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COMPARISON, EXISTENCE AND REGULARITY RESULTS FOR A CLASS OF NON–UNIFORMLY ELLIPTIC EQUATIONS

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Abstract. We prove comparison, existence and regularity results for problems whose model case is:

$$\begin{cases} -\operatorname{div}\left(\frac{Du}{(1+|u|)^{\theta}}\right)+\lambda u=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N>2$, $\theta \geqslant 0$ and $\lambda>0$.

1. Introduction

In this paper we study a class of non-uniformly elliptic problems whose model case is:

$$\begin{cases} -\operatorname{div}\left(\frac{Du}{(1+|u|)^{\theta}}\right)+\lambda u=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N>2$, $\theta \geqslant 0$, λ is a positive constant, and f is a measurable function on whose summability will be made different assumptions. The main difficulty dealing with such kind of problems is the presence of the coefficient $\frac{1}{(1+|u|)^{\theta}}$, which causes a degeneracy when u becomes large. This implies that the principal part of the operator, namely $A(u)=-\operatorname{div}\left(\frac{Du}{(1+|u|)^{\theta}}\right)$, may not be coercive on $H_0^1(\Omega)$, so that we cannot apply the classical Leray–Lions methods of [24] to prove the existence results, even if f is regular. This problem can be overcome reasoning by approximation, obtaining suitable a priori estimates which allow to pass to the limit.

A priori estimates and existence results for such kind of problems, in the case $\lambda=0$, have been obtained by several authors using different techniques (see e.g. [1], [2], [9], [11], [12], [27]). As well-known, one of these is the symmetrization method, which allows to "compare" a solution to problem (1.1) by means of the solution to a problem whose data are spherically symmetric (we recall, for example, [3], [4], [17], [36], [37], [39], for $\theta=0$ and $\lambda \neq 0$, [1], [2], for $\theta \neq 0$ and $\lambda=0$).

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In the present paper we are interested in comparing a solution to problem (1.1) with the solution of a "symmetrized" one which takes into account the influence of the zero order term, namely

$$\begin{cases} -\operatorname{div}\left(\frac{Dw}{(1+|w|)^{\theta}}\right)+\lambda w = f^{\#} \text{ in } \Omega^{\#}, \\ w=0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega^{\#}, \end{cases} \tag{1.2}$$

where $\Omega^{\#}$ is the ball centered at the origin with the same Lebesgue measure of $|\Omega|$ and $f^{\#}$ is the radially decreasing rearrangement of f (see Section 2 for precise definitions). More precisely, we will prove that

$$\int_0^s u^*(t)dt \leqslant \int_0^s w^*(t)dt, \quad \text{for all } s \in [0,|\Omega|],$$

where u^* and w^* denote the decreasing rearrangements of u and w which are the solutions to problems (1.1) and (1.2), respectively. The symmetrization method allows to obtain a priori estimates which are the main tool in order to obtain existence and regularity results.

In the hypothesis $\lambda = 0$, the cases $0 \leqslant \theta \leqslant 1$ and $\theta > 1$ are completely different. More precisely, assuming $0 \leqslant \theta < 1$, it has been proved in [1], [11] and [12], that if the datum f is in $L^p(\Omega)$, with $p \geqslant \frac{2N}{N+2-\theta(N-2)}$, then there exists a solution in the energy space $H_0^1(\Omega)$, and such a solution is bounded if $p > \frac{N}{2}$. Moreover, if f is less regular, then it is possible to prove the existence of distributional or entropy solutions (for renormalized solutions with $\theta < 1$ see [8]). We observe that if $\theta = 0$, which means that the operator is not degenerate, the quoted results coincide with the classical one (see [35]). Finally, in the limit case $\theta = 1$, it is possible to prove the existence of a bounded solution if f belongs to $L^p(\Omega)$, with $p > \frac{N}{2}$ (see, for example, [1], [2], [11]).

In the other case, with $\theta > 1$, the existence of solutions is related to a smallness assumption on the L^p-norm of f, with $p > \frac{N}{2}$. Indeed, if f is not sufficiently small, examples are given when $f \in L^\infty(\Omega)$ and the solutions u_n of problems approximating (1.1) converge to a function which is $+\infty$ on a set of positive Lebesgue measure (see [1]). Nevertheless, it is possible to adapt the definition of renormalized solution in order to obtain existence results without requiring any smallness assumption on the data (see [9]).

In general, the presence of the lower–order term, with $\lambda > 0$, may change the nature of the existence results. Indeed, if $\theta > 1$, for $f \in L^p(\Omega)$, with $p > \min\{\theta + 2, \frac{N}{2}\theta\}$, we prove the existence of solutions in the energy space $H_0^1(\Omega)$ without any smallness assumption on f (see also [11], [14]). In particular, such a solution is bounded if f is in $L^p(\Omega)$, with $p > \frac{N}{2}\theta$. On the other hand, if f is less regular, we prove the existence of an entropy solution to problem (1.1). We stress that, thanks to the presence of $\lambda > 0$, the solutions of (1.1) belong to the same Lebesgue space of the datum f. This is the crucial point in order to prove the above existence results.

We want to emphasize that if u is a positive solution of (1.1) one can perform the change of variable $v = \int_0^u (1+s)^{-\theta} ds$. If $\theta \neq 1$ problem (1.1) becomes

$$\begin{cases} -\Delta v + \lambda \left(\left(\frac{1}{(1-(\theta-1)v)} \right)^{\frac{1}{\theta-1}} - 1 \right) = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Such kind of problem has been studied in [10] with $\theta > 1$, where the existence of a bounded solution $v \in H_0^1(\Omega)$ is proved, with f nonnegative and belonging to $L^1(\Omega)$.

We also investigate the regularity of solutions by varying the summability of datum f in some Lorentz spaces $L^{p,q}$ (see Definition 3.2 in Section 3).

In particular we prove that there exist bounded solutions also for $f \in L^{\frac{N}{2}\theta,\theta}$, while if we require less summability, namely $f \in L^{\frac{N}{2}\theta,q}$ with $q > \theta$, then the solution u is in L^r, for any $r < +\infty$. Moreover, we show that in the limit case $f \in L^{\frac{N}{2}\theta,\infty}$, in general, the summability of u is not greater than the summability of f unless we impose a smallness condition on f (see Section 7).

As regards the regularity of solutions of (1.1), in the case $\lambda = 0$, we quote, for example, the papers [1], [2], [12], [20].

We stress that the parabolic problem associated to elliptic operator of (1.1) has been studied in [16] when $\lambda = 0$ and assuming a smallness hypothesis on some norm of the initial data and of the source f, and in [31], with $\lambda = 0$ and $f = 0$, without any smallness assumption on the initial data.

Finally, existence result for such kind of problems under different assumptions on the equation have been obtained by several authors. We address, for example, to [13], [15], [26], [30], [32], [33], [38].

The paper is organized as follows. In sections 2 and 3 we make precise the assumptions on the data of the problem, and we recall some basic facts about decreasing rearrangements and Lorentz spaces. In Section 4 we prove uniqueness results for weak and entropy solutions of (1.1). In Section 5 we prove the quoted comparison result. To this end, we obtain an integral inequality for solutions to problem (1.1) in terms of their decreasing rearrangements. In Section 6, using this inequality, we prove a priori estimates in Lebesgue and Sobolev spaces for solutions of suitable approximate problems of (1.1). Finally, in sections 7 and 8 we prove, respectively, the existence and regularity results quoted above.

We point out that such results are proved for solutions to a class of boundary value problems which can be written in the form

$$\begin{cases} -\operatorname{div}(a(x,u,Du)) + \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $a(x,s,\xi)$ verifies suitable conditions (see Section 2 for precise assumptions).

2. Statement of the problem and definitions of solutions

We deal with Dirichlet problems of the form

$$\begin{cases} -\operatorname{div}(a(x,u,Du)) + \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N \geqslant 3$, λ is a positive constant and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function verifying the following assumptions:

$$\frac{\alpha}{(1+|s|)^\theta}|\xi|^2 \leqslant a(x,s,\xi)\cdot\xi \tag{2.2}$$

and

$$|a(x,s,\xi)| \leqslant h(x) + c_1|s| + c_2|\xi| \tag{2.3}$$

for a.e. $x \in \Omega$, $\forall\, s \in \mathbb{R}$ and $\forall \xi \in \mathbb{R}^N$, where θ, α, c_1 and c_2 are positive constants, $h \in L^2(\Omega)$ and

$$(a(x,s,\xi) - a(x,s,\xi'))\cdot(\xi - \xi') > 0, \tag{2.4}$$

for a.e. $x \in \Omega$, $\forall\, s \in \mathbb{R}$ and $\forall \xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$. Moreover, f is a measurable function on whose summability we will make different assumptions.

In this context we deal with some classes of solutions.

DEFINITION 2.1. Let $f \in L^{\frac{2N}{N+2}}(\Omega)$. We say that $u \in H_0^1(\Omega)$ is a weak solution to problem (2.1) if

$$\int_\Omega a(x,u,Du)\cdot D\varphi\, dx + \lambda \int_\Omega u\varphi\, dx = \int_\Omega f\varphi\, dx, \quad \forall\, \varphi \in H_0^1(\Omega). \tag{2.5}$$

We observe that, in general, if the datum f is in $L^1(\Omega)$, we no longer obtain solutions in the energy space $H_0^1(\Omega)$. For this reason we need to introduce a different definition of solution.

Given $h > 0$, we denote with $T_h(s)$ the truncation function at level $\pm h$, defined as

$$T_h(s) = \begin{cases} s & \text{if } |s| \leqslant h, \\ h\,\text{sign}\, s & \text{if } |s| > h. \end{cases}$$

DEFINITION 2.2. A measurable function $u \in L^1(\Omega)$ is an entropy solution to problem (2.1) if $T_k(u) \in H_0^1(\Omega)$ for all $k > 0$ and it holds

$$\int_\Omega a(x,u,Du)\cdot DT_k(u-\varphi)\, dx + \lambda \int_\Omega uT_k(u-\varphi)dx \leqslant \int_\Omega fT_k(u-\varphi)dx \tag{2.6}$$

for any $k > 0$ and $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

We observe that first integral in the left–hand is well defined, since $|u| \leqslant k + \|\varphi\|_{L^\infty} = M$ on the set $|u - \varphi| \leqslant k$, so we have

$$\int_\Omega a(x,u,Du)\cdot DT_k(u-\varphi)\, dx = \int_\Omega a(x,T_M(u),DT_M(u))\cdot DT_k(u-\varphi)\, dx, \tag{2.7}$$

which is finite by the growth assumption on $a(x,s,\xi)$.

3. Preliminaries

We recall some definitions about decreasing rearrangement of functions.

Let Ω be a bounded open set of $\mathbb{R}^N$ and $u\colon \Omega \to \mathbb{R}$ a measurable function. If we denote with $|E|$ the Lebesgue measure of a measurable set E contained in $\mathbb{R}^N$, we define the decreasing rearrangement of u the function

$$u^*(s) = \sup\{t \geqslant 0 : \mu_u(t) > s\}, \quad s \geqslant 0,$$

where

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}|, \quad t \geqslant 0,$$

is the distribution function of u. The function μ_u is decreasing and right continuous, and u^* is the generalized inverse of μ_u. We recall that, being

$$\int_\Omega |u|^p dx = p\int_0^{+\infty} t^{p-1}\mu_u(t)dt, \quad p \geqslant 1, \tag{3.1}$$

the L^p–norm, for every $1 \leqslant p < +\infty$, is invariant with respect to rearrangement, that is

$$\|u\|_{L^p(\Omega)} = \|u^\#\|_{L^p(\Omega^\#)}.$$

Moreover, if $u \in L^\infty(\Omega)$, by definition

$$u^*(0) = \operatorname*{ess\,sup}_\Omega |u|.$$

If u and v are measurable functions, then the Hardy and Littlewood inequality states that

$$\int_\Omega u(x)v(x)\,dx \leqslant \int_0^{|\Omega|} u^*(s)v^*(s)\,ds. \tag{3.2}$$

DEFINITION 3.1. The spherically symmetric decreasing rearrangement of u is defined by

$$u^\#(x) = u^*(\omega_n|x|^n), \quad x \in \Omega^\#,$$

where $\Omega^\#$ is the ball centered at the origin having the same measure as Ω and ω_n is the measure of the unit ball in $\mathbb{R}^N$.

For general results about rearrangements we refer to [5], [21], [23], [28], [37].

DEFINITION 3.2. A measurable function $u : \Omega \to \mathbb{R}$ belongs to the Lorentz space $L^{p,q}(\Omega)$, $1 < p < +\infty$, if the quantity

$$\|u\|_{L^{p,q}} = \begin{cases} \left\{\displaystyle\int_0^{+\infty}\left[t^{1/p}u^{**}(t)\right]^q \frac{dt}{t}\right\}^{1/q}, & 0 < q < +\infty, \\ \displaystyle\sup_{0<t<+\infty} t^{1/p}u^{**}(t), & q = +\infty, \end{cases} \tag{3.3}$$

is finite, where $u^{**}(s) = s^{-1}\int_0^s u^*(\sigma)\,d\sigma$.

We remark that the quantity in (3.3) can be equivalently defined replacing u^{**} with u^{*}. We recall that the Marcinkiewicz space $M^p(\Omega)$ with $0<p<+\infty$ contains all the measurable functions u such that

$$\mu_u(t) \leqslant \frac{c}{t^p}, \quad \forall t>0,$$

for some positive constant c, and we put $\|u\|_{M^p} = \sup_{s\in[0,|\Omega|]} s^{1/p}u^*(s)$. Moreover, $M^p = L^{p,\infty}$ for $1<p<+\infty$. It is well-known that Lorentz and Marcinkiewicz spaces are related in the following way:

$$L^r \subset L^{p,1} \subset L^{p,q} \subset L^{p,p} = L^p \subset L^{p,r} \subset L^{p,\infty} = M^p \subset L^q,$$

for $1<q<p<r<+\infty$. More details on Lorentz spaces can be found, for example, in [22], or in [7].

Now we want to recall a property of Marcinkiewicz spaces which will be useful in the following.

First of all, we give a sense to the gradient of a measurable function such that the truncates of u are Sobolev functions (see [6]).

LEMMA 3.1. *Given a measurable function $u:\Omega\to\mathbb{R}$ such that for every $k>0$ the truncated function $T_k(u)$ belongs to $W^{1,1}_{loc}(\Omega)$, there exists a unique measurable function $v:\Omega\to\mathbb{R}^N$ such that*

$$DT_k(u) = v\chi_{\{|u|<k\}} \qquad \text{a.e. in } \Omega. \tag{3.4}$$

Furthermore, $u\in W^{1,1}_{loc}(\Omega)$ if and only if $v\in L^1_{loc}(\Omega)$, and then $v=Du$ in the usual weak sense.

Therefore, if $u:\Omega\to\mathbb{R}$ is such that for every $k>0$ the truncated function $T_k(u)$ belongs to $W^{1,1}_{loc}(\Omega)$, we define, Du, the weak gradient of u as the unique function v which verifies (3.4).

The following technical lemma gives a sufficient condition in order to assure that the gradient of a function belongs to some Marcinkiewicz space.

LEMMA 3.2. ([6]) *Let v be a measurable function belonging to $M^\gamma(\Omega)$ for some $\gamma\geqslant 1$, such that, for every $k\geqslant 0$, $T_k(v)$ belongs to $H^1_0(\Omega)$. Suppose that*

$$\int_{\{|v|\leqslant k\}} |Dv|^2 dx \leqslant c\,k^\sigma, \qquad \forall k>k_0, \tag{3.5}$$

for some non-negative σ, c and k_0. Then the weak gradient of v (in the sense of the above definition) is such that $|Dv|$ belongs to $M^q(\Omega)$, with $q=2\gamma/(\gamma+\sigma)$.

4. Uniqueness results

The first step in order to obtain the comparison result between the solutions to problems (2.1) and (1.2) is the proof of an uniqueness result. In this section we show that, under suitable assumptions on the operator $a(x,s,\xi)$, the problem (2.1) admits at most one solution. Such a result is an immediate consequence of the following classical pointwise comparison results. Here we study both the case of weak solutions, with $f \in L^{\frac{2N}{N+2}}(\Omega)$, and the case of entropy solutions, with $f \in L^1(\Omega)$.

THEOREM 4.1. *Let us assume that* (2.2), (2.3) *and* (2.4) *hold, and* $f \in L^{\frac{2N}{N+2}}(\Omega)$. *Suppose that the following Lipschitz condition holds:*

$$|a(x,s,\xi) - a(x,s',\xi)| \leqslant \eta|\xi||s-s'|, \tag{4.1}$$

for a.e. $x \in \Omega$, $\forall\, s,s' \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$ *and* η *is a positive constant.*

If u *and* v *are weak solutions to problem* (2.1) *with* $L^{\frac{2N}{N+2}}$ *data, respectively,* f *and* g*, then*

$$f \leqslant g \quad \text{a.e. in } \Omega \text{ implies} \quad u \leqslant v \quad \text{a.e. in } \Omega.$$

Proof. Using as test function in (2.5):

$$\varphi_\delta = \frac{1}{\delta} T_\delta(u-v)^+$$

with a fixed $\delta > 0$ and subtracting, we get

$$\frac{1}{\delta}\int_{0<u-v<\delta} (a(x,u,Du) - a(x,v,Dv)) \cdot D(u-v)\,dx \\ + \frac{\lambda}{\delta}\int_\Omega (u-v)T_\delta(u-v)^+\,dx = \frac{1}{\delta}\int_\Omega (f-g)T_\delta(u-v)^+ dx. \tag{4.2}$$

Adding and subtracting $a(x,u,Dv)$ in the first integral of (4.2), it follows that

$$\begin{aligned} &\frac{1}{\delta}\int_{0<u-v<\delta} (a(x,u,Du) - a(x,v,Dv)) \cdot D(u-v)\,dx \\ &\quad = \frac{1}{\delta}\int_{0<u-v<\delta} (a(x,u,Du) - a(x,u,Dv)) \cdot D(u-v)\,dx \\ &\qquad + \frac{1}{\delta}\int_{0<u-v<\delta} (a(x,u,Dv) - a(x,v,Dv)) \cdot D(u-v)\,dx = I_1(\delta) + I_2(\delta). \end{aligned}$$

Now by hypothesis (2.4) it follows that

$$I_1(\delta) \geqslant 0,$$

and from the Lipschitz condition (4.1) we get

$$|I_2(\delta)| \leqslant \eta \int_{0<u-v<\delta} |Dv||D(u-v)|\,dx,$$

which implies that

$$\lim_{\delta\to 0^+} I_2(\delta) = 0.$$

Hence, letting δ going to 0 in (4.2), we obtain

$$\lambda \int_\Omega (u-v)^+\,dx \leqslant 0,$$

which implies that $u \leqslant v$ a.e. in Ω.

Now we want to prove the pointwise comparison result for entropy solutions to problem (2.1) in a special form. To this end, we require an additional assumption on the principal part of the operator. More precisely, let us consider the following problem:

$$\begin{cases} -\operatorname{div}(b(x,u)Du) + \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where $f \in L^1(\Omega)$, $\lambda > 0$ and $b(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function verifying the following assumptions:

$$\frac{\alpha}{(1+|s|)^\theta} \leqslant b(x,s) \leqslant \frac{\Lambda}{(1+|s|)^\theta}, \quad \text{for a.e. } x \in \Omega, \forall\, s \in \mathbb{R}, \tag{4.4}$$

and

$$\left|\tilde b(x,s) - \tilde b(x,t)\right| \leqslant C_1 |B(|s|) - B(|t|)|, \quad \text{for a.e. } x \in \Omega,\ \forall s,t \in \mathbb{R}, \tag{4.5}$$

where α, Λ, C_1 are positive constants, $\theta > 1$, $\tilde b(x,s) = b(x,s)(1+|s|)^\theta$, and the function $B : (0,+\infty) \to \mathbb{R}$ is defined as

$$B(s) = \int_0^s \frac{\alpha}{(1+t)^\theta}\,dt.$$

We restrict our attention to non negative solutions, given by $f \geqslant 0$, and we follow the technique used in [29].

THEOREM 4.2. *Let u,v be two entropy solutions of* (4.3)*, under the assumptions* (4.4)*,* (4.5)*, with $\theta > 1$, and non negative L^1 data f and g respectively. Then $f \leqslant g$ a.e. in Ω implies $u \leqslant v$ a.e. in Ω.*

Proof. The solutions u and v are such that $|DB(u)|$ and $|DB(v)|$ are in $L^2(\Omega)$.

Indeed, taking $\varphi = B(u)$ as test function in problem (2.1) we get, by hypothesis (4.4), that

$$\int_\Omega |D(B(u)|^2\,dx + \lambda \int_\Omega uB(u)\,dx \leqslant \int_\Omega fB(u)\,dx,$$

and being $\lambda > 0$,

$$\int_\Omega |D(B(u)|^2\,dx \leqslant \bar B \|f\|_{L^1(\Omega)}, \tag{4.6}$$

where $\bar{B} = \sup_{s>0} B(s)$, which is finite since $\theta > 1$. The same reasoning holds for $|DB(v)|$.

Hence, the choice of $\varphi = T_h(u) - T_k(B(u) - B(v))^+$ as test function in (2.2) is admissible, that is

$$\int_\Omega b(x,u)Du \cdot DT_k(u - T_h(u) + T_k(B(u) - B(v))^+)dx \\ + \lambda \int_\Omega uT_k(u - T_h(u) + T_k(B(u) - B(v))^+)dx \\ \leqslant \int_\Omega fT_k(u - T_h(u) + T_k(B(u) - B(v))^+)dx. \quad (4.7)$$

We first observe that, applying the Lebesgue theorem, it is possible to pass to the limit in the second integral on the left hand side of (4.7) and in the right hand side of (4.7), as $h \to +\infty$. Moreover, splitting the first integral of (4.7) on the two sets $\{u \leqslant h\}$ and $\{u > h\}$, and using hypothesis (4.4), we have

$$\begin{aligned} &\int_\Omega b(x,u)Du \cdot DT_k(u - T_h(u) + T_k(B(u) - B(v))^+)dx \\ &\qquad \geqslant \int_{\{u \leqslant h\}} b(x,u)Du \cdot DT_k(B(u) - B(v))^+dx \\ &\qquad\quad + \int_{\{u>h, |u-h+T_k(B(u)-B(v))^+| \leqslant k\}} b(x,u)Du \cdot DT_k(B(u) - B(v))^+dx \\ &\qquad = I_1 + I_2. \end{aligned} \quad (4.8)$$

As regards the last integral I_2, by (4.4) and the Hölder inequality, it follows that

$$|I_2| \leqslant c\left(\int_{\{u>h\}} |DB(u)|^2dx\right)^{\frac{1}{2}} \left(\int_{\{u>h\}} |DT_k(B(u) - B(v))^+|^2dx\right)^{\frac{1}{2}},$$

which goes to zero since $|DB(u)|$ and $|DB(v)|$ are in $L^2(\Omega)$, and $|\{u > h\}| \to 0$ as $h \to +\infty$.

Hence the inequality (4.7), letting $h \to +\infty$, becomes

$$\int_\Omega b(x,u)Du \cdot DT_k(B(u) - B(v))^+dx + \lambda \int_\Omega uT_k(B(u) - B(v))^+dx \\ \leqslant \int_\Omega fT_k(B(u) - B(v))^+dx. \quad (4.9)$$

Similarly as before, if we take $\varphi = T_h(v) + T_k(B(u) - B(v))^+$ as test function in the equation solved by v, we get

$$\int_\Omega b(x,v)Dv \cdot DT_k(v - T_h(v) - T_k(B(u) - B(v))^+)dx \\ + \lambda \int_\Omega vT_k(v - T_h(v) - T_k(B(u) - B(v))^+)dx \\ \leqslant \int_\Omega gT_k(v - T_h(v) - T_k(B(u) - B(v))^+)dx,$$

and, as $h \to +\infty$,

$$-\int_\Omega b(x,v)Dv\cdot DT_k(B(u)-B(v))^+dx - \lambda\int_\Omega vT_k(B(u)-B(v))^+dx \leqslant -\int_\Omega gT_k(B(u)-B(v))^+dx. \quad (4.10)$$

Adding (4.9) and (4.10) and dividing by k, we have

$$\frac{1}{k}\int_\Omega (b(x,u)Du-b(x,v)Dv)\cdot DT_k(B(u)-B(v))^+dx + \frac{\lambda}{k}\int_\Omega (u-v)T_k(B(u)-B(v))^+dx \leqslant 0, \quad (4.11)$$

since $f\leqslant g$. The first integral of (4.11) can be rewritten as

$$\begin{aligned}\frac{1}{k}\int_\Omega &(\tilde b(x,u)DB(u)-\tilde b(x,v)DB(v))\cdot DT_k(B(u)-B(v))^+dx\\ &= \frac{1}{k}\int_{0<B(u)-B(v)<k}\tilde b(x,u)D(B(u)-B(v))\cdot D(B(u)-B(v))dx\\ &\quad+\frac{1}{k}\int_{0<B(u)-B(v)<k}[\tilde b(x,u)-\tilde b(x,v)]DB(v)\cdot D(B(u)-B(v))dx = J_1+J_2,\end{aligned}$$

by adding and subtracting $\tilde b(x,u)DB(v)$. Clearly, $J_1\geqslant 0$. For J_2, using condition (4.5) and Hölder inequality, we get

$$\begin{aligned}|J_2| &\leqslant C_1\int_{0<B(u)-B(v)<k}|DB(v)||D(B(u)-B(v))|dx\\ &\leqslant C_1\left(\int_{0<B(u)-B(v)<k}|DB(v)|^2dx\right)^{\frac12}\left(\int_{0<B(u)-B(v)<k}|D(B(u)-B(v))|^2dx\right)^{\frac12},\end{aligned}$$

which goes to zero by letting $k\to 0$. Hence, as $k\to 0$ in (4.11), we get

$$\lambda\int_\Omega (u-v)\,\mathrm{sign}\,(B(u)-B(v))^+dx\leqslant 0,$$

which implies that $u\leqslant v$ a.e. in Ω, being B strictly increasing.

5. Comparison result

The aim of this section is to obtain a comparison result between a solution to problem (2.1) and the solution to the following “symmetrized” problem:

$$\begin{cases} -\mathrm{div}\left(\frac{Dw}{(1+|w|)^\theta}\right)+\lambda w = f^\# & \text{in } \Omega^\#,\\ w=0 & \text{on } \partial\Omega^\#,\end{cases} \quad (5.1)$$

where $\Omega^\#$ is the ball centered at the origin with $|\Omega^\#|=|\Omega|$ and $f^\#$ is the radially symmetric decreasing rearrangement of f.

It is useful to prove the following integral inequality for entropy solutions to problem (2.1) in order to obtain the comparison result.

PROPOSITION 5.1. *Let u and w be entropy solutions of* (2.1) *and* (5.1)*, respectively, with $f \in L^1(\Omega)$. Then the following inequalities hold:*

$$\alpha \frac{d}{dt} \int_{|u| \leqslant t} \frac{|Du|^2}{(1+|u|)^\theta} dx + \lambda \int_0^{\mu_u(t)} u^*(\sigma)\, d\sigma \leqslant \int_0^{\mu_u(t)} f^*(\sigma)\, d\sigma, \tag{5.2}$$

for a.e. $t \in (0, +\infty)$, and

$$-\alpha \frac{(u^*)'(s)}{(1+u^*(s))^\theta} \leqslant N^{-2} \omega_N^{-\frac{2}{N}} s^{-\frac{2}{N'}} \int_0^s [f^*(\sigma) - \lambda u^*(\sigma)]\, d\sigma, \tag{5.3}$$

for a.e. $s \in (0, |\Omega|)$. Moreover, if $\theta > 1$, the following equality holds

$$-\alpha \frac{(w^*)'(s)}{(1+w^*(s))^\theta} = N^{-2} \omega_N^{-\frac{2}{N}} s^{-\frac{2}{N'}} \int_0^s [f^*(\sigma) - \lambda w^*(\sigma)]\, d\sigma, \tag{5.4}$$

for a.e. $s \in (0, |\Omega|)$.

Proof. Let be $t, k > 0$. Using in (2.6) as test function $\varphi = T_t(u)$, by hypothesis (2.2) we easily obtain

$$\frac{\alpha}{k} \int_{t < |u| \leqslant t+k} \frac{|Du|^2}{(1+|u|)^\theta} dx \leqslant \int_{|u| > t} (|f| - \lambda |u|)\, dx. \tag{5.5}$$

Applying the Hardy–Littlewood inequality and the properties of rearrangements, by letting $k \to 0$ we obtain

$$\alpha \frac{d}{dt} \int_{|u| \leqslant t} \frac{|Du|^2}{(1+|u|)^\theta} dx \leqslant \int_0^{\mu_u(t)} [f^*(\sigma) - \lambda u^*(\sigma)]\, d\sigma, \tag{5.6}$$

that is (5.2). In order to obtain (5.3), using the Hölder inequality, we have that

$$\frac{\alpha}{(1+t)^\theta} \left(\frac{d}{dt} \int_{|u| \leqslant t} |Du| dx \right)^2 \frac{1}{(-\mu_u'(t))} \leqslant \int_0^{\mu_u(t)} [f^*(\sigma) - \lambda u^*(\sigma)]\, d\sigma.$$

From isoperimetric inequality and the Fleming–Rishel formula, it follows that

$$\frac{\alpha}{(1+t)^\theta} \frac{1}{(-\mu_u'(t))} \leqslant N^{-2} \omega_N^{-\frac{2}{N}} (\mu_u(t))^{-\frac{2}{N'}} \int_0^{\mu_u(t)} [f^*(\sigma) - \lambda u^*(\sigma)] d\sigma,$$

choosing $t = u^*(s)$ and using the properties of rearrangements (see [3] and [34]), we get (5.3). As regards (5.4), we recall that by Theorem 4.2 the solution w of problem (5.1) is unique. This implies, by symmetry of data, that w is positive and radially symmetric. Moreover, setting $s = \omega_N |x|^N$, and writing $\tilde{w}(s) = w((s/\omega_N)^{1/N})$, we obtain

$$-\frac{\tilde{w}'(s)}{(1+\tilde{w}(s))^\theta} = \frac{s^{-2/N'}}{N^2 \omega_N^{2/N}} \int_0^s [f^*(\sigma) - \lambda \tilde{w}(\sigma)]\, d\sigma.$$

By standard arguments (see for example [18]), it is possible to show that the above integral is positive and this implies that $w = w^{\#}$. So the arguments leading to (5.3) proceed in the same way except that the inequalities are replaced by equalities, that gives (5.4).

Now we are able to prove the following comparison between concentrations.

THEOREM 5.1. *Let u and w be entropy solutions of* (2.1) *and* (5.1)*, respectively, under assumptions* (2.2)*,* (2.3) *and* (2.4)*, with $\theta > 1$ and $f \in L^1(\Omega)$. Then we have:*

$$\int_0^s u^*(\sigma)\,d\sigma \leqslant \int_0^s w^*(\sigma)\,d\sigma, \quad \forall s \in [0,|\Omega|].$$

Proof. Let us define

$$\zeta(s) = \int_0^s (u^*(\sigma) - w^*(\sigma))\,d\sigma, \quad s \in [0,|\Omega|].$$

We have

$$\zeta'(|\Omega|) = 0 \text{ and } \zeta(0) = 0.$$

We will show that

$$\zeta \leqslant 0 \quad \text{in } [0,|\Omega|].$$

By contradiction, let us suppose that there exists $\bar{s}$ such that

$$\zeta(\bar{s}) = \max_{[0,|\Omega|]} \zeta(s) > 0.$$

We proceed to distinguish two cases.

If $\bar{s} = |\Omega|$, then there exists s_1 in $[0,|\Omega|]$ such that

$$\zeta(s_1) = 0 \quad \text{and} \quad \zeta(s) > 0 \text{ in } (s_1,|\Omega|]. \tag{5.7}$$

Now choosing s in $(s_1,|\Omega|]$ by (5.3) and (5.4), we get

$$\begin{aligned}
B(u^*(s)) &= -\int_s^{|\Omega|} \frac{d}{d\sigma} B(u^*(\sigma))\,d\sigma \\
&\leqslant \left(N\omega_N^{1/N}\right)^{-2} \int_s^{|\Omega|} \sigma^{-\frac{2}{N'}} \int_0^\sigma [f^*(\tau) - \lambda u^*(\tau)]\,d\tau d\sigma \\
&< \left(N\omega_N^{1/N}\right)^{-2} \int_s^{|\Omega|} \sigma^{-\frac{2}{N'}} \int_0^\sigma [f^*(\tau) - \lambda w^*(\tau)]\,d\tau d\sigma \\
&= -\int_s^{|\Omega|} \frac{d}{d\sigma} B(w^*(\sigma))\,d\sigma = B(w^*(s)).
\end{aligned}$$

So by strict monotonicity of B,

$$u^*(s) < w^*(s)$$

in $(s_1,|\Omega|]$, which contradicts (5.7).

If $\bar{s} < |\Omega|$, there exist $s_1, s_2 \in [0, |\Omega|]$ such that

$$\zeta(s_1) = 0, \quad \zeta(s) > 0 \text{ in } (s_1, s_2) \quad \text{and} \quad \zeta'(s_2) \leqslant 0.$$

Hence, choosing s in (s_1, s_2) and using (5.3) and (5.4), we obtain

$$\begin{aligned}
B(u^*(s)) - B(u^*(s_2)) &= -\int_{s_2}^{s} \frac{d}{d\sigma} B(u^*(\sigma))\, d\sigma \\
&\leqslant \left(N\omega_N^{1/N}\right)^{-2} \int_{s_2}^{s} \sigma^{-\frac{2}{N'}} \int_0^{\sigma} [f^*(\tau) - \lambda u^*(\tau)]\, d\tau d\sigma \\
&< \left(N\omega_N^{1/N}\right)^{-2} \int_{s_2}^{s} \sigma^{-\frac{2}{N'}} \int_0^{\sigma} [f^*(\tau) - \lambda w^*(\tau)]\, d\tau d\sigma \\
&= -\int_{s_2}^{s} \frac{d}{d\sigma} B(w^*(\sigma))\, d\sigma = B(w^*(s)) - B(w^*(s_2)),
\end{aligned}$$

and being $\zeta'(s_2) = B(u^*(s_2)) - B(w^*(s_2)) \leqslant 0$, we get

$$B(u^*(s)) < B(w^*(s)) \quad \text{in } (s_1, s_2),$$

which leads to a contradiction.

REMARK 5.1. We emphasize that, using Theorem 4.1 instead of Theorem 4.2 in Proposition 5.1, the comparison result stated above holds also for weak solutions, with $f \in L^{\frac{2N}{N+2}}$, without any assumption on $\theta \geqslant 0$.

REMARK 5.2. We stress that if $\lambda = \lambda(x) \in L^\infty(\Omega)$, with $\lambda(x) \geqslant 0$, the comparison result stated above continues to hold. More precisely, if u and v are respectively the solutions to problems

$$\begin{cases} -\operatorname{div}(a(x,u,Du)) + \lambda(x)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.8}$$

where $a(x,s,\xi)$ verifies (2.2), (2.3) and (2.4), and

$$\begin{cases} -\operatorname{div}\left(\frac{Dv}{(1+|v|)^\theta}\right) + \lambda_\#(x)v = f^\# & \text{in } \Omega^\#, \\ v = 0 & \text{on } \partial\Omega^\#, \end{cases} \tag{5.9}$$

where $\lambda_\#(x) = \lambda^*(|\Omega| - \omega_N|x|^N)$, with $x \in \Omega^\#$, is the spherically symmetric increasing rearrangement of λ, then

$$\int_0^s u^*(t)dt \leqslant \int_0^s v^*(t)dt, \qquad \text{for any } s \in [0, |\Omega|].$$

(see also, for example, [4], [3], [18] in the case $\theta = 0$).

6. A priori estimates

Let us consider, for any $n>0$, the following problem:

$$\begin{cases} -\operatorname{div}(a_n(x,u_n,Du_n))+\lambda u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where $a_n(x,s,\xi)=a(x,T_n(s),\xi)$, with $x\in\Omega$, $s\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$ and $f_n\in L^\infty(\Omega)$. We observe that classical results (see, for example, [24], [25]) assure that problem (6.1) has at least one solution $u_n\in H_0^1(\Omega)\cap L^\infty(\Omega)$.

REMARK 6.1. We stress that if u_n is a weak solution of (6.1), from inequality (5.2) of Proposition 5.1 we get

$$\int_0^s u_n^*(\sigma)\,d\sigma \leqslant \frac{1}{\lambda}\int_0^s f_n^*(\sigma)\,d\sigma, \quad s\in[0,|\Omega|], \tag{6.2}$$

that implies

$$\|u_n\|_{L^p(\Omega)} \leqslant \frac{1}{\lambda}\|f_n\|_{L^p(\Omega)}, \qquad \text{for any } 1\leqslant p\leqslant +\infty, \tag{6.3}$$

or, more generally,

$$\|u_n\|_{L^{p,q}} \leqslant \frac{1}{\lambda}\|f_n\|_{L^{p,q}}, \qquad \text{for any } 1<p\leqslant +\infty, 1\leqslant q\leqslant +\infty. \tag{6.4}$$

We emphasize that, if $\lambda=0$, estimate (6.3) does not hold, in general. Moreover, the cases $0\leqslant\theta\leqslant 1$ and $\theta>1$ are completely different. Indeed, if $\theta>1$, the sequence u_n could not be bounded in any L^p space, with $p\geqslant 1$. Moreover, it may happen that u_n converges to a function which is $+\infty$ on a set of positive Lebesgue measure (see [1]). In the other case, when $0\leqslant\theta<1$, it has been proven that, if $1<p<\frac{N}{2}$,

$$\|u_n\|_{L^s} \leqslant C(\|f_n\|_{L^p}), \tag{6.5}$$

with $s=\frac{Np(1-\theta)}{N-2p}$ (see, for example, [1], [12], [2]). In the limit case, $\theta=1$, it is possible to prove the existence of a bounded solution if f belongs to $L^p(\Omega)$, with $p>\frac{N}{2}$ (see, for example, [1]).

Hence the presence of the lower–order term, with $\lambda>0$, plays a fundamental role in order to obtain the estimates (6.3). Moreover, such estimates are independent from any hypotheses on θ. We observe that, when $\theta<1$ the estimates obtained with $\lambda=0$ in [1], [12], [2], are true also if $\lambda>0$. Nevertheless, the presence of the zero–order term permits, in some cases, to gain a higher summability on the estimates of u_n. Indeed, the exponent s in (6.5) is smaller than p, when $p<\frac{N}{2}\theta$.

PROPOSITION 6.1. *Let $\theta\geqslant 1$, and suppose that u_n is a weak solution to problem* (6.1)*. If $p>\frac{N}{2}\theta$, then*

$$\|u_n\|_{L^\infty(\Omega)} \leqslant C_1, \tag{6.6}$$

and, moreover,

$$\|u_n\|_{H_0^1(\Omega)} \leqslant C_2 \tag{6.7}$$

where C_1 and C_2 are constants which continuously depend on the L^p norm of f_n.

Proof. The estimate (5.3) can be rewritten as follows:

$$-\frac{(u_n^*)'(s)}{1+u_n^*(s)} \leqslant \frac{1}{\alpha N^2 \omega_N^{2/N}} s^{-2/N'} (1+u_n^*(s))^{\theta-1} \int_0^s f_n^*(\sigma)\, d\sigma.$$

Using the Hölder inequality and integrating between 0 and $|\Omega|$, we get

$$\begin{aligned}\log(1+u_n^*(0)) &\leqslant \frac{\|f_n\|_{L^p(\Omega)}}{\alpha N^2 \omega_N^{2/N}} \int_0^{|\Omega|} s^{1/p'-2/N'} (1+u_n^*(s))^{\theta-1}\, ds \\ &\leqslant \frac{\|f_n\|_{L^p(\Omega)}}{\alpha N^2 \omega_N^{2/N}} \left[\int_0^{|\Omega|} s^{\frac{2p-N-Np}{N(p+1-\theta)}}\, ds\right]^{1-\frac{\theta-1}{p}} \|1+|u_n|\|_{L^p(\Omega)}^{\theta-1},\end{aligned}$$

and last integral is finite, being $p > \frac{N}{2}\theta$. Using (6.3) and the Minkowski inequality, the above inequality becomes

$$\log(1+u_n^*(0)) \leqslant \frac{|\Omega|^{\frac{2}{N}-\frac{\theta}{p}}}{\alpha \lambda^{\theta-1} N^2 \omega_N^{2/N}} \left(|\Omega|^{\frac{\theta-1}{p}} + \|f_n\|_{L^p(\Omega)}^{\theta-1}\right) \|f_n\|_{L^p(\Omega)},$$

which gives the estimate (6.6).

As regards the inequality (6.7), integrating between 0 and $+\infty$ the estimate (5.2) and using (6.6), we get that

$$\alpha \int_\Omega |Du_n|^2\, dx \leqslant (1+\|u_n\|_{L^\infty})^\theta \int_0^{|\Omega|} f_n^*(s) u_n^*(s) ds \leqslant C_2.$$

PROPOSITION 6.2. *Suppose that u_n is a weak solution to problem* (6.1).

(i) If $\theta+2 \leqslant p \leqslant \frac{N}{2}\theta$, then

$$\|u_n\|_{H_0^1(\Omega)} \leqslant C, \tag{6.8}$$

where the constant C continuously depends on the L^p norm of f_n.

(ii) If $\frac{\theta+2}{2} \leqslant p < \theta+2$, then

$$\|u_n\|_{W_0^{1,\beta}(\Omega)} \leqslant C, \tag{6.9}$$

where $\beta = \frac{2p}{\theta+2}$ and C is a constant which continuously depends on the L^p norm of f_n.

(iii) If $1 < p < \frac{\theta+2}{2}$, then

$$\||Du_n|^\beta\|_{L^1(\Omega)} \leqslant C, \tag{6.10}$$

where $\beta = \frac{2p}{\theta+2} < 1$ and C is a constant which continuously depends on the L^p norm of f_n.

(iv) If $p = 1$, then

$$\||Du_n|\|_{M^{\frac{2}{\theta+2}}} \leqslant C, \tag{6.11}$$

where C is a constant which continuously depends on the L^1 norm of f_n.

Proof. Let us prove *(i)*. The estimate (5.2) gives

$$\alpha \frac{d}{dt} \int_{|u_n| \leqslant t} \frac{|Du_n|^2}{(1+|u_n|)^{\theta+2-p}}\, dx \leqslant (1+t)^{p-2} \int_0^{\mu_{u_n}(t)} f_n^*(\sigma)\, d\sigma.$$

Integrating between 0 and $+\infty$ the above inequality, we get

$$\alpha \int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^{\theta+2-p}} dx \leqslant \frac{1}{p-1} \int_0^{|\Omega|} f_n^*(s) \left[(1+u_n^*(s))^{p-1} - 1 \right] ds.$$

We recall that, under our assumption, $\theta+2-p \leqslant 0$. Using Hölder inequality we obtain

$$\alpha \int_\Omega |Du_n|^2\, dx \leqslant \alpha \int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^{\theta+2-p}}\, dx \leqslant c \|1+|u_n|\|_{L^p(\Omega)}^{p-1} \|f_n\|_{L^p(\Omega)}. \tag{6.12}$$

Applying the estimate (6.3) to (6.12), we get (6.8).

As regards *(ii)*, if $\frac{\theta+2}{2} \leqslant p < \theta+2$, we get that $1 \leqslant \beta < 2$. By the Hölder inequality we get

$$\int_\Omega |Du_n|^\beta\, dx \leqslant \left(\int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^{\theta+2-p}}\, dx \right)^{\frac{\beta}{2}} \left(\int_\Omega (1+|u_n|)^p\, dx \right)^{1-\frac{\beta}{2}}.$$

The right–hand side can be estimated by a constant which depends on the L^p norm of f_n. Indeed we can reason as before for the first integral, while the second one can be estimated using (6.3).

The same arguments give the case *(iii)*.

Finally, if $p=1$, integrating between 0 and k both sides of the inequality (5.6), we get

$$\alpha \int_\Omega |DT_k(u_n)|^2\, dx \leqslant (1+k)^\theta k \|f_n\|_{L^1(\Omega)}.$$

Applying Lemma 3.2 we get a uniform bound of $|Du_n|$ in the Marcikiewicz space $M^{\frac{2}{\theta+2}}$, which proves *(iv)*.

REMARK 6.2. We emphasize that the range $\theta+2 \leqslant p \leqslant \frac{N}{2}\theta$ is non–empty if $\theta \geqslant \tilde{\theta} = \frac{4}{N-2}$.

REMARK 6.3. We stress that if $\lambda > 0$ and $\theta < 1$, it is possible to gain a better summability with respect to the case $\lambda = 0$. Indeed, taking into account Remark 6.1, if $p < \frac{N}{2}\theta$ the gradient estimates obtained in Proposition 6.2, with $\theta < 1$, are stronger than the corresponding one obtained in [1], [2], [12].

REMARK 6.4. If $\theta+2 \leqslant p \leqslant \frac{N}{2}\theta$, the argument used in the proof of the above theorem allows to prove not only that the H_0^1 norm of u_n is bounded but a more slightly stronger result, namely that

$$\int_\Omega |D\hat{B}(u_n)|^2\, dx = \int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^{\theta+2-p}}\, dx \leqslant C,$$

where $\hat{B}(s) = \int_0^s (1+t)^{-(\theta-p+2)/2} dt$ (see also [1]). On the other hand, we can apply the Sobolev embedding theorem in order to obtain an uniform bound on the $L^{(p-\theta)\frac{N}{N-2}}$ norm of u_n. Nevertheless, for $p < \frac{N}{2}\theta$, this is a weaker result than the L^p estimate (6.3).

7. Existence results

THEOREM 7.1. *Let us assume that* (2.2), (2.3) *and* (2.4) *hold.*

(*a*) *Let* $\theta \geqslant 1$. *If* $f \in L^p(\Omega)$, *with* $p > \frac{N}{2}\theta$, *then there exists a weak bounded solution* u *to problem* (2.1).

(*b*) *Let* $0 < \theta \leqslant \frac{4}{N-2}$. *If* $f \in L^p(\Omega)$, *with* $p \geqslant \theta + 2$, *then there exists a weak solution* u *to problem* (2.1).

(*c*) *If* $f \in L^1(\Omega)$, *then there exists an entropy solution* u *to problem* (2.1).

Proof. We first study the case $f \in L^\infty(\Omega)$, and we consider the approximating problems

$$\begin{cases} -\operatorname{div}(a(x, T_n(u_n), Du_n)) + \lambda u_n = f & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

As well known (see, for example, [24]), there exists a weak solution $u_n \in H_0^1(\Omega)$ of problem (7.1). By inequality (6.3) with $f_n = f$, we get the uniform estimate

$$\|u_n\|_{L^\infty(\Omega)} \leqslant \frac{1}{\lambda}\|f\|_{L^\infty(\Omega)}.$$

Hence, for n sufficiently large, we have that u_n is also a weak solution to problem (2.1).

Let now f be in $L^p(\Omega)$, $p \geqslant 1$. Using the above argument there exists $u_n \in H_0^1(\Omega)$ which solves the approximate problem

$$\begin{cases} -\operatorname{div}(a(x, u_n, Du_n)) + \lambda u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.2}$$

with $f_n = T_n(f)$.

Using (5.6), we get

$$\alpha \int_\Omega |DT_k(u_n)|^2 \, dx \leqslant (1+k)^\theta k \|f\|_{L^1(\Omega)}, \tag{7.3}$$

which is an uniform estimate with respect to n of the L^2 norms of $|DT_k(u_n)|$. Reasoning as in [6], we can conclude that there exists a measurable function u such that, up to a subsequence,

$$u_n \to u \quad \text{a.e. in } \Omega,$$

and by (7.3)

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in } H_0^1(\Omega).$$

Moreover, being

$$\int_\Omega \frac{|DT_k(u)|^2}{(1+|u|)^\theta}\,dx \leqslant \liminf_{n\to+\infty}\int_\Omega \frac{|DT_k(u_n)|^2}{(1+|u_n|)^\theta}\,dx,$$

it follows that

$$\alpha\int_\Omega |DT_k(u)|^2\,dx \leqslant (1+k)^\theta k\|f\|_{L^1(\Omega)}. \tag{7.4}$$

On the other hand, for any measurable set $E\subseteq\Omega$, by the Hardy and Littlewood inequality and (6.2) it follows that

$$\int_E |u_n|\,dx = \int_{E\cap\{|u_n|<k\}}|u_n|dx + \int_{E\cap\{|u_n|\geqslant k\}}|u_n|dx \leqslant k|E| + \int_0^{|\{|u_n|\geqslant k\}|} f^*(\sigma)\,d\sigma.$$

Now by the Hardy and Littlewood inequality and (6.2) we get that,

$$|\{|u_n|\geqslant k\}| \leqslant \frac{1}{k}\int_0^{|\{|u_n|\geqslant k\}|} u_n^*(\sigma)\,d\sigma \leqslant \frac{1}{k}\int_0^{|\{|u_n|\geqslant k\}|} f_n^*(\sigma)\,d\sigma \leqslant \frac{1}{k}\|f\|_{L^1(\Omega)},$$

which vanishes uniformly with respect to n when $k\to\infty$. Thus for any given $\varepsilon>0$, there exists k_ε such that

$$\int_0^{|\{|u_n|\geqslant k_\varepsilon\}|} f^*(\sigma)\,d\sigma \leqslant \varepsilon, \quad \forall n\in\mathbb{N},$$

and so

$$\int_E |u_n|\,dx \leqslant k_\varepsilon|E| + \varepsilon.$$

This means

$$\int_E |u_n|\,dx \leqslant \varepsilon, \quad \text{for all } \varepsilon>0 \text{ when } |E|\to 0,$$

which implies the equintegrability of u_n. By the Vitali theorem, we can conclude that $u\in L^1(\Omega)$ and

$$u_n \to u \quad \text{in } L^1(\Omega).$$

We observe that by the estimate (6.11), $|Du_n|$ is uniformly bounded in the Marcikiewicz space $M^{\frac{2}{\theta+2}}$. Moreover, using Lemma 3.1, the inequality (7.4) allows to give a sense to the gradient of u. Being $u\in L^1(\Omega)$, Lemma 3.2 and inequality (7.4) assure that $|Du|\in M^{\frac{2}{\theta+2}}$.

This allows us to claim that, up to a subsequence,

$$Du_n \to Du \quad \text{a.e. in } \Omega.$$

Such convergence result can be found, for example, in [1].

Now we consider the cases (a) and (b). Choosing $\varphi\in H_0^1(\Omega)$ as a test function in problem (7.2), we have

$$\int_\Omega a(x,u_n,Du_n)\cdot D\varphi\,dx + \lambda\int_\Omega u_n\varphi\,dx = \int_\Omega f_n\varphi\,dx. \tag{7.5}$$

In order to pass to the limit in (7.5), we observe that $\|u_n\|_{H_0^1(\Omega)}$ are uniformly bounded since the estimates (6.7) or (6.8) hold, so also $|a(x,u_n,Du_n)|$ is bounded in $L^2(\Omega)$ by hypothesis (2.3). Hence

$$a(x,u_n,Du_n) \rightharpoonup a(x,u,Du) \quad \text{in } (L^2(\Omega))^N,$$

because Du_n converges a.e. to Du in Ω. On the other hand, u_n strongly converges to u in L^2. So we can pass to the limit in (7.5), obtaining that u is a weak solution to problem (2.1). The boundedness of the solution in case (a) follows from estimate (6.6).

Now let us consider the case (c).

Fixed $k>0$ and $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$, we want to pass to the limit in the following expression

$$\int_\Omega a(x,u_n,Du_n)\cdot DT_k(u_n-\varphi)\,dx + \lambda\int_\Omega u_n T_k(u_n-\varphi)dx = \int_\Omega f_n T_k(u_n-\varphi)dx. \quad (7.6)$$

Since u_n and f_n strongly converge to u and f in L^1 respectively, and $T_k(u_n-\varphi)$ $*-$weakly converges to $T_k(u-\varphi)$, we get

$$\lim_{n\to+\infty}\int_\Omega u_n T_k(u_n-\varphi)dx = \int_\Omega u\,T_k(u-\varphi)dx,$$
$$\lim_{n\to+\infty}\int_\Omega f_n T_k(u_n-\varphi)dx = \int_\Omega f\,T_k(u-\varphi)dx.$$

As the first term of the left-hand side of (7.6) is concerned, we can split it into the sum

$$\int_{\{|u_n-\varphi|\leqslant k\}} a(x,u_n,Du_n)\cdot Du_n\,dx - \int_{\{|u_n-\varphi|\leqslant k\}} a(x,u_n,Du_n)\cdot D\varphi\,dx. \quad (7.7)$$

As regards the second term of (7.7), it can be rewritten as

$$\int_{\{|u_n-\varphi|\leqslant k\}} a(x,u_n,Du_n)\cdot D\varphi\,dx = \int_{\{|u_n-\varphi|\leqslant k\}} a(x,T_M(u_n),DT_M(u_n))\cdot D\varphi\,dx,$$

where $M = k+\|\varphi\|_{L^\infty}$. We observe that $|a(x,T_M(u_n),DT_M(u_n))|$ is bounded in $L^2(\Omega)$ by (2.3). Hence, being $T_M(u_n)\rightharpoonup T_M(u)$ in $H_0^1(\Omega)$, and $Du_n \to Du$ a.e. in Ω, we get

$$a(x,T_M(u_n),DT_M(u_n)) \rightharpoonup a(x,T_M(u),DT_M(u)) \quad \text{weakly in } (L^2(\Omega))^N.$$

Therefore

$$\lim_{n\to+\infty}\int_{\{|u_n-\varphi|\leqslant k\}} a(x,u_n,Du_n)\cdot D\varphi\,dx =$$
$$= \int_{\{|u-\varphi|\leqslant k\}} a(x,T_M(u),DT_M(u))\cdot D\varphi\,dx = \int_{\{|u-\varphi|\leqslant k\}} a(x,u,Du)\cdot D\varphi\,dx.$$

As regard the first term of (7.7), being $a(x,u_n,Du_n)\cdot Du_n \geqslant 0$ and $a(x,u_n,Du_n)\to a(x,u,Du)$ a.e., by Fatou's Lemma we get

$$\int_{\{|u-\varphi|\leqslant k\}} a(x,u,Du)\cdot Du\,dx \leqslant \liminf_{n\to+\infty}\int_{\{|u_n-\varphi|\leqslant k\}} a(x,u_n,Du_n)\cdot Du_n\,dx.$$

Putting all the terms together, we obtain that

$$\int_\Omega a(x,u,Du)\cdot DT_k(u-\varphi)\,dx+\lambda\int_\Omega u\,T_k(u-\varphi)dx\leqslant\int_\Omega f\,T_k(u-\varphi)dx,$$

which means that u is an entropy solution to problem (2.1).

REMARK 7.1. We emphasize that in order to prove the existence of a weak solution of (2.1) when $0\leqslant\theta\leqslant 1$ and $p\geqslant\frac{2N}{N(1-\theta)+2(\theta+1)}$, we can easily adapt the proofs contained, for example, in [1] or [12].

REMARK 7.2. In Theorem 7.1 we showed that, under the hypotheses (a) or (b), there exists a weak solution to problem (2.1). Otherwise,requiring instead of (2.3) the following stronger assumption:

$$|a(x,s,\xi)|\leqslant h(x)+\frac{\delta}{(1+|s|)^\theta}|\xi|, \tag{7.8}$$

with $h\in L^2(\Omega)$, $\delta>0$ and $\theta>1$, we can prove the existence of a solution u which verifies the identity (2.5) even if we assume a weaker summability on the datum f. Nevertheless, such solution could not be in the energy space $H_0^1(\Omega)$.

Indeed, taking $\varphi=B(|u_n|)\,\mathrm{sign}(u_n)$ as test function in the approximating problems (7.2) we get, by hypothesis (2.2), that

$$\int_\Omega|D(B(|u_n|)|^2\,dx+\lambda\int_\Omega|u_n|B(|u_n|)\,dx\leqslant\int_\Omega|f_n|B(|u_n|)\,dx,$$

and being $\lambda>0$,

$$\int_\Omega|D(B(|u_n|)|^2\,dx\leqslant\bar{B}\|f\|_{L^1(\Omega)}, \tag{7.9}$$

where $\bar{B}=\sup_{s>0}B(s)$, which is finite since $\theta>1$.

If f belongs to $L^p(\Omega)$, with $p\geqslant\max\{\frac{2N}{N+2},\frac{\theta+2}{2}\}$, then the hypothesis (7.8) and the estimate (7.9) assure that there exists $u\in W_0^{1,\beta}(\Omega)$, $\beta=2p/(\theta+2)$, such that $a(x,u_n,Du_n)\rightharpoonup a(x,u,Du)$ in $(L^2(\Omega))^N$. Therefore we immediately obtain that u satisfies the identity (2.5).

8. Regularity results

In this section we are interested in regularity results for solutions u to problem (2.1).

First of all, we observe that the results stated in Remark 6.1 and in the propositions 6.1 and 6.2 hold also for the solutions to problem (2.1). Hence we know how the summability of u and its gradient vary by varying the summability of the datum in Lebesgue spaces.

Now we want to study what happens when we choose the datum f in some particular Lorentz space.

We emphasize that, when $\theta \geqslant 1$, reasoning as in the proof of Proposition 6.1, if the datum f belongs to Lebesgue space L^q, with $q > \frac{N}{2}\theta$, the solutions of (2.1) are bounded. The following result assures that the solutions are bounded also when the datum f belongs to the Lorentz space $L^{\frac{N}{2}\theta,\theta}$.

THEOREM 8.1. *Under the assumptions* (2.2), (2.3) *and* (2.4), *with* $\theta \geqslant 1$, *denoting by* u *a weak solution to problem* (2.1), *the following results hold:*

(i) if $f \in L^{\frac{N}{2}\theta,\theta}$, *then* $u \in L^\infty(\Omega)$;

(ii) if $f \in L^{\frac{N}{2}\theta,q}$, *with* $\theta < q < +\infty$, *then* $u \in L^r(\Omega)$, *for any* $r < +\infty$.

Proof. We observe that the estimate (5.3) holds also for a solution u of problem (2.1). Hence we obtain

$$-\frac{(u^*)'(s)}{1+u^*(s)} \leqslant \frac{1}{\alpha N^2 \omega_N^{2/N}} s^{-2/N'} (1+u^*(s))^{\theta-1} \int_0^s f^*(\sigma)\, d\sigma.$$

Integrating between s and $|\Omega|$, we get

$$\log(1+u^*(s)) \leqslant \frac{1}{\alpha N^2 \omega_N^{2/N}} \int_s^{|\Omega|} \tau^{2/N} (1+u^*(\tau))^{\theta-1} f^{**}(\tau)\, \frac{d\tau}{\tau}, \tag{8.1}$$

where $f^{**}(s) = s^{-1} \int_0^s f^*(\sigma)\, d\sigma$.

As regards the assertion *(i)*, applying the Hölder inequality, recalling Definition 3.2 and choosing $s = 0$, we have

$$\begin{aligned}\log(1+u^*(0)) &\leqslant c\left(\int_0^{|\Omega|} s^{\frac{2}{N}} (f^{**}(s))^\theta \frac{ds}{s}\right)^{\frac{1}{\theta}} \left(\int_0^{|\Omega|} s^{\frac{2}{N}} (1+u^*(s))^\theta \frac{ds}{s}\right)^{1-\frac{1}{\theta}} \\ &= c\|f\|_{L^{\frac{N}{2}\theta,\theta}} \|1+|u|\|^{\theta-1}_{L^{\frac{N}{2}\theta,\theta}} \leqslant C,\end{aligned}$$

where the constant C continuously depends on the $L^{\frac{N}{2}\theta,\theta}$ norm of f by the estimate (6.4). Hence we get *(i)*.

In order to prove *(ii)*, applying the Hölder inequality to (8.1) and being $q > \theta$, we obtain

$$\begin{aligned}\log(1+u^*(s)) &\leqslant c\left(\int_s^{|\Omega|} \tau^{\frac{2q}{N\theta}} (f^{**}(\tau))^q \frac{d\tau}{\tau}\right)^{\frac{1}{q}} \left(\int_s^{|\Omega|} \tau^{\frac{2q'}{N\theta'}} (1+u^*(\tau))^{q'(\theta-1)} \frac{d\tau}{\tau}\right)^{1-\frac{1}{q}} \\ &\leqslant c\|f\|_{L^{\frac{N}{2}\theta,q}} \left(\int_s^{|\Omega|} \tau^{\frac{2q}{N\theta}} (1+u^*(\tau))^q \frac{d\tau}{\tau}\right)^{\frac{\theta-1}{q}} \left(\int_s^{|\Omega|} \frac{d\tau}{\tau}\right)^{1-\frac{\theta}{q}}\end{aligned}$$

Therefore, using (6.4), we get

$$\log(1+u^*(s)) \leqslant C\left(\log\frac{|\Omega|}{s}\right)^{1-\frac{\theta}{q}},$$

which means that

$$u^*(s) \leqslant \exp\left\{ C\left(\log\frac{|\Omega|}{s}\right)^{1-\frac{\theta}{q}}\right\}.$$

By easy computations the above estimate implies that u belongs to $L^r(\Omega)$, for any $r<+\infty$.
We emphasize that in the limit case $f\in L^{\frac{N}{2}\theta,\infty}$, in general, the summability of the solution u is not greater than the summability of f, as shown by the following example.

EXAMPLE 8.1. Let $\Omega = B_1(0) = \{x\in\mathbb{R}^N\colon |x|<1\}$, $\theta=2$ and $\lambda=1$. We consider the following problem,

$$\begin{cases} -\operatorname{div}\left(\frac{Du}{(1+|u|)^2}\right)+u = \frac{N}{|x|}-1 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{8.2}$$

where the datum $f(x)=\frac{N}{|x|}-1$ is such that $f\in L^{N,\infty}\setminus L^N(\Omega)$. It is not difficult to show that, being $N\geqslant 3$, the function

$$u(x)=\frac{1}{|x|}-1$$

belongs to $H_0^1(\Omega)$ and it is a weak solution to problem (8.2), but $u\notin L^N(\Omega)$.

However, this problem can be avoided by assuming a smallness assumption on the norm of the datum.

THEOREM 8.2. *Let us assume* (2.2), (2.3) *and* (2.4), *with* $\theta>1$, $f\in L^{\frac{N}{2}\theta,\infty}$, *and let* u *a weak solution to problem* (2.1). *If* $p>0$ *is such that*

$$\|f\|^{\theta}_{L^{\frac{N}{2}\theta,\infty}} < \alpha N^2\omega_N^{2/N}\left(\frac{\lambda}{2}\right)^{\theta-1}\frac{1}{p}, \tag{8.3}$$

then $u\in L^p(\Omega)$.

Proof. Reasoning as in the proof of Theorem 8.1, from (8.1) we obtain that

$$\begin{aligned}\log(1+u^*(s)) &\leqslant \frac{2^{\theta-1}}{\alpha N^2\omega_N^{2/N}}\left[\int_s^{|\Omega|}\tau^{2/N}f^{**}(\tau)\frac{d\tau}{\tau}+\int_s^{|\Omega|}\tau^{2/N}u^*(\tau)^{\theta-1}f^{**}(\tau)\frac{d\tau}{\tau}\right]\\ &\leqslant \frac{2^{\theta-1}}{\alpha N^2\omega_N^{2/N}}\left[|\Omega|^{\frac{2}{N\theta'}}\|f\|_{L^{\frac{N}{2}\theta,\infty}}+\|f\|_{L^{\frac{N}{2}\theta,\infty}}\int_s^{|\Omega|}\tau^{\frac{2}{N\theta'}}u^*(\tau)^{\theta-1}\frac{d\tau}{\tau}\right].\end{aligned}$$

Using the estimate (6.4) we have that

$$\log(1+u^*(s)) \leqslant \frac{2^{\theta-1}\|f\|_{L^{\frac{N}{2}\theta,\infty}}}{\alpha N^2\omega_N^{2/N}}\left[|\Omega|^{\frac{2}{N\theta'}}+\frac{\|f\|^{\theta-1}_{L^{\frac{N}{2}\theta,\infty}}}{\lambda^{\theta-1}}\log\frac{|\Omega|}{s}\right],$$

that means

$$u^*(s) \leqslant C\left(\frac{|\Omega|}{s}\right)^{\gamma}, \qquad \text{where } \gamma = \left(\frac{2}{\lambda}\right)^{\theta-1} \frac{\|f\|^{\theta}_{L^{\frac{N}{2}\theta,\infty}}}{\alpha N^2 \omega_N^{2/N}}.$$

This gives the thesis.

We want to stress that even if the datum f is less regular, it is possible to give a smallness assumption on f in order to assure the existence of bounded solutions.

THEOREM 8.3. *Let us assume* (2.2)*,* (2.3) *and* (2.4)*, with* $\theta > 1$*, and let* u *be a weak solution to problem* (2.1)*. If* $f \in L^{\frac{N}{2}\beta,\beta}$*, with* $1 \leqslant \beta < \theta$ *and the following smallness condition holds,*

$$\|f\|_{L^{\frac{N}{2}\beta,\beta}} \left(\frac{N}{2}|\Omega|^{\frac{2}{N}} + \frac{1}{\lambda^{\beta}}\|f\|^{\beta}_{L^{\frac{N}{2}\beta,\beta}}\right)^{1-1/\beta} < \frac{\alpha N^2 \omega_N^{2/N}}{2^{\beta-1}(\theta-\beta)}, \tag{8.4}$$

then $u \in L^{\infty}(\Omega)$*.*

Proof. We rewrite the estimate (5.3) as follows:

$$-\frac{(u^*)'(s)}{(1+u^*(s))^{\theta-\beta+1}} \leqslant \frac{1}{\alpha N^2 \omega_N^{2/N}} s^{-2/N'} (1+u^*(s))^{\beta-1} \int_0^s f^*(\sigma)\,d\sigma.$$

Integrating between s and $|\Omega|$, using the Hölder inequality and (6.2), we get

$$\begin{aligned} \tilde{B}(u^*(s)) &= \frac{1-(1+u^*(s))^{\beta-\theta}}{\theta-\beta} \\ &\leqslant \frac{1}{\alpha N^2 \omega_N^{2/N}} \left(\int_s^{|\Omega|} \tau^{\frac{2}{N}} (1+u^*(\tau))^{\beta} \frac{d\tau}{\tau}\right)^{1-1/\beta} \|f\|_{L^{\frac{N}{2}\beta,\beta}} \\ &\leqslant \frac{2^{\beta-1}}{\alpha N^2 \omega_N^{2/N}} \|f\|_{L^{\frac{N}{2}\beta,\beta}} \left(\frac{N}{2}|\Omega|^{\frac{2}{N}} + \lambda^{-\beta}\|f\|^{\beta}_{\frac{N}{2}\beta,\beta}\right)^{1-1/\beta}. \end{aligned}$$

This means that if

$$\frac{2^{\beta-1}}{\alpha N^2 \omega_N^{2/N}} \|f\|_{L^{\frac{N}{2}\beta,\beta}} \left(\frac{N}{2}|\Omega|^{\frac{2}{N}} + \lambda^{-\beta}\|f\|^{\beta}_{\frac{N}{2}\beta,\beta}\right)^{1-1/\beta} < \sup_{s\geqslant 0} \tilde{B}(s) = \frac{1}{\theta-\beta}, \tag{8.5}$$

then $u \in L^{\infty}(\Omega)$.

REMARK 8.1. We stress that, if $\beta = 1$, the smallness hypothesis (8.5) coincides with the condition given in [1]. On the other hand, if $\beta = \theta$, no smallness assumption is required to have bounded solution, as proven in Theorem 8.1.

REMARK 8.2. We explicitly observe that, for $1 \leqslant \beta < \theta$,

$$\|f\|_{\frac{N}{2}\beta,\beta} \leqslant C\|f\|_{\frac{N}{2}\theta,\infty}.$$

So it may happen that, if $f \in L^{\frac{N}{2}\theta,\infty}$, inequalities (8.3) and (8.4) hold simultaneously. In such a case Theorem 8.3 gives a stronger regularity result.

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