,sup}_{t\in[0,T]}\|\partial_t\theta\|_{2,\Omega}^{\delta}.$$

Therefore we conclude an estimate for $\partial_t\theta$ in $L^{2+\delta,\infty}(Q_T)$ and applying Lemma 3.1 we obtain $L^{2+\delta,\infty}(Q_T)\cap L^2(0,T;H^1(\Omega))\hookrightarrow L^{4+\delta}(Q_T)$.

5. Proof of Theorem 2.1

In order to apply Schauder theorem, we build an operator $\mathscr{L}$ defined on $W_4^{1,1}(Q_T)$, which maps

$$\xi\in K\mapsto\mathbf{u}=\mathbf{u}(\xi)\mapsto\theta\in W_4^{1,1}(Q_T),$$

where $K:=\{\xi\in W_4^{1,1}(Q_T):\ \nabla\xi(0)=\nabla\theta_0,\ \|\xi\|\leqslant R\}$, and $\mathbf{u}$ and θ are the solutions to the problems (16) and (17), respectively.

Step 1. Let us prove that $\mathscr{L}$ is a well defined mapping. For each $\xi \in W_4^{1,1}(Q_T)$, from the existence theory for the Navier-Stokes system there is a unique 2-dimensional solution $\mathbf{u} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{J}_0^{1,2}(\Omega))$ to (16), $\partial_t \mathbf{u} \in L^2(0,T;(\mathbf{J}_0^{1,2}(\Omega))')$ (see for example [11, 14, 25]). Hence, thanks to Proposition 4.1 we have $\nabla \mathbf{u} \in \mathbf{L}^4(Q_T)$. Thus from the existence theory for the parabolic equations there is

$$\theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)),$$

which is the unique solution to the problem (17) so that $\partial_t \theta \in L^2(0,T;H^{-1}(\Omega))$. Then, Propositions 4.2 and 4.4 guarantee the sufficient regularity to obtain $\theta \in W_4^{1,1}(Q_T)$.

Step 2. From Propositions 4.2 and 4.4, $\mathscr{L}$ maps the convex closed set K into itself, choosing $R > 0$ such that

$$R \geqslant \mathscr{H}\left(\frac{\mu_2^4}{2\mu_0^3}R, \frac{2k_2^4}{k_0^3}R\right) + \mathscr{Q}\left(\frac{\mu_2^4}{2\mu_0^3}R, \frac{2k_2^4}{k_0^3}R, \frac{\mu_2^2}{\mu_0}R, \frac{2k_2^4}{k_0^2}R, \frac{\mu_2^4}{2k_0}R\right) \tag{36}$$

under the assumption (12).

Step 3. In order to prove that $\mathscr{L}$ is compact, we take a sequence ξ_m weakly convergent to ξ in $W_4^{1,1}(Q_T)$, and corresponding solutions $\mathbf{u}_m$ and θ_m to the problems (16) and (17), respectively. The estimates (25) and (32) infer that we can extract a subsequence, still denoted by θ_m, such that

$$\nabla\theta_m \rightharpoonup \nabla\theta \text{ in } \mathbf{L}^4(Q_T), \quad \partial_t\theta_m \rightharpoonup \partial_t\theta \text{ in } L^4(Q_T).$$

From Propositions 4.2 and 4.4, we have $\nabla\theta_m$ and $\partial_t\nabla\theta_m$ bounded in $L^2(0,T;\mathbf{H}_0^1(\Omega))$ and $\mathbf{L}^2(Q_T)$, respectively. Moreover $\partial_t\theta_m$ bounded in $L^2(0,T;H_0^1(\Omega))$ and from Proposition 4.5 we get $\partial_t^2\theta_m$ bounded in $L^2(0,T;H^{-1}(\Omega))$. Then by a compactness result (cf. [22] or [24, p. 90]) we obtain

$$\nabla\theta_m \to \nabla\theta \text{ in } L^2(0,T;\mathbf{L}^q(\Omega)), \quad q < \infty,$$
$$\partial_t\theta_m \to \partial_t\theta \text{ in } L^2(0,T;L^q(\Omega)), \quad q < \infty.$$

In order to apply these strong convergences we use Proposition 3.2 to $\nabla\theta_m$ obtaining that it is bounded in $\mathbf{L}^5(Q_T)$ and consequently

$$\nabla\theta_m \to \nabla\theta \text{ in } \mathbf{L}^4(Q_T).$$

Next using Proposition 4.6, it follows

$$\partial_t\theta_m \to \partial_t\theta \text{ in } L^4(Q_T).$$

Therefore we conclude that θ is the limit solution to the problem (17).

In conclusion, Schauder Theorem guarantees the existence of at least one fixed point and accordingly there exists a strong solution $(\mathbf{u}, \theta)$ in the conditions of Theorem 2.1.

Finally let us prove that the strong solution is Hölder continuous. From Proposition 4.1, $\nabla\mathbf{u}$ belongs to $L^2(0,T;\mathbf{H}^1(\Omega)) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$. From Proposition 4.3, $\partial_t\mathbf{u}$ belongs to $L^2(0,T;\mathbf{H}^1(\Omega)) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$. In particular, applying Proposition 3.1 we have $\nabla\mathbf{u} \in \mathbf{L}^{3,\infty}(Q_T)$, that is, $\mathbf{u}$ belongs to $L^\infty(0,T;\mathbf{W}_0^{1,3}(\Omega)) \hookrightarrow L^\infty(0,T;\mathbf{C}^{0,\alpha}(\bar{\Omega}))$ for $0 < \alpha < 1/3$. Thus using Lemma 3.2 we conclude that $\mathbf{u}$ is Hölder continuous. Analogous for θ.

6. Proof of Theorem 2.2

We argue as in the proof of Theorem 2.1. Let $\mathscr{L}$ be the operator which maps

$$\xi \in K \subset W_4^{1,1}(Q_T) \mapsto \mathbf{u} = \mathbf{u}(\xi) \mapsto \theta \in W_4^{1,1}(Q_T),$$

where $\mathbf{u}$ and θ are the solutions to the problems (16) and (17) with g replaced by $-\theta|\theta|^{1/2}$, respectively.

Proceeding as in steps 1 and 2, the operator $\mathscr{L}$ is well defined considering Remarks 4.3 and 4.5. Consequently the argument of the proof of Theorem 2.1 can be followed *mutatis mutandis*. In particular, we have, for all $q<6$,

$$|\nabla\theta|^2,\ |\nabla\mathbf{u}|^2 \in L^{q/2}(Q_T); \quad \mathbf{u}\cdot\nabla\theta \in L^q(Q_T),$$

the corresponding term to the heat production $|\theta|^{1/2}\theta \in L^\infty(Q_T)$ and θ is Hölder continuous. Thus, applying the regularity theory for the heat equation

$$\partial_t\theta - k(\theta)\Delta\theta = k'(\theta)|\nabla\theta|^2 + \mu(\theta)|D\mathbf{u}|^2 - |\theta|^{1/2}\theta - \mathbf{u}\cdot\nabla\theta \quad \text{in } Q_T,$$

we find that $\theta \in W^{2,1}_{q/2}(Q_T)$.

7. Uniqueness (Proof of Theorem 2.3)

7.1. Uniqueness in Theorem 2.1

We proceed in a classical manner. We suppose the existence of two solutions $(\mathbf{u}_1,\theta_1)$ and $(\mathbf{u}_2,\theta_2)$ and we define $\mathbf{u}=\mathbf{u}_1-\mathbf{u}_2$ and $\theta=\theta_1-\theta_2$. So that $(\mathbf{u},\theta)$ verifies the following variational formulation

$$\left\{\begin{array}{c} \int_\Omega \partial_t\mathbf{u}\cdot\mathbf{v}\,dx + \int_\Omega \mu(\theta_1)D\mathbf{u}:D\mathbf{v}\,dx = \int_\Omega(\mu(\theta_2)-\mu(\theta_1))D\mathbf{u}_2:D\mathbf{v}\,dx \\ -\int_\Omega\Big(\nabla\mathbf{u}_2:\mathbf{v}\otimes\mathbf{u} + \nabla\mathbf{u}:\mathbf{v}\otimes\mathbf{u}_1\Big)dx \\ \text{a.e. } t\in[0,T] \quad \forall\mathbf{v}\in\mathbf{J}_0^{1,2}(\Omega),\ \mathbf{u}\big|_{t=0}=\mathbf{0} \text{ in } \Omega; \\ \int_\Omega(\partial_t\theta)\eta\,dx + \int_\Omega k(\theta_1)\nabla\theta\cdot\nabla\eta\,dx = \int_\Omega(k(\theta_2)-k(\theta_1))\nabla\theta_2\cdot\nabla\eta\,dx \\ -\int_\Omega\Big(\mathbf{u}\cdot\nabla\theta_2+\mathbf{u}_1\cdot\nabla\theta\Big)\eta\,dx + \int_\Omega(\mu(\theta_1)|D\mathbf{u}_1|^2-\mu(\theta_2)|D\mathbf{u}_2|^2)\eta\,dx \\ \text{a.e. } t\in[0,T]\ \forall\eta\in H_0^1(\Omega),\ \theta\big|_{t=0}=0 \text{ in } \Omega. \end{array}\right. \tag{37}$$

Taking $\mathbf{v}=\mathbf{u}$ and $\eta=\theta$, using (7) and the orthogonality property to the convective terms, and summing the resulting relations we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{2,\Omega}^2 + \frac{1}{2}\frac{d}{dt}\|\theta\|_{2,\Omega}^2 + \mu_0\|D\mathbf{u}\|_{2,\Omega}^2 + k_0\|\nabla\theta\|_{2,\Omega}^2 \\ &\quad\leqslant -\int_\Omega \nabla\mathbf{u}_2:\mathbf{u}\otimes\mathbf{u}\,dx - \int_\Omega \mathbf{u}\cdot\nabla\theta_2\theta\,dx + \int_\Omega(\mu(\theta_2)-\mu(\theta_1))D\mathbf{u}_2:D\mathbf{u}\,dx \\ &\qquad + \int_\Omega (k(\theta_2)-k(\theta_1))\nabla\theta_2\cdot\nabla\theta\,dx \\ &\qquad + \int_\Omega(\mu(\theta_1)-\mu(\theta_2))|D\mathbf{u}_1|^2\theta\,dx + \int_\Omega \mu(\theta_2)D\mathbf{u}:D(\mathbf{u}_1+\mathbf{u}_2)\theta\,dx. \end{aligned}$$

Arguing as in [11, p. 154] we get

$$I_1 := \left| \int_\Omega \nabla \mathbf{u}_2 : \mathbf{u} \otimes \mathbf{u}\, dx \right| \leqslant \|\nabla \mathbf{u}_2\|_{2,\Omega} \|\mathbf{u}\|_{4,\Omega}^2$$
$$\leqslant \frac{\mu_0}{8} \|\nabla \mathbf{u}\|_{2,\Omega}^2 + \frac{2}{\mu_0} \|\nabla \mathbf{u}_2\|_{2,\Omega}^2 \|\mathbf{u}\|_{2,\Omega}^2.$$

Analogously we have

$$I_2 := \left| \int_\Omega \mathbf{u} \cdot \nabla \theta_2 \theta\, dx \right| \leqslant \|\mathbf{u}\|_{4,\Omega} \|\nabla \theta_2\|_{2,\Omega} \|\theta\|_{4,\Omega}$$
$$\leqslant \frac{\mu_0}{8} \|\nabla \mathbf{u}\|_{2,\Omega}^2 + \frac{1}{2\mu_0} \|\nabla \theta_2\|_{2,\Omega}^2 \|\mathbf{u}\|_{2,\Omega}^2 + \frac{k_0}{8} \|\nabla \theta\|_{2,\Omega}^2 + \frac{1}{2k_0} \|\nabla \theta_2\|_{2,\Omega}^2 \|\theta\|_{2,\Omega}^2.$$

Next using the Mean Value Theorem, for every $X = (x,t) \in Q_T$ there exists ψ_X between $\theta_1(X)$ and $\theta_2(X)$ such that

$$|\mu(\theta_1) - \mu(\theta_2)| = \mu'(\psi_X)|\theta|,$$

and successively applying (7) and the Hölder's inequality

$$I_3 := \int_\Omega |\mu(\theta_1) - \mu(\theta_2)||D\mathbf{u}_2 : D\mathbf{u}| dx \leqslant \mu_2 \|\theta\|_{4,\Omega} \|D\mathbf{u}_2\|_{4,\Omega} \|D\mathbf{u}\|_{2,\Omega}.$$

Applying (14) this term can be estimated as follows

$$I_3 \leqslant \mu_2 \|\theta\|_{2,\Omega}^{1/2} \|\nabla \theta\|_{2,\Omega}^{1/2} \|\nabla \mathbf{u}_2\|_{4,\Omega} \|\nabla \mathbf{u}\|_{2,\Omega}$$
$$\leqslant \frac{2\mu_2^4}{k_0 \mu_0^2} \|\theta\|_{2,\Omega}^2 \|\nabla \mathbf{u}_2\|_{4,\Omega}^4 + \frac{k_0}{8} \|\nabla \theta\|_{2,\Omega}^2 + \frac{\mu_0}{4} \|\nabla \mathbf{u}\|_{2,\Omega}^2.$$

Analogously we get

$$I_4 := \int_\Omega (k(\theta_2) - k(\theta_1)) \nabla \theta_2 \cdot \nabla \theta\, dx$$
$$\leqslant k_2 \|\theta\|_{2,\Omega}^{1/2} \|\nabla \theta\|_{2,\Omega}^{3/2} \|\nabla \theta_2\|_{4,\Omega} \leqslant \frac{2k_2^4}{k_0^3} \|\theta\|_{2,\Omega}^2 \|\nabla \theta_2\|_{4,\Omega}^4 + \frac{3k_0}{8} \|\nabla \theta\|_{2,\Omega}^2$$

and also

$$I_5 := \int_\Omega (\mu(\theta_1) - \mu(\theta_2)) |D\mathbf{u}_1|^2 \theta\, dx$$
$$\leqslant \mu_2 \|\theta\|_{4,\Omega}^2 \|\nabla \mathbf{u}_1\|_{4,\Omega}^2 \leqslant \frac{2\mu_2^2}{k_0} \|\theta\|_{2,\Omega}^2 \|\nabla \mathbf{u}_1\|_{4,\Omega}^4 + \frac{k_0}{8} \|\nabla \theta\|_{2,\Omega}^2.$$

Finally we have

$$I_6 := \int_\Omega \mu(\theta_2) D\mathbf{u} : D(\mathbf{u}_1 + \mathbf{u}_2) \theta\, dx \leqslant \mu_1 \|\theta\|_{4,\Omega} \|\nabla(\mathbf{u}_1 + \mathbf{u}_2)\|_{4,\Omega} \|\nabla \mathbf{u}\|_{2,\Omega}$$
$$\leqslant \frac{2\mu_1^4}{k_0 \mu_0^2} \|\theta\|_{2,\Omega}^2 \|\nabla(\mathbf{u}_1 + \mathbf{u}_2)\|_{4,\Omega}^4 + \frac{k_0}{8} \|\nabla \theta\|_{2,\Omega}^2 + \frac{\mu_0}{4} \|\nabla \mathbf{u}\|_{2,\Omega}^2.$$

Then these inequalities imply

$$\frac{d}{dt}\left(\|\mathbf{u}\|_{2,\Omega}^2+\|\theta\|_{2,\Omega}^2\right)\leqslant\frac{4}{\mu_0}\left(\|\nabla\mathbf{u}_2\|_{2,\Omega}^2+\|\nabla\theta_2\|_{2,\Omega}^2\right)\|\mathbf{u}\|_{2,\Omega}^2$$
$$+\left(C(|\Omega|,k_0,k_2)\|\nabla\theta_2\|_{4,\Omega}^4+\frac{C(\mu_0,\mu_1,\mu_2)}{k_0\mu_0^2}\left(\|\nabla\mathbf{u}_2\|_{4,\Omega}^4+\|\nabla\mathbf{u}_1\|_{4,\Omega}^4\right)\right)\|\theta\|_{2,\Omega}^2.$$

Considering that $(\mathbf{u},\theta)|_{t=0}=(\mathbf{0},0)$, the Gronwall's lemma allows us to conclude that $(\mathbf{u},\theta)=(\mathbf{0},0)$.

7.2. Uniqueness in Theorem 2.2

Proceeding as in Section 7.2 we find

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{2,\Omega}^2+\frac{1}{2}\frac{d}{dt}\|\theta\|_{2,\Omega}^2+\mu_0\|D\mathbf{u}\|_{2,\Omega}^2+k_0\|\nabla\theta\|_{2,\Omega}^2$$
$$\leqslant I_1+I_2+I_3+I_4+I_5+I_6-\int_\Omega(|\theta_1|^{1/2}-|\theta_2|^{1/2})\theta_2\theta\,dx.$$

For every $t\in[0,T]$, define

$$[|\theta_1|^{1/2}+|\theta_2|^{1/2}>0]=\{x\in\Omega:\ |\theta_1(x,t)|^{1/2}+|\theta_2(x,t)|^{1/2}>0\}.$$

Then the last term in RHS of the above inequality reads

$$-\int_\Omega(|\theta_1|^{1/2}-|\theta_2|^{1/2})\theta_2\theta\,dx=-\int_{[|\theta_1|^{1/2}+|\theta_2|^{1/2}>0]}\frac{|\theta_1|-|\theta_2|}{|\theta_1|^{1/2}+|\theta_2|^{1/2}}\theta_2\theta\,dx$$
$$\leqslant\int_{[|\theta_1|^{1/2}+|\theta_2|^{1/2}>0]}\frac{|\theta_2|}{|\theta_1|^{1/2}+|\theta_2|^{1/2}}\theta^2dx\leqslant\int_\Omega|\theta_2|^{1/2}\theta^2dx\leqslant\|\theta_2\|_{\infty,\Omega}^{1/2}\|\theta\|_{2,\Omega}^2.$$

Therefore we can conclude the desired uniqueness.

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