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ANISOTROPIC PARABOLIC PROBLEMS WITH MEASURES DATA

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Abstract. In this work, we prove the existence of a weak solution of an anisotropic parabolic problem with measure data $u_t + Au + F(u, Du) = \mu$ and $u(0) = \mu_0$ with μ and μ_0 two Radon bounded measures. The operator A is a Leray-Lions operator with anisotropic growth conditions. Our approach is based on the anisotropic Sobolev inequality, a regularity result, a compactness result, and an integration by parts formula.

1. Introduction

Let us consider the following anisotropic parabolic problem:

$$(P)\qquad \begin{cases} \partial_t u + Au + F(t,x,u,Du) = \mu \ \text{ in } \ Q = \Omega\times(0,T), \\ u(0,x) = \mu_0(x) \ \text{ in } \ \Omega, \\ u = 0 \ \text{ on } \ (0,T)\times\partial\Omega, \end{cases}$$

where Ω is a smooth bounded open set of $\mathbb{R}^N$ $(N \geqslant 2)$, $T > 0$ a real number, μ is a Radon's bounded measure on Q, μ_0 a Radon's bounded measure on Ω, and A is the operator given by $Au = -\mathrm{div}(\widehat{a}(x,t,u,Du))$. Here, we suppose that $\widehat{a}(x,t,u,\xi)$ and $F(t,x,u,\xi)$ are functions verifying the following conditions.

$\widehat{a}.1)$ There exist two constants $\alpha > 0$ and $\beta > 0$ such that for a.e. $(t,x) \in Q$, all $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, the function $\widehat{a} : Q\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ satisfies the following growths:

$$\widehat{a}(t,x,u,\xi)\xi \geqslant \alpha\sum_{i=1}^{N}|\xi_i|^{p_i}, \quad \widehat{a}(t,x,u,\xi) = (a_1(t,x,u,\xi),\ldots,a_N(t,x,u,\xi)),$$

$$|a_i(t,x,u,\xi)| \leqslant \beta\Big(|g| + |u|^{\overline{p}} + \sum_{j=1}^{N}|\xi_j|^{p_j}\Big)^{1-\frac{1}{p_i}},\ g \in L^1(Q),\ i = 1,\ldots,N.$$

Here p_i are real positive numbers so that:

$$1 + \frac{N}{N+1} < p_i < \frac{\overline{p}(N+1)}{N} \quad \text{and} \quad \frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i},\ i = 1,\ldots,N.$$

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$\widehat{a}.2)$ The mapping $\widehat{a}$ is a Carathéodory function, that is to say, the function $(t,x,u,\xi)\mapsto \widehat{a}(t,x,u,\xi)$ is measurable in (t,x) for all $(u,\xi)\in\mathbb{R}\times\mathbb{R}^N$ and continuous in (u,ξ) for a.e. $(t,x)\in Q$.
$\widehat{a}.3)$ For a.e. $(t,x)\in Q$, for all $u\in\mathbb{R}$ and for all $\xi\neq\xi'$, we have

$$(\widehat{a}(t,x,u,\xi)-\widehat{a}(t,x,u,\xi'))(\xi-\xi')>0.$$

We shall assume the following growth for $F=\sum_{i=1}^N F_i$
$F)$ for each $i=1,\ldots,N$ the function F_i is a Carathéodory function and satisfies the properties:

$$uF(t,x,u,\xi)\geqslant 0,\quad \text{a.e.}\quad (t,x)\in Q,\quad \forall u\in\mathbb{R},\ \xi\in\mathbb{R}^N,$$
$$|F_i(t,x,u,\xi)|\leqslant C(|h|+|u|^{\overline{p}}+\sum_{j=1}^N|\xi_j|^{p_j})^{1-\frac{1}{p_i}},\ h\in L^1(Q),\ i=1,\ldots,N,$$

where C is a nonnegative constant.

This work is a generalization of some results on isotropic or anisotropic problems given in [2], [3], [6], [9], and [16]. None of them treat the fully parabolic anisotropic quasilinear case as we are dealing here. If $\widehat{a}$ is a non-uniformly elliptic operator, the existence of solution for isotropic parabolic equation that involved measure data is studied in [11]. If $\widehat{a}$ does not depend on (t,x) and u, namely $\widehat{a}(t,x,u,\xi)=\widehat{a}(\xi)$, $\widehat{a}(\xi)$ is the vector field whose components are $|\xi_i|^{p_i-2}\xi_i$, $i=1,\ldots,N$, $p_i>1$, then it has been proved in [5] that there exists a weak solution $u\in X_0^{1,\overrightarrow{r}}(\Omega)$, $\overrightarrow{r}=(r_1,r_2,...,r_N)$, for an anisotropic elliptic problem with (one) Radon's bounded measure data on Ω with $r_i\in[1,\frac{p_i(\overline{p}-1)N}{\overline{p}(N-1)})$. Fengquan Li in [12] introduced a definition of relaxed solutions defined by J. M. Rakotoson in [17] and new type of the functional sets for anisotropic parabolic equations. In this paper we treated the anisotropic parabolic case within a different framework of that of [10] for the formulation but we remain in the framework of Sobolev's space. A major difficulty in the parabolic case is the regularity of the time derivative. With the anisotropic conditions, we also deal with two Radon bounded measures μ and μ_0 as an initial data and $1+\frac{N}{N+1}<p_i<\frac{\overline{p}(N+1)}{N}$. We think that our estimates on the gradients are new compared to the isotropic case. To describe briefly the tools we use, first we have the anisotropic Sobolev inequality to overcome the difficulties of getting the regularity in the Lemmas 2.4, secondly we introduce two Lemmas 2.2 and 2.14 to facilitate the control of the term $\partial_t u_n$ of the regularized problem. Nevertheless, to show that the assumptions of these Lemmas are verified, we prove some regularity results derived from [4] and [20]. In this paper, we adopt mainly Rakotoson's method [16] for proving the existence of the weak solution.

Notations.

Let $1+\frac{N}{N+1}<p_i<\frac{\overline{p}(N+1)}{N}$, $i=1,...,N$ and $\overrightarrow{p}=(p_1,p_2,...,p_N)$, and

$$p_-=\min\{p_i,\ 1\leqslant i\leqslant N\},\quad p_+=\max\{p_i,\ 1\leqslant i\leqslant N\},$$
$$\mathbb{L}^{\overrightarrow{p}}(\Omega)=\prod_{i=1}^N L^{p_i}(\Omega),\quad X^{1,\overrightarrow{p}}(\Omega)=\{v\in L^{p_+}(\Omega)\mid Dv\in\mathbb{L}^{\overrightarrow{p}}(\Omega)\},$$

with the norm $|v|_{X^{1,\overrightarrow{p}}(\Omega)} = \sum_{i=1}^{N}\left(|v|_{L^{p_i}} + |D_i v|_{L^{p_i}}\right)$, $D_i v = \dfrac{\partial v}{\partial x_i}\cdot$

Let $\mathscr{D}(\Omega)$ be the space of real indefinitely differentiable functions of compact support in Ω. We introduce the closure of $\mathscr{D}(\Omega)$ with respect to the above norm:

$$\overline{\mathscr{D}(\Omega)} = X_0^{1,\overrightarrow{p}}(\Omega) = \left\{v \in X^{1,\overrightarrow{p}}(\Omega) \mid v = 0 \quad \text{on} \quad \partial\Omega\right\} \subset W_0^{1,p_-}(\Omega).$$

The norm on this space is $|v|_{X_0^{1,\overrightarrow{p}}(\Omega)} = \sum_{i=1}^{N}\left(\int_\Omega |D_i v|^{p_i}\,dx\right)^{1/p_i}$ and its dual is denoted by $\left(X_0^{1,\overrightarrow{p}}(\Omega)\right)'$. The overline on a space shall denote the closure with respect to a given norm. We define also

$$X = \mathbb{L}^{\overrightarrow{p}}(0,T;X_0^{1,\overrightarrow{p}}(\Omega)) = \Big\{v : [0,T] \to X_0^{1,\overrightarrow{p}}(\Omega) \quad \text{measurable and} \\ \sum_{i=1}^{N}\left(\int_0^T\int_\Omega |D_i v|^{p_i}\,dxdt\right)^{1/p_i} < \infty\Big\}.$$

The norm on this space is $|v|_X = \sum_{i=1}^{N}\left(\int_0^T\int_\Omega |D_i v|^{p_i}\,dxdt\right)^{1/p_i}$.

We note that

$$\mathbb{L}^{\overrightarrow{p}}(0,T;X_0^{1,\overrightarrow{p}}(\Omega)) \subset L^{p_-}(0,T;W_0^{1,p_-}(\Omega)).$$

The above norms $|\cdot|_{X^{1,\overrightarrow{p}}}$, $|\cdot|_{X_0^{1,\overrightarrow{p}}}$ are equivalent on $X_0^{1,\overrightarrow{p}}(\Omega)$. Next we define the following norm on $\mathscr{D}(\Omega)$, for i,

$$||v||_i = |v|_{L^{p_i}(\Omega)} + |D_i v|_{L^{p_i}(\Omega)},$$

we set

$$W_{x_i,0}^{1,p_i}(\Omega) = \overline{\mathscr{D}}(\Omega)^{||\cdot||_i}.$$

Then one has the following lemma.

LEMMA 1.1. *There is $c_i > 0$ such that:*

$$|\varphi|_{L^{p_i}} \leqslant c_i |D_i\varphi|_{L^{p_i}}, \ \ \forall\, \varphi \in W_{x_i,0}^{1,p_i}(\Omega).$$

Proof. Indeed, let $\varphi \in \mathscr{D}(\Omega)$ and $x \in \operatorname{supp}(\varphi)$, then

$$\varphi(x) = \int_{-\infty}^{x_i} D_i\varphi(x_1,x_2,\ldots,x_{i-1},t,x_{i+1},\ldots,x_N)dt.$$

Let $\hat{x}_i(t) = (x_1,x_2,\ldots,x_{i-1},t,x_{i+1},\ldots,x_N)$, and $a > 0$ be such that $\operatorname{supp}(\varphi) \subset [-a,a]^N = R_a$, we have

$$\varphi(x) = \int_{-a}^{x_i} D_i\varphi(\hat{x}_i(t))dt,$$

then

$$|\varphi(x)| \leqslant \int_{-a}^{x_i} |D_i\varphi(\hat{x}_i(t))|dt \leqslant \int_{-a}^{a} |D_i\varphi(\hat{x}_i(t))|dt$$
$$\leqslant (2a)^{1-\frac{1}{p_i}} \left(\int_{-a}^{a} |D_i\varphi(\hat{x}_i(t))|^{p_i} dt \right)^{\frac{1}{p_i}},$$

so that

$$|\varphi(x)|^{p_i} \leqslant C_i \int_{-a}^{a} |D_i\varphi(\hat{x}_i(t))|^{p_i}\, dt.$$

We integrate on R_a to get

$$\int_{R_a} |\varphi(x)|^{p_i}\, dx \leqslant c_i \int_{R_a} |D_i\varphi|^{p_i}\, dx,$$

so that

$$\int_{\Omega} |\varphi(x)|^{p_i}\, dx \leqslant c_i \int_{\Omega} |D_i\varphi|^{p_i}\, dx, \ \ \forall\varphi \in \mathscr{D}(\Omega).$$

With the density of $\mathscr{D}(\Omega)$ in $W^{1,p_i}_{x_i,0}(\Omega)$, we have

$$\int_{\Omega} |\varphi(x)|^{p_i}\, dx \leqslant c_i \int_{\Omega} |D_i\varphi|^{p_i}\, dx, \ \ \forall\varphi \in W^{1,p_i}_{x_i,0}(\Omega),$$

from which we derive

$$\int_{Q} |\varphi(x)|^{p_i}\, dx \leqslant c_i \int_{Q} |D_i\varphi|^{p_i}\, dx, \ \ \forall\varphi \in W^{1,p_i}_{x_i,0}(\Omega). \tag{1}$$

Such inequality has been already used in a frame of elliptic anisotropic problem (see for instance [13]). We shall denote by C or c_j various constants depending only on the structure of $\widehat{a}$, μ, μ_0, Ω and T, for $j \in \mathbb{N}$. If necessary, the index might specify the dependence with respect to a variable.

COROLLARY 1.1. *There hold true:*

$$\mathbb{L}^{\overrightarrow{p}}(0,T;X_0^{1,\overrightarrow{p}}(\Omega)) = \bigcap_{i=1}^{N} L^{p_i}(0,T;W^{1,p_i}_{x_i,0}(\Omega)),$$

$$X_0^{1,\overrightarrow{p}}(\Omega) = \bigcap_{i=1}^{N} W^{1,p_i}_{x_i,0}(\Omega).$$

2. The case $1+\frac{N}{N+1} < p_i < \frac{\overline{p}(N+1)}{N}$ and $F = 0$

In this section we precise the notion of weak solution of problem (P). We use a sequence of problems (P_n) and we show the existence of solutions (u_n) of (P_n).

2.1. Notations and hypotheses

Let T be a real positive number, Ω an open bounded set of $\mathbb{R}^N$, and let the cylinder $Q=(0,T)\times\Omega$. We denote by $\mathscr{M}(\Omega)$ the set of bounded Radon measures on Ω, and $\mathscr{M}(Q)$ the space of bounded Radon measures on Q. We suppose that $\mu_0\in\mathscr{M}(\Omega)$, $\mu\in\mathscr{M}(Q)$, and the real numbers p_i satisfy

$$\frac{\overline{p}(N+1)}{N} > p_i > 2-\frac{1}{N+1}, \quad i=1,\ldots,N.$$

We define the sign function as:

$$\operatorname{sign}(\sigma)=\begin{cases}1, & \text{if } \sigma>0,\\ 0, & \text{if } \sigma=0,\\ -1, & \text{if } \sigma<0.\end{cases}$$

Putting $K=((0,T)\times\partial\Omega)\cup(\Omega\times\{T\})$, we denote $\mathscr{D}(\mathbb{R}^{N+1},K)$ the set of functions defined by

$$\mathscr{D}(\mathbb{R}^{N+1},K)=\{\varphi\in\mathscr{D}(\mathbb{R}^{N+1}) \mid \exists \text{ a neighborhood } V \text{ of } K \text{ s. t. } \varphi=0 \text{ in } V\}.$$

For convenience for the reader, we start with the case $F=0$.

DEFINITION 2.1. A function u is a *weak solution* of problem (P) if:

$$\begin{aligned}&u\in L^1(0,T;W_0^{1,1}(\Omega)),\ \widehat{a}(t,x,u,Du)\in(L^1(Q))^N \text{ and}\\ &-\int_Q u\partial_t\varphi\,dxdt-\int_\Omega\varphi(0,x)\,d\mu_0+\int_Q\widehat{a}(t,x,u,Du)D\varphi\,dxdt\\ &\qquad\qquad=\int_Q\varphi(t,x)\,d\mu, \quad \forall\varphi\in\mathscr{D}(\mathbb{R}^{N+1},K).\end{aligned}$$

Our main result is the following.

THEOREM 2.1. *Let*

$$1+\frac{N}{N+1}<p_i<\frac{\overline{p}(N+1)}{N}, \ \ i=1,\ldots,N,$$

with $\overline{p}\leqslant N+\frac{N}{N+1}$, *and* $\widehat{a}$ *an operator which verifies* $(\widehat{a}.1)$-$(\widehat{a}.3)$. *Then the problem* (P) *has at least one weak solution* $u\in\mathbb{L}^{\overrightarrow{q}}(0,T;X_0^{1,\overrightarrow{q}}(\Omega))$ *for all* $q_i\in[1,\frac{p_i}{\overline{p}}(\overline{p}-\frac{N}{N+1}))$, *with* $\overrightarrow{q}=(q_1,q_2,\ldots,q_N)$.

Notice that the regularity of u given in the previous Theorem 2.1 guarantees that $\widehat{a}(t,x,u,Du)\in(L^1(Q))^N$.

The proof of this Theorem needs several steps: First, we approximate the problem (P) with sequence of problems (P_n) having smooth solutions (u_n). Then, after deriving uniform estimates on u_n, we pass to the limit using a compactness results as in [16].

2.2. Approximation of (P)

Let (μ_{0n}) (resp. (μ_n)) be a sequence of $\mathscr{D}(\Omega)$ (resp. $\mathscr{D}(Q)$) which converges to μ_0 (resp. μ) in $\mathscr{D}'(\Omega)$ (resp. $\mathscr{D}'(Q)$) and which verifies the inequalities

$$\|\mu_{0n}\|_{L^1(\Omega)} \leqslant \|\mu_0\|_{\mathscr{M}(\Omega)} \quad \text{and} \quad \|\mu_n\|_{L^1(Q)} \leqslant \|\mu\|_{\mathscr{M}(Q)}, \quad \forall n \geqslant 1.$$

We define the problems (P_n) by:

$$(P_n) \qquad \begin{cases} \partial_t u_n - \operatorname{div}(\widehat{a}(t,x,u_n,Du_n)) = \mu_n \text{ in } Q, \\ u_n(0,x) = \mu_{0n}(x) \text{ in } \Omega, \\ u_n = 0 \text{ on } (0,T)\times\Omega. \end{cases}$$

The existence of a solution u_n in $X \cap L^\infty(0,T;L^2(\Omega))$ of the problem (P_n) is classical, see for instance [4] and [14]. The regularity of $u \in C\big([0,T];L^2(\Omega)\big)$ can be justified using approximation argument as below in Lemma 2.2.

Remarks.

1. Since $p_i > 1 + \frac{N}{N+1}$, then

$$1 + \frac{N}{N+1} - \frac{2N}{N+2} = \frac{3N+2}{(N+1)(N+2)} > 0,$$

which implies $W_0^{1,p_i}(\Omega) \subset L^2(\Omega), \forall i \in \{1,2,...,N\}$. With the definition of $X_0^{1,\overrightarrow{p}}(\Omega)$, we also have

$$X_0^{1,\overrightarrow{p}}(\Omega) \subset W_0^{1,p_-}(\Omega) \subset L^2(\Omega),$$

with continuous and dense imbedding.

2. The solution u_n of problem (P_n) is in $L^{\overline{p}}(Q)$, this because, there exists $j \in \{1,...,N\}$ so that $p_j \geqslant \overline{p}$. Then $u_n \in L^{\overline{p}}(Q)$.

3. The operator A maps X into $\sum\limits_{i=1}^{N} L^{p_i'}(0,T;(W_{x_i,0}^{1,p_i}(\Omega))')$, $p_i' = p_i/(p_i-1)$. In fact, if for $u \in X$, we put

$$Au = -\operatorname{div}(\widehat{a}(x,t,u,Du)),$$

we get

$$\begin{aligned} \|Au\|_{X'} &= \sup_{\substack{\varphi\in X \\ \|\varphi\|\leqslant 1}} \Big| \sum_{i=1}^{N} \int_Q a_i(t,x,u,Du) D_i\varphi \, dxdt \Big| \\ &\leqslant \beta \sup_{\substack{\varphi\in X \\ \|\varphi\|\leqslant 1}} \sum_{i=1}^{N} \int_Q \Big(|g| + |u|^{\overline{p}} + \sum_{j=1}^{N} |D_j u|^{p_j} \Big)^{1-\frac{1}{p_i}} |D_i\varphi| \, dxdt. \end{aligned}$$

And, by using Hölder's inequality we obtain

$$\begin{aligned} \|Au\|_{X'} &\leqslant \beta \sup_{\substack{\varphi\in X \\ \|\varphi\|\leqslant 1}} \sum_{i=1}^{N} \Big(\int_Q \Big(|g| + |u|^{\overline{p}} + \sum_{j=1}^{N} |D_j u|^{p_j} \Big) dxdt \Big)^{1/p_i'} \Big(\int_Q |D_i\varphi|^{p_i} dxdt \Big)^{1/p_i} \\ &\leqslant \beta \sum_{i=1}^{N} \Big(\int_Q \Big(|g| + |u|^{\overline{p}} + \sum_{j=1}^{N} |D_j u|^{p_j} \Big) dxdt \Big)^{1/p_i'}. \end{aligned}$$

We deduce, in particular, that the solution $u_n \in X$ of (P_n) satisfies

$$\partial_t u_n \in \sum_{i=1}^{N} L^{p_i'}(0,T;(W_{x_i,0}^{1,p_i}(\Omega))') + L^1(Q),$$

where $L^{p_i'}\Big(0,T;\big(W_{x_i,0}^{1,p_i}(\Omega)\big)'\Big)$ denotes the dual of $L^{p_i}\big(0,T;W_{x_i,0}^{1,p_i}(\Omega)\big)$.

LEMMA 2.1. *We have:*

$$Au \in \sum_{i=1}^{N} L^{p_i'}\Big(0,T;\big(W_{x_i,0}^{1,p_i}(\Omega)\big)'\Big) \text{ for } u \in X.$$

We shall use the following integration by parts to derive uniform estimates on u_n.

LEMMA 2.2. *Let* $v \in X$ *with* $\partial_t v \in \sum_{i=1}^{N} L^{p_i'}(0,T;(W_{x_i,0}^{1,p_i}(\Omega))') + L^1(Q)$ *and* $v \in C\big([0,T];L^1(\Omega)\big)$. *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *a Lipschitz bounded function, with* $\phi(0) = 0$. *Then, we have in* $\mathscr{D}'(0,T)$

$$\begin{aligned}\int_0^t \langle v'(\sigma), \phi(v(\sigma))\rangle\, d\sigma = &\int_\Omega dx \int_0^{v(t,x)} \phi(\sigma)\, d\sigma \\ &- \int_\Omega dx \int_0^{v(0,x)} \phi(\sigma)\, d\sigma, \quad \forall t \in [0,T],\end{aligned}$$

where the brackets $\langle \cdot, \cdot \rangle$ *denote a duality product between* $(X_0^{1,\overrightarrow{p}}(\Omega))' + L^1(\Omega)$ *and* $X_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$, $p_i' = p_i/(p_i - 1)$, $i = 1,\ldots,N$.

Before giving the proof of this Lemma, we note that it extends a result given in [16] and recall that if f is a function in $L^\gamma(0,T;L^\gamma(\Omega))$ for $\gamma \geqslant 1$, the sequence (f_n) defined by

$$f_n(t,x) = n \int_t^{t+\frac{1}{n}} f(s,x)ds, \quad t \in \Big[0, T - \frac{1}{n}\Big]$$

verifies, for all $0 \leqslant t_0 \leqslant t_1 \leqslant T - \frac{1}{n}$, the inequality

$$\int_{t_0}^{t_1} \int_\Omega |f_n(t,x)|^\gamma \, dxdt \leqslant \int_{t_0}^{T} \int_\Omega |f(t,x)|^\gamma \, dxdt, \quad \forall \gamma \geqslant 1,$$

which implies strong convergence of the sequence (f_n) in $L^\gamma(0,T-\eta;L^\gamma(\Omega))$ to f for all $\gamma \geqslant 1$ and $\eta \geqslant \frac{1}{n}$.

Proof. [Proof of Lemma 2.2] Fix $t' \in [0,T]$. Since the function

$$[0,T] \ni t \longmapsto \int_{t'}^{t} \langle v'(\sigma), \Phi(v(\sigma))\rangle\, d\sigma \in \mathbb{R}$$

is in $W^{1,1}(0,T)\subset C(0,T)$, it is enough to show that for all $\eta>0$, we have in $\mathscr{D}'(0,T-\eta)$,

$$\int_{t'}^{t}\langle v'(\sigma),\phi(v(\sigma))\rangle d\sigma=\int_\Omega dx\int_0^{v(t,x)}\phi(\sigma)d\sigma-\int_\Omega dx\int_0^{v(t',x)}\phi(\sigma)d\sigma.$$

For a function $v\in X$ such that $v'\in L^1(0,T;L^1(\Omega))$ the result is classical. Let $v\in X\cap C\big([0,T];L^1(\Omega)\big)$, with

$$v'=\sum_{i=1}^{N}w^i+h,\quad w^i\in L^{p_i'}(0,T;(W^{1,p_i}_{x_i,0}(\Omega))'),\quad \text{and}\quad h\in L^1(Q).$$

Consider the sequence (v_n) defined by

$$v_n(t)=n\int_t^{t+1/n}v(s)\,ds,\quad v\in\bigcap_{i=1}^{N}L^{p_i}(0,T-\eta,W^{1,p_i}_{x_i,0}(\Omega)),\ \eta\geqslant\frac{1}{n}\cdot$$

We see that the sequence (v_n) converges to v in $\bigcap_{i=1}^{N}L^{p_i}(0,T-\eta,W^{1,p_i}_{x_i,0}(\Omega))$ strongly, $v_n'\in L^1(0,T-\eta,L^1(\Omega))$ with

$$v_n'=\sum_{i=1}^{N}w_n^i+h_n,\ w_n^i(t)=n\int_t^{t+1/n}w^i(s)\,ds\quad\text{and}\quad h_n(t)=n\int_t^{t+1/n}h(s)\,ds.$$

Moreover, for all $i=1,\dots,N$ we have:

$$\begin{gathered}w_n^i\to w^i\quad\text{strongly in}\quad L^{p_i'}(0,T-\eta;(W^{1,p_i}_{x_i,0}(\Omega))'),\\ h_n\to h\quad\text{strongly in}\quad L^1(0,T-\eta,L^1(\Omega)).\end{gathered}$$

In fact, we have

$$|D_iv_n(t)-D_iv(t)|\leqslant n\int_t^{t+1/n}|D_iv(s)-D_iv(t)|\,ds,$$

then, with the Hölder inequality, we get

$$|D_iv_n(t)-D_iv(t)|^{p_i}\leqslant n^{p_i}\Big(\frac{1}{n}\Big)^{p_i-1}\int_t^{t+1/n}|D_iv(s)-D_iv(t)|^{p_i}\,ds.$$

Integrating first on Ω and after on the interval $[0,T-\eta]$, we obtain

$$\begin{aligned}&\int_0^{T-\eta}\int_\Omega|D_iv_n(t)-D_iv(t)|^{p_i}\,dxdt\\&\qquad\leqslant n\int_0^{T-\eta}\int_t^{t+1/n}\Big(\int_\Omega|D_iv(s)-D_iv(t)|^{p_i}dx\Big)\,dsdt.\end{aligned}$$

Remark that the sequence of functions

$$[0,T-\eta] \ni t \mapsto n\int_t^{t+1/n}\Big(\int_\Omega |D_i v(s) - D_i v(t)|^{p_i}\,dx\Big)\,ds$$

are in $L^1(0,T-\eta)$ and converges strongly in this space to 0 for all $i=1,\dots,N$. Therefore, we have

$$\|v_n - v\|_X \to 0 \quad \text{as} \quad n\to+\infty.$$

For the convergence of w_n^i in $L^{p_i'}(0,T-\eta;(W^{1,p_i}_{x_i,0}(\Omega))')$, we write

$$\begin{aligned}
\|w_n^i(t) - w^i(t)\|_{(W^{1,p_i}_{x_i,0}(\Omega))'} &= \sup_{\substack{\varphi\in W^{1,p_i}_{x_i,0}(\Omega)\\ \|\varphi\|_i\leqslant 1}} |\langle w_n^i(t) - w^i(t), \varphi(t)\rangle| \\
&\leqslant n \sup_{\substack{\varphi\in W^{1,p_i}_{x_i,0}(\Omega)\\ \|\varphi\|_i\leqslant 1}} \int_t^{t+1/n} |\langle w^i(s) - w^i(t), \varphi(t)\rangle|\,ds \\
&\leqslant n\int_t^{t+1/n} \|w^i(s) - w^i(t)\|_{(W^{1,p_i}_{x_i,0}(\Omega))'}\,ds.
\end{aligned}$$

Since $w^i\in L^{p_i'}(0,T-\eta;(W^{1,p_i}_{x_i,0}(\Omega))')$, arguing as before, we deduce that

$$w_n^i \to w^i \quad \text{strongly in} \quad L^{p_i'}(0,T-\eta;(W^{1,p_i}_{x_i,0}(\Omega))'),\ i=1,\dots,N.$$

For the convergence of (h_n) to h, we use the following inequality and conclude as before

$$\int_0^{T-\eta}\int_\Omega |h_n(t) - h(t)|\,dxdt \leqslant n\int_0^{T-\eta}\int_t^{t+1/n}\int_\Omega |h(s)-h(t)|\,dxdsdt.$$

To end, writing for all $t,t'\in[0,T-\eta]$,

$$\begin{aligned}
&\sum_{i=1}^N \int_{t'}^t \langle w_n^i(\sigma), \Phi(v_n(\sigma))\rangle d\sigma + \int_{t'}^t d\sigma \int_\Omega h_n(\sigma,x)\Phi(v_n(\sigma,x))\,dx \\
&\quad = \int_{t'}^t \langle v_n'(\sigma), \Phi(v_n(\sigma))\rangle d\sigma = \int_\Omega dx\int_0^{v_n(t,x)}\Phi(\sigma)d\sigma - \int_\Omega dx\int_0^{v_n(t',x)}\Phi(\sigma)d\sigma,
\end{aligned}$$

for subsequence of (v_n), denoted in the same way, which converges to v a.e. in $(0,T-\eta)\times\Omega$, and using a standard argument, we get the desired result using the fact that $v\in C\big([0,T],L^1(\Omega)\big)$.

2.3. Uniform estimates

LEMMA 2.3. *If $p_i > 1+\frac{N}{N+1}$, $i=1,\dots,N$, the sequence (u_n) of solutions of problems (P_n) remains in a bounded set of $L^\infty(0,T;L^1(\Omega))$.*

Proof. For fixed $\nu > 0$, we define the function S_ν for all $\sigma \in \mathbb{R}$ by

$$\mathrm{S}_\nu(\sigma) = \begin{cases} \operatorname{sign}\sigma, & \text{if } |\sigma| > \nu, \\ \frac{\sigma}{\nu}, & \text{if } |\sigma| \leqslant \nu. \end{cases}$$

Taking $\mathrm{S}_\nu(u_n)$ as a test function in (P_n) and integrating over the interval $[0,t] \subset [0,T]$, we find:

$$\int_0^t \langle \partial_t u_n, \mathrm{S}_\nu(u_n)\rangle\, dt + \int_0^t \int_\Omega \widehat{a}(t,x,u_n,Du_n) Du_n \mathrm{S}'_\nu(u_n)\, dxdt = \int_0^t \int_\Omega \mu_n \mathrm{S}_\nu(u_n)\, dxdt.$$

Using Lemma 2.2, the fact that $\mathrm{S}'_\nu \geqslant 0$ and $|\mathrm{S}_\nu| \leqslant 1$, we obtain, after dropping the nonnegative term,

$$\begin{aligned} \int_\Omega \int_0^{u_n(t,x)} \mathrm{S}_\nu(\sigma)\, d\sigma dx &\leqslant \|\mu_n\|_{L^1(Q)} + \int_\Omega \int_0^{u_n(0,x)} \mathrm{S}_\nu(\sigma)\, d\sigma dx \\ &\leqslant \|\mu_n\|_{L^1(Q)} + \|\mu_{0n}\|_{L^1(\Omega)}. \end{aligned} \tag{2}$$

To go further, we notice that for any $\alpha \in \mathbb{R}$

$$\int_0^\alpha \mathrm{S}_\nu(\sigma)\, d\sigma = \begin{cases} \frac{\alpha^2}{2\nu}, & \text{if } |\alpha| \leqslant \nu, \\ -\frac{\nu}{2} + |\alpha|, & \text{if } |\alpha| > \nu, \end{cases} \quad \text{and} \quad \lim_{\nu\to 0} \int_0^\alpha \mathrm{S}_\nu(\sigma)\, d\sigma = |\alpha|.$$

Now, letting ν goes to 0 in relation (2) and using the Lebesgue's dominated convergence theorem, we have $\forall t \in [0,T]$,

$$\int_\Omega |u_n(t,x)|\, dx \leqslant \|\mu_n\|_{L^1} + \|\mu_{0n}\|_{L^1} \leqslant \|\mu\|_{\mathscr{M}(Q)} + \|\mu_0\|_{\mathscr{M}(\Omega)} = c_0.$$

This finishes the proof of the Lemma.

In the sequel, if $\overrightarrow{r} = (r_1, r_2, ..., r_N)$, we write $\overrightarrow{r} > 1$ to mean that $r_i > 1$, $i = 1, ..., N$.

LEMMA 2.4. *Let p_i and q_i be such that*

$$1 + \frac{N}{N+1} < p_i < \frac{\overline{p}(N+1)}{N} \ \textit{and}\ \ q_i \in \Big[1, \frac{p_i}{\overline{p}}\big(\overline{p} - \frac{N}{N+1}\big)\Big),\ i = 1, \ldots, N,$$

with $\overline{p} \leqslant N + \frac{N}{N+1}$, where $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i}$. Then:

- *(u_n) remains in a bounded set of $L^{\overline{q}}(Q)$, where $\frac{1}{\overline{q}} = \frac{1}{N}\sum_{i=1}^N \frac{1}{q_i}$,*
- *$(D_i u_n)$ remains in a bounded set of $L^{q_i}(Q)$.*

Proof. We can assume that $\frac{q_i}{p_i} = \frac{\overline{q}}{\overline{p}}$. If not, we set $\theta = \max\{\frac{q_i}{p_i}, i = 1, \ldots, N\}$ and replace q_i by θp_i. Observe that, since $\theta p_i \geqslant q_i$, the fact that $(D_i u_n)$ remains in a bounded set of $L^{\theta p_i}(Q)$ implies the result.

From now on, we set $q_i = \theta p_i$, $\theta = \frac{\overline{q}}{\overline{p}}$ and $\theta \in [\frac{1}{p_+}, \frac{1}{\overline{p}}(\overline{p} - \frac{N}{N+1}))$.

To carry on the proof, we need the following Lemmas 2.5, 2.6,..., 2.10.

LEMMA 2.5. *Let $\overline{q}$ be the harmonic of q_i, i.e., $\frac{1}{\overline{q}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{q_i}$. Then:*

1. $\overline{q} < N$ provided that $\overline{p} \leqslant N + \frac{N}{N+1}$,

2. setting $\overline{q}^* = \frac{N\overline{q}}{N-\overline{q}}$ there exists a constant $C > 0$ such that

$$\int_Q |u_n(x,t)|^d dxdt \leqslant C\int_0^T ||u_n(\cdot,t)||^{\overline{q}}_{L^{\overline{q}^*}(\Omega)} dt,$$

where $d = \overline{q}\frac{N+1}{N}\cdot$

Proof. [Proof of Lemma 2.5] First, note that $\overline{q}^* = \frac{N\overline{q}}{N-\overline{q}} > 1$. Second, using the interpolation inequality and the Lemma 2.3, we have

$$\begin{aligned}\|u_n(.,t)\|_{L^d(\Omega)} &\leqslant \|u_n(.,t)\|^{1-\tau}_{L^1(\Omega)}\|u_n(.,t)\|^{\tau}_{L^{\overline{q}^*}(\Omega)}\\ &\leqslant C\|u_n(.,t)\|^{\tau}_{L^{\overline{q}^*}(\Omega)},\ \ \tau = \frac{1-d}{1-\overline{q}^*}\frac{\overline{q}^*}{d}\cdot\end{aligned} \tag{3}$$

Choosing $d = \overline{q}\frac{N+1}{N}$, we see that

$$\tau = \frac{N}{N+1} < 1 \quad \text{and} \quad \frac{\overline{q}^*(1-d)}{1-\overline{q}^*} = \overline{q}.$$

Now, integrating (3) on $[0,T]$, we obtain

$$\int_Q |u_n(x,t)|^d\, dxdt \leqslant C\int_0^T \|u_n(.,t)\|^{d\tau}_{L^{\overline{q}^*}(\Omega)} dt = C\int_0^T \|u_n(.,t)\|^{\overline{q}}_{L^{\overline{q}^*}(\Omega)} dt. \tag{4}$$

LEMMA 2.6. *There exists a constant $C > 0$ depending only $\overrightarrow{q}$ and N such that for all $v \in \mathbb{L}^{\overrightarrow{q}}(0,T;X_0^{1,\overrightarrow{q}}(\Omega))$ there holds true:*

$$\|v\|_{L^{\overline{q}}(0,T;L^{\overline{q}^*}(\Omega))} \leqslant C\prod_{i=1}^{N}\Big(\int_0^T\int_\Omega |D_i v|^{q_i}\, dxdt\Big)^{\frac{1}{q_i N}}.$$

Proof. [Proof of Lemma 2.6] The proof needs the following anisotropic Sobolev inequality.

LEMMA 2.7. (see [21]).) *Let $\overrightarrow{\alpha} = (\alpha_1,\ldots,\alpha_N) \geqslant 1$ and $u \in X_0^{1,\overrightarrow{\alpha}}(\Omega)$. Then*

$$\|u\|_{L^s(\Omega)} \leqslant C\Big(\prod_{i=1}^{N}\|D_i u\|_{L^{\alpha_i}(\Omega)}\Big)^{\frac{1}{N}}, \tag{5}$$

where $s = \overline{\alpha}^* = \frac{N\overline{\alpha}}{N-\overline{\alpha}}$ if $\overline{\alpha} < N$ with $\overline{\alpha}$ given by $\frac{1}{\overline{\alpha}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\alpha_i}$. The constant C depends on N and α_i, $i = 1,\ldots,N$. Furthermore, if $\overline{\alpha} \geqslant N$, the inequality (5) is true for all $s \geqslant 1$ and C depends on s and $|\Omega|$.

Let us apply this last Lemma with $\overrightarrow{\alpha}=\overrightarrow{q}$. We get

$$\int_0^T \Big(\int_\Omega |v|^{\overline{q}^*}\,dx\Big)^{\overline{q}/\overline{q}^*} dt \leqslant C\int_0^T \prod_{i=1}^N \Big(\int_\Omega |D_i v|^{q_i}\,dx\Big)^{\frac{\overline{q}}{q_i N}} dt.$$

The fact that $\sum_{i=1}^N \frac{\overline{q}}{q_i N}=\overline{q}\frac{1}{N}\sum_{i=1}^N \frac{1}{q_i}=1$ and the generalized Hölder inequality, lead to

$$\|v\|_{L^{\overline{q}}(0,T;L^{\overline{q}^*}(\Omega))} \leqslant C\prod_{i=1}^N \Big(\int_0^T \int_\Omega |D_i v|^{q_i}\,dxdt\Big)^{\frac{1}{q_i N}}.$$

This finishes the proof of Lemma 2.6.

LEMMA 2.8. *There exists a constant $C>0$ (independent of n) such that*

$$y_{ni}=\int_Q |D_i u_n|^{q_i} dxdt \leqslant C\Big(1+\int_Q |u_n|^d dxdt\Big)^{1-\theta},$$

with $d=\overline{q}\frac{N+1}{N}$, $\theta=\frac{\overline{q}}{\overline{p}}\cdot$

Proof. [Proof of Lemma 2.8] Since $\overline{p}>\overline{q}+\frac{N}{N+1}$, we have $\eta=d(\frac{1-\theta}{\theta})>1$. Let us choose as a test function in (P_n):

$$\Phi_n(u_n)=\int_0^{u_n}\frac{d\sigma}{(1+|\sigma|)^\eta}.$$

Then, using Lemma 2.3 and the fact that

$$|\Phi_\eta(u_n)|_{L^\infty(Q)} \leqslant \int_{-\infty}^{+\infty}\frac{d\sigma}{(1+|\sigma|)^\eta}<+\infty,$$

we deduce

$$\int_Q \frac{|D_i u_n|^{p_i}}{(1+|u_n|)^\eta}dxdt \leqslant c_\eta, \quad c_\eta=\text{constant independent of } n. \tag{6}$$

Next, writing

$$y_{ni}=\int_Q \frac{|D_i u_n|^{q_i}}{(1+|u_n|)^{\eta\theta}}(1+|u_n|)^{\eta\theta}$$

and using Hölder inequality, the inequality (6) gives

$$y_{ni}\leqslant C_\eta^\theta\Big(\int_Q (1+|u_n|)^{\eta\frac{\theta}{1-\theta}}\Big)^{1-\theta*}.$$

Since $\eta\frac{\theta}{1-\theta}=d$, we get the Lemma.

Next, we set $\mathbb{T}_n=\prod_{i=1}^N y_{ni}^{\frac{1}{q_i}}$. Then from Lemma 2.5 and Lemma 2.6, we have

LEMMA 2.9. *There exists a constant $C > 0$ (independent of n) such that*

$$\int_Q |u_n|^d dxdt \leqslant C\mathbb{T}_n^{\frac{\overline{q}}{N}}.$$

Proof. [Proof of Lemma 2.9] From Lemma 2.5,

$$\int_Q |u_n|^d dxdt \leqslant C\|u_n\|^{\overline{q}}_{L^{\overline{q}}(0,T;L^{\overline{q}^*}(\Omega))} \tag{7}$$

and Lemma 2.6, we get that

$$\|u_n\|^{\overline{q}}_{L^{\overline{q}}(0,T;L^{\overline{q}^*}(\Omega))} \leqslant C\mathbb{T}_n^{\frac{\overline{q}}{N}}. \tag{8}$$

The combination of these two relations, it gives the result.

LEMMA 2.10. *There exists a constant $C > 0$ (independent of n) such that*

$$\mathbb{T}_n \leqslant C, \quad \forall n \geqslant 1.$$

Proof. [Proof of Lemma 2.10] From Lemma 2.8, we have

$$\mathbb{T}_n = \prod_{i=1}^{N} y_{ni}^{\frac{1}{q_i}} \leqslant C^{\frac{N}{\overline{q}}}\left(1 + \int_Q |u_n|^d dxdt\right)^{\frac{(1-\theta)N}{\overline{q}}}. \tag{9}$$

Using Lemma 2.9, we have from (9)

$$\mathbb{T}_n \leqslant C\Big(1 + \mathbb{T}_n^{\frac{\overline{q}}{N}}\Big)^{\frac{(1-\theta)N}{\overline{q}}} \leqslant c(1 + \mathbb{T}_n^{1-\theta}). \tag{10}$$

Since $1-\theta < 1$, Lemma 2.10 follows from (10).

COROLLARY 2.1. (of Lemma 2.10) *There exists a constant $C > 0$ (independent of n) such that*

$$||u_n||_{L^{\overline{q}}(0,T;L^{\overline{q}*}(\Omega))} \leqslant C, \quad \forall n \geqslant 1.$$

Now, Lemmas 2.8, 2.9, 2.10 and its corollary imply the proof of Lemma 2.4.

Next we show that (u_n') is in a bounded set of $L^{r_-}(0,T;(X_0^{1,\overrightarrow{r}'}(\Omega))') + L^1(Q)$ for some $\overrightarrow{r} > 1$.

LEMMA 2.11. *Let*

$$r_i \in \left[\frac{1}{p_i - 1}, \frac{p_i}{(p_i-1)\overline{p}}(\overline{p} - \frac{N}{N+1})\right), \quad i = 1,\ldots,N.$$

We can choose $r_i > 1$, for

$$1 + \frac{N}{N+1} < p_i < \frac{\overline{p}(N+1)}{N}.$$

The sequence (u_n') remains in a bounded set of $L^{r_-}(0,T;(X_0^{1,\overrightarrow{r}'}(\Omega))') + L^1(Q)$, $\overrightarrow{r}' = (r_1', r_2', \ldots, r_N')$, r_i' is the conjugate r_i.

Proof. For all n we have

$$u_n' = \operatorname{div}(\widehat{a}(t,x,u_n,Du_n)) + \mu_n,$$

knowing that (μ_n) is in a bounded set of $L^1(Q)$, we have to show that

$$v_n = \operatorname{div}(\widehat{a}(t,x,u_n,Du_n))$$

belongs to a bounded set of $L^{r_-}(0,T;(X_0^{1,\overrightarrow{r}'}(\Omega))')$, with $\overrightarrow{r} > 1$. Indeed, setting for $t \in (0,T)$, $v_n(t) = v_n$, we have for $r_i > 1$ (given as in Lemma 2.11),

$$\begin{aligned}\|v_n\|_{(X_0^{1,\overrightarrow{r}'}(\Omega))'} &= \sup_{\substack{\varphi\in X_0^{1,\overrightarrow{r}'}(\Omega)\\ \|\varphi\|\leqslant 1}} \Big|\int_\Omega \sum_{i=1}^N a_i(t,x,u_n,Du_n)D_i\varphi dx\Big| \\ &\leqslant \beta \sup_{\substack{\varphi\in X_0^{1,\overrightarrow{r}'}(\Omega)\\ \|\varphi\|\leqslant 1}} \sum_{i=1}^N \int_\Omega \Big(|g| + |u_n|^{\overline{p}} + \sum_{j=1}^N |D_ju_n|^{p_j}\Big)^{1-\frac{1}{p_i}} |D_i\varphi| dx.\end{aligned}$$

By using Hölder inequality, we have

$$\begin{aligned}&\|v_n\|_{(X_0^{1,\overrightarrow{r}'}(\Omega))'} \\ &\leqslant \beta \sup_{\substack{\varphi\in X_0^{1,\overrightarrow{r}'}(\Omega)\\ \|\varphi\|\leqslant 1}} \sum_{i=1}^N \Big(\int_\Omega |D_i\varphi|^{r_i'}\Big)^{1/r_i'} \left(\int_\Omega \Big(|g| + |u_n|^{\overline{p}} + \sum_{j=1}^N |D_ju_n|^{p_j}\Big)^{(1-\frac{1}{p_i})r_i}\right)^{1/r_i} \\ &\leqslant \beta \sum_{i=1}^N \left(\int_\Omega \Big(|g| + |u_n|^{\overline{p}} + \sum_{j=1}^N |D_ju_n|^{p_j}\Big)^{(1-\frac{1}{p_i})r_i} dx\right)^{1/r_i},\end{aligned}$$

then

$$\|v_n\|^{r_-}_{(X_0^{1,\overrightarrow{r}'}(\Omega))'} \leqslant C \sum_{i=1}^N \left(\int_\Omega \Big(|g| + |u_n|^{\overline{p}} + \sum_{j=1}^N |D_ju_n|^{p_j}\Big)^{(1-\frac{1}{p_i})r_i} dx\right)^{r_-/r_i}. \tag{11}$$

We set

$$G_i(x,t) = \Big(|g| + |u_n|^{\overline{p}} + \sum_{j=1}^N |D_ju_n|^{p_j}\Big)^{(1-\frac{1}{p_i})r_i}(x,t).$$

Let σ be such that

$$\frac{r_i(p_i-1)}{p_i} < \sigma < 1 - \frac{1}{\overline{p}}\frac{N}{N+1} < 1.$$

This is possible since we have

$$1 < r_i < \frac{p_i}{(p_i-1)\overline{p}}(\overline{p} - \frac{N}{N+1}).$$

Then

$$\sigma p_i \in \left[1, \frac{p_i}{\overline{p}}(\overline{p} - \frac{N}{N+1})\right), \quad \frac{(p_i-1)r_i}{\sigma p_i} < 1 \text{ and } \sigma < 1,$$

we integrate relation (11) on $[0,T]$ and we apply Hölder inequality to derive

$$\int_0^T \|v_n\|^{r_-}_{(X_0^{1,\overrightarrow{r}'}(\Omega))'} dt \leqslant C \sum_{i=1}^N \left(\int_Q G_i(x,t)dxdt \right)^{\frac{r_-}{r_i}}. \tag{12}$$

Using the inequality that

$$\left(\sum a_i\right)^\sigma \leqslant \sum a_i^\sigma \quad \text{for } \sigma \in [0,1]$$

and writing $G_i = G_i^{\frac{\sigma}{\sigma}}$, from (12) and the Hölder inequality we deduce

$$\int_0^T \|v_n\|^{r_-}_{(X_0^{1,\overrightarrow{r}'}(\Omega))'} dt \leqslant C \sum_{i=1}^N \left(\int_Q \left(|g|^\sigma + |u_n|^{\overline{p}\sigma} + \sum_{j=1}^N |D_j u_n|^{\sigma p_j}\right) dxdt \right)^{\frac{(p_i-1)r_-}{\sigma p_i}}. \tag{13}$$

Using the inequality (13) and Lemma 2.4, we have

$$\int_0^T \|v_n\|^{r_-}_{(X_0^{1,\overrightarrow{r}'}(\Omega))'} dt \leqslant C.$$

LEMMA 2.12. *There exists a subsequence (still denoted by (u_n)) which converges a.e. to a function $u \in L^1(Q)$, and weakly in $\mathbb{L}^{\overrightarrow{q}}(0,T;X_0^{1,\overrightarrow{q}}(\Omega))$.*

Proof. Let $\overrightarrow{q} = (q_1, q_2, ..., q_N)$, the sequence (u_n) is in a bounded set of the space $\mathbb{L}^{\overrightarrow{q}}(0,T;X_0^{1,\overrightarrow{q}}(\Omega))$ for all $q_i \in [1, \frac{p_i}{\overline{p}}(\overline{p} - \frac{N}{N+1}))$ and (u'_n) remains in a bounded set of the space $L^{r_-}(0,T;(X_0^{1,\overrightarrow{r}'}(\Omega))') + L^1(Q)$. As

$$W_0^{1,r'_-}(\Omega) \subset X_0^{1,\overrightarrow{r}'}(\Omega), \; (r'_- \text{ the Hölder conjugate of } r_- \text{ and } \overrightarrow{r}' = (r'_1, ..., r'_N))$$

and the fact that the imbedding is continuous and dense, we see that

$$L^{r_-}(0,T;(X_0^{1,\overrightarrow{r}'}(\Omega))') + L^1(Q) \subset L^{r_-}(0,T;W^{-1,r_-}(\Omega)) + L^1(Q), \; r_- > 1.$$

As (u_n) remains in a bounded set of $\mathbb{L}^{\overrightarrow{q}}(0,T;X_0^{1,\overrightarrow{q}}(\Omega))$ and the sequence $(\partial_t u_n)$ remains in a bounded set of $L^{r_-}(0,T;W^{-1,r_-}(\Omega)) + L^1(Q)$, a result given in [19] shows that the sequence (u_n) converges strongly to a function u in $L^1(Q)$, it shows the existence of a subsequence (u_n) converging a.e. to u in Q.

Now we consider the following family of functions $(\Phi_k)_{k>0}$:

- Φ_k is a twice differentiable function, Φ'_k, Φ''_k are bounded on $\mathbb{R}$.
- $\Phi_k(\sigma) = \sigma$ if $|\sigma| \leqslant k$, and $\Phi'_k(\sigma) = 0$ if $|\sigma| \geqslant k + (1/k)$, $0 < \Phi'_k < 1$ on the interval $(k, k+(1/k)) \cup (-(k+(1/k)), -k)$.

The construction of this family $(\Phi_k)_{k>0}$ can be made explicitly. For example we have:

- for $\sigma \in [0,k]$, $\Phi_k(\sigma) = \sigma$;
- for $\sigma \in (k, k+1/(2k))$, $\Phi_k(\sigma) = l_1(\sigma) + l_2(\sigma) + l_3(\sigma)$, with

$$l_1(\sigma) = 1/2(k+1/(2k)-\sigma) - 2k^3(k+1/k)(\sigma-k) - 1/(2k),$$
$$l_2(\sigma) = 2k^2(k+1/k)[(\sigma+1/(2k))^2 - (k+1/(2k))^2] + k[(k+1/(2k))^2 - \sigma^2],$$
$$l_3(\sigma) = -2k^2/3[(\sigma+1/(2k))^3 - (k+1/(2k))^3];$$

- for $\sigma \in [k+1/(2k), k+1/k)$, $\Phi_k(\sigma) = L_1(\sigma) + L_2(\sigma) + L_3(\sigma)$, with

$$L_1(\sigma) = 2k(k+1/k)(\sigma-k-1/(2k)) - 2k^3(k+1/k)(k+1/k-\sigma) - 1/(2k),$$
$$L_2(\sigma) = 2k^2(k+1/k)[(k+1/k)^2 - \sigma^2],$$
$$L_3(\sigma) = -2k^2/3[(k+1/k)^3 - \sigma^3];$$

- for $\sigma \in [k+1/k, +\infty)$,

$\Phi_k(\sigma) = k + 1(2k)$;

- for $\sigma < 0$, we take $\Phi_k(\sigma) = -\Phi_k(-\sigma)$. Another construction of Φ_k can be also made by using a suitable convolution (personal communication by J. M. Rakotoson).

Taking $u_n^k = \Phi_k(u_n)$ and $u^k = \Phi_k(u)$, we obtain the following result.

LEMMA 2.13. *(u_n^k) remains in a bounded set of X and (u_n^k) converges a.e. to u^k as $n \to +\infty$, and weakly in X.*

Proof. Consider as a test function $\Psi(u_n(t,x)) = \sum_{j=1}^N \int_0^{u_n(t,x)} |\Phi_k'(\sigma)|^{p_j}\, d\sigma$. Using the Lemma 2.2, and the coercivity of A, we obtain

$$\alpha \sum_{i,j=1}^N \int_Q |D_i u_n|^{p_i} |\Phi_k'(u_n)|^{p_j}\, dxdt$$
$$\leqslant (\|\mu_n\|_{L^1(Q)} + \|\mu_{0n}\|_{L^1(\Omega)}) \sum_{i=1}^N \int_{-\infty}^{+\infty} \Phi_k'(\sigma)^{p_i} d\sigma \leqslant c_k.$$

So that the sequence (u_n^k) is in a bounded set of X, because

$$\sum_{i=1}^N \int_Q |D_i(\Phi_k(u_n))|^{p_i}\, dxdt \leqslant \frac{c_k}{\alpha}, \quad (c_k = \text{constant depending of } k).$$

Since the space X is reflexive ($\overrightarrow{p} > 1$ and finite), and u_n converges to u a.e. thus u_n^k converges to u^k a.e. and weakly in X. In particular we have

$$\lim_{n\to+\infty} Du_n^k = Du^k \quad \text{weakly in} \quad \prod_{i=1}^N L^{p_i}(Q).$$

LEMMA 2.14. *Let $\Phi \in C^2(\mathbb{R})$ with Φ, Φ' and Φ'' are bounded on $\mathbb{R}$, and let $v \in X$ so that $\partial_t v \in \sum_{i=1}^N L^{p_i'}(0,T;(W_{x_i,0}^{1,p_i}(\Omega))') + L^1(Q)$. Then*

$$\langle \partial_t \Phi(v), \varphi \rangle_{\mathscr{D}'(Q),\mathscr{D}(Q)} = \langle \partial_t v, \varphi \Phi'(v) \rangle, \quad \forall\, \varphi \in \mathscr{D}(Q),$$

where the last bracket of duality is between $\sum_{i=1}^N L^{p_i'}(0,T;(W_{x_i,0}^{1,p_i}(\Omega))') + L^1(Q)$ and X.

The proof of this Lemma can be like the proof of Lemma 2.2, or see [16]. As an application of the preceding Lemma 2.14, we have:

LEMMA 2.15. *For all $k > 0$, all n, $(u_n^k)'$ is in $\sum_{i=1}^N L^{p_i'}(0,T;(W_{x_i,0}^{1,p_i}(\Omega))') + L^1(Q)$ and we have the following equality in $\mathscr{D}'(Q)$,*

$$(u_n^k)' = \operatorname{div}\left((\Phi_k'(u_n)\widehat{a}(t,x,u_n,Du_n))\right) \\ -\widehat{a}(t,x,u_n,Du_n)Du_n\Phi_k''(u_n) + \mu_n\Phi_k'(u_n). \tag{14}$$

Proof. Let $\varphi \in \mathscr{D}(Q)$. With Lemma 2.14 for (Φ_k) and u_n we have

$$\langle \partial_t \Phi_k(u_n), \varphi \rangle_{\mathscr{D}'(Q),\mathscr{D}(Q)} = \langle \partial_t u_n, \varphi \Phi_k'(u_n) \rangle.$$

So we take $v = \varphi \Phi_k'(u_n)$ as test function in (P_n), we get

$$\langle \partial_t \Phi_k(u_n), \varphi \rangle_{\mathscr{D}'(Q),\mathscr{D}(Q)} = -\int_Q \widehat{a}(t,x,u_n,Du_n)D\varphi\Phi_k'(u_n)\,dxdt \\ -\int_Q \widehat{a}(t,x,u_n,Du_n)Du_n\varphi\Phi_k''(u_n)\,dxdt + \int_Q \mu_n\varphi\Phi_k'(u_n)\,dxdt$$

this is a relation with (14). We deduce from (14), the regularity of $(u_n^k)'$.

LEMMA 2.16. *For all $k > 0$, there exists a function θ_k such that for all $\varepsilon > 0$, we have*

$$\limsup_n \int_{\{|u_n - u^k| \leqslant \varepsilon\}} \widehat{a}(t,x,u_n,Du_n)(Du_n - Du^k)\,dxdt \leqslant \theta_k(\varepsilon)$$

with $\lim_{\varepsilon \to 0} \theta_k(\varepsilon) = 0$.

Proof. Let $\varepsilon > 0$ fixed. For $\varepsilon > 0$, let T_ε be the truncation at level $-\varepsilon$ and ε. This Lipschitz function satisfies $T_\varepsilon(0) = 0$ and

$$T_\varepsilon'(\sigma) = \begin{cases} 1, & |\sigma| \leqslant \varepsilon, \\ 0, & |\sigma| > \varepsilon. \end{cases}$$

For all m, n, we choose $v_{n,m}^k = T_\varepsilon(u_n - u_m^k)$ as a test function in (P_n), we have

$$\int_0^T \langle u_n', T_\varepsilon(u_n - u_m^k) \rangle\,dt \\ + \int_{|u_n - u_m^k| \leqslant \varepsilon} \widehat{a}(t,x,u_n,Du_n)(Du_n - Du_m^k)\,dxdt \leqslant \varepsilon \|\mu\|_{\mathscr{M}(Q)}. \tag{15}$$

We write,

$$\int_0^T \langle u_n', T_\varepsilon(u_n - u_m^k)\rangle\, dt = \int_0^T \langle u_n' - (u_m^k)', T_\varepsilon(u_n - u_m^k)\rangle\, dt$$
$$+ \int_0^T \langle (u_m^k)', T_\varepsilon(u_n - u_m^k)\rangle\, dt \equiv I_1 + I_2. \qquad (16)$$

The fact that u_m is in $X \cap C([0,T];L^1(\Omega))$ implies that the function u_m^k is in the same space. From Lemma 2.15, $(u_m^k)'$ is in $\sum_{i=1}^N L^{p_i'}(0,T;(W_{x_i,0}^{1,p_i}(\Omega))') + L^1(Q)$, with use of Lemma 2.2, we obtain

$$I_1 = \int_\Omega dx \int_0^{(u_n - u_m^k)(T,x)} T_\varepsilon(\sigma)\, d\sigma - \int_\Omega dx \int_0^{(u_n - u_m^k)(0,x)} T_\varepsilon(\sigma)\, d\sigma.$$

As (u_n) is bounded in $L^\infty(0,T,L^1(\Omega))$, we have

$$\left|I_1\right| \leqslant \varepsilon \int_\Omega (|u_n(0,x)| + |u_m^k(0,x)|)\, dx + \varepsilon \int_\Omega (|u_n(T,x)| + |u_m^k(T,x)|)\, dx \leqslant c\varepsilon. \qquad (17)$$

For I_2, using (14) we can write: $I_2 \equiv J_{1n}^m - J_{2n}^m - J_{3n}^m$, where

$$J_{1n}^m = \int_Q \Phi_k'(u_m)\mu_m T_\varepsilon(u_n - u_m^k)\, dxdt,$$
$$J_{2n}^m = \int_Q \Phi_k''(u_m)\widehat{a}(t,x,u_m,Du_m)Du_m T_\varepsilon(u_n - u_m^k)\, dxdt,$$
$$J_{3n}^m = \int_Q \Phi_k'(u_m)\widehat{a}(t,x,u_m,Du_m)D(T_\varepsilon(u_n - u_m^k))\, dxdt.$$

] • For estimating J_{1n}^m, we use the fact that the sequence (μ_m) is bounded in $L^1(Q)$, Φ_k' is bounded on $\mathbb{R}$ and $|T_\varepsilon| \leqslant \varepsilon$, so that we have

$$|J_{1n}^m| \leqslant c_k\varepsilon. \qquad (18)$$

- For J_{2n}^m, taking $b = k + 1/k$. With the definition of Φ_k and u_m^k, we can write

$$J_{2n}^m = \sum_{i=1}^N \int_Q \Phi_k''(u_m^b) a_i(t,x,u_m^b,Du_m^b) D_i u_m^b T_\varepsilon(u_n - u_m^k)\, dxdt$$
$$\leqslant c_k\varepsilon\beta \sum_{i=1}^N \int_Q \Big(|g| + |u_m^b|^{\overline{p}} + \sum_{j=1}^N |D_j u_m^b|^{p_j}\Big)^{1-\frac{1}{p_i}} |D_i u_m^b|\, dxdt$$
$$\leqslant c_k\varepsilon \sum_{i=1}^N \Big(\int_Q |D_i u_m^b|^{p_i}\, dxdt\Big)^{\frac{1}{p_i}} \Big(\int_Q \Big(|g| + |u_m^b|^{\overline{p}} + \sum_{j=1}^N |D_j u_m^b|^{p_j}\Big)\, dxdt\Big)^{\frac{1}{p_i'}}.$$

With Lemma 2.13, we get

$$|J_{2n}^m| \leqslant c_k\, \varepsilon. \qquad (19)$$

- For J_{3n}^m, we consider the following sets:

$$E_{n\varepsilon} = \{(t,x) \mid |u_n - u^k| \neq \varepsilon\}, \ \ E_{n\varepsilon}^c = \{(t,x) \mid |u_n - u^k| = \varepsilon\},$$
$$E_{nm} = \{(t,x) \mid |u_n - u_m^k| < \varepsilon\}, \ \ E_n = \{(t,x) \mid |u_n - u^k| < \varepsilon\}.$$

We write J_{3n}^m like $J_{3n}^m = J_{3n}^{m1} + J_{3n}^{m2}$, where

$$J_{3n}^{m1} = \int_{E_{nm}\cap E_{n\varepsilon}} \Phi_k'(u_m)\widehat{a}(t,x,u_m,Du_m)(Du_n - Du_m^k)\,dxdt,$$
$$J_{3n}^{m2} = \int_{E_{nm}\cap E_{n\varepsilon}^c} \Phi_k'(u_m)\widehat{a}(t,x,u_m,Du_m)(Du_n - Du_m^k)\,dxdt.$$

We begin with J_{3n}^{m1}, that we write as $J_{3n}^{m1} = J_{3n}^{m11} - J_{3n}^{m12}$, where:

$$J_{3n}^{m11} = \int_{E_{nm}\cap E_{n\varepsilon}} \Phi_k'(u_m)\widehat{a}(t,x,u_m,Du_m)Du_n\,dxdt$$

and

$$J_{3n}^{m12} = \int_{E_{nm}\cap E_{n\varepsilon}} \Phi_k'(u_m)\widehat{a}(t,x,u_m,Du_m)Du_m^k\,dxdt.$$

By using the properties of the functions Φ_k, we see that

$$J_{3n}^{m11} = \int_Q \Phi_k'(u_m)\widehat{a}(t,x,u_m^b,Du_m^b)Du_n\chi_{E_{nm}\cap E_{n\varepsilon}}(t,x)\,dxdt,$$

where χ_E denotes the characteristic function of a set E. With the Lemma 2.13, the sequence $(\widehat{a}(u_m^b,Du_m^b))_m$ remains in a bounded set of the space $\prod_{i=1}^N L^{p_i'}(Q)$.

We can extract a subsequence, still denoted $(\widehat{a}(u_m^b,Du_m^b))_m$ which converges weakly to a limit denoted M_k and $(a_i(u_m^b,Du_m^b))_m$ converges weakly to M_k^i in $L^{p_i'}(Q)$, $M_k = (M_k^1,...,M_k^N)$. The sequence (u_m) converges a.e. to u, we deduce that the sequence $(\Phi_k'(u_m)\chi_{E_{nm}\cap E_{n\varepsilon}})_m$ converges a.e on Q to $\Phi_k'(u)\chi_{E_n}$. So, $(Du_n\Phi_k'(u_m)\chi_{E_{nm}\cap E_{n\varepsilon}})_m$ converges strongly in $\prod_{i=1}^N L^{p_i}(Q)$ as $m\to+\infty$. So we can write

$$\lim_{m\to+\infty} J_{3n}^{m11} = \int_{E_n} \Phi_k'(u)M_kDu_n\,dxdt. \tag{20}$$

- For J_{3n}^{m12}, we need the following lemma.

LEMMA 2.17. *We have:*

$$\limsup_m \int_{E_{nm}\cap E_{n\varepsilon}} \Phi_k'(u_m)^2\widehat{a}(t,x,u_m^b,Du_m^b)Du_m^b\,dxdt \geqslant \int_{E_n} \Phi_k'(u)^2M_kDu\,dxdt.$$

Proof. Consider the following non-negative quantity:

$$\Delta(u_m^b,u^b)=\Phi_k'(u_m)^2(\widehat{a}(t,x,u_m^b,Du_m^b)-\widehat{a}(t,x,u_m^b,Du^b))(Du_m^b-Du^b).$$

This term is in $L^1(Q)$, because u_m^b and u^b are in X, (for that we use Young's inequality). We develop $\Delta(u_m^b,u^b)$, we see that

$$\begin{aligned}\Phi_k'(u_m)^2\widehat{a}(t,x,u_m^b,Du_m^b)Du_m^b \geqslant \Phi_k'(u_m)^2\Big\{&\widehat{a}(t,x,u_m^b,Du_m^b)Du^b\\ &+\widehat{a}(t,x,u_m^b,Du^b)Du_m^b-\widehat{a}(t,x,u_m^b,Du^b)Du^b\Big\}.\end{aligned}$$

By integrating over $E_{nm}\cap E_{n\varepsilon}$ and taking the $\limsup$ as m goes to infinity, we derive easily

$$\begin{aligned}\limsup_m \int_{E_{nm}\cap E_{n\varepsilon}} &\Phi_k'(u_m)^2\widehat{a}(t,x,u_m^b,Du_m^b)Du_m^b\,dxdt\\ &\geqslant \int_{E_n}\Phi_k'(u)^2M_kDu^b\,dxdt.\end{aligned}$$

We have used the convergence a.e. of (u_m) to u in Q, the weak convergence of $(\widehat{a}(t,x,u_m^b,Du_m^b))_m$ to M_k in $\prod_{i=1}^N L^{p_i'}(Q)$ and the weak convergence of (Du_m^b) to Du^b in $\prod_{i=1}^N L^{p_i}(Q)$. So we have the result.

For J_{3n}^{m12}, we can remark that $Du_m^k=\Phi_k'(u_m)Du_m^b$, J_{3n}^{m12} can be written as

$$J_{3n}^{m12}=\int_{E_{nm}\cap E_{n\varepsilon}}\Phi_k'(u_m)^2\widehat{a}(t,x,u_m^b,Du_m^b)Du_m^b\,dxdt.$$

With Lemma 2.17, we have

$$\limsup_m J_{3n}^{m12}\geqslant\int_{E_n}\Phi_k'(u)^2M_kDu\,dxdt.$$

As $\liminf(-\cdot)=-\limsup(\cdot)$, the equality (20) and the last inequality give

$$\begin{aligned}\liminf_m J_{3n}^{m1}&=\liminf_m(J_{3n}^{m11}-J_{3n}^{m12})\\ &\leqslant\liminf_m J_{3n}^{m11}-\limsup_m J_{3n}^{m12}\\ &\leqslant\int_{E_n}\Phi_k'(u)M_kDu_n\,dxdt-\int_{E_n}\Phi_k'(u)^2M_kDu\,dxdt.\end{aligned}\tag{21}$$

- For J_{3n}^{m2}, we write

$$\begin{aligned}J_{3n}^{m2}&=\int_{E_{nm}\cap E_{n\varepsilon}^c}\Phi_k'(u_m)\widehat{a}(t,x,u_m,Du_m)(Du_n-Du_m^k)\,dxdt\\ &=\int_{E_{nm}\cap E_{n\varepsilon}^c}\Phi_k'(u_m)\widehat{a}(t,x,u_m^b,Du_m^b)Du_n\,dxdt\\ &\quad-\int_{E_{nm}\cap E_{n\varepsilon}^c}\Phi_k'(u_m)^2\widehat{a}(t,x,u_m,Du_m)Du_m\,dxdt.\end{aligned}$$

We drop the non negative term and as $Du_n = Du^k$ a.e. on the set $E^c_{n\varepsilon}$, we obtain

$$J^{m2}_{3n} \leqslant \int_{E_{nm}\cap E^c_{n\varepsilon}} \widehat{a}(t,x,u^b_m,Du^b_m)\Phi'_k(u_m)Du^k\,dxdt.$$

Since the sequence $(a_i(t,x,u^b_m,Du^b_m))_m$ is in a bounded set of $L^{p'_i}(Q)$, and $E_{nm}\cap E^c_{n\varepsilon} \subseteq E^c_{n\varepsilon}$, the Hölder inequality yields

$$J^{m2}_{3n} \leqslant c_k \sum_{i=1}^{N} \Big[\int_{E^c_{n\varepsilon}} \Phi'_k(u_m)^{p_i}|D_i u^k|^{p_i}\,dxdt\Big]^{1/p_i}.$$

As the function Φ'_k is bounded, the pointwise convergence of (u_m) to u on Q and the Lebesgue's dominated convergence theorem imply

$$\liminf_m J^{m2}_{3n} \leqslant c_k \sum_{i=1}^{N} \Big[\int_{E^c_{n\varepsilon}} \Phi'_k(u)^{p_i}|D_i u^k|^{p_i}\,dxdt\Big]^{1/p_i}.$$

Now we can split J^m_{3n} as $J^m_{3n} = J^{m1}_{3n} + J^{m2}_{3n}$, so we have from (21) and the last inequality

$$\liminf_m J^m_{3n} \leqslant S^1_n + S^2_n, \tag{22}$$

where

$$S^1_n = \int_{E_n} \Phi'_k(u)M_k Du_n\,dxdt - \int_{E_n} \Phi'_k(u)^2 M_k Du\,dxdt,$$

$$S^2_n = c_k \sum_{i=1}^{N} \Big[\int_{E^c_{n\varepsilon}} \Phi'_k(u)^{p_i}|D_i u^k|^{p_i}\,dxdt\Big]^{1/p_i}.$$

We want show that $\limsup_n(\liminf_m J^m_{3n}) \leqslant \theta_k(\varepsilon)$ with $\lim_{\varepsilon\to 0}\theta_k(\varepsilon) = 0$. For that we write

$$\begin{aligned} S^1_n = &\int_{\{|u|\leqslant k\}\cap E_n} M_k(Du_n - Du^{b+1})\,dxdt \\ &+ \int_{\{|u|>k\}\cap E_n} \Phi'_k(u)M_k Du_n\,dxdt - \int_{\{|u|>k\}\cap E_n} \Phi'_k(u)^2 M_k Du^{b+1}\,dxdt, \end{aligned}$$

since $\Phi'_k \equiv 1$ on the interval $[-k,k]$, $\Phi'_k(\sigma) = 0$ for $|\sigma| \geqslant b$. So we have

$$\begin{aligned} S^1_n = &\int_{\{|u|\leqslant k\}\cap E_n} M_k(Du_n - Du^{b+1})\,dxdt \\ &+ \int_{\{|u|>k\}\cap E_n} \Phi'_k(u)M_k(Du_n - Du^{b+1})\,dxdt \\ &+ \int_{\{|u|>k\}\cap E_n} M_k Du^{b+1}(1-\Phi'_k(u))\Phi'_k(u)\,dxdt. \end{aligned}$$

As u_n converges a.e. to u on Q and Du^{b+1}_n, ($u_n = u^{b+1}_n$ on E_n, $\varepsilon \leqslant 1$), converge weakly to Du^{b+1} in $\prod_{i=1}^N L^{p_i}(Q)$, the two first integrals in the expression of S^1_n converge to 0 when $n \to +\infty$. While for the third integral we note that

$$S^{11}_n = \int_{\{k<|u|<b\}\cap E_n} M_k Du^{b+1}(1-\Phi'_k(u))\Phi'_k(u)\,dxdt, \quad \text{with } b = k+1/k.$$

Using Lemma 2.13 and Hölder inequality we get

$$
\begin{aligned}
|S_n^{11}| &\leqslant \sum_{i=1}^{N}\Big(\int_{\{k<|u|<b\}\cap E_n}|M_k^i|^{p_i'}dxdt\Big)^{1/p_i'}\cdot\Big(\int_{\{k<|u|<b\}\cap E_n}|D_iu^{b+1}|^{p_i}\,dxdt\Big)^{1/p_i}\\
&\leqslant \sum_{i=1}^{N}\Big(\int_{\{k<|u|<b\}\cap E_n}|M_k^i|^{p_i'}\,dxdt\Big)^{1/p_i'}\cdot\Big(\int_{Q}|D_iu^{b+1}|^{p_i}\,dxdt\Big)^{1/p_i}\\
&\leqslant c_k\sum_{i=1}^{N}\Big(\int_{\{k<|u|<b\}\cap E_n}|M_k^i|^{p_i'}\,dxdt\Big)^{1/p_i'}
\end{aligned}
$$

so we have

$$
\limsup_n |S_n^{11}| \leqslant c_k\sum_{i=1}^{N}\Big(\int_{\{k<|u|<b\}\cap\{|u-u^k|\leqslant\varepsilon\}}|M_k^i|^{p_i'}\,dxdt\Big)^{1/p_i'} = \theta_k^{11}(\varepsilon). \tag{23}
$$

Since one has

$$
|\{k<|u|<b\}\cap\{|u-u^k|\leqslant\varepsilon\}|\to 0 \quad \text{when} \quad \varepsilon\to 0,
$$

then

$$
\lim_{\varepsilon\to 0}\theta_k^{11}(\varepsilon)=0.
$$

For the term S_n^2, we have

$$
S_n^2 = c_k\sum_{i=1}^{N}\Big[\int_{\{|u_n-u^k|=\varepsilon\}}\Phi_k'(u)^{p_i}|D_iu^k|^{p_i}dxdt\Big]^{1/p_i}.
$$

Since u_n converge to u a.e. in Q, we get

$$
\lim_n S_n^2 = \theta_k^{12}(\varepsilon) \quad \text{and} \quad \lim_{\varepsilon\to 0}\theta_k^{12}(\varepsilon)=0. \tag{24}
$$

We combine the inequalities (20) to (24), we obtain

$$
\limsup_n \liminf_m J_{3n}^m \leqslant \theta_k^{11}(\varepsilon)+\theta_k^{12}(\varepsilon), \tag{25}
$$

where $\theta_k^{11}(\varepsilon)$ and $\theta_k^{12}(\varepsilon)$ are given by relation (23) and (24) respectively.
End of the proof of Lemma 2.16. By using (15) and (16), (17), (18), and (19), we find

$$
\begin{aligned}
&\int_{\{|u_n-u_m^k|\leqslant\varepsilon\}}\widehat{a}(t,x,u_n,Du_n)(Du_n-Du_m^k)\,dxdt\\
&\qquad\qquad \leqslant \varepsilon\|\mu\|_{\mathscr{M}(Q)}-I_1-I_2\leqslant c_k\cdot\varepsilon+J_{3n}^m.
\end{aligned} \tag{26}
$$

Let us begin the computation of the limit with respect to m on the left hand side of this last inequality. We recall that on the set $|u_n-u_m^k|\leqslant\varepsilon\leqslant 1$, we have $u_n=u_n^{b+1}$ and then

$$
\begin{aligned}
&\int_{\{|u_n-u_m^k|\leqslant\varepsilon\}}\widehat{a}(t,x,u_n,Du_n)(Du_n-Du_m^k)\,dxdt\\
&\qquad = \int_Q \widehat{a}(t,x,u_n^{b+1},Du_n^{b+1})D(T_\varepsilon(u_n-u_m^k))\,dxdt.
\end{aligned}
$$

The sequence $(T_\varepsilon(u_n-u_m^k))_m$ is strongly convergent in $L^{p_i}(Q)$ for all $i=1,...,N$ to $T_\varepsilon(u_n-u^k)$ (we use Lebesgue's dominated convergence theorem). The function $T_\varepsilon(u_n-u^k)$ is in X. To see that, we consider the sequences $(D_i(T_\varepsilon(u_n-u_m^k))_m$, for $i=1,...,N$, where

$$D_i(T_\varepsilon(u_n-u_m^k)) = \begin{cases} D_iu_n^{b+1}-D_iu_m^k, & |u_n-u_m^k|\leqslant\varepsilon\leqslant 1,\\ 0, & |u_n-u_m^k|>\varepsilon,\end{cases}$$

which remains bounded with the Lemma 2.13, in the reflexive set $L^{p_i}(Q)$ so that it exists a weakly convergent subsequence in $L^{p_i}(Q)$ whose limit is necessarily $D_i(T_\varepsilon(u_n-u^k))$, $i=1,...,N$ so that

$$\begin{aligned}\lim_{m\to+\infty}&\int_{\{|u_n-u_m^k|\leqslant\varepsilon\}}\widehat{a}(t,x,u_n,Du_n)(Du_n-Du_m^k)\,dxdt\\ &=\lim_{m\to+\infty}\int_Q\widehat{a}(t,x,u_n^{b+1},Du_n^{b+1})D(T_\varepsilon(u_n-u_m^k))\,dxdt\\ &=\int_{\{|u_n-u^k|\leqslant\varepsilon\}}\widehat{a}(t,x,u_n,Du_n)(Du_n-Du^k)dxdt.\end{aligned}$$

With the inequalities (25) and (26), we obtain

$$\limsup_n\liminf_m\int_{\{|u_n-u_m^k|\leqslant\varepsilon\}}\widehat{a}(t,x,u_n,Du_n)(Du_n-Du_m^k)\,dxdt\leqslant\theta_k(\varepsilon),$$

where $\theta_k(\varepsilon)=c_k\varepsilon+\theta_k^{11}(\varepsilon)+\theta_k^{12}(\varepsilon)$ and $\lim_{\varepsilon\to0}\theta_k(\varepsilon)=0$. So that we have the proof of Lemma 2.16.

2.4. An important Lemma of compactness

LEMMA 2.18. *Let (u_n) a sequence of X with following properties.*

(i) There exists $\overrightarrow{q}=(q_1,q_2,...,q_N)$, $q_i\in[1,\frac{p_i}{\overline{p}}(\overline{p}-\frac{N}{N+1}))$, such that (u_n) remains in a bounded set of $\mathbb{L}^{\overrightarrow{q}}(0,T;X_0^{1,\overrightarrow{q}}(\Omega))$, (u_n) converges weakly and pointwise to u.

(ii) For all $k>k_0\geqslant0$, $(u_n^k)=(\Phi_k(u_n))$ remains in a bounded set of X as n goes to infinity.

(iii) For all $k>k_0\geqslant0$, there exists a function θ_k so that for all $\varepsilon\in]0,\varepsilon_0)$, $\varepsilon_0>0$, we have

$$\limsup_n\int_{\{|u_n-u^k|\leqslant\varepsilon\}}\widehat{a}(t,x,u_n,Du_n)(Du_n-Du^k)\,dxdt\leqslant\theta_k(\varepsilon),$$

with $\lim_{\varepsilon\to0}\theta_k(\varepsilon)=0$.
Then, there exists a subsequence (still denoted (u_n)) so that

$$Du_n\to Du\quad a.e.\ on\ Q,$$

and for all sequence, we have:

$$D_iu_n\to D_iu\quad \textit{strongly in } L^s(Q),\ \forall s\in[1,q_i),\ \forall q_i\in\Big[1,\frac{p_i}{\overline{p}}\Big(\overline{p}-\frac{N}{N+1}\Big)\Big).$$

Proof. The proof of this Lemma can be make like the proof of Lemma 3 in [16].
Remark. The compactness results on anisotropic problem can be found in [8].

2.5. Passage to the limit for the approximate problems

Let $q_i \in [1, \frac{p_i}{\overline{p}}(\overline{p} - \frac{N}{N+1}))$, $i = 1, \dots, N$. On one hand, with Lemma 2.4, the sequence (u_n) is in a bounded set of $\mathbb{L}^{\overrightarrow{q}}(0,T;X_0^{1,\overrightarrow{q}}(\Omega))$ and we have $u_n \to u$ a.e. in Q and weakly to u in $\mathbb{L}^{\overrightarrow{q}}(0,T;X_0^{1,\overrightarrow{q}}(\Omega))$. On the other hand, with Lemma 2.13, u_n^k remains in a bounded of X and with Lemma 2.16, we have for all $k > 0$,

$$\limsup_n \int_{\{|u_n - u^k| \leqslant \varepsilon\}} \widehat{a}(t,x,u_n,Du_n)(Du_n - Du^k)\,dxdt \leqslant \theta_k(\varepsilon),\ \forall \varepsilon > 0.$$

This shows that we have all the hypotheses of Lemma 2.18 for (u_n) and u. We can have a subsequence (u_n) such that

$$u_n \to u \quad \text{and} \quad Du_n \to Du \quad \text{a.e. in } Q.$$

Now, let $\varphi \in \mathscr{D}(\mathbb{R}^{N+1}, K)$. We have

$$-\int_Q u_n \partial_t \varphi\,dxdt - \int_\Omega \varphi(0,x)\mu_{0n}\,dx + \int_Q \widehat{a}(t,x,u_n,Du_n)D\varphi\,dxdt = \int_Q \varphi(t,x)\mu_n(t,x)\,dxdt. \tag{27}$$

As $Du_n \to Du$ a.e. in Q, $u_n \to u$ a.e. in Q and by the assumption $(\widehat{a}.2)$, we have

$$a_i(t,x,u_n,Du_n) \to a_i(t,x,u,Du) \quad \text{a.e. in } Q,\ i = 1, \dots, N. \tag{28}$$

By the assumption $(\widehat{a}.1)$, from (28), Lemma 2.4 and the Vitali's theorem, we derive for all $i = 1, \dots, N$,

$$a_i(t,x,u_n,Du_n) \to a_i(t,x,u,Du),\ \forall r_i \in \left[1, \frac{p_i}{(p_i - 1)\overline{p}}\left(\overline{p} - \frac{N}{N+1}\right)\right),$$

strongly in $L^{r_i}(Q)$. We can easily pass to the limit in (27). The Theorem 2.1 is so proved.

3. The case $1 + \frac{N}{N+1} < p_i < \frac{\overline{p}(N+1)}{N}$ and $F \neq 0$

In this section, we add the nonlinear term F and we consider the following problem

$$(P') \qquad \begin{cases} \partial_t u - \operatorname{div}(\widehat{a}(x,t,u,Du)) + F(t,x,u,Du) = \mu \ \text{ in } \ Q, \\ u(0,x) = \mu_0(x) \ \text{ in } \ \Omega, \\ u = 0 \ \text{ on } \ (0,T) \times \partial\Omega, \end{cases}$$

where μ is in $\mathscr{M}(Q)$ and $\mu_0 \in \mathscr{M}(\Omega)$.

DEFINITION 3.1. A function u is a weak solution of problem (P') if:

$$u \in L^1(0,T;W_0^{1,1}(\Omega)),\ \widehat{a}(t,x,u,Du) \in (L^1(Q))^N \text{ and}$$
$$-\int_Q u\partial_t\varphi\,dxdt - \int_\Omega \varphi(0,x)\,d\mu_0 + \int_Q \widehat{a}(t,x,u,Du)D\varphi\,dxdt$$
$$+\int_Q F(t,x,u,Du)\varphi\,dxdt = \int_Q \varphi(t,x)\,d\mu, \quad \forall\varphi \in \mathscr{D}(\mathbb{R}^{N+1},K).$$

The proof is similar to the preceding case so we sketch it.

3.1. Approximation of problem (P')

Let (μ_{0n}) (resp. (μ_n)) be a sequence of $\mathscr{D}(\Omega)$ (resp. $\mathscr{D}(Q)$) which converges to μ_0 (resp. μ) in $\mathscr{D}'(\Omega)$ (resp. $\mathscr{D}'(Q)$) and which verifies the inequality

$$\|\mu_{0n}\|_{L^1(\Omega)} \leqslant \|\mu_0\|_{\mathscr{M}(\Omega)} \quad \text{and} \quad \|\mu_n\|_{L^1(Q)} \leqslant \|\mu\|_{\mathscr{M}(Q)}, \quad \forall n \geqslant 1.$$

We approach the problem (P') by the sequence of problems (P'_n):

$$\begin{cases} \partial_t u_n - \operatorname{div}(\widehat{a}(x,t,u_n,Du_n)) + F(t,x,u_n,Du_n) = \mu_n \ \text{ in } \ Q, \\ u_n(0,x) = \mu_{0n}(x) \ \text{ in } \ \Omega, \\ u_n = 0 \ \text{ on } \ (0,T)\times\partial\Omega. \end{cases}$$

For the existence of the solution $u_n \in X \cap C\big([0,T];L^2(\Omega)\big)$ of problem (P'_n) is classical, see [14] for instance.

We can establish all the estimations that we have done for the sequence of solutions of problems (P_n), using the following remark.

REMARK 3.1. Let ϕ is a non-decreasing function from $\mathbb{R}$ into $\mathbb{R}$, and $\phi(0) = 0$. Then we have $\phi(u)F(t,x,u,\xi) \geqslant 0$.

A consequence of this remark, is that the following Lemmas can be proved exactly as before

LEMMA 3.1. *Let p_i and q_i be such that:*

$$1 + \frac{N}{N+1} < p_i < \frac{\overline{p}(N+1)}{N}, \ \textit{where} \ \overline{p} \leqslant N + \frac{N}{N+1},$$
$$q_i \in \Big[1, \frac{p_i}{\overline{p}}(\overline{p} - \frac{N}{N+1})\Big),\ i = 1,\ldots,N.$$

Then the sequence (u_n) remains in a bounded set of $L^{\overline{q}}(Q)$ and in a bounded set of $\mathbb{L}^{\overrightarrow{q}}(0,T;X_0^{1,\overrightarrow{q}}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$, with $\overrightarrow{q} = (q_1,q_2,\ldots,q_N)$.

Thanks to Lemma 3.1, we have that $F_i(u_n,Du_n)$ remains in a bounded set of $L^{r_i}(Q)$, for some $\overrightarrow{r}>1$ (given as in Lemma 2.11). We may also assume that u_n converges weakly to some function u in $\mathbb{L}^{\overrightarrow{q}}(0,T;X_0^{1,\overrightarrow{q}}(\Omega))$, for all $q_i\in[1,\frac{p_i}{\overline{p}}(\overline{p}-\frac{N}{N+1}))$. The sequence u_n' remains in a bounded set of $L^{r_-}(0,T;W^{-1,r_-}(\Omega))+L^1(Q)$. Using a result given in [19] (see also [18]), we can see u_n converges strongly to u in $L^1(Q)$, so that we have the existence of a sequence (u_n) converges to u almost everywhere.

Now we consider the following family of functions $(\Phi_k)_{k>0}$:

- Φ_k is a twice differentiable function, Φ_k', Φ_k'' are bounded on $\mathbb{R}$,
- $\Phi_k(\sigma)=\sigma$ if $|\sigma|\leqslant k$, and $\Phi_k'(\sigma)=0$ if $|\sigma|\geqslant k+(1/k)$, $0<\Phi_k'<1$ on the set $(k,k+(1/k))\cup(-(k+(1/k)),-k)$.

Let $u_n^k=\Phi_k(u_n)$ and $u^k=\Phi_k(u)$. So, with Lemma 2.13, we have the following result.

LEMMA 3.2. *(u_n^k) remains in a bounded set of X, and if (u_n^k) converges to u^k, we have $u^k\in X$. Furthermore, $(u_n^k)'$ is in $\sum_{i=1}^N L^{p_i'}(0,T;(W_{x_i,0}^{1,p_i}(\Omega))')+L^1(Q)$, for all $k>0$, and we have the following equality in $\mathscr{D}'(Q)$:*

$$
\begin{aligned}
(u_n^k)' = \operatorname{div}((\Phi_k'(u_n)\widehat{a}(t,x,u_n,Du_n))) - \widehat{a}(t,x,u_n,Du_n)Du_n\Phi_k''(u_n)\\
-F(t,x,u_n,Du_n)\Phi_k'(u_n)+\mu_n\Phi_k'(u_n).
\end{aligned}
$$

Using Lemmas 2.4 to 2.15, the following result can be proved exactly as before.

LEMMA 3.3. *For all $k>0$, there exists a function θ_k such that for all $\varepsilon>0$, we have*

$$
\limsup_n \int_{\{|u_n-u^k|\leqslant\varepsilon\}} \widehat{a}(t,x,u_n,Du_n)(Du_n-Du^k)\,dxdt \leqslant \theta_k(\varepsilon),
$$

with $\lim\limits_{\varepsilon\to 0}\theta_k(\varepsilon)=0$.

Using the compactness result we deduce that u_n converges strongly to u in the space $\mathbb{L}^{\overrightarrow{q}}(0,T;X_0^{1,\overrightarrow{q}}(\Omega))$, for all $q_i\in[1,\frac{p_i}{\overline{p}}(\overline{p}-\frac{N}{N+1}))$, so $\widehat{a}(t,x,u_n,Du_n)$ converges to $\widehat{a}(t,x,u,Du)$ strongly in $L^1(Q)^N$.

With the growth condition on F_i, $F_i(u_n,Du_n)$ remains in a bounded set of $L^{r_i}(Q)$, for some $\overrightarrow{r}>0$, and it converges a.e to $F_i(u,Du)$, we derive from Vitali's theorem that $F_i(u_n,Du_n)$ converges strongly to $F_i(u,Du)$ in $L^1(Q)$ for $i=1,\dots,N$.

Since u_n satisfies the condition for any $\varphi\in\mathscr{D}(\mathbb{R}^{N+1},K)$, we have

$$
\begin{aligned}
-\int_Q u_n\partial_t\varphi\,dxdt - \int_\Omega \varphi(0,x)\mu_{0n}\,dx + \int_Q \widehat{a}(t,x,u_n,Du_n)D\varphi\,dxdt\\
+\int_Q F(t,x,u_n,Du_n)\varphi\,dxdt = \int_Q \varphi(t,x)\mu_n\,dxdt,
\end{aligned}
$$

we can easily pass to the limit in this relation. That shows the following theorem.

THEOREM 3.1. *Let p_i be such that*

$$1+\frac{N}{N+1} < p_i < \frac{\overline{p}(N+1)}{N}, \ \text{where} \ \overline{p} \leqslant N+\frac{N}{N+1}.$$

Let $\widehat{a}$ be an operator satisfying $(\widehat{a}.1)$-$(\widehat{a}.3)$ and let F satisfy (F). Then the problem (P') has at least one weak solution $u \in \mathbb{L}^{\overrightarrow{q}}(0,T;X_0^{1,\overrightarrow{q}}(\Omega))$ for all $q_i \in [1, \frac{p_i}{\overline{p}}(\overline{p}-\frac{N}{N+1}))$, $i=1,\ldots,N$, with $\overrightarrow{q}=(q_1,q_2,...,q_N)$.

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REFERENCES

[1] Y. ATIK, *T-ensembles locaux et problèmes quasi-linéaires dégénérés à donnée mesure sur un ouvert quelconque*, Thèse de Doctorat, Université de Poitiers, 1993.

[2] M. BENDAHMANE, M. LANGLAIS, M. SAAD, *On some anisotropic reaction-diffusion systems with L^1-data modeling the propagation of an epidemic disease*, Nonlinear Analysis, **54** (2003), 617–636.

[3] M. BENDAHMANE, KENNETH H. KARLSEN, *Renormalized entropy solutions for quasi-linear anisotropic degenerate parabolic equations*, SIAM J. Math. Anal., **36**, 2 (2004), 405–422.

[4] H. BRÉZIS, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publishing company- Amsterdam-London, 1973.

[5] L. BOCCARDO, T. GALLOUËT, P. MARCELLINI, *Anisotropic Equations in L^1*, Differential and Integral Equations, **9** (1996), 209–212.

[6] L. BOCCARDO, A. DALL'AGLIO, T. GALLOUËT, AND L. ORSINA, *Nonlinear Parabolic Equations With Measure Data*, Journal of Functional Analysis, **147** (1997), 237–258.

[7] H. BRÉZIS, *Analyse Fonctionnelle, théorie et applications*, Masson, Paris, 1987.

[8] A. EL HAMIDI, J. M. RAKOTOSON, *On a perturbed anisotropic equation with a critical exponent*, Ricerche di Matematica, **55** (2006), 55–69.

[9] F. LI, *Anisotropic Elliptic Equations in L^{m*}*, Journal of Convex Analysis, **8** (2001), 417–422.

[10] F. LI AND H. ZHAO, *Anisotropic parabolic equations with measure data*, J. Partial Differ. Equations, **14**, 1 (2001), 21–30.

[11] F. LI, *Nonlinear degenerate parabolic equations with measure data*, Nonlinear Analisis, **61** (2005), 1269–1282.

[12] F. LI, *Anisotropic parabolic equations with measure data II*, Math. Nachr., **279**, 13-14 (2006), 1585–1596.

[13] I. FRAGALA, F. GAZZOLA, B. KAWOHL, *Existence and nonexistence results for anisotropic quasilinear elliptic equations*, Ann. I. H. Poincaré, **21** (2004), 715–734.

[14] J. L. LIONS, *Quelques méthodes de résolution des problèmes au limites*, Dunod, Paris, 1969.

[15] F. MOKHTARI, *Etude d'un problème non linéaire du type elliptique-parabolique à données mesures: existence, régularité et comportement de solutions au voisinage des points singuliers*, Mémoire de Magister, Ecole Normale Supérieure, Vieux Kouba, Alger, Algérie, 2002.

[16] J. M. RAKOTOSON, *A compactness lemma for quasilinear problems: application to parabolic equations*, J. Funct. Analysis, **106** (1992), 358–374.

[17] J. M. RAKOTOSON, *T-sets and relaxed solutions for parabolic equations*, J. Differ. Equations, **111**, 2 (1994), 458–471.

[18] J. SIMON, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura. App., **146** (1987), 65–96.

[19] R. TEMAM, *Navier Stokes equations, theory and numerical analysis*, North-Holland, Amsterdam, 1984.
[20] R. TEMAM, *Infinite Dimensional Dynamical Sysytems in Mechanics and Physics*, Springer-Verlag, 1988.
[21] M. TROISI, *Theoremi di inclusione per Spazi di Sobolev non isotropi*, Ricerche Mat., **18** (1969), 3–24.

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