Differential
Equations
& Applications
Volume 2, Number 1 (2010), 15–25

NONLOCAL PROBLEMS FOR DELAY INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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(*Communicated by J. Henderson*)

Abstract. In this paper we study the existence of mild solutions for a class of first-order delay integrodifferential equations with nonlocal condition in a Banach space. The results are established by the application of the theory of resolvent operators, the contraction mapping principle and the Schaefer theorem. An example is presented in the end to show the applications of the obtained results.

1. Introduction

In this paper, we will investigate the existence of mild solutions for the following first-order delay integrodifferential equations with nonlocal condition:

$$\begin{aligned} &x'(t) = A(t)\left(x(t) + \int_0^t H(t,s)x(s)ds\right) + \int_0^t F(t,s,x(\sigma(s)))ds, \quad t \in J, \\ &x(0) + g(x) = x_0, \end{aligned} \tag{1.1}$$

where $J = [0,b]$, the unknown $x(\cdot)$ takes values in the Banach space X, and $x_0 \in X$. Here $A(t)$ is a closed linear operator on X with dense domain $D(A)$, which is independent of t. $H(t,s), t,s \in J$, is a bounded operator in X. The nonlinear operators $F : \Delta \times X \to X$, $g : C(J,X) \to X$, $\sigma : J \to J$, are given functions. Here Δ denotes the set $\{(t,s) : 0 \leqslant s \leqslant t \leqslant b\}$.

The nonlocal Cauchy problem was first considered by Byszewski. As pointed out by Byszewski [2] the study of Cauchy problem with nonlocal conditions is of significance since they have applications in problems in physics and other areas of applied mathematics. Subsequently, many authors are devoted to studying of nonlocal problems. For example, Deng [8], Byszewski and Akca [3], Ntouyas and Tsamatos [15], Lin and Liu [13], Liang *et al.* [14], Benchohra *et al.* [4], Balachandran [5], Aizicovici and Mckibben [1], Chandrasekaran [7].

Mathematics subject classification (2010): 34K30, 34A60, 34G20.
Keywords and phrases: delay integrodifferential equations, fixed point theorem, nonlocal conditions.

Paper DEA-02-02

Recently, an existence of mild solutions to nonautonomous integro-differential equations with nonlocal condition in a Banach space has been done by Lin and Ezznbi [12]. In addition, Lin and Liu [13] studied the nonlocal Cauchy problem by using resolvent operators. For other results on nonlocal Cauchy problems, we refer the interested reader to [6] and the references therein.

The purpose of this paper is to prove the existence of mild solutions for delay integrodifferential equations with nonlocal condition in a Banach space. Our approach is different from the ones in [1-8, 12-15] and consists in applying resolvent operators, Banach's contraction principle and Schaefer fixed-point theorem.

This paper will be organized as follows. In Section 2, we will recall briefly some preliminaries fact which will be used in paper. Section 3 is devoted to the existence of mild solutions of problem (1.1). Finally, a concrete example is presented in Section 4 to show the application of our main results.

2. Preliminaries

In this section, we shall introduce some basic definitions and lemmas which are used throughout this paper.

Let $(X, \|\cdot\|)$ be a Banach space. $C(J,X)$ is the Banach space of continuous functions from J into X with the norm

$$\| x \|_\infty = \sup\{\| x(t) \| : t \in J\},$$

and $B(X)$ denotes the Banach space of bounded linear operators from X to X.

A measurable function $x : J \to X$ is Bochner integrable if and only if $\| x \|$ is Lebesgue integrable (For properties of the Bochner integral see Yosida [17]). $L^1(J,X)$ denotes the Banach space of measurable functions $x : J \to X$ which are Bochner integrable normed by

$$\| x \|_{L^1} = \int_0^b \| x(t) \| \, dt \quad \text{for all} \quad x \in L^1(J,X).$$

DEFINITION 2.1. A resolvent operator for problem (1.1) is a bounded operator valued function $R(t,s) \in B(X), 0 \leqslant s \leqslant t \leqslant b$, the space of bounded linear operators on X, having the following properties:

(a) $R(t,s)$ is strongly continuous in s and t, $R(s,s) = I, 0 \leqslant s \leqslant b, \| R(t,s) \| \leqslant Me^{\beta(t-s)}$ for some constants M and β.

(b) $R(t,s)Y \subset Y, R(t,s)$ is strongly continuous in s and t on Y.

(c) For each $x \in X, R(t,s)x$ is continuously differentiable in $s \in J$ and

$$\frac{\partial R}{\partial s}(t,s)x = -R(t,s)A(s)x - \int_s^t R(t,\tau)H(\tau,s)A(s)x d\tau.$$

(d) For $x \in X$, and $s \in J, R(t,s)x$ is continuously differentiable in $t \in [s,b]$ and

$$\frac{\partial R}{\partial t}(t,s)x = A(t)R(t,s)x + \int_s^t H(t,\tau)A(\tau)R(\tau,s)xd\tau$$

with $\frac{\partial R}{\partial s}(t,s)x$ and $\frac{\partial R}{\partial t}(t,s)x$ are strongly continuous on $0 \leqslant s \leqslant t \leqslant b$. Here, $R(t,s)$ can be deduced from the evolution operator of the generator $A(t)$.

DEFINITION 2.2. A continuous function $x(\cdot) : J \to X$ is a said to be a mild solution to problem (1.1) if for all $x_0 \in X$, it satisfies the following integral equation

$$x(t) = R(t,0)[x_0 - g(x)] + \int_0^t R(t,s) \int_0^s F(s,\tau,x(\sigma(\tau)))d\tau ds. \tag{2.1}$$

LEMMA 2.1. (Schaefer's fixed point theorem [9]) *Let E be a normed linear space. Let $Q : E \to E$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set and let*

$$\zeta(Q) = \{x \in E \,:\, x = \lambda Qx \ \textit{for some}\ \ 0 < \lambda < 1\}.$$

Then either $\zeta(Q)$ is unbounded or Q has a fixed point.

Further we assume the following hypotheses:

(H1) The resolvent operator $R(t,s)$ is compact for $t,s > 0$.
(H2) The function $F : \Delta \times X \to X$ is continuous, $F(t,s,0) = 0$, and it satisfies the Lipschitz continuous with respect to x, i.e.,

$$\| F(t,s,x_1) - F(t,s,x_2) \| \leqslant L(t,s,\| x_1 \|,\| x_2 \|) \| x_1 - x_2 \|, \quad (t,s) \in \Delta, x_1, x_2 \in X,$$

where $L : J \times J \times [0,\infty) \times [0,\infty) \to (0,\infty)$ and is monotonically nondecreasing with respect to the second and third arguments.
(H3) There exists a continuous function $p : J \times J \to (0,\infty)$ such that

$$\| F(t,s,x) \| \leqslant p(t,s)\Theta(\| x \|), \quad (t,s) \in \Delta, x \in X,$$

where $\Theta : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.
(H4) $\sigma : J \to J,$ is continuous functions such that $\sigma(t) \leqslant t$.
(H5) (i) The function $g(\cdot) : C(J,X) \to X$ is continuous and there exists a $\delta \in (0,b)$ such that $g(\phi) = g(\psi)$ for any $\phi, \psi \in C := C(J,X)$ with $\phi = \psi$ on $[\delta,b]$.
(ii) There is a constant $c > 0$ such that

$$0 \leqslant \lim_{\|\phi\| \to \infty} \sup \frac{\| g(\phi) \|}{\| \phi \|} \leqslant c \quad \text{and} \quad Me^{\beta b}c < 1.$$

3. Main result

THEOREM 3.1. *If hypotheses* $(H1)$-$(H5)$ *are satisfied, then the nonlocal Cauchy problem* (1.1) *has at least one mild solution on* J, *provided that*

$$\int_{\delta+1}^{\infty} \frac{ds}{s+\Theta(s)} = \infty. \tag{3.1}$$

Proof. Let $L_0 > 0$ is a constant chosen so that

$$q := \sup_{t\in J} \left\{ Mb\int_0^t e^{-(L_0+\beta)(t-s)} L(s,s,\|\phi\|,\|\psi\|)ds < 1 \right\},$$

and we introduce in the space $C(J,X)$ the equivalent norm defined as

$$\|\phi\|_V := \sup_{t\in J}\{ e^{-L_0 t} \|\phi(t)\| \}.$$

Then, it is easy to see that $V := (C(J,X), \|\cdot\|_V)$ is a Banach space. Fix $v \in C(J,X)$ and for $t\in J, \phi\in V$, we now define an operator

$$(Q_v\phi)(t) = R(t,0)[x_0 - g(v)] + \int_0^t R(t,s)\int_0^s F(s,\tau,\phi(\sigma(\tau)))d\tau ds. \tag{3.2}$$

Since $R(\cdot,0)(x_0-g(v)) \in C(J,X)$, so, it follows from (H1)-(H4) that $(Q_v\phi)(t) \in V$ for all $\phi\in V$. Let $\phi,\psi\in V$, we have

$$\begin{aligned}
& e^{-L_0 t} \| (Q_v\phi)(t) - (Q_v\psi)(t) \| \\
&\leqslant e^{-L_0 t}\int_0^t \left\| R(t,s)\left[\int_0^s F(s,\tau,\phi(\sigma(\tau)))d\tau - \int_0^s F(s,\tau,\psi(\sigma(\tau)))d\tau\right]\right\| ds \\
&\leqslant M\int_0^t e^{-L_0 t} e^{\beta(t-s)} \\
&\qquad \left[\int_0^s L(s,\tau,\|\phi(\sigma(\tau))\|,\|\psi(\sigma(\tau))\|)\,\|\phi(\sigma(\tau))-\psi(\sigma(\tau))\|\,d\tau\right] ds \\
&\leqslant M\int_0^t e^{-L_0 t} e^{\beta(t-s)}\left[\int_0^s L(s,\tau,\|\phi\|,\|\psi\|)\,\|\phi(\tau)-\psi(\tau)\|\,d\tau\right] ds \\
&\leqslant Mb\int_0^t e^{-L_0 t} e^{\beta(t-s)} L(s,s,\|\phi\|,\|\psi\|)\,\|\phi(s)-\psi(s)\|\,ds \\
&\leqslant Mb\int_0^t e^{-(L_0+\beta)(t-s)} L(s,s,\|\phi\|,\|\psi\|)ds\,\|\phi-\psi\|_V \\
&\leqslant q\,\|\phi-\psi\|_V, \ \ t\in J,
\end{aligned}$$

which implies that

$$\| Q_v\phi - Q_v\psi \|_V \leqslant q\,\|\phi-\psi\|_V, \quad \phi,\psi\in V.$$

Hence, Q_v is a strict contraction. By Banach's contraction principle we conclude that Q_v has a unique fixed point $\phi_v \in V$ and Eq. (3.2) has a unique mild solution on $[0,b]$. Set

$$\tilde{v}(t) := \begin{cases} v(t), & \text{if } t \in (\delta, b], \\ v(\delta), & \text{if } t \in [0,\delta]. \end{cases}$$

From (3.2), we have

$$\phi_{\tilde{v}}(t) = R(t,0)[x_0 - g(\tilde{v})] + \int_0^t R(t,s) \int_0^s F(s,\tau,\phi_{\tilde{v}}(\sigma(\tau)))d\tau ds. \tag{3.3}$$

Consider the map, $P : C_\delta = C([\delta,b],X) \to C_\delta$ defined by

$$(Pv)(t) = \phi_{\tilde{v}}(t), \ \ t \in [\delta,b]. \tag{3.4}$$

We shall show that P satisfy all conditions of Lemma 2.1. The proof will be given in several steps.

Step 1. The set $\Omega = \{v \in C_\delta : \lambda \in (0,1), v = \lambda P(v)\}$ is bounded. Indeed, let $\lambda \in (0,1)$ and let $v \in C_\delta$ be a possible solution of $v = \lambda P(v)$ for some $0 < \lambda < 1$. This implies by (3.3) and (3.4) that for each $t \in [\delta,b]$ we have

$$v(t) = \lambda\phi_{\tilde{v}}(t) = \lambda R(t,0)[x_0 - g(\tilde{v})] + \lambda \int_0^t R(t,s) \int_0^s F(s,\tau,\phi_{\tilde{v}}(\sigma(\tau)))d\tau ds. \tag{3.5}$$

From the condition (H5)(ii), we conclude that there exist $0 < \rho < c$ and $\gamma > 0$ such that, for all $\| \phi \| > \gamma$,

$$\| g(\phi) \| < \rho \| \phi \| \quad \text{and} \quad Me^{\beta b}\rho < 1. \tag{3.6}$$

We define

$$E_1 = \{\phi : \| \phi \| \leqslant \gamma\}, \quad E_2 = \{\phi : \| \phi \| > \gamma\},$$
$$M_1 = \max\{\| g(\phi) \|, \phi \in E_1\}.$$

Therefore,

$$\| g(\phi) \| \leqslant M_1 + \rho \| \phi \| . \tag{3.7}$$

It is from (H1)-(H4), (3.5) and (3.7) that for each $t \in [\delta,b]$ we have $\| v(t) \| \leqslant \| \phi_{\tilde{v}}(t) \|$ and

$$\begin{aligned} e^{-\beta t} \| \phi_{\tilde{v}}(t) \| &\leqslant e^{-\beta t} \| R(t,0)[x_0 - g(\tilde{v})] \| + e^{-\beta t} \int_0^t \left\| R(t,s) \int_0^s F(s,\tau,\phi_{\tilde{v}}(\tau)))d\tau \right\| ds \\ &\leqslant M(\| x_0 \| + M_1 + \rho \| \tilde{v} \|) + M \int_0^t e^{-\beta s} \left[\int_0^s p(s,\tau)\Theta(\| \phi_{\tilde{v}}(\sigma(\tau)) \|)d\tau \right] ds \\ &\leqslant M(\| x_0 \| + M_1 + \rho \| \tilde{v} \|) + M \int_0^t e^{-\beta s} \left[\int_0^s p(s,\tau)\Theta(\| \phi_{\tilde{v}}(\tau) \|)d\tau \right] ds \\ &\leqslant M(\| x_0 \| + M_1 + \rho \| \phi_{\tilde{v}} \|) + Mb \int_0^t e^{-\beta s} p(s,s)\Theta(\| \phi_{\tilde{v}}(s) \|)ds. \end{aligned}$$

We consider the function η defined by

$$\eta(t) := \sup\{\| \phi_{\tilde{v}}(s) \| : 0 \leqslant s \leqslant t\}, t \in (0,b].$$

By the previous inequality we have for $t \in [\delta, b]$,

$$(e^{-\beta t} - M\rho)\eta(t) \leqslant M(\| x_0 \| + M_1) + Mb\int_0^t e^{-\beta s} p(s,s)\Theta(\eta(s))ds.$$

Denoting the right-hand side of the above inequality by $w(t)$. Then we have:

$$(e^{-\beta t} - M\rho)\eta(t) \leqslant w(t) \;\text{ for all }\; t \in [\delta, b],$$
$$w(\delta) = M(\| x_0 \| + M_1) + Mb\int_0^\delta e^{-\beta s} p(s,s)\Theta(\eta(s))ds,$$
$$w'(t) \leqslant Mbe^{-\beta t} p(t,t)\Theta(\eta(t)) \leqslant Mbe^{-\beta t} p(t,t)\Theta\left(\frac{e^{\beta t}}{1 - M\rho e^{\beta t}} w(t)\right), \quad t \in [\delta, b].$$

Then, for each $t \in [\delta, b]$ we have

$$\begin{aligned}\left(\frac{e^{\beta t}}{1 - M\rho e^{\beta t}} w(t)\right)' &= \frac{\beta e^{\beta t}}{(1 - M\rho e^{\beta t})^2} w(t) + \frac{e^{\beta t}}{1 - M\rho e^{\beta t}} w'(t)\\ &\leqslant \frac{\beta e^{\beta t}}{(1 - M\rho e^{\beta t})^2} w(t) + \frac{Mb}{1 - M\rho e^{\beta t}} p(t,t)\Theta\left(\frac{e^{\beta t}}{1 - M\rho e^{\beta t}} w(t)\right)\\ &\leqslant \max\left\{\frac{\beta}{1 - M\rho e^{\beta t}}, \frac{Mb}{1 - M\rho e^{\beta t}} p(t,t)\right\}\left[\frac{e^{\beta t}}{1 - M\rho e^{\beta t}} w(t)\right.\\ &\qquad \left. + \Theta\left(\frac{e^{\beta t}}{1 - M\rho e^{\beta t}} w(t)\right)\right].\end{aligned}$$

This implies for each $t \in [\delta, b]$ that

$$\int_D^T \frac{du}{u + \Theta(u)} \leqslant \int_\delta^t \max\left\{\frac{\beta}{1 - M\rho e^{\beta s}}, \frac{Mb}{1 - M\rho e^{\beta s}} p(s,s)\right\} ds < \infty,$$

where:

$$D = \frac{e^{\beta\delta}}{1 - M\rho e^{\beta\delta}} w(\delta) \quad\text{and}\quad T = \frac{e^{\beta t}}{1 - M\rho e^{\beta t}} w(t).$$

Thus from (3.1) there exists a constant d such that $w(t) \leqslant d, t \in [\delta, b]$, and hence $\| v \| \leqslant \| \phi_{\tilde{v}} \| \leqslant d^*$ where d^* depends only on the functions p and Θ. This shows that Ω is bounded.

Step 2. P maps bounded sets into equicontinuous sets of C_δ. Let

$$v \in C_r(\delta) := \{\phi \in C([\delta, b], X); \sup_{\delta \leqslant t \leqslant b} \| \phi(t) \| \leqslant r\}, r > 0,$$

and $v \in C_r(\delta)$, then, if $\delta \leqslant t_1 < t_2 \leqslant b$, we have

$$\begin{aligned}
&\| Pv(t_2) - Pv(t_1) \| \\
&\quad \leqslant \| [R(t_2,0) - R(t_1,0)][x_0 - g(\tilde{v})] \| + \int_0^{t_2} \left\| R(t_2,s) \int_0^s F(s,\tau,\phi_{\tilde{v}}(\sigma(\tau)))d\tau \right. \\
&\qquad \left. - \int_0^{t_1} R(t_1,s) \int_0^s F(s,\tau,\phi_{\tilde{v}}(\sigma(\tau)))d\tau \right\| ds \\
&\quad \leqslant \| R(t_2,0) - R(t_1,0) \| [\| x_0 \| + M_1 + \rho r] \\
&\qquad + \int_0^{t_1} \| R(t_2,s) - R(t_1,s) \| \left\| \int_0^s F(s,\tau,\phi_{\tilde{v}}(\sigma(\tau)))d\tau \right\| ds \\
&\qquad + Me^{\beta t_2} \int_{t_1}^{t_2} e^{-\beta s} \left\| \int_0^s F(s,\tau,\phi_{\tilde{v}}(\sigma(\tau)))d\tau \right\| ds.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\left\| \int_0^s F(s,\tau,\phi_{\tilde{v}}(\sigma(\tau)))d\tau \right\| \\
&\quad \leqslant \int_0^s \| F(s,\tau,\phi_{\tilde{v}}(\sigma(\tau)))d\tau - F(s,\tau,0) \| + \| F(s,\tau,0) \| d\tau \\
&\quad \leqslant \int_0^s L(s,\tau,\| \phi_{\tilde{v}}(\sigma(\tau)) \|,0) \| \phi_{\tilde{v}}(\sigma(\tau)) \| d\tau \\
&\quad \leqslant \int_0^s L(s,\tau,\| \phi_{\tilde{v}} \|,0) \| \phi_{\tilde{v}}(\tau)) \| d\tau \\
&\quad \leqslant \int_0^s L(s,s,\| \phi_{\tilde{v}} \|,0) \sup_{s\in[\delta,b]} \| \phi_{\tilde{v}}(\tau)) \| d\tau \leqslant M_L br.
\end{aligned}$$

We see that $\| Pv(t_2) - Pv(t_1) \|$ tends to zero independently of $v \in C_r(\delta)$ as $t_2 - t_1 \to 0$, since the compactness of $R(t,s)$ for $t,s > 0$, implies the continuity in the uniform operator topology. Thus the family of functions $\{(Pv) : v \in C_r(\delta)\}$ is equicontinuous on $[\delta, b]$.

Step 3. The set $\{P(v)(t) : v \in C_r(\delta)\}$ is relatively compact in X. Let $\delta < t \leqslant s \leqslant b$ be fixed and ε a real number satisfying $0 < \varepsilon < t$, for $v \in C_r(\delta)$, we define

$$(P_\varepsilon v)(t) = R(t,0)[x_0 - g(\tilde{v})] + \int_0^{t-\varepsilon} R(t,s) \int_0^s F(s,\tau,\phi_{\tilde{v}}(\sigma(\tau)))d\tau ds.$$

Using the compactness of $R(t,s)$ for $t,s > 0$, we obtain the set $\{(P_\varepsilon v)(t) : v \in Y_r(\delta)\}$

is precompact $v \in C_r(\delta)$ for every $\varepsilon, 0 < \varepsilon < t$. Moreover for every $v \in C_r(\delta)$ we have

$$\begin{aligned}
&\| (Pv)(t) - (P_\varepsilon v)(t) \| \\
&\leqslant \int_{t-\varepsilon}^{t} \left\| R(t,s) \int_0^s F(s,\tau,\phi_{\tilde{v}}(\sigma(\tau)))d\tau \right\| ds \\
&\leqslant Me^{\beta b} \int_{t-\varepsilon}^{t} e^{-\beta s} \left\| \int_0^s F(s,\tau,\phi_{\tilde{v}}(\sigma(\tau)))d\tau \right\| ds \\
&\leqslant Me^{\beta b} \int_{t-\varepsilon}^{t} e^{-\beta s} M_L b r ds.
\end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $\{(Pv) : v \in C_r(\delta)\}$. Hence the set $\{(Pv) : v \in C_r(\delta)\}$ is a precompact in X.

Step 4. $P : C_\delta \to C_\delta$ is continuous. From (3.2), (3.3) and (H1)-(H4), we deduce that for $v_1, v_2 \in C_r(\delta)$, $t \in (0,b]$,

$$\begin{aligned}
&e^{-\beta t} \| \phi_{\tilde{v}_1}(t) - \phi_{\tilde{v}_2}(t) \| \\
&\quad\leqslant e^{-\beta t} \| R(t,0)[g(\tilde{v}_1) - g(\tilde{v}_2)] \| \\
&\qquad + e^{-\beta t} \int_0^t \left\| R(t,s) \left[\int_0^s F(s,\tau,\phi_{\tilde{v}_1}(\sigma(\tau)))d\tau - \int_0^s F(s,\tau,\phi_{\tilde{v}_2}(\sigma(\tau)))d\tau \right] \right\| ds \\
&\quad\leqslant M \| g(\tilde{v}_1) - g(\tilde{v}_2) \| + M \int_0^t e^{-\beta s} \\
&\qquad \left[\int_0^s L(s,\tau,\| \phi_{\tilde{v}_1}(\sigma(\tau)) \|, \| \phi_{\tilde{v}_2}(\sigma(\tau)) \|) \| \phi_{\tilde{v}_1}(\sigma(\tau)) - \phi_{\tilde{v}_2}(\sigma(\tau)) \| d\tau \right] ds \\
&\quad\leqslant M \| g(\tilde{v}_1) - g(\tilde{v}_2) \| \\
&\qquad + M \int_0^t e^{-\beta s} \left[\int_0^s L(s,\tau,\| \phi_{\tilde{v}_1} \|, \| \phi_{\tilde{v}_2} \|) \| \phi_{\tilde{v}_1}(\tau) - \phi_{\tilde{v}_2}(\tau) \| d\tau \right] ds \\
&\quad\leqslant M \| g(\tilde{v}_1) - g(\tilde{v}_2) \| \\
&\qquad + M \int_0^t L(s,s,\| \phi_{\tilde{v}_1} \|, \| \phi_{\tilde{v}_2} \|) \sup_{s \in [\delta,b]} e^{-\beta s} \| \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \| d\tau \Big] ds \\
&\quad\leqslant M \| g(\tilde{v}_1) - g(\tilde{v}_2) \| + MM_L b \int_0^t \sup_{s \in [\delta,b]} e^{-\beta s} \| \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \| ds.
\end{aligned}$$

Using the Gronwall's inequality, that for t, v_1, v_2 as above

$$\sup_{t \in [\delta,b]} e^{-\beta t} \| \phi_{\tilde{v}_1}(t) - \phi_{\tilde{v}_2}(t) \| \leqslant Me^{MM_L b^2} \| g(\tilde{v}_1) - g(\tilde{v}_2) \|,$$

for all $t \in [\delta, b]$, which implies that

$$\| Pv_1 - Pv_2 \| \leqslant Me^{(MM_L b + \beta)b} \| g(\tilde{v}_1) - g(\tilde{v}_2) \|,$$

for all $t \in [\delta, b]$, $v_1, v_2 \in C_r(\delta)$. Therefore, P is continuous.

These arguments enable us to conclude that P is completely continuous. We can now apply Lemma 2.1 to conclude that P has at least fixed point $\tilde{v}_* \in C_\delta$. Let $x = \phi_{\tilde{v}_*}$. Then, we have

$$x(t) = R(t,0)[x_0 - g(\tilde{v}_*)] + \int_0^t R(t,s) \int_0^s F(s, \tau, \phi_{\tilde{v}}(\sigma(\tau))) d\tau ds. \tag{3.8}$$

Noting that $x = \phi_{\tilde{v}_*} = (P\tilde{v}_*)(t) = \tilde{v}_*,\ t \in [\delta, b]$. By (H5)(i), we obtain

$$g(x) = g(\tilde{v}_*).$$

This implies that x is Q has a fixed point in $C_\delta \subset C(J,X)$. Hence, problem (1.1) has a mild solution and completes the proof of Theorem 3.1.

REMARK 3.1. (H5) is satisfied if there exist constants $b_1, b_2,$ such that

$$\| g(\phi) \| \leqslant b_1 + b_2 \| \phi \|, \quad \phi \in C(J,X),$$

or there exist constants c_1 and $c_2, \mu \in [0,1)$, such that

$$\| g(\phi) \| \leqslant c_1 + c_2 \| \phi \|^{\mu}, \quad \phi \in C(J,X).$$

REMARK 3.2. As we all know, most of work discussed a related nonlinear nonlocal Cauchy problem when g satisfied Lipschitz-type condition, or convex and compact on a given ball [1-8, 12, 13, 15]. In this paper, we consider the case in which g is continuous but without imposing severe compactness conditions and convexity.

REMARK 3.3. Condition (H5) on g in the above theorem is an extension of the corresponding conditions in paper [14].

4. Application

To illustrate the application of the obtained results of this paper, we study the following example in this section:

$$\begin{aligned} z_t(t,x) &= \frac{\partial^2}{\partial x^2}\left[a_0(t)z(t,x) + \int_0^t l(t,s)z(s,x)ds\right] \\ &\quad + \int_0^1 \frac{1}{(1+t^2)(1+s)}[z^2(\sin s, x) + \sin z^2(s,x)]ds, \\ z(t,0) &= z(t,\pi) = 0, \\ z(0,x) &+ \int_\delta^1 [z(s,x) + \log(1+|z(s,x)|)]ds = z_0(x), \quad 0 \leqslant t \leqslant 1,\ 0 \leqslant x \leqslant \pi, \end{aligned} \tag{4.1}$$

where $\delta > 0$, $z_0(x) \in X = L^2([0,\pi])$ and $z_0(0) = z_0(\pi) = 0$. Here, the functions $a_0(t)$ and $l(t,s)$ are continuous on $0 \leqslant t \leqslant b$ and $0 \leqslant s \leqslant t \leqslant b$, respectively.

Let $X = L^2([0,\pi])$ and the operators $A(t)$ be defined by

$$A(t)w = a_0(t)w''$$

with the domain $D(A) = \{w \in X : w, w''$ are absolutely continuous, $w'' \in X, w(0) = w(1) = 0\}$, then $A(t)$ generates an evolution system and $R(t,s)$ can be deduced from the evolution systems [10, 11, 16] such that $R(t,s)$ is compact and $\| R(t,s) \| \leqslant Me^{\beta(t-s)}$ for some constants M and β.

Define respectively $F : [0,1] \times [0,1] \times X \to X$, and $g : C([0,1],X) \to X$ by

$$\int_0^t F(t,s,z(\sigma(s)))(x)ds = \int_0^1 \frac{1}{(1+t^2)(1+s)}[z^2(\sin s,x) + \sin z^2(s,x)]ds,$$

and

$$g(z)(x) = \int_\delta^1 [z(s,x) + \log(1 + |z(s,x)|)]ds, \quad x \in C([0,1],X).$$

We have

$$\| F(t,s,z) \| \leqslant \left\| \frac{1}{(1+t^2)(1+s)}[z^2(\sin s,x) + \sin z^2(s,x)]ds \right\| \leqslant \frac{2}{(1+t^2)(1+s)} \| z \|^2,$$

and

$$\begin{aligned} &\| F(t,s,z_1) - F(t,s,z_2) \| \\ &\leqslant \frac{1}{(1+t^2)(1+s)} \left\| z_1^2(\sin s,x) + \sin z_1^2(s,x) - z_2^2(\sin s,x) - \sin z_2^2(s,x) \right\| \\ &\leqslant \frac{1}{(1+t^2)(1+s)} [\| z_1 + z_2 \| + 1] \| z_1 - z_2 \| . \end{aligned}$$

Then Eq. (4.1) takes the abstract form (1.1). Moreover, suppose that condition $Me^{\beta b}(1 - \delta) < 1$ hold. Further, all the other conditions stated in Theorem 3.1 are satisfied. Hence, problem (4.1) has a mild solution on $[0,1]$.

REFERENCES

[1] S. AIZICOVICI, M. MCKIBBEN, *Existence results for a class of abstract nonlocal Cauchy problems*, Nonlinear Anal., **39** (2000), 649–668.

[2] L. BYSZEWSKI, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl., **162** (1991), 494–505.

[3] L. BYSZEWSKI, H. AKCA, *Existence of solutions of a semilinear functional-differential evolution nonlocal problem*, Nonlinear Anal., **34** (1998), 65–72.

[4] M. BENCHOHRA, E. GATSORI, J. HENDERSON, S.K. NTOUYAS, *Nondensely defined evolution impulsive differential inclusions with nonlocal conditions*, J. Math. Anal. Appl., **286** (2003), 307–325.

[5] K. BALACHANDRAN, J. Y. PARK, M. CHANDRASEKARAN, *Nonlocal Cauchy problems for delay integrodifferential equations of sobolev type in Banach spaces*, Appl. Math. Lett., **15** (2002), 845–854.

[6] M. BENCHOHRA, E. P. GATSORI AND S. K. NTOUYAS, *Controllability results for semilinear evolution inclusions with nonlocal conditions*, J. Optim. Theory Appl., **118** (2003), 493–513.

[7] M. CHANDRASEKARAN, *Nonlocal Cauchy problem for quasilinear integrodifferential equations in Banach spaces*, Electron. J. Diff. Eqns., **2007** (2007), 1–6.

[8] K. DENG, *Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions*, J. Math. Anal. Appl., **179** (1993), 630–637.

[9] J. DUGUNDJI AND A. GRANAS, *Fixed point theory*, Monografie Mat. PWN, Warsaw, 1982.

[10] R. GRIMMER AND J.H. LIU, *Integrated semigroups and integrodifferential equations*, Semigroup Forum, **48** (1994), 79–95.

[11] R. GRIMMER, *Resolvent operators for integral equations in Banach space*, Trans. Amer. Math. Soc., **48** (1982), 333–349.

[12] J. H. LIU AND K. EZZINBI, *Non-autonomous integrodifferential equations with nonlocal conditions*, J. Integral Equ. Appl., **15** (2003), 79–93.

[13] Y. LIN, J.H. LIU, *Semilinear integrodifferential equations with nonlocal Cauchy problem*, Nonlinear Anal., **26** (1996), 1023–1033.

[14] J. LIANG, J.H. LIU, T.J. XIAO, *Nonlocal Cauchy problems governed by compact operator families*, Nonlinear Anal., **57** (2004), 183–189.

[15] S.K. NTOUYAS, P.CH. TSAMATOS, *Global existence for semilinear evolution equations with nonlocal conditions*, J. Math. Anal. Appl., **210** (1997), 679–687.

[16] J. PRUSS, *On resolvent operators for linear integrodifferential equations of Volterra type*, J. Integral Equations, **5** (1983), 211–236.

[17] K.YOSIDA, *Functional Analysis*, sixth ed., Springer, Berlin, 1980.

(Received March 15, 2009)

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