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BOUNDARY BEHAVIOR FOR SOLUTIONS OF SINGULAR QUASI–LINEAR ELLIPTIC EQUATIONS

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Abstract. In this paper, for $1 \leqslant \gamma \leqslant 3$ our main purpose is to consider the quasilinear elliptic equation: $\operatorname{div}(|\nabla u|^{m-2}\nabla u) + (m-1)u^{-\gamma} = 0$ on a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N > 1$. We get some first-order estimates of a nonnegative solution u satisfying $u = 0$ on $\partial\Omega$. For $\gamma = 1$, we find the estimate: $\lim_{x\to\partial\Omega} u(x)/p(\delta(x)) = 1$, where $p(r) \approx r\sqrt[m]{m\log(1/r)}$ near $r = 0$, $\delta(x)$ denotes the distance from x to $\partial\Omega$. For $1 < \gamma \leqslant 3$, we obtain

$$\lim_{x\to\partial\Omega} \frac{u(x)}{(b_\gamma \delta(x))^{\frac{m}{\gamma+(m-1)}}} = 1,$$

where $b_\gamma = \frac{\gamma+(m-1)}{m}\left(\frac{m}{\gamma-1}\right)^{\frac{1}{m}}$.

1. Introduction

Let $N > 1$ and let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. For $\gamma > 0$, we consider the Dirichlet singular problem

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u) + (m-1)u^{-\gamma} = 0. \tag{1.1}$$

Equations of the above form are mathematical models occurring in the study of the p-Laplace system, generalized reaction-diffusion theory, non-Newtonian fluid theory ([2], [19]), non-Newtonian filtration ([15]) and the turbulent flow of a gas in porous medium ([13]). In the non-Newtonian fluid theory, the quantity m is a characteristic of the medium. Media with $m > 2$ are called dilatant fluids and those with $m < 2$ are called pseudoplastics. If $m = 2$, they are Newtonian fluids.

When $m = 2$, the following Dirichlet singular problem

$$\Delta u + u^{-\gamma} = 0 \ \text{ in } \Omega, \ \ u = 0 \ \text{ on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ has been extensively studied. We refer the reader to [8], [9], [10], and [20]. In [10], it is proved that problem (1.2) has a classical solution $u \in C^0(\overline{\Omega})$ and that, near the boundary $\partial\Omega$, it satisfies

$$\lambda p(\delta(x)) \leqslant u(x) \leqslant \Lambda p(\delta(x)). \tag{1.3}$$

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Here $p(r)$ is a (positive) solution to the problem

$$p'' + p^{-\gamma} = 0, \ \ p(0) = 0, \tag{1.4}$$

and λ, Λ are two suitable positive constants. If $\gamma > 1$, one can take $p(r) = (b_\gamma r)^{\frac{2}{\gamma+1}}$ with

$$b_\gamma = \frac{\gamma+1}{\sqrt{2(\gamma-1)}}. \tag{1.5}$$

For $\gamma > 1$, the estimate (1.5) has been improved in [8], where it is shown that

$$|u(x) - (b_\gamma \delta(x))^{\frac{2}{\gamma+1}}| \leqslant \beta \delta(x), \tag{1.6}$$

where β is a suitable constant. For $\gamma > 3$, inequality (1.6) has been made more precise in [20], where the estimate

$$|u(x) - (b_\gamma \delta(x))^{\frac{2}{\gamma+1}}| \leqslant \beta(\delta(x))^{\frac{\gamma+3}{\gamma+1}} \tag{1.7}$$

has been proved.

In [9], one can take $p(r)$ satisfying $\int_0^{p(r)} \frac{dt}{\sqrt{2\log(1/t)}} = r$ such that: for $\gamma = 1$ the solution to (1.2) satisfies

$$|u(x) - p(\delta(x))| < \beta \delta(x)(\log \frac{1}{\delta(x)})^{-\varepsilon}, \ \varepsilon < 1/2,$$

for $1 < \gamma < 3$,

$$|u(x) - (b_\gamma \delta(x))^{\frac{2}{\gamma+1}}| \leqslant \beta(\delta(x))^{\frac{2\gamma}{\gamma+1}},$$

and for $\gamma = 3$,

$$|u(x) - (2\delta(x))^{\frac{1}{2}}| \leqslant \beta(\delta(x))^{\frac{3}{2}} \log \frac{1}{\delta(x)} .$$

In [17] the singular boundary-value problem

$$\Delta u(x) + q(x)u(x)^{-\gamma} = 0 \ \text{ in } \Omega, \ \ u = 0 \ \text{ on } \partial\Omega, \tag{1.8}$$

is considered, where Ω is a sufficiently regular bounded domain in $\mathbb{R}^N$, $N \geqslant 1$, and q is a sufficiently regular function which is positive in $\overline{\Omega}$. It's shown that equation (1.8) can have a classical solution but not a weak solution. In the case $N = 1$, this problem arises in certain problems in fluid mechanics and pseudoplastic flow [8], [20]. The N-dimensional problem (1.8) has been studied in [10] for general regions. Its results are divided into two sections; first, the existence of solutions is proved by an upper-lower solution method, and later, in an extremely complicated way, using localization near the boundary, the boundary is deduced.

In [3] problem of the type

$$\Delta u = f(u) \ \text{ in } D, \ \ u(x) \to \infty \ \text{ as } \ x \to \partial D, \tag{1.9}$$

where $f:\mathbb{R}\to\mathbb{R}^+$ is a nondecreasing function is considered. The solutions of (1.9) are called large solutions. If $\partial\Omega$ is bounded and satisfies an inner and outer sphere condition, J. Keller [16] has shown that such solutions exist if and only if

$$\int_0^\infty \frac{dt}{\sqrt{2F(t)}} < \infty, \text{ where } F'(t) = f(t).$$

The asymptotic behavior of the large solutions near the boundary has been studied in a series of papers [1], [4], [5], [6], [7], [12], [14], and [18]. It turns out that the first order approximation depends only on the distance $\delta(x) = \mathrm{dist}\{x, \partial D\}$ and not on the geometry of the domain D. The expressions for the asymptotic behavior are particulary simple for power nonlinearities $f(t) = t^p$, namely

$$u(x) = (\gamma\delta)^{-\frac{2}{p-1}}(1+o(1)) \text{ as } x\to\partial D,\ \gamma = \frac{p-1}{\sqrt{2(p+1)}},$$

and for the exponential function $f(t) = e^t$, where

$$u(x) = \log\frac{2}{\delta^2}(1+o(1)) \text{ as } x\to\partial D.$$

The problem of the second order effects was first discussed by Lazer and Mckenna [18]. They proved that for $f(t) = t^p$ with $p > 3$ and $f(t) = e^t$,

$$\lim_{x\to\partial\Omega}(u(x) - (\gamma\delta)^{-\frac{2}{p-1}}) = 0 \text{ and } \lim_{x\to\partial\Omega}(u(x) - \log\frac{2}{\delta^2}) = 0.$$

Let $\partial D \in C^4$ be compact and let $\bar{x}$ be the nearest point to x on ∂D. Let $H(\bar{x})$ denote the mean curvature of ∂D at $\bar{x}$. C. Bandle in [3] proved that for $f(t) = t^p$ with $p > 1$ and $f(t) = e^t$,

$$u(x) = (\gamma\delta)^{-\frac{2}{p-1}}(1 + \frac{(N-1)H(\bar{x})}{p+3}\delta + o(\delta)) \text{ as } x\to\partial D,$$

and

$$u(x) = \log\frac{2}{\delta^2} + (N-1)H(\bar{x})\delta + o(\delta) \text{ as } x\to\partial D.$$

It was shown in [23] that problem

$$\mathrm{div}(|\nabla u|^{m-2}\nabla u) + q(x)u^{-\gamma} = 0,\ x\in\mathbb{R}^N,$$

has a positive entire solution if $q\in C(\mathbb{R}^+)$, $0 \leqslant \gamma < p-1$, for any

$$0 < \varepsilon < (N-p)(p-1-|r|)/(p-1),$$

such that

$$\int_1^\infty r^{p+\varepsilon-1} + [(N-p)|r|/(p-1)]q(r)dr < \infty,$$

and for $r\in(0,1)$, $\delta < 1$, $q(r) = O(r^{-\delta})$.

Motivated by the results of the above cited papers, we further study the boundary behaviour of solutions to singular elliptic problems (1.1), the results of the semilinear problem are extended to the quasilinear ones. We can find the related part results for $m = 2$ in [9]. By a modification of the method given in [9], we obtain the following main results.

2. The case when $\gamma = 1$

2.1. The radial case

Introduce the function $p = p(r) : (0,1) \to \mathbb{R}^+$ such that

$$\int_0^{p(r)} \frac{dt}{\sqrt[m]{m\log(1/t)}} = r. \tag{2.1}$$

Note that $p(r)$ satisfies $(|p'|^{m-2}p')' + (m-1)p^{-1} = 0,\ p(0) = 0.$ We will prove that

$$\lim_{r\to 0} \frac{p(r)}{r\sqrt[m]{m\log(1/r)}} = 1. \tag{2.2}$$

First we show that for r small,

$$p(r) < r\sqrt[m]{m\log(1/r)}. \tag{2.3}$$

Indeed, (2.3) can be written as

$$\int_0^{p(r)} \frac{dt}{\sqrt[m]{m\log(1/t)}} = r < \int_0^{r\sqrt[m]{m\log(1/r)}} \frac{dt}{\sqrt[m]{m\log(1/t)}},$$

and the latter inequality holds because

$$1 < \frac{\sqrt[m]{m\log(1/r)} - \left(\sqrt[m]{m\log(1/r)}\right)^{1-m}}{\sqrt[m]{m\log(1/(r\sqrt[m]{m\log(1/r)}))}},$$

for r small. Now let us take $\varepsilon > 0$ small. We claim that for r near zero, we have

$$p(r) > (1-\varepsilon)r\sqrt[m]{m\log(1/r)}.$$

This inequality can be written as

$$\int_0^{p(r)} \frac{dt}{\sqrt[m]{m\log(1/t)}} = r > \int_0^{(1-\varepsilon)r\sqrt[m]{m\log(1/r)}} \frac{dt}{\sqrt[m]{m\log(1/t)}},$$

and the latter inequality holds because

$$1 > (1-\varepsilon)\frac{\sqrt[m]{m\log(1/r)} - \left(\sqrt[m]{m\log(1/r)}\right)^{1-m}}{\sqrt[m]{m\log(1/((1-\varepsilon)r\sqrt[m]{m\log(1/r)}))}},$$

for r small. Therefore, (2.2) holds.

For $N > 1$ we will investigate the problem

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u) + (m-1)u^{-1} = 0 \ \text{ in } B(R),\ \ u = 0 \ \text{ on } \partial B(R),$$

where $B(R)$ is a ball centered at the origin and with radius R. Denoting its solution by $u(x) = z(r), r = |x|,$ we have

$$(|z'|^{m-2}z')' + \frac{N-1}{r}|z'|^{m-2}z' + (m-1)z^{-1} = 0, \;\; z'(0) = z(R) = 0. \tag{2.4}$$

Multiplying by z' and integrating over $(0,r)$ we find

$$|z'|^m + \frac{m}{m-1}(N-1)\int_0^r \frac{|z'|^m}{t}dt = m\log(z(0)/z). \tag{2.5}$$

Equation (2.5) implies that $|z'|^m \to \infty$ as $r \to R$. By (2.4) we find

$$r^{(N-1)}|z'|^{m-2}z' = -(m-1)\int_0^r t^{N-1}z^{-1}dt,$$

therefore, $z' < 0,$ this equation implies

$$\frac{|z'|^{m-2}z'}{r} > -\frac{m-1}{N}z^{-1}.$$

Using the last inequality and (2.4) we find

$$0 = (|z'|^{m-2}z')' + \frac{N-1}{r}|z'|^{m-2}z' + (m-1)z^{-1} > (|z'|^{m-2}z')' + \frac{1}{N}(m-1)z^{-1}.$$

Hence, $(|z'|^{m-2}z')' < 0.$ Then by (2.4) again we find

$$\frac{N-1}{r}|z'|^{m-2}z' + (m-1)z^{-1} > 0.$$

Since $z'(r) < 0,$ this inequality means $\frac{N-1}{r}|z'|^m + (m-1)\frac{z'}{z} < 0.$ Hence,

$$\frac{d|z'|^m}{dr} = -\frac{m}{m-1}(N-1)\frac{|z'|^m}{r} - m\frac{z'}{z} > -\frac{m}{m-1}(N-1)\frac{|z'|^m}{r} + \frac{m}{m-1}\frac{N-1}{r}|z'|^m = 0.$$

Therefore, by a result of Lazer-McKenna [9, Lemma 2.1] we have

$$\lim_{r\to R}\frac{\int_0^r \frac{|z'|^m}{t}dt}{|z'|^m} = 0.$$

The latter result and (2.5) yield

$$\lim_{r\to R}\frac{m\log(z(0)/z)}{|z'|^m} = 1$$

and

$$\lim_{r\to R}\frac{-z'}{\sqrt[m]{m\log(1/z)}} = 1. \tag{2.6}$$

Given $\varepsilon > 0$, there exists $r_\varepsilon < R$ such that

$$\frac{-z'}{\sqrt[m]{m\log(1/z)}} > 1-\varepsilon,\ \forall r \in (r_\varepsilon, R).$$

Integrating over (r,R) and recalling that $z(R)=0$, we find

$$\int_0^{z(r)} \frac{dt}{\sqrt[m]{m\log(1/t)}} > (1-\varepsilon)(R-r). \tag{2.7}$$

Equation (2.1) with r replaced by $R-r$ becomes

$$\int_0^{p(R-r)} \frac{dt}{\sqrt[m]{m\log(1/t)}} = R-r. \tag{2.8}$$

Combining (2.8) and (2.7) we get

$$\begin{aligned}\int_0^{z(r)} \frac{dt}{\sqrt[m]{m\log(1/t)}} &> (1-\varepsilon)\int_0^{p(R-r)} \frac{dt}{\sqrt[m]{m\log(1/t)}}\\ &= \int_0^{(1-\varepsilon)p(R-r)} \frac{dt}{\sqrt[m]{m\log((1-\varepsilon)/t)}} > \int_0^{(1-\varepsilon)p(R-r)} \frac{dt}{\sqrt[m]{m\log(1/t)}}.\end{aligned}$$

Hence we have

$$z(r) > (1-\varepsilon)p(R-r),\ \ \forall r \in (r_\varepsilon, R). \tag{2.9}$$

Now we investigate the problem

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u) + (m-1)u^{-1} = 0 \text{ in } B(R,\bar{R}),\ \ u=0 \text{ on } \partial B(R,\bar{R}),$$

where $B(R,\bar{R})$ is the annulus centered at the origin and with radii R, $\bar{R}$. Denoting its solution by $u(x)=w(r), r=|x|$, we have

$$(|w'|^{m-2}w')' + \frac{N-1}{r}|w'|^{m-2}w' + (m-1)w^{-1} = 0,\ \ w'(R_0) = w(R) = 0, \tag{2.10}$$

where R_0 is a particular number with $R < R_0 < \bar{R}$. Integrating (2.10) over (r,R_0) we find

$$|w'|^m = \frac{m}{m-1}(N-1)\int_r^{R_0} \frac{|w'|^m}{t}dt + m\log\Big(\frac{w(R_0)}{w}\Big). \tag{2.11}$$

By (2.11) it follows that $|w'(r)|^m \to \infty$ as $r \to R$ and $d(w')^m/dr < 0$, by Lazer-McKenna [9, Lemma 2.1] we find

$$\lim_{r\to R} \frac{\int_r^{R_0} \frac{|w'|^m}{t}dt}{|w'|^m} = 0.$$

The latter result and (2.11) yield

$$\lim_{r\to R} \frac{w'(r)}{\sqrt[m]{m\log(1/w)}} = 1.$$

Hence, given $\varepsilon > 0$ one finds r_ε with $R < r_\varepsilon < R_0$ such that

$$\frac{w'(r)}{\sqrt[m]{m\log(1/w)}} < 1+\varepsilon,\ \forall r \in (R, r_\varepsilon).$$

Integration over (R,r) yields

$$\int_0^{w(r)} \frac{dt}{\sqrt[m]{m\log(1/t)}} < (1+\varepsilon)(r-R). \tag{2.12}$$

Equation (2.1) with r replaced by $r-R$ becomes

$$\int_0^{p(r-R)} \frac{dt}{\sqrt[m]{m\log(1/t)}} = r-R. \tag{2.13}$$

Combining (2.12) with (2.13) we find

$$\begin{aligned}\int_0^{w(r)} \frac{dt}{\sqrt[m]{m\log(1/t)}} &< (1+\varepsilon)\int_0^{p(r-R)} \frac{dt}{\sqrt[m]{m\log(1/t)}} \\ &= \int_0^{(1+\varepsilon)p(r-R)} \frac{dt}{\sqrt[m]{m\log((1+\varepsilon)/t)}} < \int_0^{(1+\varepsilon)p(r-R)} \frac{dt}{\sqrt[m]{m\log(1/t)}}.\end{aligned}$$

Hence we have

$$w(r) < (1+\varepsilon)p(r-R),\ \ \forall r \in (R, r_\varepsilon). \tag{2.14}$$

2.2. General domains

In this subsection the domain Ω is assumed to be bounded, smooth and satisfy a uniform interior and exterior sphere solution. Consider the problem

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u) + (m-1)u^{-1} = 0 \text{ in } \Omega,\ \ u = 0 \text{ on } \partial\Omega. \tag{2.15}$$

Now we introduce the following comparison principles according to Lemma 2.2 and Lemma 2.6 in [11].

LEMMA 1. *Let z be solution to*

$$(|z'|^{m-2}z')' + (m-1)z^{-1} > 0,\ \ 0 \leqslant r < R,\ \ z'(0) = z(R) = 0,$$

and let u be solution to

$$(|u'|^{m-2}u')' + (m-1)u^{-1} = 0,\ \ 0 \leqslant r < R,\ \ u'(0) = u(R) = 0,$$

then $z(r) < u(r)$ in $[0,R)$.

Proof. We argue by contradiction. If the statement of the lemma does not hold the $z(r_0) - u(r_0) \geqslant 0$ for some $r_0 \in [0,R)$. Let r_1 be a point of maximum of $z(r) - u(r)$

in[0,R]. Since $z(R)-u(R)=0$, we may assume that $r_1 \in [0,R)$. So $z'(r_1)-u'(r_1)=0$ and $z(r_1)-u(r_1) \geqslant 0$. But this contradicts the inequality

$$(|z'|^{m-2}z')'-(|u'|^{m-2}u')'+(m-1)[z^{-1}-u^{-1}]>0.$$

The lemma is proved.

LEMMA 2. *Let w be solution to*

$$(|w'|^{m-2}w')'+(m-1)w^{-1}<0,\ \ 0<r<R_0,\ \ w(R)=w'(R_0)=0,$$

and let u be solution to

$$(|u'|^{m-2}u')'+(m-1)u^{-1}=0,\ \ 0<r<R_0,\ \ u(R)=u'(R_0)=0,$$

then $w(r)>u(r)$ in $(R,R_0]$.

In the following we denote by x a point of Ω and by $\delta(x)$ the distance from x to $\partial\Omega$.

THEOREM 2.1. *The solution $u(x)$ to problem (2.15) satisfies* $\lim_{x\to\partial\Omega}\frac{u(x)}{p(\delta(x))}=1$, *where $p(r)$ is the function defined in* (2.1).

Proof. Now let Ω be a bounded domain with a smooth boundary, and let $P\in\partial\Omega$. We can consider a small ball $B=B(R)$ contained in Ω and tangent to $\partial\Omega$ in P. Furthermore, we can consider a suitable annulus $A=A(R,\bar{R})$ containing Ω and such that the inner boundary is tangent to $\partial\Omega$ in P. We may assume that the radius R of the ball B_R is equal to the inner radius of the annulus $A(R,\bar{R})$. If z,u,w are the solutions to problem (2.15) respectively in B,Ω and A then we have $z(x)\leqslant u(x)\leqslant w(x)$, in Ω. Using these inequalities (2.9) and (2.14) we get

$$(1-\varepsilon)p(\delta(x))\leqslant u(x)\leqslant (1+\varepsilon)p(\delta(x)).$$

Since ε is arbitrary, the theorem follows.

3. The case when $1<\gamma\leqslant 3$

3.1. The radial case

For $N>1$ we will investigate the problem

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u)+(m-1)u^{-\gamma}=0 \text{ in } \Omega,\ \ u=0 \text{ on } \partial\Omega. \tag{3.1}$$

If Ω is a ball of radius R and if we denote by $z(r)$ the corresponding solution to problem (3.1) we have

$$(|z'|^{m-2}z')'+\frac{N-1}{r}|z'|^{m-2}z'+(m-1)z^{-\gamma}=0,\ \ z'(0)=z(R)=0. \tag{3.2}$$

Problem (3.2) has a positive decreasing solution. If we multiply equation (3.2) by mz' and integrate over $(0,r)$, we find

$$(m-1)|z'|^m + m(N-1)\int_0^r \frac{|z'|^m}{t}dt = \frac{m(m-1)}{(\gamma-1)}(z^{1-\gamma} - z^{1-\gamma}(0)). \tag{3.3}$$

Equation (3.3) implies that $|z'|^m \to \infty$ as $r \to R$. Moreover, since by Lazer-McKenna [9, Lemma 2.1] we have

$$\lim_{r\to R} \frac{\int_0^r \frac{|z'|^m}{t}dt}{|z'|^m} = 0,$$

equation (3.3) yields

$$\lim_{r\to R} \frac{|z'|^m}{z^{1-\gamma}} = \frac{m}{\gamma-1}. \tag{3.4}$$

By using the de l'Hospital rule and (3.4) we find

$$\lim_{r\to R} \frac{z^{\frac{\gamma+(m-1)}{m}}}{R-r} = \lim_{r\to R} \frac{\gamma+(m-1)}{m} z^{\frac{\gamma-1}{m}}(-z') \leqslant \frac{\gamma+(m-1)}{m}(\frac{m}{\gamma-1})^{\frac{1}{m}} = b_\gamma. \tag{3.5}$$

We have found the well-known expansion

$$z^{\frac{\gamma+(m-1)}{m}} = b_\gamma(R-r)(1+o(1)). \tag{3.6}$$

Now let Ω be the annulus $B(R,\bar{R})$. If $w(r)$ denotes the corresponding solution to our problem, we have

$$(|w'|^{m-2}w')' + \frac{N-1}{r}|w'|^{m-2}w' + (m-1)w^{-\gamma} = 0, \ \ w'(R_0) = w(R) = 0, \tag{3.7}$$

where R_0 is a particular number with $R < R_0 < \bar{R}$. The function $w(r)$ is increasing in (R,R_0). Integrating over (r,R_0) we find

$$|w'|^m = \frac{m}{m-1}(N-1)\int_r^{R_0} \frac{|w'|^m}{t}dt + m\frac{1}{1-\gamma}(w^{1-\gamma}(R_0) - w^{1-\gamma}(r)). \tag{3.8}$$

By (3.8) it follows that $w'(r) \to \infty$ as $r \to R$. Moreover, using Lazer-McKenna [9, Lemma 2.1] we find

$$\lim_{r\to R} \frac{\int_r^{R_0} \frac{|w'|^m}{t}dt}{|w'|^m} = 0,$$

the latter result and (3.8) yield

$$\lim_{r\to R} \frac{|w'|^m}{w^{1-\gamma}} = \frac{m}{\gamma-1}. \tag{3.9}$$

Using the de l'Hospital rule and (3.9) we find

$$\lim_{r\to R} \frac{w^{\frac{\gamma+(m-1)}{m}}}{r-R} = \lim_{r\to R} \frac{\gamma+(m-1)}{m} w^{\frac{\gamma-1}{m}} w' \leqslant \frac{\gamma+(m-1)}{m}(\frac{m}{\gamma-1})^{\frac{1}{m}} = b_\gamma. \tag{3.10}$$

We have found the well-known expansion

$$w^{\frac{\gamma+(m-1)}{m}} = b_\gamma(r-R)(1+o(1)). \tag{3.11}$$

3.2. General domains

The domain Ω is assumed to be bounded, smooth and satisfy a uniform interior and exterior sphere condition. Consider the problem (3.1).

THEOREM 3.1. The solution $u(x)$ to problem (3.1) satisfies

$$\lim_{x\to\partial\Omega}\frac{u(x)}{(b_\gamma\delta(x))^{\frac{m}{\gamma+(m-1)}}}=1.$$

Proof. Proceed as in the proof of Theorem 2.1 and use the same comparison principle.

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