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GLOBAL HÖLDER SOLUTIONS FOR ABSTRACT NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We study the existence of global α-Hölder mild solutions and S-classical solutions for a class of abstract neutral functional differential equations defined on whole the real axis. Some concrete applications to ordinary and partial differential equations are considered.

1. Introduction

In this paper we study the existence of α-Hölder solutions for a class of abstract neutral functional differential equations of the form

$$\frac{d}{dt}[x(t)+g(t)x_t] = Ax(t)+f(t)x_t, \quad t\in\mathbb{R}. \tag{1.1}$$

In this system, A is the infinitesimal generator of an hyperbolic C_0-semigroup of bounded linear operators $(T(t))_{t\geqslant 0}$ on a Banach space $(X,\|\cdot\|)$, the history $x_t(\theta) := x(t+\theta)$, $x_t : (-\infty,0]\to X$, belongs to some abstract phase space $\mathscr{B}$ defined axiomatically and $g, f :\in C(\mathbb{R};\mathscr{L}(\mathscr{B},X)$, where $\mathscr{L}(\mathscr{B},X)$ is the space of bounded linear operators from Z into W.

The literature related ordinary neutral functional differential equations is very extensive and we refer the reader to the book by Hale & Lunel [7] for details. Concerning partial neutral functional differential equations on bounded intervals we cite Hale [8], Wu [18, 19, 20], Adimy [1] and Hernández et all [10] for finite delay equations, and Hernández et all [9, 13] and Hernández [11] for the case of equations with infinite delay. To the best of our knowledge, the problem of the existence of solutions defined on the whole real axis (in particular, the existence of α-Hölder mild solutions and regular solutions) for equations in the abstract form (1.1) is an untreated topic in the literature, and it is the main motivation of this paper.

Abstract neutral differential equations described in the abstract from (1.1) arise, for instance, in the theory of heat conduction in fading memory materials. In the classic theory of heat conduction, it is assumed that the internal energy and the heat flux

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depends linearly on the temperature $u(\cdot)$ and on its gradient $\nabla u(\cdot)$. Under these conditions, the classical heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [6, 17], the internal energy and the heat flux are described as functionals of $u(\cdot)$ and $u_x(\cdot)$. The following equation, see [2, 4, 5, 15], has been frequently used to describe this phenomena

$$\frac{d}{dt}\left[u(t,x)+\int_{-\infty}^{t} k_1(t-s)u(s,x)ds\right] = c\triangle u(t,x)+\int_{-\infty}^{t} k_2(t-s)\triangle u(s,x)ds,$$
$$u(t,x)=0, \quad x\in\partial\Omega.$$

In this equation, $\Omega\subset\mathbb{R}^n$ is open, bounded and has smooth boundary, $(t,x)\in[0,\infty)\times\Omega$, $u(t,x)$ represents the temperature in x at the time t, the letter c denotes a physical constant and $k_i:\mathbb{R}\to\mathbb{R}$, $i=1,2$, are the internal energy and the heat flux relaxation respectively. By assuming that $k_2\equiv 0$, we can represent the above system in the abstract form (1.1). In the last section, we consider two additional concrete applications to ordinary and partial differential equations.

We include now some definitions, properties and technicalities. In this paper, the pair $(X,\|\cdot\|)$ is a Banach space and A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(T(t))_{t\geqslant 0}$, on $(X,\|\cdot\|)$. We also assume $\sigma(A)\cap i\mathbb{R}=\emptyset$ and $\sigma_+(A)=\{\lambda\in\sigma(A):\mathrm{Re}(\lambda)>0\}$ is bounded. Under these conditions, the sets $\sigma_-(A)=\{\lambda\in\sigma(A):\mathrm{Re}(\lambda)<0\}$ and $\sigma_+(A)=\{\lambda\in\sigma(A):\mathrm{Re}(\lambda)>0\}$ are closed, disjoint and there exists $\delta>0$ such that

$$\sup\{\mathrm{Re}(\lambda):\lambda\in\sigma_-(A)\}<-\delta<0<\delta<\inf\{\mathrm{Re}(\lambda):\lambda\in\sigma_+(A)\}.$$

Let $\Omega\subset\mathbb{R}^2$ be an open bounded set with smooth boundary $\partial\Omega$ such that $\sigma_+(A)\subset\Omega\subseteq\mathbb{C}_+=\{\lambda\in\mathbb{C}:\mathrm{Re}(\lambda)>0\}$ and $P:X\to X$ be the operator defined by

$$Px=\frac{1}{2\pi i}\int_{\partial\Omega}R(\mu;A)x\,d\mu,\ x\in X,$$

with $\partial\Omega$ oriented counterclockwise. In the next result, $X_1=P(X)$, $X_2=(I-P)(X)$ and $A_1:X_1\to X$, $A_2:D(A_2)=\{x\in D(A):x\in X_2, Ax\in X_2\}\to X_2$, are the operators defined by $A_1x=Ax$ for $x\in X_1$ and $A_2y=Ay$ for $y\in D(A_2)$. The next result can be found in [16].

PROPOSITION 1.1. *The following properties are valid.*

(a) *The operator P is a projection, $P(X)\subset D(A^n)$ for all $n\in\mathbb{N}$, $T(t)Px=PT(t)x$ for all $x\in X$ and $T(t)X_i\subset X_i$ for $i=1,2$, and every $t\geqslant 0$.*

(b) $A_1(X_1)\subset X_1$, $\sigma(A_1)=\sigma_+(A)$ *and* $R(\lambda:A_1)=R(\lambda:A)|_{X_1}$ *for all* $\lambda\in\rho(A_1)$. *Moreover, A_1 is the generator of a C_0-group $(T_{A_1}(t))_{t\geqslant 0}$ on X_1 and $T_{A_1}(t)=T(t)|_{X_1}$ for every $t\geqslant 0$.*

(c) $\sigma(A_2)=\sigma_-(A)$, $R(\lambda:A_2)=R(\lambda:A)|_{X_2}$ *for all* $\lambda\in\rho(A_2)$, *A_2 is the generator of a uniformly stable analytic semigroup $(T_{A_2}(t))_{t\geqslant 0}$ on X_2, $T_{A_2}(t)=T(t)|_{X_2}$ for every $t\geqslant 0$ and $T(t)=T_{A_1}(t)+T_{A_2}(t)$ for each $t\geqslant 0$.*

(d) *There are* $\gamma > 0$ *and positive constants* C_i, d_i, $i \in \mathbb{N}$, *such that*

$$\|A^i T(-t)P\| \leqslant d_i e^{-\omega t} \text{ and } \|A^i T(t)(I-P)\| \leqslant C_i e^{-\gamma t} \text{ for all } t \geqslant 0.$$

From Lunardi [16] we also remark the following result. In this result, $[D(A)]$ represent the domain of A endowed with the graph norm.

PROPOSITION 1.2. *Let* $f \in L^\infty(\mathbb{R}, X)$ *and assume* $u \in C^1(\mathbb{R}, X) \cap C(\mathbb{R}, [D(A)])$ *is a bounded solution of the equation*

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R}.$$

Then

$$u(t) = \int_{-\infty}^{t} T(t-\tau)(I-P)f(\tau)d\tau - \int_{t}^{\infty} T(t-\tau)Pf(\tau)d\tau, \qquad t \in \mathbb{R}.$$

In this work we employ an axiomatic definition for the phase space $\mathscr{B}$, which is similar to the one used in [14]. Specifically, $\mathscr{B}$ is a vector space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathscr{B}}$ such that the next axioms hold.

(A) If $x : (-\infty, \sigma + a) \mapsto X$, $a > 0$, $\sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathscr{B}$, then for every $t \in [\sigma, \sigma + a)$ the following hold:
(i) x_t is in $\mathscr{B}$;
(ii) $\|x(t)\| \leqslant H\|x_t\|_{\mathscr{B}}$;
(iii) $\|x_t\|_{\mathscr{B}} \leqslant K(t-\sigma)\sup\{\|x(s)\| : \sigma \leqslant s \leqslant t\} + M(t-\sigma)\|x_\sigma\|_{\mathscr{B}}$,
where $H > 0$ is a constant; $K, M : [0, \infty) \mapsto [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.
$(A1)$ For the function $x(\cdot)$ in (A), the function $t \to x_t$ is continuous from $[\sigma, \sigma + a)$ into $\mathscr{B}$.
(B) The space $\mathscr{B}$ is complete.
$(C2)$ If $(\varphi^n)_{n \in \mathbb{N}}$ is a uniformly bounded sequence in $C((-\infty, 0], X)$ given by functions with compact support and $\varphi^n \to \varphi$ in the compact-open topology, then $\varphi \in \mathscr{B}$ and $\|\varphi^n - \varphi\|_{\mathscr{B}} \to 0$ as $n \to \infty$.

REMARK 1.1. In this paper we suppose that $\mathfrak{L}$ is a positive constant such that $\|\varphi\|_{\mathscr{B}} \leqslant \mathfrak{L}\sup_{\theta \leqslant 0}\|\varphi(\theta)\|$ for each $\varphi \in \mathscr{B}$ bounded continuous. We cite [14, Proposition 7.1.1] for additional details related this condition.

EXAMPLE 1.1. *The phase space* $C_r \times L^p(\eta, X)$ Let $r \geqslant 0$, $1 \leqslant p < \infty$ and let $\eta : (-\infty, -r] \mapsto \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5)-(g-6) in the terminology of [14]. Briefly, this means that η is locally integrable and there exists a non-negative locally bounded function ζ on $(-\infty, 0]$ such that $\eta(\xi + \theta) \leqslant \zeta(\xi)\eta(\theta)$ for all $\xi \leqslant 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero.

The space $\mathscr{B} = C_r \times L^p(\eta, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \mapsto X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable on $(-\infty, 0)$ and $\eta\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(\eta, X)$ is given by

$$\| \varphi \|_{\mathscr{B}} := \sup\{\|\varphi(\theta)\| : -r \leqslant \theta \leqslant 0\} + \left(\int_{-\infty}^{-r} \eta(\theta)\|\varphi(\theta)\|^p d\theta\right)^{1/p}.$$

The space $\mathscr{B} = C_r \times L^p(\eta, X)$ satisfies axioms (A), (A1) and (B). Moreover, when $r = 0$ and $p = 2$, we can take:

$$H = 1,\ K(t) = 1 + \left(\int_{-t}^{0} \eta(\theta)\, d\theta\right)^{1/2} \quad \text{and} \quad M(t) = \gamma(-t)^{1/2} \ \text{ for } t \geqslant 0,$$

(see [14, Theorem 1.3.8] for details).

In this paper, $C_b(I; Z)$ is the space of all the bounded continuous functions from an interval $I \subset \mathbb{R}$ into Z provided with the sup-norm denoted by $\| \cdot \|_{C_b(I;Z)}$ and $C_b^{\beta}(I; Z)$, $\beta \in (0, 1)$, represents the space formed for all the β-Hölder Z-valued bounded continuous functions from I into Z with the norm

$$\| \xi \|_{C_b^{\gamma}(I;Z)} = \| \xi \|_{C_b(I;Z)} + [| \xi |]_{C_b^{\gamma}(I;Z)},$$

where $[| \xi |]_{C_b^{\gamma}(I;Z)} = \sup_{t,s \in I; t \neq s} \frac{\|\xi(s) - \xi(t)\|_Z}{(t-s)^{\gamma}}$. In the rest of this paper, we will write simply $\| \cdot \|_{C_b(Z)}$ and $\| \cdot \|_{C_b^{\alpha}(Z)}$ when non confusion on the interval I arise.

This paper has tree sections. In the next section we discuss the existence of Hölder mild solutions and classical solutions for the equation (1.1). In the last section, we consider some concrete applications to ordinary and partial differential equations.

2. Existence of solution

In this section, we discuss the existence of C^{α}-Hölder mild and S-classical solutions for the equation (1.1). Motivated from Proposition 1.2 and the development in [9, 12], we introduce the following of mild solution for (1.1). In the rest of this paper, we denote by g_u and f_u the functions given by $g_u(t) = g(t)u_t$ and $f_u(t) = f(t)u_t$.

DEFINITION 2.1. A function $u \in C_b^{\alpha}(\mathbb{R}; X)$ is called of C^{α}-Hölder mild solution of the neutral equation (1.1) if

$$\begin{aligned} u(t) = -g_u(t) - \int_{-\infty}^{t} AT(t-s)(I-P)g_u(s)ds + \int_{t}^{\infty} AT(t-s)Pg_u(s)ds \\ + \int_{-\infty}^{t} T(t-s)(I-P)T(t-s)f_u(s)ds - \int_{t}^{\infty} T(t-s)Pf_u(s)ds,\ \forall t \in \mathbb{R}. \end{aligned}$$

DEFINITION 2.2. A function $u \in C_b^{\alpha}(\mathbb{R}; X)$ is called of S-classical solution of (1.1) if $u \in C_b(\mathbb{R}, [D(A)])$, the functions $t \to u(t) + g(t, u_t)$ belongs to $C_b^1(\mathbb{R}, X)$ and the equation (1.1) is verified.

To establish our results, we always assume that the following condition is verified.

($\mathrm{H_g}$) There are a Banach space $(Y, \|\cdot\|_Y)$ continuously included in X and a function $H \in L^1((0,\infty),(0,\infty))$ such that $\| AT(t)(I-P) \|_{\mathscr{L}(Y,X)} \leqslant e^{-\gamma t}H(t)$ for all $t \geqslant 0$ and $g \in C([0,a] \times \mathscr{B}; Y)$.

REMARK 2.1. The assumption ($\mathrm{H_g}$) is linked to the integrability of the function $s \to AT(t-s)g_u(s)$. We note that except trivial cases, the operator function $AT(\cdot)$ is not integrable in the operator topology on $[0,b]$, for $b > 0$. See [10, 11] for additional details related this type of conditions in the theory of neutral equations.

To establish the main result of this section we need some preliminary lemmas.

LEMMA 2.1. *Let $g \in C_b^\alpha(\mathbb{R};Y)$ and $\Upsilon_i g : \mathbb{R} \to X$, $i = 1,2$ be the functions given by*

$$\Upsilon_1 g(t) = \int_{-\infty}^{t} AT(t-s)(I-P)g(s)ds, \quad t \in \mathbb{R},$$
$$\Upsilon_2 g(t) = \int_{t}^{\infty} AT(t-s)Pg(s)ds, \quad t \in \mathbb{R}.$$

Then $\Upsilon_i g \in C_b^\alpha(\mathbb{R};X)$, $i = 1,2$, and:

$$\| \Upsilon_1 g \|_{C_b(X)} \leqslant \| g \|_{C_b(Y)} \| e^{-\gamma(\cdot)}H(\cdot) \|_{L^1([0,\infty))},$$
$$\| \Upsilon_2 g \|_{C_b(X)} \leqslant \frac{d_1}{\gamma} \| g \|_{C_b(X)},$$
$$[| \Upsilon_1 g |]_{C_b^\alpha(X)} \leqslant [| g |]_{C^\alpha(Y)} \| e^{-\gamma(\cdot)}H(\cdot) \|_{L^1([0,\infty))},$$
$$[| \Upsilon_2 g |]_{C_b^\alpha(X)} \leqslant \frac{d_1}{\gamma} [| g |]_{C^\alpha(X)}.$$

Proof. From the estimate

$$\begin{aligned}
&\int_{-\infty}^{t} \|AT(t-s)(I-P)g(s)\| \, ds \\
&\qquad \leqslant \int_{-\infty}^{t} \| AT(t-s)(I-P) \|_{L(Y,X)} \| g(s) \|_Y \, ds \\
&\qquad \leqslant \int_{-\infty}^{t} e^{-\gamma(t-s)}H(t-s) \| g \|_{C_b(Y)} \, ds \\
&\qquad \leqslant \| g \|_{C_b(Y)} \| e^{-\gamma(\cdot)}H(\cdot) \|_{L^1([0,\infty))},
\end{aligned}$$

and the Bochner's criterion for integrable functions we obtain that the function $s \to AT(t-s)(I-P)g(s)$ is integrable on $(-\infty,t]$ for all $t \in \mathbb{R}$, $\Upsilon_1 g \in C_b(X)$ and

$$\| \Upsilon_1 g \|_{C_b(X)} \leqslant \| g \|_{C_b(Y)} \| e^{-\gamma(\cdot)}H(\cdot) \|_{L^1([0,\infty))}.$$

On the other hand, for $s<t$ we get

$$\begin{aligned}\| \Upsilon_1 g(t)-\Upsilon_1 g(s) \| &\leqslant \int_0^\infty \| AT(\tau)(I-P)(g(t-\tau)-g(s-\tau)) \| \, d\tau \\ &\leqslant [\| g \|]_{C^\alpha(Y)} \int_0^\infty e^{-\gamma\tau} H(\tau) d\tau (t-s)^\alpha,\end{aligned}$$

which implies that $\Upsilon_1 g \in C^\alpha(\mathbb{R},X)$ and

$$[\| \Upsilon_1 \|]_{C_b^\alpha(X)} \leqslant [\| g \|]_{C^\alpha(Y)} \| e^{-\gamma(\cdot)} H(\cdot) \|_{L^1([0,\infty))} . \tag{2.1}$$

This complete the proof related the properties of Υ_1. The rest of the proof is similar, we omit the details. □

Proceeding as in the proof of Lemma 2.1, we can show the following result.

LEMMA 2.2. *Let $f \in C_b^\alpha(\mathbb{R};X)$ and $\Upsilon_i : \mathbb{R} \to X$, $i=3,4$ be the functions given by*

$$\begin{aligned}\Upsilon_3 f(t) &= \int_{-\infty}^t T(t-s)(I-P)f(s)ds, \quad t\in\mathbb{R},\\ \Upsilon_4 f(t) &= \int_t^\infty T(t-s)Pf(s)ds, \quad t\in\mathbb{R}.\end{aligned}$$

Then $\Upsilon_i \in C_b^\alpha(\mathbb{R};X)$, $\| \Upsilon_3 f \|_{C_b(X)} \| \leqslant \frac{C_0}{\gamma} \| f \|_{C_b(X)}$, $\| \Upsilon_4 fu \|_{C_b(X)} \leqslant \frac{d_0}{\gamma} \| f \|_{C_b(X)}$, $[\| \Upsilon_3 f \|]_{C_b^\alpha(X)} \leqslant [\| f \|]_{C_b^\alpha(X)} \frac{C_0}{\gamma}$ and $[\| \Upsilon_4 f \|]_{C_b^\alpha(X)} \leqslant \frac{d_0}{\gamma} [\| f \|]_{C_b^\alpha(X)}$.

In Theorem 2.1 below, we establish the existence of C^α-Hölder mild solutions for (1.1). In this result, i_c represent the inclusion map from Y into X and $\mathfrak{L}$ is the constant introduced in Remark 1.1.

THEOREM 2.1. *Assume $g \in C_b^\alpha(\mathbb{R},\mathscr{L}(\mathscr{B},Y))$, $f \in C_b^\alpha(\mathbb{R},\mathscr{L}(\mathscr{B},X))$ and*

$$\begin{aligned}&\mathfrak{L} \| g \|_{C_b^\alpha(\mathscr{L}(\mathscr{B},Y))} (\| i_c \|_{\mathscr{L}(Y,X)} + \| e^{-\gamma(\cdot)} H(\cdot) \|_{L^1([0,\infty))} + \frac{d_1}{\gamma}) \\ &\qquad + \mathfrak{L} \| f \|_{C_b^\alpha(\mathscr{L}(\mathscr{B},X))} (\frac{C_0+d_0}{\gamma}) < 1.\end{aligned}$$

Then there exists a unique mild solution $u \in C_b^\alpha(\mathbb{R};X)$ of the neutral equation (1.1).

Proof. Let $\Gamma : C_b^\alpha(\mathbb{R};X) \to C_b^\alpha(\mathbb{R};X)$ be the map defined by

$$\begin{aligned}\Gamma u(t) = -g_u(t) &- \int_{-\infty}^t AT(t-s)(I-P)g_u(s)ds + \int_t^\infty AT(t-s)Pg_u(s)ds \\ &+ \int_{-\infty}^t T(t-s)(I-P)f_u(s)ds - \int_t^\infty T(t-s)Pf_u(s)ds.\end{aligned}$$

From the assumption on $f(\cdot)$ and $g(\cdot)$, and Lemmas 2.1 and 2.2 we infer that Γ well defined.

In order to prove that Γ is a contraction on $C_b^\alpha(\mathbb{R};X)$, we take $u,v \in C_b^\alpha(\mathbb{R};X)$ and $w = u - v$. If in the Lemmas 2.1 and 2.2 we use g_w and f_w in the place of g and f, we get

$$\begin{aligned}
&\| \Upsilon_1 g_w \|_{C_b^\alpha(X)} + \| \Upsilon_2 g_w \|_{C_b^\alpha(X)} + \| \Upsilon_3 f_w \|_{C_b^\alpha(X)} + \| \Upsilon_4 f_w \|_{C_b^\alpha(X)} \\
&\quad \leqslant \| i_c \|_{\mathscr{L}(Y,X)} \| g \|_{C_b^\alpha(\mathscr{L}(\mathscr{B},Y))} \| w \|_{C_b^\alpha(X)} \\
&\qquad + \| g_w \|_{C_b^\alpha(\mathscr{L}(Y))} (\| e^{-\gamma(\cdot)} H(\cdot) \|_{L^1([0,\infty))} + \frac{d_1}{\gamma}) + \| f_w \|_{C^\alpha(X)} (\frac{C_0 + d_0)}{\gamma} \\
&\quad \leqslant \| g \|_{C_b^\alpha(\mathscr{L}(\mathscr{B},Y))} (\| i_c \|_{\mathscr{L}(Y,X)} + \| e^{-\gamma(\cdot)} H(\cdot) \|_{L^1([0,\infty))} + \frac{d_1}{\gamma}) \mathfrak{L} \| w \|_{C_b^\alpha(X)} \\
&\qquad + \| f \|_{C^\alpha(\mathscr{L}(\mathscr{B},X))} (\frac{C_0 + d_0}{\gamma}) \mathfrak{L} \| w \|_{C_b^\alpha(X)},
\end{aligned}$$

which implies there exists $\Theta \in (0,1)$ such that $\| \Gamma u - \Gamma v \|_{C_b^\alpha(X)} \Theta \| u - v \|_{C_b^\alpha(X)}$ for all $u,v \in C_b^\alpha(\mathbb{R};X)$. Thus, Γ is a contraction and there exists a unique fixed point $u(\cdot)$ of Γ. Obviously, $u(\cdot)$ is a C^α-mild solution of (1.1). The proof is complete. □

We finish this section with the following result on the existence of S-classical solutions for (1.1).

THEOREM 2.2. *Assume* $g \in C_b^\alpha(\mathbb{R}, \mathscr{L}(\mathscr{B}, [D(A)]))$, $f \in C_b^\alpha(\mathbb{R}, \mathscr{L}(\mathscr{B}, X))$ *and*

$$\mathfrak{L} \left[\| g \|_{C_b^\alpha(\mathscr{L}(\mathscr{B},[D(A)]))} (\| i_c \|_{\mathscr{L}([D(A)],X)} + \frac{C_0 + d_0}{\gamma}) + \| f \|_{C_b^\alpha(\mathscr{L}(\mathscr{B},X))} (\frac{C_0 + d_0}{\gamma}) \right] < 1.$$

Then there exists a unique mild solution $u \in C_b^\alpha(\mathbb{R};X)$ *of the neutral equation* (1.1).

Proof. Since $H \equiv 1$, from Theorem 2.1 there exists a unique C^α-Hölder mild solution $u(\cdot)$ of (1.1). Let $y : \mathbb{R} \to X$ be the function given by $y(t) = u(t) + g_u(t)$. Noting that g is $D(A)$-valued, we see that

$$\begin{aligned}
y(t) = &\int_{-\infty}^t T(t-s)(I-P)[-Ag_u(s) + f_u(s)]ds \\
&- \int_t^\infty T(t-s)P[-Ag_u(s) + f_u(s)]ds, \quad \forall\, t \in \mathbb{R}.
\end{aligned} \tag{2.2}$$

Moreover, since the functions $-Ag_u$ and f_u are α-Hölder continuous on $\mathbb{R}$, from (2.2) it is easy to show that $y \in C_b(\mathbb{R}, [D(A)]) \cap C_b^1(\mathbb{R}, X)$ and $y'(t) = Ay(t) - Ag_u(t) + f_u(t)$ for all $t \in \mathbb{R}$. Finally, by noting that $u(t) = -g_u(t) + y(t) \in D(A)$ for all $t \in \mathbb{R}$, we get

$$\begin{aligned}
\frac{d}{dt}(u(t) + g_u(t)) &= Ay(t) - Ag_u(t) + f_u(t) \\
&= A(u(t) + g_u(t)) - Ag_u(t) + f_u(t) \\
&= Au(t) + f_u(t), \quad t \in \mathbb{R},
\end{aligned} \tag{2.3}$$

which complete the proof that $u(\cdot)$ is a S-classical solutions of (1.1). □

3. Applications

In this section we consider some applications of our abstract results. To begin, we consider an ordinary differential equation described on a finite dimensional space. We note that in this case our results are easily applicable since A is a bounded linear operator and the operator function $AT(\cdot)$ is integrable on bounded intervals (the condition H_g is verified with $Y = X$).

Motivated by the development in [3], we consider the neutral equation

$$\frac{d}{dt}\left[u(t) - \lambda \int_{-\infty}^{t} C(t,t-s)u(s)ds\right] = Au(t) + \lambda \int_{-\infty}^{t} B(t,t-s)u(s)ds - p(t) + q(t), \quad (3.1)$$

which arises in the study of the dynamics of income, employment, value of capital stock, and cumulative balance of payment, see [3] for details. In this equation, λ is a real number, the state $u(t) \in \mathbb{R}^n$, $C(\cdot), B(\cdot)$ are $n \times n$ matrix continuous functions, A is a constant $n \times n$ matrix, $p(\cdot)$ represents the government intervention and $q(\cdot)$ the private initiative.

To treat this equation, we take $\mathscr{B} = C_0 \times L^p(\eta, X)$ with $X = \mathbb{R}^n$, see Example 1.1, and we assume $\sigma(A) \cap i\mathbb{R} = \emptyset$. In what follows, $\gamma, d_i, \mathfrak{L}$ and C_i are as in the Section 1. Next, we suppose

$$L_g^1(t) = (\int_{-\infty}^{0} \mid C(t,\tau) \mid^2 \eta(\tau)d\tau)^{\frac{1}{2}} \text{ and } L_f^1(t) = (\int_{-\infty}^{0} \mid B(t,\tau) \mid^2 \eta(\tau)d\tau)^{\frac{1}{2}}$$

are finite for all $t \in \mathbb{R}$ and there are $c, b \in C(\mathbb{R}, [0,\infty))$ and $\alpha \in (0,1)$ such that

$$\begin{aligned} \mid C(t,\tau) - C(s,\tau) \mid &\leqslant c(\tau) \mid t-s \mid^{\alpha}, \ t,s,\tau \in \mathbb{R}, \\ \mid B(t,\tau) - B(s,\tau) \mid &\leqslant b(\tau) \mid t-s \mid^{\alpha}, \ t,s,\tau \in \mathbb{R}. \end{aligned}$$

We also assume $L_g^2 = (\int_{-\infty}^{0} \mid c(t,\tau) \mid^2 \eta(\tau)d\tau)^{\frac{1}{2}}$ and $L_f^1 = (\int_{-\infty}^{0} \mid b(t,\tau) \mid^2 \eta(\tau)d\tau)^{\frac{1}{2}}$ are finite.

To describe the equation (3.1) in the abstract form (1.1), we introduce the maps $f, g : \mathbb{R} \times \mathscr{B} \to X$ defined by:

$$g(t,\psi) = -\lambda \int_{-\infty}^{0} C(t,-s)\psi(s)ds \text{ and } f(t,\psi) = \int_{-\infty}^{0} B(t,-s)\psi(s)ds - p(t) + q(t).$$

The proof of the next result, which is a consequence of Theorem 2.1, is omitted. In this result, we said that $u \in C_b(\mathbb{R};X)$ is a C^{α}-mild solution of (3.1) if $u(\cdot)$ is a C^{α}-mild solution of the associated abstract equation (1.1).

PROPOSITION 3.1. *Assume the above conditions are verified,* $p, q \in C_b^{\alpha}(\mathbb{R})$ *and*

$$\mathfrak{L}\left[\left(\sup_{t\in\mathbb{R}} L_g^1(t) + L_g^2\right) \parallel A \parallel (1 + \frac{1}{\gamma} + \frac{d_0}{\gamma}) + \left(\sup_{t\in\mathbb{R}} L_f^1(t) + L_f^2\right)(\frac{C_0 + d_0}{\gamma})\right] < 1.$$

Then there exists a unique C^{α}*-mild solution of* (3.1).

To complete this section, we consider briefly the partial differential system

$$\frac{\partial}{\partial t}\left[u(t,\xi)+\int_{-\infty}^{0}\int_{0}^{\pi} b_0(t)b_1(s,\eta,\xi)u(t+s,\eta)d\eta ds\right]$$
$$=\frac{\partial^2}{\partial\xi^2}u(t,\xi)+a_0(t)a_1(\xi)u(t,\xi)+\int_{-\infty}^{0} a_2(t)a_3(s)u(t+s,\xi)ds, \quad (3.2)$$

$$u(t,0)=u(t,\pi)=0, \quad (3.3)$$

for $(t,\xi)\in\mathbb{R}\times[0,\pi]$, which arise in control systems which are described by abstract functional differential equations with feedback control governed by proportional integro-differential law, see [9, Examples 4.2] for details.

Next, $X=L^2([0,\pi])$ and let A be the operator given by $Ax=x''$ with domain $D(A):=\{x\in X: x''\in L^2([0,\pi]),\, x(0)=x(\pi)=0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geqslant 0}$ on X. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n\in\mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(\xi):=(\frac{2}{\pi})^{\frac{1}{2}}\sin(n\xi)$. In addition, the set $\{z_n : n\in\mathbb{N}\}$ is an orthonormal basis for X and

$$T(t)x=\sum_{n=1}^{\infty}e^{-n^2t}\langle x,z_n\rangle z_n \text{ for } x\in X \text{ and } Ax=-\sum_{n=1}^{\infty}n^2\langle x,z_n\rangle z_n \text{ for } x\in D(A).$$

From the above, we see that $\| T(t)\|\leqslant e^{-t}$ for all $t\geqslant 0$.

We can also define the fractional powers of A, see [16] for details. For $x\in X$ and $\alpha\in(0,1)$,

$$(-A)^{-\alpha}x=\sum_{n=1}^{\infty}\frac{1}{n^{2\alpha}}\langle x,z_n\rangle z_n,$$

the operator $(-A)^{\alpha}: D((-A)^{\alpha})\subseteq X\to X$ is given by

$$(-A)^{\alpha}x=\sum_{n=1}^{\infty}n^{2\alpha}\langle x,z_n\rangle z_n, \text{ for all } x\in D((-A)^{\alpha}),$$

where

$$D((-A)^{\alpha})=\{x\in X: \sum_{n=1}^{\infty}n^{2\alpha}\langle x,z_n\rangle z_n\in X\}.$$

Moreover, for $\alpha=\frac{1}{2}$ we have that:

$$\|(-A)^{-1/2}\|=1 \text{ and } \| (-A)^{\frac{1}{2}}T(t)\|\leqslant\frac{1}{\sqrt{2}}e^{\frac{-t}{2}}t^{-\frac{1}{2}} \text{ for all } t>0.$$

As in the first example, we take $\mathscr{B}=C_0\times L^p(\eta,X)$ as phase space. Next, we suppose that the following condition hold.

(i) The functions $\frac{\partial^i}{\partial\zeta^i}b_1(\tau,\eta,\zeta)$, $i=0,1$ are Lebesgue measurable, $b_1(\tau,\eta,\pi)=0$, $b_1(\tau,\eta,0)=0$ for every (τ,η), $b_0\in C_b^{\alpha}(\mathbb{R})$ for some $\alpha\in(0,1)$ and

$$L_g:=\max\{\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\eta(\tau)\left(\frac{\partial^i}{\partial\zeta^i}b_1(\tau,\eta,\zeta)\right)^2 d\eta d\tau d\zeta : i=0,1\}<\infty.$$

(ii) The functions a_i are continuous, $a_0, a_2 \in C_b^{\alpha}(\mathbb{R},\mathbb{R})$ and

$$L_f = \| a_1 \| + (\int_{-\infty}^{0} | a_3(s) |^{\frac{1}{2}} \eta(s) ds)^{\frac{1}{2}} < \infty.$$

Under these conditions, we can define the functions $f, g : C(\mathbb{R} \times \mathscr{B}; X)$ by

$$g(t,\psi)(\xi) = b_0(t) \int_{-\infty}^{0} \int_{0}^{\pi} b_1(s,\eta,\xi)\psi(s,\eta)d\eta ds,$$
$$f(t,\psi)(\xi) = a_0(t)a_1(\xi)\psi(0,\xi) + a_2(t)\int_{-\infty}^{0} a_3(s)\psi(s,\xi)ds,$$

which permit to describe the system (3.2)-(3.3) in the abstract form (1.1). Moreover, from (i) and (ii), it follows that $g \in C_b^{\alpha}(\mathbb{R}; \mathscr{L}(\mathscr{B}, X_{\frac{1}{2}}))$, $f \in C^{\alpha}(\mathbb{R} \times \mathscr{B}; \mathscr{L}(X))$ and

$$\| g \|_{C_b^{\alpha}(\mathbb{R},\mathscr{L}(\mathscr{B},X_{\frac{1}{2}}))} \leqslant \| b_0 \|_{C_b^{\alpha}(\mathbb{R})} L_g,$$
$$\| f(t,\cdot) \|_{C_b^{\alpha}(\mathbb{R},\mathscr{L}(\mathscr{B},X))} \leqslant (\| a_0 \|_{C_b^{\alpha}(\mathbb{R})} + \| a_1 \|_{C_b^{\alpha}(\mathbb{R})}) L_f, \tag{3.4}$$

for each $t \in \mathbb{R}$.

In the next result, which is a consequence of Theorem 2.1, we said that a function $u \in C_b(\mathbb{R}; X)$ is a α-Hölder mild solution of (3.2)-(3.3) if $u(\cdot) \in C_b^{\alpha}(\mathbb{R}; X)$ and $u(\cdot)$ is a mild solution of the associated abstract equation (1.1). The numbers $\gamma, \mathfrak{L}$ and C_i are as in the first section.

PROPOSITION 3.2. *If* $\mathfrak{L}\left[\| b_0 \|_{C_b^{\alpha}(\mathbb{R})} \frac{L_g}{\sqrt{2}} + (\| a_0 \|_{C_b^{\alpha}(\mathbb{R})} + \| a_1 \|_{C_b^{\alpha}(\mathbb{R})}) L_f C_0\right] < 1$, *then there exits a unique α-Hölder mild solution of* (3.2)-(3.3).

Next, we said that a function $u(\cdot)$ is a S-classical solution of (3.2)-(3.3) if $u(\cdot)$ is a S-classical solution of the associated equation (1.1).

PROPOSITION 3.3. *Assume that the above conditions are verified. Suppose, in addition, that the functions* $\frac{\partial^i}{\partial \zeta^i} b_1(\tau,\eta,\zeta)$, $i = 0,1,2$ *are Lebesgue measurable and*

$$L_g^1 := \max\left\{ \int_0^{\pi} \int_{-\infty}^{0} \int_0^{\pi} \eta(\tau) \left(\frac{\partial^i}{\partial \zeta^i} b_1(\tau,\eta,\zeta) \right)^2 d\eta d\tau d\zeta : i = 0,1,2 \right\} < \infty. \tag{3.5}$$

Then there exists a unique S-classical solution of (3.2)-(3.3).

Proof. Under condition (3.5), the function g verifies the condition H_g with $Y = [D(A)]$. Now, the assertion follows from Theorem 2.2. □

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